UNIQUENESS OF PIECEWISE SMOOTH WEAK SOLUTIONS OF MULTIDIMENSIONAL DEGENERATE PARABOLIC EQUATIONS[1]

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Proposed Running head: Piecewise smooth weak solutions

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Abstract

We study the degenerate parabolic equation $u_t + \nabla \cdot f = \nabla \cdot (Q\nabla u) + g$, where $(x, t) \in \mathbb{R}^N \times \mathbb{R}^+$, the flux $\vec{f}$, the viscosity coefficient Q and the source term g depend on (x, t, u) and Q is nonnegative definite. Due to the possible degeneracy, weak solutions are considered. In general, these solutions are not uniquely determined by the initial data and, therefore, additional conditions must be imposed in order to guarantee uniqueness. We consider here the subclass of piecewise smooth weak solutions, i.e., continuous solutions which are C^2-smooth everywhere apart from a closed nowhere dense collection of smooth manifolds. We show that the solution operator is L^1-stable in this subclass and, consequently, that piecewise smooth weak solutions are uniquely determined by the initial data.

1 Introduction

Consider the equation

$$u_t + \nabla \cdot f = \nabla \cdot (Q\nabla u) + g, \qquad (x,t) = (\vec{x},t) \in \mathbb{R}^N \times \mathbb{R}^+, \qquad \nabla = \vec{\nabla} = \partial/\partial\vec{x} \ , \tag{1.1}$$

where f (the flux) denotes a vector field,

$$f = \vec{f}(x,t,u) = (f_1(x,t,u), ..., f_N(x,t,u)) \ ,$$

$g = g(x,t,u)$ is a scalar source term and $Q = Q(x,t,u) = (Q_{i,j}(x,t,u))_{i,j=1}^N$ (the viscosity coefficient) is nonnegative definite, i.e.,

$$Q = Q^T \ \text{ and } \xi^T Q(x,t,u)\xi \geq 0 \qquad \forall (x,t,u) \in \mathbb{R}^N \times \mathbb{R}^+ \times \mathbb{R} \ , \ \xi \in \mathbb{R}^N \ . \tag{1.2}$$

f_i $(1 \leq i \leq N)$, $Q_{i,j}$ $(1 \leq i,j \leq N)$ and g are assumed to be smooth functions of (x,t,u).

It is well known [3, Theorem 13] that if equation (1.1) is uniformly parabolic,

$$\xi^T Q(x,t,u)\xi \geq \varepsilon > 0 \qquad \forall (x,t,u) \in \mathbb{R}^N \times \mathbb{R}^+ \times \mathbb{R} \ \text{ and } \ |\xi| = 1 \ ,$$

the corresponding Cauchy problem admits a unique classical solution. We, on the other hand, are interested here in the case where $Q(x,t,u)$ may become singular, (1.2). Such equations are called *degenerate* parabolic and examples include the porous media equation,

$$u_t = \triangle(|u|^{m-1}u) \quad , \quad m > 2 \ ,$$

or hyperbolic conservation laws,

$$u_t + \nabla \cdot f = g$$

(the reader who is interested in the theory of degenerate parabolic equations is referred to [1] and the references therein). In this case, classical solutions usually do not exist and, therefore, weak solutions are sought:

Definition 1 *A bounded function $u(x,t)$ is a weak solution of (1.1), subject to the Cauchy data*

$$u(x,0) = u_0(x) \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \ , \tag{1.3}$$

if $Q\nabla u$ exists in the sense of distributions and

$$\iint_{\mathbb{R}^N\times\mathbb{R}^+} [u\phi_t + f\cdot\nabla\phi - (Q\nabla u)\cdot\nabla\phi + g\phi]dxdt = -\int_{\mathbb{R}^N} u_0\phi(\cdot,0)dx \qquad \forall\phi\in C_0^\infty(\mathbb{R}_x^N\times\mathbb{R}_t) \ . \tag{1.4}$$

Remark. For any domain D, $C_0^\infty(D)$ denotes the space of smooth functions which are compactly supported in D^o, the interior of D; i.e., $\phi\in C_0^\infty(D)$ if $\phi\in C^\infty(D)$ and $\overline{supp\phi}\subset D^o$.

It is well known that the Cauchy problem (1.1)–(1.3) admits weak solutions [4]. However, due to the possible degeneracy, weak solutions are not always uniquely determined by the initial data. In order to have uniqueness, further assumptions should be imposed on the weak solution. In other words, uniqueness holds only in *subclasses* of the class of weak solutions.

Volpert and Hudjaev [4] proved uniqueness of weak solutions of the Cauchy problem (1.1)–(1.3) in the subclass of *generalized solutions*:

Definition 2 *A weak solution $u(x,t)$ is a generalized (or entropy) solution if it has a bounded variation, if $Q^{\frac{1}{2}}\nabla u$ exists in the sense of distributions and is locally square integrable, and if for any nonnegative $\phi\in C_0^\infty(\mathbb{R}_x^N\times\mathbb{R}_t^+)$ and any constant $c\in\mathbb{R}$ the following inequality holds:*

$$\begin{aligned}\iint_{\mathbb{R}^N\times\mathbb{R}^+} \mathrm{sgn}(u-c)\cdot\Big[(u-c)\phi_t+(g(x,t,u)-\nabla\cdot f(x,t,c))\phi+ \\ (f(x,t,u)-f(x,t,c))\cdot\nabla\phi - Q(x,t,u)\nabla u\cdot\nabla\phi\Big]dxdt \geq 0 \ .\end{aligned} \tag{1.5}$$

It seems to be a part of the folklore that if the weak solution is sufficiently regular, then it is uniquely determined by its initial value. Our goal in this note is to show that by replacing the *entropy condition* (1.5) with a *regularity condition*, one may still prove uniqueness. To this end, we define the following:

Definition 3 *A function $v(x,t)$ is called piecewise smooth if:*

(a) $v(x,t) \in C^0(\mathbb{R}_x^N \times \mathbb{R}_t) \cap C^2((\mathbb{R}_x^N \times \mathbb{R}_t^+) \setminus \Omega)$ where Ω, the irregular set, is a closed nowhere dense collection of smooth manifolds;

(b) at irregular points $(x,t) \in \Omega$ where the normal space to Ω is defined, the one-sided limits of ∇v along normal directions exist.

In most all physical applications, the solutions of equation (1.1) are piecewise smooth in the sense of Definition 3. This is why it is this type of piecewise smoothness which is assumed – sometime implicitly – in many finite-dimensional computations of such problems.

In the following section we prove uniqueness in the subclass of piecewise smooth weak solutions by showing that the solution operator is L^1-stable in that subclass.

2 Proof of main result

If u is a piecewise smooth weak solution of (1.1)–(1.3), we let Ω denote its irregular set, i.e., the *closed* set in which u is not smooth. This set, by assumption (Definition 3), is a nowhere dense collection of smooth manifolds. Hence, the tangent space is well-defined almost everywhere in Ω (it is not defined only in points of intersection of different manifolds) and, consequently, we can speak of normal directions to Ω. In the following proposition we show that even though ∇u may be discontinuous along Ω, $Q\nabla u$ is continuous in normal directions to Ω:

Proposition 1 *Define, for all $t \geq 0$, $\Omega(t) := \Omega \cap (\mathbb{R}^N \times \{t\})$. Then, for almost all (x,t) in $\Omega(t)$,*

$$\langle Q\nabla u\rangle(x,t) \cdot n = 0 \ , \tag{2.1}$$

where $n = \vec{n}(x,t) \in \mathbb{R}^N$ is a normal vector to $\Omega(t)$ and $\langle \ \cdot \ \rangle$ denotes the jump in the direction n, i.e., $\langle v\rangle(x,t) = v(x + 0 \cdot n, t) - v(x - 0 \cdot n, t)$.

Remarks.

1. The meaning of 'for almost all (x,t) in $\Omega(t)$' is as follows: on each of the manifolds which compose $\Omega(t)$, equality (2.1) holds H_k-almost everywhere where k is the dimension of the manifold and H_k is the k-dimensional Hausdorff measure on the manifold.

2. If $\Omega(t)$ is locally of co-dimension $N-k \geq 1$ in $\mathbb{R}^N$, equality (2.1) holds for all $n \in \mathcal{N}_x(\Omega(t))$, where $\mathcal{N}_x(\Omega(t))$ is the $N-k$-dimensional local normal space to $\Omega(t)$ at the point (x,t).

Proof. Let Γ be an N-dimensional manifold in Ω and let P be a point on Γ. Since Ω is nowhere dense, there exists a closed ball $B \subset \mathbb{R}^N \times \{t: \ t>0\}$, centered at P, such that $B \cap \Omega = B \cap \Gamma$ (unless P happens to be in an intersection of Γ with another manifold of Ω, but the set of such points is of zero measure in $\Omega(t)$). Therefore, Γ splits B into two components, B_1 and B_2, in the interior of which u is smooth.

Let ϕ be a test function in $C_0^\infty(B)$. Then, by (1.4),

$$0 = \iint_B [u\phi_t + f \cdot \nabla\phi - (Q\nabla u)\cdot\nabla\phi + g\phi]dxdt = \sum_{j=1}^{2} I_j \ , \tag{2.2}$$

where

$$I_j = \iint_{B_j} [u\phi_t + f \cdot \nabla\phi - (Q\nabla u)\cdot\nabla\phi + g\phi]dxdt \ ,$$

and j stands henceforth for $j=1,2$. Since u satisfies equation (1.1) in the strong sense in B_j^o, we get that

$$I_j = \iint_{B_j} [(u\phi)_t + \nabla\cdot((f - Q\nabla u)\phi)]dxdt \ . \tag{2.3}$$

We introduce the following notations:

- $\Gamma_B = \Gamma \cap B$ is the inner boundary between B_1 and B_2.

- $\vec{\nu}_j = \vec{\nu}_j(x,t) \in \mathbb{R}^{N+1}$ is the outer unit normal to B_j at $(x,t) \in \partial B_j$.

- $n_j = \vec{n}_j \in \mathbb{R}^N$ and $m_j \in \mathbb{R}$ are, respectively, the spatial and time components of $\vec{\nu}_j$,

$$\vec{\nu}_j = \begin{pmatrix} n_j \\ m_j \end{pmatrix} . \tag{2.4}$$

Next, we define

$$\Gamma_B^j(\varepsilon_j) := \Gamma_B - \varepsilon_j \begin{pmatrix} n_j(P) \\ 0 \end{pmatrix} .$$

$\Gamma_B^j(\varepsilon_j)$ is, therefore, a translation of Γ_B along the normal direction to $\Gamma_B \cap (\mathbb{R}^N \times \{t\})$ at P, towards the interior of B_j. Γ_B is the internal part of ∂B_j (the external part of ∂B_j is $\partial B_j \cap \partial B$); by replacing Γ_B with $\Gamma_B^j(\varepsilon_j)$, B_j shrinks into a new domain, denoted $B_j(\varepsilon_j)$.

We now consider the integrals

$$I_j(\varepsilon_j) = \iint_{B_j(\varepsilon_j)} [(u\phi)_t + \nabla \cdot ((f - Q\nabla u)\phi)] dxdt . \tag{2.5}$$

Applying The Divergence Theorem in (2.5), we get that

$$I_j(\varepsilon_j) = \int_{\partial B_j(\varepsilon_j)} \left\{ \left[\begin{pmatrix} f - Q\nabla u \\ u \end{pmatrix} \cdot \vec{\nu}_j^{\varepsilon_j} \right] \phi \right\} (x,t) dS(x,t) , \tag{2.6}$$

where $\vec{\nu}_j^{\varepsilon_j}(x,t) \in \mathbb{R}^{N+1}$ is the outer unit normal to $B_j(\varepsilon_j)$ at $(x,t) \in \partial B_j(\varepsilon_j)$. Since ϕ vanishes on ∂B, we get that

$$I_j(\varepsilon_j) = \int_{\Gamma_B(\varepsilon_j)} \left\{ \left[\begin{pmatrix} f - Q\nabla u \\ u \end{pmatrix} \cdot \vec{\nu}_j^{\varepsilon_j} \right] \phi \right\} (x,t) dS(x,t) , \tag{2.7}$$

or, after the changes of variables $x \mapsto x - \varepsilon_j n_j(P)$,

$$I_j(\varepsilon_j) = \int_{\Gamma_B} \left\{ \left[\begin{pmatrix} f - Q\nabla u \\ u \end{pmatrix} \cdot \vec{\nu}_j^{\varepsilon_j} \right] \phi \right\} (x - \varepsilon_j n_j(P), t) dS(x,t) . \tag{2.8}$$

We now let $\varepsilon_j \to 0$. Since u, $f(u)$ and ϕ are continuous and $\vec{\nu}_j^{\varepsilon_j}(x - \varepsilon_j n_j(P), t) = \vec{\nu}_j(x,t)$, we conclude that

$$I_j = \lim_{\varepsilon_j \to 0} I_j(\varepsilon_j) = \int_{\Gamma_B} \left[\begin{pmatrix} f(u(x,t)) - (Q\nabla u)(x - 0 \cdot n_j(P), t) \\ u(x,t) \end{pmatrix} \cdot \vec{\nu}_j(x,t) \right] \phi(x,t) dS(x,t) . \tag{2.9}$$

Since $\vec{\nu}_1 = -\vec{\nu}_2$ on Γ_B, we get, using (2.9), (2.2) and (2.4), that

$$\int_{\Gamma_B} \Big\{ [\ (Q\nabla u)(x + 0 \cdot n_1(P), t) - (Q\nabla u)(x - 0 \cdot n_1(P), t)\ ] \cdot n_1(x,t) \Big\} \phi(x,t) dS(x,t) = 0 . \tag{2.10}$$

Note that since $\vec{\nu}_1$ is a normal vector to Ω in $\mathbb{R}^{N+1}$, n_1 is a normal vector to $\Omega(t)$ in $\mathbb{R}^N$. Finally, by letting $supp\phi$ shrink to P we conclude that (2.1) holds at P.

Let us now consider manifolds $\Gamma \subset \Omega$ of dimension $k < N$. Let P be a point in $\Gamma(t) = \Gamma \cap (\mathbb{R}^N \times \{t\})$ and $\vec{n} \in \mathbb{R}^N$ be any normal vector to $\Gamma(t)$ at P. Then, there exists an N-dimensional manifold, $\tilde{\Gamma} \subset \mathbb{R}^{N+1}$, such that $\Gamma \subset \tilde{\Gamma}$ and $\vec{n}$ is the normal vector to $\tilde{\Gamma}(t) = \tilde{\Gamma} \cap (\mathbb{R}^N \times \{t\})$ at P. Repeating our arguments, as before, for $\tilde{\Gamma}$, we conclude that (2.1) holds in this case as well. $\square$

A consequence of Proposition 1 is that the solution operator of (1.1) is L^1-stable in the class of piecewise smooth weak solutions. Before proving that, we state and prove the following lemma:

Lemma 1 *Let D be a bounded domain in $\mathbb{R}^N$ and $w = w(x)$ be a smooth function such that*

$$w\Big|_D \geq 0 \quad \text{and} \quad w\Big|_{\partial D} = 0 \ . \tag{2.11}$$

Let $Q = Q(x)$ be a $N \times N$ nonnegative definite matrix function. Then if n is the outer unit normal to ∂D,

$$(Q(x)\nabla w) \cdot n\Big|_{\partial D} \leq 0 \ . \tag{2.12}$$

Proof. Let x_0 be a point in ∂D. We make the change of variables, $x \mapsto \tilde{x} = Px$, where P is an orthogonal diagonalizer for $Q(x_0)$, i.e.,

$$PQ(x_0)P^T = \Lambda = diag\{\lambda_1, ..., \lambda_N\} \quad , \quad \lambda_i \geq 0 \ , \ \ 1 \leq i \leq N \ . \tag{2.13}$$

Denoting the gradient with respect to the new variables by $\tilde{\nabla} = \partial/\partial\tilde{x}$ and the new outer unit normal vector to ∂D by $\tilde{n}$, we have that

$$\tilde{\nabla} = P\nabla \quad \text{and} \quad \tilde{n} = Pn \ . \tag{2.14}$$

Using (2.13)–(2.14) we get that

$$(Q\nabla w) \cdot n\Big|_{x=x_0} = (\Lambda\tilde{\nabla}w) \cdot \tilde{n}\Big|_{\tilde{x}=Px_0} = \sum_{i=1}^{N} \lambda_i \frac{\partial w}{\partial \tilde{x}_i}\tilde{n}_i\Big|_{\tilde{x}=Px_0} \ . \tag{2.15}$$

However, assumption (2.11) implies that

$$\frac{\partial w}{\partial \tilde{x}_i}\tilde{n}_i \le 0 \qquad 1 \le i \le N \ , \tag{2.16}$$

at $\tilde{x} = Px_0$. Hence, since the eigenvalues λ_i are nonnegative, (2.13), we get by (2.15)–(2.16) that inequality (2.12) holds at $x = x_0$. □

Theorem 1 *(L^1-Stability). Let u and v be two piecewise smooth weak solutions of (1.1)–(1.2). Let $M_T^{\pm}$ be such that*

$$M_T^- \le u(x,t), v(x,t) \le M_T^+ \qquad \forall (x,t) \in \mathbb{R}^N \times [0,T] \ , \tag{2.17}$$

and assume that

$$u(\cdot,t) - v(\cdot,t) \in L^1(\mathbb{R}^N) \qquad \forall t \in [0,T] \ . \tag{2.18}$$

Then

$$\|u(\cdot,t) - v(\cdot,t)\|_{L^1} \le e^{\gamma t}\|u(\cdot,0) - v(\cdot,0)\|_{L^1} \qquad \forall t \in [0,T] \ , \tag{2.19}$$

where

$$\gamma = \gamma(T) := \sup_{\mathbb{R}^N \times [0,T] \times [M_T^-, M_T^+]} g_u(x,t,u) \ . \tag{2.20}$$

The proof of this theorem is motivated by the classical proof of P.D. Lax [2, p. 14] of uniqueness of L^1 piecewise smooth entropy solutions of hyperbolic conservation laws, $u_t + f(u)_x = 0$.

Proof. For every $t \ge 0$, we divide the space $\mathbb{R}^N$ to sub-domains, $\mathbb{R}^N = \cup_k D_k(t)$, so that

$$\sigma_k[u(\cdot,t) - v(\cdot,t)]\Big|_{D_k(t)} \ge 0 \qquad \forall k \tag{2.21}$$

where $\sigma_k = \pm 1$ is a signature coefficient and

$$u(\cdot,t) = v(\cdot,t)\Big|_{\partial D_k(t)} \qquad \forall k \ . \tag{2.22}$$

Using (2.21) and (2.22) we conclude that

$$\frac{d}{dt}\|u(\cdot,t) - v(\cdot,t)\|_{L^1} = \tag{2.23}$$

$$= \frac{d}{dt} \sum_k \sigma_k \int_{D_k(t)} [u(x,t) - v(x,t)]dx = \sum_k \sigma_k \int_{D_k(t)} [u_t(x,t) - v_t(x,t)]dx := \sum_k I_k \ .$$

We show below that all the terms in the last sum in (2.23) are nonpositive. We concentrate on terms I_k which correspond to bounded sub-domains $D_k(t)$. The modification for unbounded sub-domains is straightforward.

First, let us assume that both $u(\cdot,t)$ and $v(\cdot,t)$ are smooth in $D_k(t)^o$. Therefore, both u and v satisfy equation (1.1) in the strong sense there and we conclude that

$$I_k = -\sigma_k \int_{D_k(t)} \nabla \cdot [f(x,t,u) - f(x,t,v)]dx + \sigma_k \int_{D_k(t)} \nabla \cdot [Q(x,t,u)\nabla u - Q(x,t,v)\nabla v]dx + \sigma_k \int_{D_k(t)} [g(x,t,u) - g(x,t,v)]dx := I_k^1 + I_k^2 + I_k^3 \ . \tag{2.24}$$

The first term on the right hand side of (2.24) is zero, due to The Divergence Theorem and equality (2.22):

$$I_k^1 = -\sigma_k \int_{\partial D_k(t)} [f(s,t,u) - f(s,t,v)] \cdot n ds = 0 \ ; \tag{2.25}$$

$n \in \mathbb{R}^N$ denotes here and henceforth the outer unit normal to $D_k(t)$. As for the second term, it equals, by The Divergence Theorem, to

$$I_k^2 = \sigma_k \int_{\partial D_k(t)} [Q(s,t,u)\nabla u - Q(s,t,v)\nabla v] \cdot n ds \ .$$

Since $u = v$ on $\partial D_k(t)$, (2.22), it may be written as

$$I_k^2 = \int_{\partial D_k(t)} [\tilde{Q}(s,t)\nabla w] \cdot n ds \ ,$$

where $\tilde{Q}(s,t) = Q(s,t,u = u(s,t))$ and $w = \sigma_k(u - v)$. Since, by (2.21)–(2.22), w is nonnegative in $D_k(t)$ and vanishes on $\partial D_k(t)$, and $\tilde{Q}(s,t) \geq 0$, Lemma 1 implies that

$$I_k^2 \leq 0 \ . \tag{2.26}$$

Using The Mid-value Theorem, (2.21) and (2.20) for the last term on the right hand side of (2.24), we get that

$$I_k^3 \leq \gamma \int_{D_k(t)} |u - v| dx \qquad \forall t \in [0,T] \ . \tag{2.27}$$

Combining (2.24)–(2.27) we conclude that

$$I_k \leq \gamma \int_{D_k(t)} |u-v|dx \qquad \forall t \in [0,T] \ . \tag{2.28}$$

Next, we handle those sub-domains, $D_k(t)$, in the interior of which u or v are not smooth. Let Ω_u and Ω_v denote the irregular sets of u and v, respectively. Assume that $D_k(t)^o$ is intersected by one of the manifolds of Ω_u, Γ,

$$D_k(t)^o \cap \Omega_u = D_k(t)^o \cap \Gamma \neq \emptyset \tag{2.29}$$

and that

$$D_k(t)^o \cap \Omega_v = \emptyset \ . \tag{2.30}$$

The case where $D_k(t)^o$ is intersected by more than one manifold of either of the two irregular sets, is treated in a similar manner, as we explain later on.

If the dimension of Γ is less than N, we embed it in a N-dimensional manifold, still denoted by Γ. Therefore, $S := \Gamma \cap D_k(t)$, splits $D_k(t)$ into two components, $D_k^1(t)$ and $D_k^2(t)$, and in view of (2.29)–(2.30) u and v satisfy equation (1.1) in the strong sense in $D_k^j(t)^o$, $j=1,2$. Therefore,

$$I_k = \sigma_k \int_{D_k(t)} [u_t(x,t) - v_t(x,t)]dx = \sum_{j=1}^{2} \left\{ \sigma_k \int_{D_k^j(t)} \nabla \cdot [f(x,t,v) - f(x,t,u)]dx + \right. \tag{2.31}$$

$$\left. \sigma_k \int_{D_k^j(t)} \nabla \cdot [Q(x,t,u)\nabla u - Q(x,t,v)\nabla v]dx + \sigma_k \int_{D_k^j(t)} [g(x,t,u) - g(x,t,v)]dx \right\} \ .$$

Let n_j denote the outer unit normal to $D_k^j(t)$. Note that on S, the interface between $D_k^1(t)$ and $D_k^2(t)$, $n_1 = -n_2$, and that on $\partial D_k^j(t) \setminus S$, n_j coincides with n, the outer unit normal to $D_k(t)$. Therefore, using The Divergence Theorem and equality (2.22), the first term on the right hand side of (2.31) vanishes:

$$\sum_{j=1}^{2} \sigma_k \int_{D_k^j(t)} \nabla \cdot [f(x,t,v) - f(x,t,u)]dx = \sum_{j=1}^{2} \sigma_k \int_{\partial D_k^j(t)} [f(s,t,v) - f(s,t,u)] \cdot n_j ds =$$

$$\sigma_k \left\{ \int_S [f(s,t,v) - f(s,t,u)] \cdot (n_1 + n_2) ds + \int_{\partial D_k(t)} [f(s,t,v) - f(s,t,u)] \cdot n ds \right\} = 0 \ .$$

As for the second term, it is nonpositive:

$$\sum_{j=1}^{2} \sigma_k \int_{D_k^j(t)} \nabla \cdot [Q(x,t,u)\nabla u - Q(x,t,v)\nabla v] dx = \tag{2.32}$$

$$\sum_{j=1}^{2} \sigma_k \int_{\partial D_k^j(t)} [Q(s,t,u)\nabla u - Q(s,t,v)\nabla v] \cdot n_j ds =$$

$$\int_{\partial D_k(t)} [\tilde{Q}(s,t)\nabla w] \cdot n ds + \sigma_k \int_S \langle Q(s,t,u)\nabla u\rangle \cdot n_1 ds - \sigma_k \int_S \langle Q(s,t,v)\nabla v\rangle \cdot n_1 ds \ ,$$

where, as before, $\tilde{Q}(s,t) = Q(s,t,u = u(s,t))$, $w = \sigma_k(u-v)$ and $\langle \ \cdot \ \rangle$ denotes the jump across S in the normal direction, n_1. The first term on the right hand side of (2.32) is nonpositive, in light of Lemma 1, while the other two terms vanish in view of Proposition 1.

Since the last term on the right hand side of (2.31) may be bounded as in (2.27), we conclude that inequality (2.28) holds in such sub-domains as well.

If $D_k(t)$ is intersected by any number of manifolds from either Ω_u or Ω_v, it may be decomposed into $D_k(t) = \cup_{j\in J} D_k^j(t)$, so that both u and v are smooth in $D_k^j(t)^o$, $j \in J$, and the proof goes along the same lines as before.

To summarize all of the above, inequality (2.28) holds for all k. Hence, we get from (2.23) that

$$\frac{d}{dt}\|u(\cdot,t) - v(\cdot,t)\|_{L^1} \le \gamma \cdot \|u(\cdot,t) - v(\cdot,t)\|_{L^1} \qquad \forall t \in [0,T] \ ,$$

which implies (2.19). □

Corollary 1 *(Uniqueness). The Cauchy problem for equation (1.1)–(1.2) admits at most one $L^1(\mathbb{R}^N)$ piecewise smooth weak solution.*

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