Contests with Ties

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Abstract

We study two-player all-pay contests in which there is a positive probability of a tied outcome. We show that the players' efforts in equilibrium do not depend on the expected prize in the case of a tie given that this prize is smaller than the prize for winning. The implications of this result are twofold. First, in symmetric one-stage contests, the designer who wishes to maximize the expected total effort should not award a prize in the case of a tie which is larger than one-third of the prize for winning. Second, in multi-stage contests, the designer should not limit the number of stages (tie-breaks) but should allow the contest to continue until a winner is decided.

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1 Introduction

In winner-take-all contests, all contestants including those who do not win the prize, incur costs as a result of their efforts, but only the winner receives the prize. Winner-take-all contests have been applied to rent-seeking (Hillman and Samet (1987) and Hillman and Riley (1989)); lobbying activities (Becker (1983) and Che and Gale (1998)); R&D races (Dasgupta (1986)); political contests (Snyder (1989)); promotions in labor markets (Rosen (1986)); and military and biological wars of attrition (O'Neil (1986)). Some winner-take-all contests are decided by objective valuations such as tests of skill and ability but others are resolved on the basis of considerably more subjective evaluations. If the performances of the best contestants are the same or almost the same, but not distinguishable, the contests end without either side winning. Such a situation is referred to as a tie. Ties usually occur in sports competitions, but they also occur in many different kinds of contests. For example, in political elections with more than two candidates, if no candidate succeeds to achieve a majority, a runoff is often held between the two candidates receiving the highest numbers of votes. Ties also can occur for such contests as the Nobel prize, but in that case the prize is awarded to several people who share it equally.[1]

In this paper we address some important questions related to ties. For example, what is the optimal prize in the case of a tie in relation to the prize for winning?

[1]For example, in 1994 the Nobel prize in economics was awarded to J. Harsany, J. Nash and R. Selten for their pioneering analysis of equilibria in the theory of noncooperative games.

Should the policy be as in chess where both contestants earn half a point in the case of a tie and the winner earns one point, or, alternatively, as in soccer where three points are awarded for a win and one for a tie? Furthermore, in the case of a tie, should prizes be awarded at all?

We investigate the optimal policy in contests with ties by applying the model of all-pay contests (auctions) in which the player with the highest effort wins the prize.[2] To allow a positive probability of a tie we assume that the sets of possible strategies are finite. The designer of the contest therefore must determine not only the size of the prize in the case that a single player exerts the highest effort, but also the size of the prizes in the case of a tie when several players exert the highest effort. In this case the designer actually determines the probability of winning for each of the tied winners where the sum of these probabilities may be less than one.[3]

In Section 2 we analyze the equilibrium strategies in all-pay contests with complete information where the sets of strategies are finite. Baye, Kovenock and de Vries (1994) showed that in this model some of the equilibrium strategies are similar to those of the standard all-pay contest where the sets of strategies are not finite (see, for example, Hillman and Riley (1989) and Baye, Kovenock and de Vries (1993, 1996)). However, we show that there is a meaningful distinction between both models. For example,

[2] There is a deterministic relation between effort and output, and therefore we can assume them to be equal.

[3] Other works on all-pay auctions in which the designer determines the optimal reward structure include Barut and Kovenock (1998) and Moldovanu and Sela (2001, 2006).

in our asymmetric two-player contest the probability of the stronger player to win the contest is not necessarily higher than the probability of the weaker player to win the contest, and the expected payoff of the weaker player is not necessarily zero. Moreover, in the case of a contest with more than two players there is always an equilibrium in which only two players are active (they exert some effort) and the rest of the players are passive (they do not exert any effort), and we show, in contrast to the standard model, that being passive may actually be profitable.

In Section 3 we investigate the optimal value of the prize in the case of a tie for the contest designer who wishes to maximize the expected players' total effort, or equivalently, the optimal probability of winning for each player who exerts the highest effort. We show that in symmetric two-player contests if the sum of the winners' probabilities to win the contest in a tie is less than one, independent of the value of these probabilities of winning, there is a unique symmetric equilibrium. Consequently, the size of the prize in the case of a tie does not affect the players' efforts and therefore it is inefficient to award a prize for each player with a probability higher than one-third[4] since if this probability is equal to one-half, the symmetric equilibrium is not unique and there is uncertainty about the expected players' efforts. Moreover, if offering a prize causes disutility for the contest designer, the contest designer should not award a prize at all in the case of a tie.

In Section 4 we study multi-stage all-pay contests with tie-breaks. In the case of

[4] The money unit in our model is equal to 1. When ε is the smallest money unit, our results show that awarding a prize for each player with a probability smaller than $\frac{1}{2+\epsilon}$ is inefficient.

a tie, the players continue to compete until one of the players exerts the highest effort and wins the contest. In the last stage, each player wins with the same probability. We assume that the players' valuations are decreasing in stages. We show that there is a unique symmetric sub-game perfect equilibrium in a symmetric two-player contest where the players' strategies do not depend on their valuations in the next stages, although there is a positive probability at each stage that the contest will proceed to the next stage. Gershkov and Perry (2006) showed that in two-stage contests with midterm reviews, if there is a tie-break in the first stage, players exert higher efforts in the second stage. In contrast, in our model, the number of stages do not affect the players' strategies in each stage and a contest designer seeking to maximize total effort should not limit the number of stages (tie-breaks), but should allow the contest to continue until a winner is decided.

In Section 5 we assume a positive tie distance such that a tie is declared if the difference between the players' efforts is within the tie distance. In contrast to Eden (2006) who showed in a different environment that there are cases in which it is optimal to impose a positive tie distance, the characterization of the equilibrium strategies in our model reveals that increasing the tie distance decreases the players' expected efforts. Furthermore, in the case of a tie, if the tie distance is not small (larger than the money unit) then the players' expected efforts depend on the size of the prize.

Section 6 contains several concluding comments, and the Appendix presents most of the proofs.

2 Contests with winners

Consider n players competing for a single prize in a one-stage contest. The value of winning in the contest for player i is v_i. Valuations are common knowledge. We model the match between the players as in an all-pay auction: each player exerts an effort $x \in \{0, 1, 2, 3, ...\}$,[5] all players pay their costs of effort and the player with the highest effort wins. For simplicity, we postulate a deterministic relation between effort and output, and assume them to be equal. In the case of a tie in which $h \leq n$ players exert the same highest effort, we assume that each of the players wins with probability $1/h$. In a two-player contest, if players exert efforts $x_i, x_j \in \{0, 1, 2, 3, ...\}$ then the payoff for player i is given by

$$u_i(x_i, x_j) = \begin{cases} -x_i & \text{if } x_i < x_j \\ \frac{1}{2}v_i - x_i & \text{if } x_i = x_j \\ v_i - x_i & \text{if } x_i > x_j \end{cases}$$

The main difference between this model and the standard all-pay contest (auction) where the set of possible efforts is not finite (see Hillman and Riley (1989) and Baye, Kovenock and de Vries (1993, 1996)) is that in the standard model the probability of a tie (the players exert the same effort) is zero, while in our model there is a positive

[5] All the results in this paper hold for any size of the money unit $\epsilon > 0$; that is, for every set of efforts $\{\epsilon, 2\epsilon, 3\epsilon, ...\}$.

probability for a tie in the contest.

We consider first the symmetric two-player contests in which the players have the same valuation. In this case there is an equilibrium which is similar to the unique equilibrium in the standard all-pay contest.

Proposition 1 *(Baye, Kovenock and de Vries, 1994) Consider two players with the same valuation v who compete in an all-pay contest for a unique prize. Then, there is a symmetric mixed strategy equilibrium where each player chooses every effort $x \in \{0, 1, ..., v-1\}$ with the same probability $p_x = \frac{1}{v}$. In this case, the expected payoff of each player is* 0.5.

In contrast to the standard all pay contest, all-pay contests where the sets of actions are finite do not necessarily have a unique equilibrium, as we can see in the following example.

Example 2 *Consider two players with the same valuation v who compete in an all-pay contest for a unique prize. If $v = 2k$, where k be any positive integer, then there is a symmetric mixed strategy equilibrium where each player chooses every effort $x \in \{0, 2, 4, ..., v-2\}$ with the same probability $p_x = \frac{2R}{v}$, and he chooses every effort $x \in \{1, 3, ..., v-1\}$ with the same probability $p_x = \frac{2(1-R)}{v}$. In this case, the expected payoff of every player is $R, 0 \leq R \leq 1$.*

Note that in contrast to the symmetric standard model of all-pay contests, in our symmetric model the expected payoff of both players is not zero. These results hold independently of the size of the money unit (which in our model is equal to 1). However, when the money unit approaches zero our results are consistent with the standard all-pay contest.

The generalization of our model of all-pay contests for the case of $n > 2$ players is simple since there is always an equilibrium in which only two players are active (they exert some efforts) and the rest of the players are passive (they exert no efforts). However, in contrast to the standard model, in our model being passive may actually be profitable.

Proposition 3 *Consider $n > 2$ symmetric players with the same valuation v who compete in an all-pay contest for a unique prize.*

1. If $v = 2k + 1$ and k be any positive integer, there is an equilibrium in which $n - 2$ players do not choose any effort. The other two players use the symmetric mixed strategy equilibrium where the probability of every effort $x \in \{2, 4, ..., v - 1\}$ is $p_x = \frac{4}{nv}$, the probability of every effort $x \in \{1, 3, ..., v - 2\}$ is $p_x = \frac{2n-4}{nv}$, and the probability of $x = 0$ is $p_0 = 1 - \left[\frac{v-1}{2}\frac{4}{nv} + \frac{v-1}{2}\frac{2n-4}{nv}\right]$. In this case, the expected payoff of each of the active players is $\frac{1}{n}$ and the expected payoff of each of the passive players is $\frac{1}{vn}$.

2. If $v = 2k$ and k be any positive integer, there is an equilibrium in which $n - 2$ players do not choose any effort. The other two players use the symmetric mixed strategy equilibrium where the probability of every effort $x \in \{1, 3, ..., v - 1\}$ is $p_x = \frac{2}{v}$. In this case, the expected payoff of each player is zero.

According to Proposition 3, when players have positive payoffs, the expected payoffs of the two active players do not depend on the players' valuation v, while the expected payoffs of the $n-2$ passive players may decrease in their valuations. The seller's payoff, however, increases both in v and in n.

Ties do not necessarily occur in all-pay contests with finite sets of actions. In contrast to contests with symmetric players, if the players have different valuations, then there are contests in which ties are not possible.

Proposition 4 *Consider two asymmetric players, $v_1 > v_2$, who compete in an all-pay contest for a unique prize.*

1. Let $v_2 = 2k$ and k be any positive integer. Then, there is a mixed strategy equilibrium where player 1 chooses every effort $x_1 \in \{1, 3, 5, ..., v_2 - 1\}$ with the same probability $p_{x_1} = \frac{2}{v_2}$, and player 2 chooses $x_2 = 0$ with probability $q_0 = \frac{v_1 - v_2 + 2}{v_1}$ and every effort $x_2 \in \{2, 4, ..., v_2 - 2\}$ with the same probability $q_{x_2} = \frac{2}{v_1}$. The expected payoff of player 1 is $v_1 - v_2 + 1$ and the expected payoff of player 2 is 0.

2. Let $v_2 = 2k + 1$ and k be any positive integer. Then, there is a mixed strategy equilibrium where player 1 chooses the effort $x_1 = v_2$ with probability $p_{v_2} = \frac{1}{v_2}$ and every effort $x_1 \in \{1, 3, 5, ..., v_2 - 2\}$ with the same probability $p_{x_1} = \frac{2}{v_2}$. Player 2 chooses the effort $x_2 = 0$ with probability $q_0 = \frac{v_1 - v_2 + 1}{v_1}$ and every effort $x_2 \in \{2, 4, ..., v_2 - 1\}$ with the same probability $q_{x_2} = \frac{2}{v_1}$. The expected payoff of player 1 is $v_1 - v_2$ and the expected payoff of player 2 is 0.

The following example shows that ties are possible even in asymmetric contests, and that in a two-player contest the probability of the strong player to win the contest is not necessarily higher than the probability of the weak player to win. Moreover, the expected payoff of the weak player is not necessarily zero.

Example 5 *Consider an all-pay contest with two asymmetric players where* $v_1 = 4$ *and* $v_2 = 3$*. Then there is a mixed strategy equilibrium where player* 2 *chooses every effort* $x_2 \in \{0, 2\}$*with the same probability* $\frac{1}{2}$*, and player* 1 *chooses every effort* $x_1 \in \{0, 1, 2\}$*with the same probability* $\frac{1}{3}$*. The probability of winning is the same for both players although the expected payoff of player* 1 *is equal to* 1 *and is larger than the expected payoff of player* 2 *which is equal to* 0.5*.*

3 Contests with or without winners

Consider two players competing for a single prize in a one-stage all-pay contest. Each player exerts an effort $x \in \{0, 1, 2, 3, ...\}$, both players pay their costs of effort and the player with the highest effort wins. In the case of a tie in which both players exert the same highest effort, we assume that the sum of the players' probabilities of winning is less than 1. That is, if the probability of winning of each player is $\frac{1}{m}, m \in \{3, ...\}$ and players exert efforts $x_i, x_j \in \{0, 1, 2, 3, ...\}$, the payoff for player i is

$$u_i(x_i, x_j) = \begin{cases} -x_i & \text{if } x_i < x_j \\ \frac{1}{m}v_i - x_i & \text{if } x_i = x_j \\ v_i - x_i & \text{if } x_i > x_j \end{cases}$$

In a two-player contest in which there is a positive probability that no player wins the contest, we have only one symmetric equilibrium that is independent of the players' probabilities of winning.

Theorem 6 *Consider two players with the same valuation v who compete in an all-pay contest for a unique prize. If the probability of winning of each player in the case of a tie is smaller than one-half, then, independent of the players' probabilities of winning, there is a unique symmetric equilibrium in which each player chooses every effort $x \in \{ 0, 1, ..., v-1\}$ with the same probability $p_x = \frac{1}{v}$.*

Proof. Denote by $p_x \geq 0$ the probability that each player chooses an effort of $x \geq 0$. Then, the expected payoff of a player that chooses an effort of x is:

$$v(p_0 + p_1 + ... + p_{x-1} + \frac{p_x}{m}) - x \quad , \; p_i \geq 0 \text{ for all } i \leq x$$

Lemma 7 *In every symmetric equilibrium $p_x \neq 0$, $0 \leq x \leq v-1$.*

Proof. See the Appendix.

The system of linear equations that describes the symmetric equilibrium strategies can be written in the following matrix form:

$$A * y = b \tag{1}$$

where

$$A = \begin{bmatrix} \frac{v}{m} & 0 & 0 & 0 & 0 & . & . & -1 \\ v & \frac{v}{m} & 0 & 0 & 0 & . & . & -1 \\ v & v & \frac{v}{m} & 0 & 0 & . & . & -1 \\ v & v & v & \frac{v}{m} & 0 & . & . & -1 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 1 & 1 & 1 & 1 & 1 & . & . & 0 \end{bmatrix}_{vxv} \quad y = \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ . \\ . \\ p_{v-1} \\ k \end{bmatrix}_{vx1} \quad b = \begin{bmatrix} 0 \\ 1 \\ 2 \\ . \\ . \\ v_{-1} \\ 1 \end{bmatrix}_{vx1}$$

Formally, matrix A is given by:

$a_{x,x} = \frac{v}{m}$ for all $x < v$ and $a_{x,x} = 0$ for $x = v$.

$a_{x,y} = v$ for all $v > x > y$ and $a_{x,y} = 0$ for all $x < y < v$.

$a_{v,y} = 1$ for all $y < v$, $a_{x,v} = -1$ for all $x < v$ and $a_{v,v} = 0$.

The vector y is given by:

$y_x = p_x$ for all $x < v$ and $y_v = k$ where k is each player's expected payoff.

The vector b is given by:

$y_x = x - 1$ for all $x < v$ and $y_v = 1$.

It can be verified that independent of the value of m, the system of linear equations (1) has the solution $p_x = \frac{1}{v}$ for all $x \in \{0, 1, 2, ..., v-1\}$ and $k = \frac{1}{m}$. The following Lemma completes the proof of Theorem 6.

Lemma 8 *The system of linear equations (1) has a unique symmetric solution.*

Proof. See the Appendix. Q.E.D

By Theorem 6, if $m > 3$, the probability of each player to win the contest in the case of a tie or, alternatively, the expected prize in the case of a tie, does not affect his strategy. Thus, in symmetric contests, the contest designer who wishes to maximize the players' expected total effort does not have an incentive for awarding the prize with a probability higher than one-third for each of the two players in the case of a tie. This is because if total prizes in cases of tie and win are the same, then there will be uncertainty about the outcome of the contest and, in particular, there will be uncertainty about the players' expected efforts. Moreover, if we assume that the prize causes a disutility for the contest designer, we can conclude that in a contest with two symmetric players, the contest designer should not award any prize.

Dechenaux, Kovenock and Lugovskyy (2003) showed that if there is no prize in the case of a tie ($m = \infty$), two asymmetric players will never choose the same action. Below we show that this result holds even in the case where prizes are awarded in the case of a tie.

Proposition 9 *Consider an all-pay contest with two asymmetric players where $v_1 > v_2$, and where the probability of each player to win the contest in the case of a tie is smaller than one-half. Then, there is no equilibrium with a tie. Moreover, the expected payoff of player 2 is zero and the expected payoff of player 1 is either $v_1 - v_2$ or $v_1 - v_2 + 1$.*

Proof. See the Appendix.

Note that by Example 5 we show that in contests with winners (the sum of the players' probabilities of winning is 1) the expected payoffs of both players might be positive even if they are asymmetric. Proposition 9 shows that when the sum of the players' probabilities of winning is less than 1, only the stronger player has a positive expected payoff similar to the standard asymmetric model where the sets of efforts are not finite.

4 Multi-stage contests

Consider two players competing for a single prize in a T-stage all-pay contest, $T \geq 1$. Both players pay their costs of effort, the player with the highest effort wins and the contest is over. However, in the case of a tie where players exert the same effort in stage $t < T$, they compete again in the next stage until one of the players wins the contest (one player exerts a higher effort than his opponent). In the case of a tie in stage T, we assume that each of the players wins with probability $p \leq 1/2$. The value of winning in the contest for player i at stage $t \geq 0$ is $v_{i,t} = v_i - kt$, where k is any positive integer and $kT < v_i$ for all i. Valuations in every stage are common knowledge. If the players exert efforts of $x_{i,t}, x_{j,t}$ in stage t then the payoff for player i in this stage is given by

$$u_i(x_{i,t}, x_{j,t}) = \begin{cases} v_{i,t} - x_{i,t} \text{ if } x_{i,t} > x_{j,t} \\ E_{i,t} - x_{i,t} \text{ if } x_i = x_j \\ -x_{i,t} \text{ if } x_{i,t} < x_{j,t} \end{cases}$$

where $E_{i,t}$ is the expected payoff for player i after stage t and $E_{i,T} = \frac{1}{m}v_{i,T}, m \geq 2$.

Proposition 10 *Consider two players with the same valuation who compete in a T-stage contest for a unique prize. Then, in every symmetric sub-game perfect equilibrium, for every $t < T$, each player chooses every effort $x_t \in \{0, 1, ..., v_t - 1\}$ with the same probability $(v_t)^{-1}$. In stage T, if the probability of winning of each player in the case of a tie is smaller than one half, then, independent of the players' probabilities of winning, each player has a unique symmetric equilibrium strategy in which he chooses an effort $x_T \in \{0, 1, ..., v_T - 1\}$ with the same probability $(v_T)^{-1}$.*

Proof. By the assumption of symmetric equilibrium strategies and $1 \leq k$, we obtain that $E_{i,t} < \frac{1}{2}v_t$ for every $t \geq 0$. Since the expected payoff of each player in the case of a tie is smaller than half of his valuation in every stage, the result is obtained by Theorem 6. Q.E.D.

The conclusion from Proposition 10 is that the contest designer who maximizes the players' total effort should not limit the number of stages (tie-breaks) since the length of the contest does not have any effect on the players' strategies at each stage $t \leq T$ of the contest.

The application of Proposition 9 implies that in the case of asymmetric T-stage contests with tie-breaks, the prize in the case of a tie is not necessarily relevant since players exert different levels of efforts such that the contest will already be decided in the first stage.

Proposition 11 *Consider two asymmetric players, $v_i \neq v_j$, that compete in a T-stage all-pay contest for a unique prize. Then, there is a sub-game perfect equilibrium* *where the contest is decided in the first stage.*

5 Contests with a tie distance

Consider two players competing for a single prize in a one-stage all-pay contest. Each player exerts an effort $x \in \{0, 1, 2, 3, ...\}$ and both players pay their costs of effort. A player with the highest effort wins if the distance between his effort and his opponent's effort is larger than $d > 0$. The case in which the distance between the efforts of the players is smaller or equal to d is defined as a tie. In the case of a tie, the sum of the players' probabilities of winning is less than or equal to 1. Assume that the probability of winning of each player in the case of a tie is $\frac{1}{m}, m \in \{2, ...\}$ and $d = 1$. If players exert efforts $x_i, x_j \in \{0, 1, 2, 3, ...\}$, then the payoff for player i is

$$u_i(x_i, x_j) = \begin{cases} -x_i & \text{if } x_i < x_j - 1 \\ \frac{1}{m}v_i - x_i & \text{if } |x_i - x_j| \leq 1 \\ v_i - x_i & \text{if } x_i > x_j + 1 \end{cases}$$

Proposition 12 *Consider two players with the same valuation v who compete in an all-pay contest with a tie distance of $d = 1$. Assume that the probability of winning of each player in the case of a tie is 0.5. Then,*

1. If $v = 2k$ where $k > 1$ is a positive integer, there is a symmetric mixed strategy equilibrium where each player chooses every effort $x \in \{\ 0, 2, 4, ..., v - 2\}$ with the same probability $p_x = \frac{2}{v}$. In this case, the expected payoff of each player is 1.

2. If $v = 2k + 1$ where $k > 1$ is a positive integer, there is a symmetric mixed strategy equilibrium where the probability of every effort $x \in \{1, 2, 3, ..., v - 4\}$ is $p_x = \frac{1}{v}$, and the probability of every effort $x \in \{0, v - 3\}$ is $p_x = \frac{2}{v}$. In this case, the expected payoff of each player is 1.5.

Although the symmetric equilibrium strategies given in Proposition 12 are not unique, by comparing them with the equilibrium strategies given in Proposition 1 we can see that increasing the tie distance decreases the players' expected efforts.[6] In contrast to Theorem 6, we now show that if the tie distance is positive and larger than the money unit, the players' equilibrium efforts depend on the expected prize.

Proposition 13 *Consider two players with the same valuation v who compete in an all-pay contest for a unique prize, and no prize is awarded in the case of a tie. If the tie distance is $d = 1$, then,*

1. If $v = 2k$ where $k > 1$ is a positive integer, there is a unique symmetric mixed strategy equilibrium where each player chooses every effort $x \in \{\ 0, 2, 4, ..., v-2\}$ with the same probability $p_x = \frac{2}{v}$.

2. If $v = 2k + 1$ where $k > 1$ is a positive integer, there is a unique symmetric mixed strategy equilibrium where the probability of every effort $x \in \{0, 2, ..., v - 3\}$ is $p_x = \frac{2}{v}$ and the probability of $x = v - 1$ is $p_x = \frac{1}{v}$.

Proof. See the appendix.

[6] Eden (2006) showed in a different environment that there are contests for which a positive tie distance may increase the players' efforts.

If no prize is awarded in the case of a tie, the symmetric players' equilibrium strategies given in Proposition 13 are unique and do not hold (if v is odd) for any case in which a prize is awarded in the case of a tie. Thus, the expected prize in the case of a tie does affect the players' expected efforts in equilibrium and it is not clear what the optimal prize is, if the designer is not able to distinguish among different efforts.

A simple application of Proposition 13 yields that if the probability of winning for each one of the players in the case of a tie is small enough, then there is no symmetric equilibrium if the tie distance is positive. Then, although the players are symmetric, we can find only asymmetric equilibrium strategies, as in the following example.

Example 14 *Consider two players with the same valuation $v = 5$ who compete in an all-pay contest with a tie distance of $d = 1$. The probability of winning of each player in the case of a tie is $\frac{1}{3}$. Then there is a mixed strategy equilibrium where player 1 chooses the effort $x_1 = 0$ with probability $\frac{3}{5}$ and every effort $x_1 \in \{1, 4\}$ with probability $\frac{1}{5}$. Player 2 chooses every effort $x_2 \in \{0, 3\}$ with probability $\frac{1}{5}$ and $x_2 = 2$ with probability $\frac{3}{5}$. The expected payoff of player 1 is $\frac{1}{3}$ and the expected payoff of player 2 is $\frac{4}{3}$.*

6 Concluding Remarks

In 1982 the English soccer league changed one of the basic rules in the European soccer leagues by deciding to award three points for a win rather than two points, while continuing to award one point for a tie. This policy was adopted in France in 1995, in Germany and Spain in 1996 and afterwards in the rest of Europe. The results of this paper support the legitimacy of that decision.

We showed that in the special case of one-stage contests, if the prize in the case of a tie is smaller than the prize in the case of a win (i.e., the sum of the players' probabilities of winning is smaller than one), then the size of the prize in the case of a tie does not affect the players' efforts. Thus, if the money unit is equal to one, the prize in the case of a tie should be smaller or equal to one-third of the prize for winning. Given that the prize in soccer leagues (points) does not cause a disutility to the contest designer, this suggests that the decision of the English soccer league to award three points for a win rather than two points may induce an increase of the total effort exerted by the players. Indeed, the introduction of the award of three points for a win in the English league made a dramatic jump in the average number of goals by away teams (see, Dobson and Goddard (2001).

7 Appendix

7.1 Proof of Lemma 7

We prove Lemma 7 by the following four lemmas:

Lemma A: For every $v-1 > x > 0$, there is no symmetric equilibrium $p = (p_0, p_1, , ..., p_{v-1}, p_v)$ where $p_x \neq 0$ and $p_{x+1} = p_{x-1} = 0$.

Proof: Suppose that Lemma A does not hold, that is, $p_x \neq 0$, $v-1 > x > 0$ and $p_{x+1} = p_{x-1} = 0$. Since each player weakly prefers the effort of x to $x-1$ (since x is supposed to be played with positive probability in equilibrium) we have

$$-1 + p_x \frac{v}{m} \geq 0. \tag{2}$$

Likewise, since each player weakly prefers the effort of x to $x+1$ we have

$$1 + p_x \frac{v}{m} \geq p_x v. \tag{3}$$

Note that $m > 2$. Thus, we obtain from (3) that

$$p_x \frac{v}{m} < 1 \tag{4}$$

But the inequalities in (2) and (4) contradict each other.

Lemma B: For every $v-1 > x > 0$ and $0 < j < v-1-x$, there is no symmetric equilibrium $p = (p_0, p_1, , ..., p_{v-1}, p_v)$ where $p_{x+i} \neq 0$, $i = 0, ..., j$, and $p_{x+j+1} = p_{x-1} = 0$.

Proof: Suppose that Lemma B does not hold, that is, $p_{x+i} \neq 0, i = 0, ..., j$ and $p_{x+j+1} = p_{x-1} = 0$ for every $v-1 > x > 0$ and $0 < j < v-1-x$. Since each player weakly prefers the effort of x to $x-1$, we have

$$-1 + p_x \frac{v}{m} \geq 0 \tag{5}$$

Now, since each player is indifferent to the efforts of x and $x+1$ (both are supposed to be played with positive probability in equilibrium) we have

$$1 + p_x \frac{v}{m} = p_x v + p_{x+1} \frac{v}{m} \tag{6}$$

By (6) and $m > 2$, we obtain that $\frac{v}{m}(p_x + p_{x+1}) < 1$ and therefore

$$p_x \frac{v}{m} < 1 \tag{7}$$

But, the inequalities in (5) and (7) contradict each other.

Lemma C: For every $x < v - 1$, there is no symmetric equilibrium $p = (p_0, p_1, , ..., p_{v-1}, p_v)$ where $p_i \neq 0$ for all $0 \leq i \leq x$ and $p_{x+1} = 0$.

Proof: Suppose that Lemma C does not hold, that is, $p_i \neq 0$ for all $0 \leq i \leq x < v - 1$ and $p_{x+1} = 0$. The expected payoff from an effort of 0 is $p_0 \frac{v}{m}$. Since the players weakly prefer effort of 0 and 1 we have

$$p_0 \frac{v}{m} + 1 \geq p_0 v + p_1 \frac{v}{m}. \tag{8}$$

By equation (8) since $m > 2$ we obtain that $\frac{v}{m}(p_0 + p_1) < 1$, and therefore $p_0 \frac{v}{m} < 1$. That is, the expected payoff of each player is smaller than 1. But this is a contradiction, since every player can choose an effort of $v - 1$ which gives him a payoff of 1 for sure.

Lemma D: For all $x > 0$, there is no symmetric equilibrium $p = (p_0, p_1, , ..., p_{v-1}, p_v)$ where $p_{v-i} \neq 0$, $i = 1, ..., v - x$ and $p_{x-1} = 0$.

Proof: Suppose that Lemma D does not hold, that is, $p_{v-i} \neq 0$, for all $i = 1, ..., v - x$, and $p_{x-1} = 0$. Since each player weakly prefers the effort of x to $x - 1$ we have

$$p_x \frac{v}{m} \geq 1. \tag{9}$$

Since the players are indifferent to the efforts of x and $x+1$ we have

$$p_x \frac{v}{m} + 1 = p_x v + p_{x+1} \frac{v}{m} \tag{10}$$

Note that $m > 2$. Thus, equation (10) implies that

$$p_x \frac{v}{m} + p_{x+1} \frac{v}{m} < 1. \tag{11}$$

By (9)+(11) and the fact that $p_{x+1} > 0$, we obtain the contradiction $1 \leq p_x \frac{v}{m} < 1$.

Combining the above four lemmas yields that in any symmetric equilibrium $p = (p_0, p_1, , ..., p_{v-1}, p_v)$, $p_x \neq 0$ for every $0 \leq x \leq v-1$.

7.2 Proof of Lemma 8

In the following we show that the solution of (1) that satisfies the condition of Lemma 7 is unique. If $v = 1$ the equation system (1) has the reduced form

$$\begin{bmatrix} \frac{v}{m} & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} p_0 \\ k \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

In this case ($v = 1$) there is a unique solution since $\det(A_1) = \det \begin{bmatrix} \frac{v}{m} & -1 \\ 1 & 0 \end{bmatrix} = 1$.

If $v = 2$ the equation system (1) has the reduced form

$$\begin{bmatrix} \frac{v}{m} & 0 & -1 \\ v & \frac{v}{m} & -1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ k \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

In this case ($v = 2$) there is a unique solution since $\det(A_2) = \det \begin{bmatrix} \frac{v}{m} & 0 & -1 \\ v & \frac{v}{m} & -1 \\ 1 & 1 & 0 \end{bmatrix} =$ $2\frac{v}{m} - v < 0$.

For every $v \geq 1$ we have

$$\det(A_v) = (-1)^{v+1} \sum_{s=0}^{v-1} b_s$$

where

$$b_s = (-1)^s \left[(\frac{v}{m})^s (v - \frac{v}{m})^{v-1-s} \right]$$

Note that $b_s > 0$ for $s = 0, 2, 4, ...$ and $b_s < 0$ for $s = 1, 3, 5, ...$Since $(v - \frac{v}{m}) > \frac{v}{m}$ we obtain that $|b_s| > |b_{s+1}|$ for all $s \geq 0$. Therefore for every $v \geq 1$, $\det(A_v) \neq 0$.

7.3 Proof of Proposition 9

We wish to show that in an all-pay contest with two asymmetric players where $v_1 > v_2$, there is no equilibrium with a tie. First, we show that the smallest level of effort of both players in equilibrium are not identical and afterwards that any level of effort of both players in equilibrium are also not identical.

Let p_x^i be the probability that player i chooses the effort of x. Suppose that for $i = 1, 2$, $p_x^i \neq 0$, $x \geq 0$, and $p_y^i = 0$ for all $y < x$. Player 1, who exerts the effort of x, has the expected payoff

$$v_1 \frac{p_x^2}{m} \geq 1 \quad (12)$$

Since player 1 weakly prefers x to $x+1$ we have

$$v_1 \frac{p_x^2}{m} + 1 \geq v_1(p_x^2 + \frac{p_{x+1}^2}{m}) \tag{13}$$

Since $p_{x+1}^2 \geq 0$ and $m > 2$, equation (13) implies that

$$v_1 \frac{p_x^2}{m} < 1 \tag{14}$$

But equation (14) contradicts equation (12). Thus, the smallest level of effort of both players are not identical, and therefore the smallest effort of player 2 is necessarily $x = 0$ and his expected payoff is zero. The highest effort of player 1 is necessarily larger or equal to $v_2 - 1$ and his expected payoff is larger $(v_1 - v_2 + 1)$ or equal to the difference in the players' valuations $(v_1 - v_2)$.

We now show that both players do not exert identical efforts in equilibrium, that is, ties are not possible at all. Suppose there is a tie, and the tie with the lowest effort is $x > 0$. Then if $p_{x-1}^2 = 0$, we have the same argument as the one above according to which the smallest level of effort of both players are not identical. Otherwise, if $p_{x-1}^2 \neq 0$ $(p_{x-1}^1 = 0)$ since player 2 weakly prefers x to $x-1$ we have

$$v_2 \frac{p_x^1}{m} \geq 1. \tag{15}$$

Similarly, since player 2 weakly prefers x to $x+1$ we have

$$v_2 \frac{p_x^1}{m} + 1 \geq v_2(p_x^1 + \frac{p_{x+1}^1}{m}). \tag{16}$$

Thus, since $m > 2$, equation (16) implies that

$$v_2 \frac{p_x^1}{m} < 1 \tag{17}$$

Combining inequalities (15) and (17) yields a contradiction.

7.4 Proof of Proposition 13

Consider two players with the same valuation v who compete in an all-pay contest for a unique prize where no prize is awarded in the case of a tie ($m = \infty$). Let the tie distance be $d = 1$. Assume that players have symmetric equilibrium strategies $p = (p_0, p_1, ..., p_{v-1}, p_v)$ where $p_x \geq 0$, $0 \leq x \leq v$.

Lemma E: $p_0 \neq 0$.

Proof: If $p_x = 0$, $x = 0, 12, ...j$ and $p_j \neq 0$ it is better for each player to choose $x = j - 1$ instead of $x = j$ since by doing so a player does not change his probability to win the prize and he decreases his cost of effort.

Lemma F: $p_1 = 0$.

Proof: A player who exerts an effort of $x = 1$ does not have a chance to win the prize.

Lemma G: $p_i > 0$, $i = 2, 3, ...$only if $\sum_{x=0}^{i-2} p_x = \frac{i}{v}$.

Proof: If $\sum_{x=0}^{i-2} p_x < \frac{i}{v}$ the expected payoff of a player will be negative if he chooses $x = i$, and if $\sum_{x=0}^{i-2} p_x > \frac{i}{v}$ the expected payoff of a player will be positive. However, this contradicts Lemma E according to which each player chooses $x = 0$ with positive probability and therefore each player has an expected payoff of zero in equilibrium.

Lemma H: $p_2 \neq 0$ and $p_0 = \frac{2}{v}$.

Proof: if $p_x = 0$, $x = 2, ...j$ and $p_j \neq 0$, it is better for each player to choose $x = j - 1$ instead of $x = j$ and therefore $p_2 \neq 0$. Then, by Lemmas F and G we have

$p_0 = \frac{2}{v}$.

Lemma I: $p_3 = 0$.

Proof: If $p_2 \neq 0$ it is clear that $p_3 = 0$ since the expected payoff from $x = 3$ will be smaller than $x = 2$.

Lemma J: $p_4 > 0$ and $p_2 = \frac{2}{v}$.

Proof: If $p_x = 0$, $x = 4, ...j$ and $p_j \neq 0$, it is better for each player to choose $x = j - 1$ instead of $x = j$ and therefore $p_4 \neq 0$. Then, by Lemmas G, H and I we have $p_2 = \frac{2}{v}$.

Hence, by induction we obtain that if $v = 2k$, $p_x = \frac{2}{v}$ for every effort $x \in \{0, 2, 4, v - 2\}$ and if $v = 2k + 1$, $p_x = \frac{2}{v}$ for every effort $x \in \{0, 2, 4, v - 3\}$ an'd $p_{v-1} = 1 - \sum_{i=0}^{v-3} \frac{2}{v} = \frac{1}{v}$.

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