Approximating minimum-cost connectivity problems via uncrossable bifamilies

Zeev Nutov
The Open University of Israel

We give approximation algorithms for the Survivable Network problem. The input consists of a graph $G = (V, E)$ with edge/node-costs, a node subset $S \subseteq V$, and connectivity requirements $\{r(s, t) : s, t \in T \subseteq V\}$. The goal is to find a minimum cost subgraph H of G that for all $s, t \in T$ contains $r(s, t)$ pairwise edge-disjoint st-paths such that no two of them have a node in $S \setminus \{s, t\}$ in common. Three extensively studied particular cases are: Edge-Connectivity Survivable Network ($S = \emptyset$), Node-Connectivity Survivable Network ($S = V$), and Element-Connectivity Survivable Network ($r(s, t) = 0$ whenever $s \in S$ or $t \in S$). Let $k = \max_{s,t\in T} r(s, t)$. In Rooted Survivable Network there is $s \in T$ such that $r(u, t) = 0$ for all $u \neq s$, and in the Subset k-Connected Subgraph problem $r(s, t) = k$ for all $s, t \in T$.

For edge-costs, our ratios are $O(k \log k)$ for Rooted Survivable Network and $O(k^2 \log k)$ for Subset k-Connected Subgraph. This improves the previous ratio $O(k^2 \log n)$, and for constant values of k settles the approximability of these problems to a constant.

For node-costs, our ratios are:

—$O(k \log |T|)$ for Element-Connectivity Survivable Network, matching the best known ratio for Edge-Connectivity Survivable Network.
—$O(k^2 \log |T|)$ for Rooted Survivable Network and $O(k^3 \log |T|)$ for Subset k-Connected Subgraph, improving the ratio $O(k^8 \log^2 |T|)$.
—$O(k^4 \log^2 |T|)$ for Survivable Network; this is the first non-trivial approximation algorithm for the node-costs version of the problem.

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1. INTRODUCTION

1.1 Survivable Network problems

In network design connectivity problems the goal is to find a low cost subgraph that satisfies prescribed connectivity requirements. When only connectedness is required between certain pairs of nodes, some classic examples are: Shortest Path, Minimum Spanning Tree, Steiner Tree, and Steiner Forest. Corresponding examples that al-

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Authors' address: Zeev Nutov, Department of Computer Science, The Open University of Israel, Raanana, Israel. E-mail: nutov@openu.ac.il.

low high connectivity requirements are: Min-Cost k-Flow, k-Edge/Node-Connected Subgraph, Subset k-Edge/Node-Connected Subgraph, and Edge/Node-Connectivity Survivable Network, respectively.

For an edge set I on node set V let $V(I) = \bigcup_{uv \in I}\{u,v\}$ denote the set of end-nodes of the edges in I. Given node-costs $\{c(v) : v \in V\}$, let $c(I) = c(V(I))$ be the *node-cost* of I. For a subset S of nodes in a graph H, let $\lambda_H^S(s,t)$ denote the S-*connectivity* between s and t in H, namely, the maximum number of pairwise edge-disjoint st-paths in H so that no two of them have a node in $S \setminus \{s,t\}$ in common. We consider the following fundamental problem on undirected graphs, that includes as a special case the problems mentioned above.

Survivable Network

Instance: A graph $G = (V,E)$ with edge/node-costs, $S \subseteq V$, and S-connectivity requirements $\{r(s,t) : s,t \in T \subseteq V\}$.

Objective: Find a minimum cost subgraph H of G such that $\lambda_H^S(s,t) \geq r(s,t)$ for all $s,t \in T$.

Extensively studied particular cases of Survivable Network are: Edge-Connectivity Survivable Network ($S = \emptyset$), Node-Connectivity Survivable Network ($S = V$), and Element-Connectivity Survivable Network ($r(s,t) = 0$ whenever $s \in S$ or $t \in S$). Edge-Connectivity Survivable Network is also called Steiner Network in the literature, c.f. [Jain 2001], and various variants of Survivable Network are also referred to as the Survivable Network Design Problem (SNDP) in the literature, c.f. [Goemans et al. 1994; Ravi and Williamson 1997]. Element-Connectivity Survivable Network is essentially the edge-connectivity version of the problem on hypergraphs, studied in the 90s by Frank, Benczur, and many others; see e.g. [Nutov 2009a] and the references therein. We note that Survivable Network can be reduced to its node-connectivity variant by elementary constructions. Thus all our results for the node-connectivity variant extend to the S-connectivity one, and we simply write Survivable Network to mean Node-Connectivity Survivable Network. In Rooted Survivable Network there is $s \in T$ such that $r(u,t) = 0$ for all $u \neq s$ and in Subset k-Connected Subgraph $r(s,t) = k$ for all $s,t \in T$. The latter problem generalizes the k-Connected Subgraph problem; see [Nutov 2009c; Fackharoenphol and Laekhanukit 2008] and the references therein.

1.2 Previous and related work

We refer the reader to a survey [Kortsarz and Nutov 2007] on Survivable Network problems with various edge-costs and connectivity requirements, and here mention some literature relevant to this paper. For an instance of Survivable Network let $k = \max_{s,t \in T} r(s,t)$ denote the maximum requirement. The first approximation algorithms for the problem appeared in the 90s for the Steiner Forest problem – the case $k = 1$. Agrawal, Klein, and Ravi [Agrawal et al. 1995] gave a 2-approximation for edge-costs (see also [Goemans and Williamson 1995; Goemans et al. 1994] for a more general result and a simpler analysis), and [Klein and Ravi 1995] gave an $O(\log n)$-approximation for node-costs. The latter ratio is essentially (up to constants) the best possible, as the node-costs version is Set-Cover hard [Klein and Ravi 1995].

For $k \geq 2$, a line of research initiated by Frank, Goemans and Williamson, and others, was to study a more general setting of edge-covering the "set-function" arising from the Survivable Network variant. For example, Edge-Connectivity Survivable Network can be formulated as a Set-Function Edge-Cover problem as follows. An edge e covers a set X if it has exactly one endnode in X. Let $\delta_H(X)$ denote the set of edges in a graph H that cover X. By Menger's Theorem a subgraph H of G is a feasible solution to an Edge-Connectivity Survivable Network instance if, and only if $|\delta_H(X)| \geq f(X)$ for all $X \subseteq V$, where $f(X) = \max\{r(s,t) : |X \cap \{s,t\}| = 1\}$. This set-function f is *weakly supermodular*, namely,

$$f(X)+f(Y) \leq \max\{f(X\cap Y)+f(X\cup Y), f(X\setminus Y)+f(Y\setminus X)\} \text{ for all } X,Y \subseteq V .$$

A set-family $\mathcal{F}$ is *uncrossable* if for any $X,Y \in \mathcal{F}$ we have $X \cap Y, X \cup Y \in \mathcal{F}$ or $X \setminus Y, Y \setminus X \in \mathcal{F}$. It is known (c.f. [Goemans et al. 1994]) that the problem of edge-covering a weakly supermodular set-function f can be decomposed into $f_{\max} = \max_{X \subseteq V} f(X)$ problems of edge-covering an uncrossable set-family.

The seminal paper [Jain 2001], and numerous papers preceding it, considered Edge-Connectivity Survivable Network with edge-costs, and developed novel tools for approximating minimum cost edge-covers of several types of set-functions and families. [Jain 2001] gave a 2-approximation algorithm for edge-covering a weakly-supermodular set-function using the iterative rounding method. Earlier, [Goemans et al. 1994] gave a combinatorial (primal-dual/local-ratio) 2-approximation algorithm for the special case of uncrossable set-families. The 2-approximation of [Jain 2001] for Edge-Connectivity Survivable Network was extended to element-connectivity by Fleischer, Jain, and Williamson [Fleischer et al. 2006] and by Cheriyan, Vempala, and Vetta [Cheriyan et al. 2006].

Recently, progress has also been made for node-costs. Generalizing the algorithm of [Klein and Ravi 1995] for Survivable Network with $k = 1$, [Nutov 2010b] developed an $O(\log |V|)$-approximation algorithm for edge-covering an uncrossable set-family by a minimum *node-cost* edge set. For node-costs, this algorithm implies an $O(k \log |T|)$-approximation algorithm for Edge-Connectivity Survivable Network, and also for Node-Connectivity Survivable Network with $k \leq 2$. In [Nutov 2010b] is given an evidence that for large values of k, even the simplest version of Edge-Connectivity Survivable Network with node-costs when $r(s,t) \neq 0$ for only one pair s,t, the so called Node-Weighted k-Flow problem, may not admit a polylogarithmic approximation ratio. Specifically, the reduction in [Nutov 2010b] shows that a ratio ρ for the Node-Weighted k-Flow problem implies ratio $1/2\rho^2$ for the Densest k-Subgraph problem: given a graph G and an integer k, find a k-node subgraph of G with maximum number of edges. This problem has been studied extensively, and the currently best known ratio for it is $O\left(n^{1/4+\varepsilon}\right)$ [Bhaskara et al. 2010].

We survey some results for Survivable Network with edge-costs. A hardness result of [Kortsarz et al. 2004] suggests that Subset k-Connected Subgraph is unlikely to admit a polylogarithmic approximation; this is so even when the input graph is complete and the costs are in $\{0,1\}$ [Nutov 2009a]. Chakraborty, Chuzhoy, and Khanna [Chakraborty et al. 2008] extended this to $\Omega(k^\varepsilon)$-hardness for any $k \geq k_0$, where k_0 and $\varepsilon > 0$ are universal constants. [Lando and Nutov 2009] proved that for $k = n/2 + k'$ the approximability of the undirected Survivable Network variant is the same (up to a factor of 2) as that of the directed one with maximum requirement k'. This is so also for Rooted Survivable Network. The directed variant of Rooted Survivable Network includes as a special case, when $k' = 1$, the Directed Steiner Tree problem. The latter is not known to admit a polylogarithmic approximation, but admits an $O(n^{\varepsilon})$-approximation scheme [Charikar et al. 1999]; for $k' = 2$ no sublinear approximation for the directed rooted variant is known. On the positive side, the best known ratios for Survivable Network problems were: $O(k^3 \log n)$ for Survivable Network [Chuzhoy and Khanna 2009], $O(k^2 \log n)$ for Subset k-Connected Subgraph by [Chuzhoy and Khanna 2008], and $O(k^2 \log n)$ for Rooted Survivable Network [Chuzhoy and Khanna 2008] and [Nutov 2009d]. Survivable Network also admits an $O(\log k)$-approximation for metric edge-costs [Cheriyan and Vetta 2007]. In contrast, for node-costs, non-trivial approximation ratios were known only for rooted requirements; $O(k^8 \log^2 n)$ by [Chuzhoy and Khanna 2008].

1.3 Uncrossable bifamilies and Survivable Network Augmentation problems

As was mentioned, Edge-Connectivity Survivable Network can be formulated as a Set-Function Edge-Cover problem with weakly supermodular set function f. For other Survivable Network problems, a similar formulation can be given in terms of *setpairs* instead of sets [Frank and Jordán 1995; Fleischer et al. 2006; Cheriyan and Vempala 2001]. Following [Frank 2009], we will use the following equivalent formulation.

Definition 1.1. *An ordered pair $\hat{X} = (X, X^+)$ of subsets of a groundset V is called a* biset *if $X \subseteq X^+$; X is the* inner part *and X^+ is the* outer part *of $\hat{X}$. Let $\Gamma(\hat{X}) = X^+ \setminus X$. The intersection and the union of bisets $\hat{X}, \hat{Y}$ is naturally defined by $\hat{X} \cap \hat{Y} = (X \cap Y, X^+ \cap Y^+)$ and $\hat{X} \cup \hat{Y} = (X \cup Y, X^+ \cup Y^+)$. The biset $\hat{X} \setminus \hat{Y}$ is defined by $\hat{X} \setminus \hat{Y} = (X \setminus Y^+, X^+ \setminus Y) = \hat{X} \cap (V \setminus Y^+, V \setminus Y)$.*

An edge e covers a biset $\hat{X}$ if it has one endnode in X and the other in $V \setminus X^+$. For an edge-set or a graph H and a biset $\hat{X}$ on a node set V let $\delta_H(\hat{X})$ denote the set of edges in H covering $\hat{X}$. Given an instance of Survivable Network let

$$r(\hat{X}) = \max\{r(s,t) : |X \cap \{s,t\}| = |X^+ \cap \{s,t\}| = 1\} .$$

By the S-connectivity version of Menger's Theorem, a subgraph H of G is a feasible solution to a Survivable Network instance if, and only if $|\delta_H(\hat{X})| \geq f(\hat{X})$ for all bisets $\hat{X}$ on V, where here f is a *biset-function* defined by

$$f(\hat{X}) = r(\hat{X}) - |\Gamma(\hat{X})| \ \text{ if } \ \Gamma(\hat{X}) \subseteq S$$

and $f(\hat{X}) = 0$ otherwise.

We study biset-families arising from Rooted Survivable Network and Element-Connectivity Survivable Network instances. For applications considered in this paper it suffices to consider biset-families $\mathcal{F}$ that are:

- *bijective* – $X = Y$ implies $X^+ = Y^+$ for any $\hat{X}, \hat{Y} \in \mathcal{F}$.
- *monotone* – $X \subseteq Y$ implies $X^+ \subseteq Y^+$ for any $\hat{X}, \hat{Y} \in \mathcal{F}$.

A biset-family $\mathcal{F}$ is called a *bifamily* if it is bijective, monotone, and $X, V \setminus X^+$ are both nonempty for every $\hat{X} \in \mathcal{F}$.

Definition 1.2. *Given a bifamily $\mathcal{F}$ on V and a set $T \subseteq V$ of terminals, we say that $\hat{X}, \hat{Y} \in \mathcal{F}$ are* T-dependent *if $X \cap T \subseteq \Gamma(\hat{Y})$ or if $Y \cap T \subseteq \Gamma(\hat{X})$, and $\hat{X}, \hat{Y}$ are* T-independent *otherwise. We say that $\mathcal{F}$ is* T-uncrossable *if $X \cap T \neq \emptyset$ for all $\hat{X} \in \mathcal{F}$, and if $\hat{X} \cap \hat{Y}, \hat{X} \cup \hat{Y} \in \mathcal{F}$ or $\hat{X} \setminus \hat{Y}, \hat{Y} \setminus \hat{X} \in \mathcal{F}$ for any T-independent $\hat{X}, \hat{Y} \in \mathcal{F}$. We say that $\mathcal{F}$ is* uncrossable *if it is V-uncrossable and any $\hat{X}, \hat{Y} \in \mathcal{F}$ are V-independent (equivalently, a bifamily $\mathcal{F}$ is uncrossable if $\hat{X} \cap \hat{Y}, \hat{X} \cup \hat{Y} \in \mathcal{F}$ or $\hat{X} \setminus \hat{Y}, \hat{Y} \setminus \hat{X} \in \mathcal{F}$ for any $\hat{X}, \hat{Y} \in \mathcal{F}$).*

Let Survivable Network Augmentation be the restriction of Survivable Network to instances where the input graph G contains a subgraph J of cost 0 such that $\lambda^S_J(s,t) \geq \max\{r(s,t) - 1, 0\}$ for all $s, t \in T$. Namely, we seek to increase the connectivity by 1 between certain pairs. Formally, this version is as follows.

Survivable Network Augmentation
Instance: A graph $G = (V, E)$ with edge/node-costs, a graph $J = (V, E_J)$, $S \subseteq V$, and a set $\mathcal{T}$ of node pairs from a set $T \subseteq V$ of terminals.
Objective: Find a minimum cost edge-set $I \subseteq E \setminus E_J$ so that $\lambda^S_{J \cup I}(s,t) \geq \lambda^S_J(s,t) + 1$ for all $\{s,t\} \in \mathcal{T}$.

Given an instance of Survivable Network Augmentation we assume that $E \cap E_J = \emptyset$, by setting $E \leftarrow E \setminus E_J$. Menger's Theorem for S-connectivity (c.f. [Kortsarz and Nutov 2007, Theorem 3.1]) states that for any $s, t \in V$

$$\lambda^S_J(s,t) = \min\{C : C \subseteq E_J \cup S \setminus \{s,t\}, J \setminus C \text{ has no } uv\text{-path}\} .$$

Namely, $\lambda^S_J(s,t)$ equals the minimum size of a "mixed cut" C of edges and nodes in S whose deletion separates between s and t. Let us say that a biset $\hat{X} = (X, X^+)$ is *tight* if there exists $\{s,t\} \in \mathcal{T}$ such that $|X \cap \{s,t\}| = |X^+ \cap \{s,t\}| = 1$, $\Gamma(\hat{X}) \subseteq S$, and $|\Gamma(\hat{X})| + |\delta_J(\hat{X})| = \lambda^S_J(s,t)$. By Menger's Theorem, I is a feasible solution to Survivable Network Augmentation if, and only if, I covers the family $\mathcal{F}(J, S, \mathcal{T})$ of tight bisets; see [Kortsarz and Nutov 2007]. It is not hard to verify that if $\hat{X}$ is tight then $\Gamma(\hat{X})$ must be the set of neighbors in $S \setminus \{s,t\}$ of X in the graph J. This implies that $\mathcal{F}(J, S, \mathcal{T})$ is a bifamily. This bifamily is uncrossable for Element-Connectivity Survivable Network, by [Fleischer et al. 2006; Cheriyan et al. 2006]. In the case of rooted requirements, it is sufficient to cover the bifamily $\{\hat{X} \in \mathcal{F}(J, S, \mathcal{T}) : s \notin X^+\}$, where s is the root. This bifamily is T-uncrossable for Rooted Survivable Network by [Nutov 2009d]. We therefore consider the following generic problem which includes Survivable Network Augmentation problems.

Bifamily Edge-Cover
Instance: A graph $G = (V, E)$ with edge/node-costs and a bifamily $\mathcal{F}$ on V.
Objective: Find a minimum cost edge-cover $I \subseteq E$ of $\mathcal{F}$.

A polynomial time implementation of our algorithms requires two assumptions, that certain queries related to $\mathcal{F}$ can be answered in polynomial time. We need some definitions to describe these assumptions.

Given an edge set I on V (I is a partial edge-cover of $\mathcal{F}$), the *residual bifamily* $\mathcal{F}_I$ *of* $\mathcal{F}$ (w.r.t. I) consists of all members of $\mathcal{F}$ that are not covered by the edges of I. It is easy to verify that if $\mathcal{F}$ is T-uncrossable, so is $\mathcal{F}_I$, for any I, c.f. [Fleischer et al. 2006] for the particular case of uncrossable bifamilies.

DEFINITION 1.3. *A set* $C \in \{X : \hat{X} \in \mathcal{F}\}$ *is a* core *of a bifamily* $\mathcal{F}$*, or* C *is an* $\mathcal{F}$-core*, if* C *does not contain two distinct inclusion-minimal members of the set-family* $\{X : \hat{X} \in \mathcal{F}\}$*. An inclusion-minimal (inclusion-maximal) core is a* min-core *(*max-core*). Let* $\mathcal{C}_{\mathcal{F}}$ *(*$\mathcal{M}_{\mathcal{F}}$*) denote the set-family of min-cores (max-cores) of* $\mathcal{F}$*.*

ASSUMPTION 1. *Given the inner part* X *of a biset* $\hat{X} \in \mathcal{F}$*, the outer part* X^+ *of* $\hat{X}$ *can be computed in polynomial time.*

ASSUMPTION 2. *For any edge set* I *on* V*, the families* $\mathcal{C}_{\mathcal{F}_I}$ *of min-cores and* $\mathcal{M}_{\mathcal{F}_I}$ *of max-cores of* $\mathcal{F}_I$ *can be computed in polynomial time.*

Using standard max-flow min-cut methods, it is easy to see that Assumptions 1 and 2 hold for the family $\mathcal{F}$ of tight bisets, c.f. [Nutov 2009d; 2012a]. Summarizing, we have the following.

COROLLARY 1.1. Element-Connectivity Survivable Network Augmentation *is a particular case of* Bifamily Edge-Cover *with uncrossable* $\mathcal{F}$*, and* Rooted Survivable Network Augmentation *is a particular case of* Bifamily Edge-Cover *with* T*-uncrossable* $\mathcal{F}$*. Furthermore, in both cases, Assumptions 1 and 2 hold for* $\mathcal{F}$*.*

1.4 Our results

Our first result is the following decomposition, which is obtained by an improved analysis of the algorithm from [Nutov 2009d].

THEOREM 1.2. *There exists a polynomial time algorithm that, given a* T*-uncrossable bifamily* $\mathcal{F}$*, sequentially finds* $4\gamma + \lceil \lg(\lfloor \gamma/2 \rfloor + 1) \rceil = O(\gamma)$ *uncrossable sub-bifamilies of* $\mathcal{F}$ *such that the union of their edge-covers is an edge-cover of* $\mathcal{F}$*, where* $\gamma = \gamma(\mathcal{F}, T) = \max_{\hat{X}, \hat{Y} \in \mathcal{F}} |\Gamma(\hat{X}) \cap Y \cap T| \geq 1$*. In particular, if* Bifamily Edge-Cover *with uncrossable* $\mathcal{F}$ *admits a* ρ*-approximation algorithm, then* Bifamily Edge-Cover *with* T*-uncrossable* $\mathcal{F}$ *admits an* $O(\rho\gamma)$*-approximation algorithm.*

Let $\tau(\mathcal{F})$ denote the optimal value of a standard LP-relaxation for Bifamily Edge-Cover, namely

$$\tau(\mathcal{F}) = \min \left\{ \sum_{e \in E} c_e x_e : \sum_{e \in \delta_E(\hat{U})} x_e \geq 1 \ \forall \hat{U} \in \mathcal{F}, \ x_e \geq 0 \ \forall e \in E \right\} .$$

For edge-costs, Bifamily Edge-Cover with uncrossable $\mathcal{F}$ admits a polynomial time algorithm that computes an edge-cover of $\mathcal{F}$ of cost $\leq 2\tau(\mathcal{F})$. In [Fleischer et al. 2006; Cheriyan et al. 2006] such an algorithm uses the iterative rounding method, and applies to a more general biset-function edge-cover problem. A combinatorial algorithm that relies on Assumptions 1, 2 only can be found in [Nutov 2009d].

For node-costs, Bifamily Edge-Cover with uncrossable $\mathcal{F}$ includes the Set-Cover problem, and thus is $\Omega(\log n)$-hard to approximate. The only approximation algorithm known was for set-families; [Nutov 2010b] gives an $O(\log |\mathcal{C}_{\mathcal{F}}|)$-approximation algorithm. In this paper we prove the following generalization:

THEOREM 1.3. *For node-costs,* Bifamily Edge-Cover *with uncrossable* $\mathcal{F}$ *admits an* $O(\log |\mathcal{C}_{\mathcal{F}}|)$*-approximation algorithm.*

From Theorems 1.2 and 1.3 we obtain the main result of this paper.

THEOREM 1.4. Bifamily Edge-Cover *with T-uncrossable bifamily $\mathcal{F}$ admits the following approximation algorithms.*

—*For edge-costs, an algorithm that computes a solution of cost $O(\gamma) \cdot \tau(\mathcal{F})$.*

—*For node-costs, an $O(\gamma \log |T|)$-approximation algorithm.*

We now consider some applications of Theorem 1.4. For Rooted Survivable Network, two different $O(k^2 \log n)$-approximation algorithms were suggested independently in [Chuzhoy and Khanna 2008] and [Nutov 2009d]. A particularly elegant and simple approach was suggested recently by [Chuzhoy and Khanna 2009]. They showed that Rooted Survivable Network can be decomposed into p instances of Element-Connectivity Survivable Network, where $p = p(|T|, k)$ is the minimum number of subsets $T_1, \ldots, T_p$ of T, such that for every pair (t, Q) with $Q \subset T$, $|Q| = k$, $t \in V \setminus Q$, there exists T_i with $t \in T_i$ and $Q \cap T_i = \emptyset$. Chuzhoy and Khanna proved that $p = O(k^2 \log |T|)$. A factor of $\log |T|$ is unavoidable here even for $k = 1$. However, for $k \leq 2$ Survivable Network admits a constant ratio approximation algorithm [Ravi and Williamson 1997; Fleischer et al. 2006; Cheriyan et al. 2006]. Hence it seems reasonable that Survivable Network with edge-costs admits an approximation ratio that depends on k only. This was proved recently in [Nutov 2012a] for the special case when the input graph G is complete and the costs are in $\{0, 1\}$. Here we prove this for Rooted Survivable Network and Subset k-Connected Subgraph with arbitrary costs, by deducing it from Theorem 1.4.

The following statement will be proved later in Section 4.

PROPOSITION 1.5. *Suppose that for edge-costs,* Bifamily Edge-Cover *with T-uncrossable $\mathcal{F}$ admits a polynomial time algorithm that computes a solution of cost $\leq \rho(\gamma) \cdot \tau(\mathcal{F})$, where ρ is a monotone non-decreasing function. Then* Rooted Survivable Network *admits a polynomial time algorithm that computes a solution of cost $\leq$ $\mathsf{opt} \cdot \sum_{\ell=1}^{k} \frac{\rho(\ell)}{k-\ell+1} \leq \rho(k) \cdot H(k) \cdot \mathsf{opt}$, where* opt *denotes the optimal solution cost for* Rooted Survivable Network.

Combining Theorem 1.4 with Proposition 1.5 we prove in Section 4 the following result.

THEOREM 1.6. Survivable Network *problems admit the following approximation ratios:*

—*For edge-costs, $O(k \log k)$ for* Rooted Survivable Network *and $O(k^2 \log k)$ for* Subset k-Connected Subgraph.

—*For node-costs, $O(k \log |T|)$ for* Element-Connectivity Survivable Network*, $O(k^2 \log |T|)$ for* Rooted Survivable Network*, $O(k^3 \log |T|)$ for* Subset k-Connected Subgraph*, and $O(k^4 \log^2 |T|)$ for* Survivable Network.

For constant values of k, this settles the approximability of Rooted Survivable Network and of Subset k-Connected Subgraph with edge-costs to a constant, and of Element-Connectivity Survivable Network, Rooted Survivable Network, and Subset k-Connected Subgraph with node-costs to $O(\log |T|)$.

Theorems 1.2, 1.3, and 1.6, are proved in Sections 2, 3, and 4, respectively. Conclusions, recent developments, and open problems are given in Section 5.

2. PROOF OF THEOREM 1.2

Let us say that a bifamily $\mathcal{F}$ is *simple* if the inner part of every member of $\mathcal{F}$ is a core. Note that the sub-bifamily $\{\hat{X} \in \mathcal{F} : X \text{ is an } \mathcal{F}\text{-core}\}$ of any bifamily $\mathcal{F}$ is always simple, and we will prove that it is T-uncrossable if $\mathcal{F}$ is. To prove Theorem 1.2, we will show how to decompose any simple T-uncrossable bifamily into a small number of uncrossable bifamilies, see Lemma 2.3 to follow. We start with the following useful property of min-cores.

LEMMA 2.1. *Let $C \in \mathcal{C}_\mathcal{F}$ be a min-core of a T-uncrossable bifamily $\mathcal{F}$. If $\hat{C}$ and $\hat{X} \in \mathcal{F}$ are T-independent, then $C \subseteq X$ or $C \cap X^+ = \emptyset$. In particular, the min-cores of $\mathcal{F}$ are pairwise disjoint on T.*

PROOF. Since $\hat{C}$ and $\hat{X}$ are T-independent, $\hat{C} \cap \hat{X} \in \mathcal{F}$ or $\hat{C} \setminus \hat{X} \in \mathcal{F}$. Thus one of the sets $C \cap X$ or $C \setminus X^+$ is an inner part of a biset in $\mathcal{F}$. If the statement in the lemma does not hold, then these sets are strictly contained in C. This contradicts the minimality of C. □

For a bifamily $\mathcal{F}$ on V and $C \subseteq V$ let

$$\mathcal{F}(C) = \{\hat{X} \in \mathcal{F} : X \supseteq C, X \text{ is an } \mathcal{F}\text{-core}\} .$$

DEFINITION 2.1. *Let us say that a bifamily $\mathcal{F}$ is a* ring-bifamily *if $\mathcal{F}$ has a unique min-core and if $\hat{X} \cap \hat{Y}, \hat{X} \cup \hat{Y} \in \mathcal{F}$ holds for any $\hat{X}, \hat{Y} \in \mathcal{F}$.*

Note that any ring-bifamily also has a unique max-core. We need the following fundamental property of cores.

LEMMA 2.2. *Let $\mathcal{F}$ be a T-uncrossable bifamily and let X, Y be $\mathcal{F}$-cores.*

(i) *If X, Y contain the same min-core $C \in \mathcal{C}_\mathcal{F}$ (namely, if $\hat{X}, \hat{Y} \in \mathcal{F}(C)$) then $\hat{X}, \hat{Y}$ are T-independent and $\hat{X} \cap \hat{Y}, \hat{X} \cup \hat{Y} \in \mathcal{F}(C)$.*
(ii) *If X, Y contains distinct min-cores $C_X, C_Y \in \mathcal{C}_\mathcal{F}$, respectively, and if $\hat{X}, \hat{Y}$ are T-independent, then $\hat{X} \setminus \hat{Y} \in \mathcal{F}(C_X)$ and $\hat{Y} \setminus \hat{X} \in \mathcal{F}(C_Y)$.*

Consequently, the bifamily $\{\hat{X} \in \mathcal{F} : X \text{ is an } \mathcal{F}\text{-core}\}$ is also T-uncrossable, and for every min-core $C \in \mathcal{C}_\mathcal{F}$ of $\mathcal{F}$ the following holds: $\mathcal{F}(C)$ is a ring-bifamily, there is a unique max-core M containing C, and $\mathcal{F}(C) = \{\hat{X} \in \mathcal{F} : X \subseteq M\}$.

PROOF. We prove (i). Let $\hat{X}, \hat{Y} \in \mathcal{F}(C)$. As $X \cap Y \cap T \supseteq C \cap T \neq \emptyset$, $\hat{X}, \hat{Y}$ are T-independent. We cannot have $\hat{X} \setminus \hat{Y}, \hat{Y} \setminus \hat{X} \in \mathcal{F}$, as then X will contain two disjoint sets $C, X \setminus Y^+$ that are inner parts of bisets in $\mathcal{F}$, contradicting that X is a core. Hence $\hat{X} \cap \hat{Y}, \hat{X} \cup \hat{Y} \in \mathcal{F}$. Clearly, $X \cap Y$ is a core. It remains to show that $X \cup Y$ is a core. Otherwise, there is a min-core $C' \subseteq X \cup Y$, $C' \neq C$. Then $X \cap T$ or $Y \cap T$ contains a node from $C' \cap T$, say $X \cap C' \cap T \neq \emptyset$. Thus $\hat{X}, \hat{C}'$ are T-independent, and hence $C' \subseteq X$, by Lemma 2.1. But then X contains two distinct cores C, C'. This contradicts that X is a core.

We prove (ii). If (ii) does not hold, then $\hat{X} \cap \hat{Y}, \hat{X} \cup \hat{Y} \in \mathcal{F}$, hence $X \cap Y$ contains some $C \in \mathcal{C}_\mathcal{F}$. If $C \neq C_X$ then X contains two distinct min-cores C, C_X, and if $C \neq C_Y$ then Y contains two distinct min-cores C, C_Y. In both cases we obtain a contradiction.

The last statement of the lemma easily follows from (i) and (ii). □

Later, we will prove the following decomposition lemma.

LEMMA 2.3. *Let $\mathcal{F}$ be a T-uncrossable bifamily with $|C \cap T| \geq q$ for all $C \in \mathcal{C}_{\mathcal{F}}$. If $q \geq \gamma + 1$ then any $\hat{X}, \hat{Y} \in \mathcal{F}$ are T-independent and thus $\mathcal{F}$ is uncrossable. If $\mathcal{F}$ is simple, then $\mathcal{C}_{\mathcal{F}}$ can be partitioned into at most $2\lfloor \gamma/q \rfloor + 1$ parts such that for every part $\mathcal{C}$, the bifamily $\bigcup_{C \in \mathcal{C}} \mathcal{F}(C)$ is uncrossable. Furthermore, given the families $\mathcal{C}_{\mathcal{F}}$ and $\mathcal{M}_{\mathcal{F}}$ of min-cores and max-cores of $\mathcal{F}$, such a partition can be found in polynomial time.*

The following lemma enables us to estimate a progress made towards covering $\mathcal{F}$, if we cover the sub-bifamily $\{\hat{X} \in \mathcal{F} : X \text{ is an } \mathcal{F}\text{-core}\}$ of $\mathcal{F}$.

LEMMA 2.4. *Suppose that the min-cores of a bifamily $\mathcal{F}$ are pairwise disjoint on a subset T of V, and that $|C \cap T| \geq q$ for all $C \in \mathcal{C}_{\mathcal{F}}$. If an edge-set I covers the bifamily $\{\hat{X} \in \mathcal{F} : X$ is an $\mathcal{F}$-core$\}$ then $|C' \cap T| \geq 2q$ for every min-core C' of $\mathcal{F}_I$.*

PROOF. Let C' be an $\mathcal{F}_I$-core. As I covers $\{\hat{X} \in \mathcal{F} : X \text{ is an } \mathcal{F}\text{-core}\}$, C' contains at least two min-cores of $\mathcal{F}$. These two min-cores have no node of T in common, and each of them contains at least q nodes from T, by the assumption. The statement follows. □

Now we deduce Theorem 1.2 from Lemmas 2.1, 2.3, and 2.4. The algorithm starts with $I = \emptyset$, and while $q = \min\{|C \cap T| : C \in \mathcal{C}_{\mathcal{F}_I}\} \leq \lceil (\gamma+1)/2 \rceil = \lfloor \gamma/2 \rfloor + 1$, it adds to I a cover of the bifamily $\{\hat{X} \in \mathcal{F}_I : X \text{ is an } \mathcal{F}_I\text{-core}\}$. Then, at the last iteration, the algorithm adds to I a cover of the entire residual bifamily $\mathcal{F}_I$. Note that at the beginning of an iteration, the bifamily $\{\hat{X} \in \mathcal{F}_I : X \text{ is an } \mathcal{F}_I\text{-core}\}$ is simple and T-uncrossable, by Lemma 2.1, and thus can be partitioned into at most $2\lfloor \gamma/q \rfloor + 1$ uncrossable bifamilies, by Lemma 2.3. Initially, $q \geq 1$, and q is at least doubled during each iteration, by Lemmas 2.1 and 2.4. At the beginning of the last iteration, we have $q \geq \gamma + 1$, and then the residual bifamily $\mathcal{F}_I$ is uncrossable, by Lemma 2.3. Consequently, the total number of uncrossable bifamies we cover is bounded by

$$\begin{aligned}
\sum_{p=0}^{\lceil \lg(\lfloor \gamma/2 \rfloor + 1) \rceil} (2\lfloor \gamma/2^p \rfloor + 1) + 1 &\leq 2 + \lceil \lg(\lfloor \gamma/2 \rfloor + 1) \rceil + 2\gamma \sum_{p=0}^{\lceil \lg(\lfloor \gamma/2 \rfloor + 1) \rceil} (1/2)^p \\
&\leq 2 + \lceil \lg(\lfloor \gamma/2 \rfloor + 1) \rceil + 2\gamma \frac{1 - 1/2^{\lg(\lfloor \gamma/2 \rfloor + 1) + 1}}{1 - 1/2} \\
&\leq 2 + \lceil \lg(\lfloor \gamma/2 \rfloor + 1) \rceil + 4\gamma \left(1 - \frac{1}{2(\lfloor \gamma/2 \rfloor + 1)}\right) \\
&\leq 4\gamma + \lceil \lg(\lfloor \gamma/2 \rfloor + 1) \rceil = O(\gamma) \ .
\end{aligned}$$

In the rest of this section we prove Lemma 2.3. Let $\mathcal{F}$ be a T-uncrossable bifamily on V. If $q \geq \gamma + 1$ then $\mathcal{F}$ is uncrossable, since any $\hat{X}, \hat{Y} \in \mathcal{F}$ are T-independent; otherwise, there are $\hat{X}, \hat{Y} \in \mathcal{F}$ such that $|X \cap \Gamma(\hat{Y}) \cap T| \geq \gamma + 1$, contradicting the definition of γ.

Now assume that $\mathcal{F}$ is simple. Let $\mathcal{C}_{\mathcal{F}} = \{C_1, \ldots, C_\nu\}$ be the family of min-cores of $\mathcal{F}$ and let M_i be the (unique, by Lemma 2.2) max-core containing C_i.

Definition 2.2. *We say that* $C_i, C_j \in \mathcal{C}_\mathcal{F}$ *are* strongly T-independent *if both* $\hat{M}_i, \hat{C}_j$ *are* T*-independent and* $\hat{M}_j, \hat{C}_i$ *are* T*-independent.*

Note that $\mathcal{F}(C_i) = \{\hat{X} \in \mathcal{F} : X \subseteq M_i\}$ for every i, by Lemma 2.2. Hence if C_i, C_j are strongly T-independent then any $\hat{X}, \hat{Y} \in \mathcal{F}$ with $X \subseteq M_i$ and $Y \subseteq M_j$ are T-independent, by the monotonicity of $\mathcal{F}$. Thus we have the following.

Corollary 2.5. *Let* $C_i, C_j \in \mathcal{C}_\mathcal{F}$ *be strongly* T*-independent. Then for any* $\hat{X}, \hat{Y} \in \mathcal{F}$ *with* $X \subseteq M_i$ *and* $Y \subseteq M_j$ *we have* $\hat{X} \cap \hat{Y}, \hat{X} \cup \hat{Y} \in \mathcal{F}$ *if* $i = j$*, and* $\hat{X} \setminus \hat{Y}, \hat{Y} \setminus \hat{X} \in \mathcal{F}$ *if* $i \neq j$*. Thus for any* $\mathcal{C} \subseteq \mathcal{C}_\mathcal{F}$*, if the members of* $\mathcal{C}$ *are pairwise strongly* T*-independent, then the bifamily* $\bigcup_{C \in \mathcal{C}} \mathcal{F}(C) = \{\hat{X} \in \mathcal{F} : X \subseteq M_i, C_i \in \mathcal{C}\}$ *is uncrossable.*

Consequently, the following lemma finishes the proof of Lemma 2.3.

Lemma 2.6. *If* $|C_i \cap T| \geq q$ *for all* i*, then* $\mathcal{C}_\mathcal{F}$ *admits a partition into at most* $2\lfloor \gamma/q \rfloor + 1$ *parts such that the members of each part are pairwise strongly* T*-independent, and given the families* $\mathcal{C}_\mathcal{F}, \mathcal{M}_\mathcal{F}$ *such a partition can be found in polynomial time.*

Proof. Construct an auxiliary directed graph $\mathcal{J}$ as follows. The node set of $\mathcal{J}$ is $\mathcal{C}_\mathcal{F}$. Add an arc C_iC_j if $T \cap C_i \subseteq \Gamma(\hat{M}_j)$. The indegree of every node in $\mathcal{J}$ is at most $\lfloor \gamma/q \rfloor$, by Lemma 2.2(i). This implies that every subgraph of the underlying graph of $\mathcal{J}$ has a node of degree $\leq 2\lfloor \gamma/q \rfloor$. A graph is *d-degenerate* if every subgraph of it has a node of degree $\leq d$. It is known that any d-degenerate graph can be colored in polynomial time with $(d+1)$ colors. Hence $\mathcal{J}$ is $(2\lfloor \gamma/q \rfloor + 1)$-colorable, and such a coloring can be computed in polynomial time. Consequently, $\mathcal{C}_\mathcal{F}$ can be partitioned in polynomial time into $2\lfloor \gamma/q \rfloor + 1$ independent sets, as required. □

The proof of Lemma 2.3, and thus also of Theorem 1.2, is complete.

3. PROOF OF THEOREM 1.3

The proof of Theorem 1.3 is along the lines of the proofs of [Nutov 2010a, Theorem 1.3] and [Nutov 2010b, Theorem 1.4], where *set-families* are considered. The latter two theorems rely on [Nutov 2010a, Theorem 2.3] and [Nutov 2010b, Theorem 2.3], respectively, that established a certain "spider-decomposition" of covers of *set-families*. We present a full proof only of an analogous "spider-decomposition" theorem for bifamilies (Theorems 3.1 and 3.5 to follow); the other parts of the proof of Theorem 1.3 are essentially identical to the ones in [Nutov 2010a; 2010b].

We start by extending the concept of "spider-covers", introduced in [Nutov 2010a; 2010b], from set-families to bifamilies. For $s \in V$ and $C \in \mathcal{C}_\mathcal{F}$ let

$$\mathcal{F}(s, C) = \{\hat{X} \in \mathcal{F}(C) : s \notin X^+\} \ .$$

Note that by Lemma 2.2(i), $\mathcal{F}(C)$, and thus also $\mathcal{F}(s, C)$, is a ring-bifamily if $\mathcal{F}$ is an uncrossable (or even a T-uncrossable) bifamily.

Definition 3.1. *Let* $\mathcal{F}$ *be a bifamily on* V *and let* $\mathcal{C} \subseteq \mathcal{C}_\mathcal{F}$*. We say that an edge-set* S *on* V *is an* $\mathcal{F}(s, \mathcal{C})$-spider-cover *(for illustration see Fig. 1) if* $s \in V(S)$ *and if* S *can be partitioned into* $\mathcal{F}(s, C)$*-covers* $\{P_C : C \in \mathcal{C}\}$ *such that the node sets* $\{V(P_C) \setminus \{s\} : C \in \mathcal{C}\}$ *are pairwise disjoint. We say that* S *is an* $\mathcal{F}(\mathcal{C})$-spider-cover *(or simply a* spider-cover*, if* $\mathcal{C}$ *is clear from the context) if the following holds:*

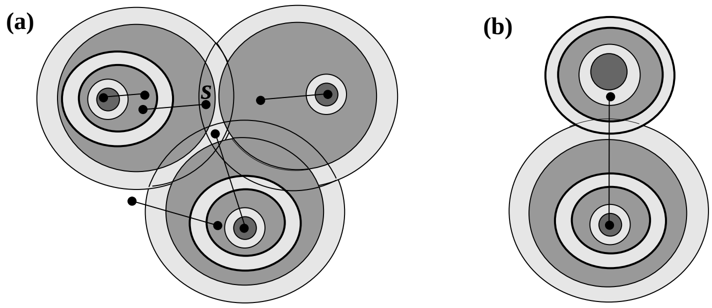

Fig. 1. Examples of $\mathcal{F}(s,\mathcal{C})$-spider-covers. The inner parts of the bisets are shown by darker gray circles, min-cores in $\mathcal{C}$ are shown by darkest gray circles.

—*If* $|\mathcal{C}| \geq 2$ *then there exists* $s \in V$ *(a* center *of the spider-cover) such that* S *is an* $\mathcal{F}(s,\mathcal{C})$*-spider-cover.*

—*If* $|\mathcal{C}| = 1$*, say* $\mathcal{C} = \{C\}$*, then* S *covers* $\mathcal{F}(C)$*.*

Equivalently, for $|\mathcal{C}| \geq 2$, an $\mathcal{F}(\mathcal{C})$-spider-cover S with a chosen center $s \in V(S)$ is a union of $\mathcal{F}(s,C)$-covers $\{P_C : C \in \mathcal{C}\}$ so that only s can be a common end-node of two of them. Note that there might be $C \in \mathcal{C}$ such that P_C does not cover $\hat{C}$. This may happen if $|\mathcal{C}| \geq 2$ and $s \in C^+$ for some $C \in \mathcal{C}$; then $\mathcal{F}(s,C) = \emptyset$ and $P_C = \emptyset$ is an $\mathcal{F}(s,C)$-cover, although no edge in P_C covers $\hat{C}$ itself (see Fig. 1(b)).

Definition 3.2. *Let* $\mathcal{F}$ *be a bifamily on* V *and let* $\mathcal{C} \subseteq \mathcal{C}_{\mathcal{F}}$*. We say that a collection* $\mathcal{S} = \{S_1,\ldots,S_h\}$ *of edge-sets* spider-covers $\mathcal{C}$ *if the following holds:*

—*The node-sets* $V(S_1),\ldots,V(S_h)$ *are pairwise disjoint.*

—$\mathcal{C}$ *admits a partition* $\{\mathcal{C}_1,\ldots,\mathcal{C}_h\}$ *such that each* S_t *is an* $\mathcal{F}(\mathcal{C}_t)$*-spider-cover.*

In [Nutov 2010a] *directed* covers of *intersecting set-families* are considered, when $X,Y \in \mathcal{F}$ and $X \cap Y \neq \emptyset$ implies $X \cap Y, X \cup Y \in \mathcal{F}$, and for every $X \in \mathcal{F}$ there should be an edge in I *entering* X. For this case, [Nutov 2010a, Theorem 2.3] states that any cover I of $\mathcal{F}$ admits a "tail-disjoint" subpartition that spider-covers a subfamily $\mathcal{C} \subseteq \mathcal{C}_{\mathcal{F}}$ of size at least $|\mathcal{C}| \geq \lceil 2|\mathcal{C}_{\mathcal{F}}|/3 \rceil$; in the setting of [Nutov 2010a] this bound is the best possible. [Nutov 2010b, Theorem 2.3] states that any (undirected) cover I of an uncrossable *set-family* $\mathcal{F}$ admits a subpartition that spider-covers the entire family $\mathcal{C}_{\mathcal{F}}$ of $\mathcal{F}$-cores. In the case of bifamilies, the situation is more involved, and by extending the method from [Nutov 2010a] we prove the following.

Theorem 3.1. *Let* $\mathcal{F}$ *be a simple bifamily such that the* $\mathcal{F}$*-cores are pairwise disjoint and such that* $\mathcal{F}(C)$ *is a ring-bifamily for every* $C \in \mathcal{C}_{\mathcal{F}}$*. Then any cover* I *of* $\mathcal{F}$ *admits a subpartition that spider-covers a subfamily* $\mathcal{C} \subseteq \mathcal{C}_{\mathcal{F}}$ *of* $\mathcal{F}$*-cores of size at least* $|\mathcal{C}| \geq \lceil 2|\mathcal{C}_{\mathcal{F}}|/3 \rceil$*.*

We now prove Theorem 3.1, and at the end of this section briefly describe how Theorem 3.1 implies Theorem 1.3.

The proofs of the next two statements, Claim 3.2 and Lemma 3.3, are similar to the proofs of Lemma 2.6 and Corollary 2.5 from [Nutov 2010a], respectively, where *directed* covers of ring-set-families were considered.

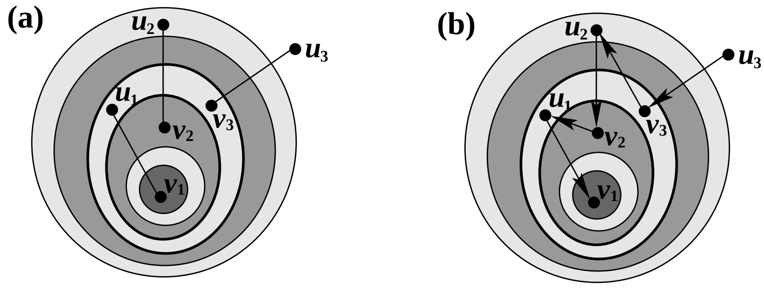

Fig. 2. (a) Illustration to Lemma 3.3. (b) Construction of the path P_C.

Claim 3.2. *Let I be an inclusion-minimal cover of a ring-bifamily $\mathcal{F}$ and let C be the min-core of $\mathcal{F}$. Then $|\delta_I(\hat{C})| = 1$.*

Proof. Clearly, $|\delta_I(\hat{C})| \geq 1$ since I covers $\mathcal{F}$ and since $\hat{C} \in \mathcal{F}$. Suppose to the contrary that there are distinct $e, f \in \delta_I(\hat{C})$. By the minimality of I, there are $\hat{W}_e, \hat{W}_f \in \mathcal{F}$ such that $\delta_I(\hat{W}_e) = \{e\}$ and $\delta_I(\hat{W}_f) = \{f\}$. There is an edge in I covering $\hat{W}_e \cup \hat{W}_f$, because $\hat{W}_e \cup \hat{W}_f \in \mathcal{F}$. This edge must be one of e, f, because if for arbitrary bisets $\hat{X}, \hat{Y}$ an edge covers $\hat{X} \cup \hat{Y}$ then it also covers one of $\hat{X}, \hat{Y}$. Each of e, f covers $\hat{W}_e \cap \hat{W}_f$, because each of e, f has an endnode in C and $C \subseteq W_e \cap W_f$. Consequently, one of e, f covers both $\hat{W}_e \cap \hat{W}_f$ and $\hat{W}_e \cup \hat{W}_f$. However, if for arbitrary bisets $\hat{X}, \hat{Y}$ an edge covers both $\hat{X} \cap \hat{Y}, \hat{X} \cup \hat{Y}$ then it cover both $\hat{X}$ and $\hat{Y}$. Hence one of e, f covers both $\hat{W}_e, \hat{W}_f$. This is a contradiction, since $\delta_I(\hat{W}_e) = \{e\}$, $\delta_I(\hat{W}_f) = \{f\}$, and $e \neq f$. □

Lemma 3.3. *Let I be an inclusion-minimal cover of a ring-bifamily $\mathcal{F}$. There exists an ordering $e_1, \ldots, e_q$ of I and a nested set-family $C_1 \subset \cdots \subset C_q$ of sets in $\{X : \hat{X} \in \mathcal{F}\}$ such that for every $j = 1, \ldots, q$ the following holds (see Fig. 2(a)):*

(i) *C_j is the min-core of $\mathcal{F}_{I_{j-1}}$, where $I_j = \{e_1, \ldots, e_j\}$ and $I_0 = \emptyset$, and e_j is the unique edge in I covering $\hat{C}_j$.*

(ii) *If $e_j = u_j v_j$ where $v_j \in C_j$, then I_j is an $\mathcal{F}(u_j, C)$-cover and I_{j-1} is an $\mathcal{F}(v_j, C)$-cover, where C is the min-core of $\mathcal{F}$.*

Proof. Let $C_1 = C$. By Claim 3.2 there is a unique edge $e_1 \in I$ covering $\hat{C}_1$. Let $e_1 = u_1 v_1$, where $v_1 \in C_1$. Then clearly $I_0 = \emptyset$ is an $\mathcal{F}(v_1, C)$-cover and $I_1 = \{e_1\}$ is an $\mathcal{F}(u_1, C)$-cover. Thus if e_1 covers $\mathcal{F}$ we are done. Otherwise, let C_2 be the min-core of $\mathcal{F}_{I_1}$. Then $C_1 \subset C_2$. Let $e_2 = u_2 v_2$ be the edge in I covering $\hat{C}_2$, where $v_2 \in C_2$. As C_2 is the min-core of $\mathcal{F}_{I_1}$ and $v_2 \in C_2$, it follows that I_1 is an $\mathcal{F}(v_2, C)$-cover and $I_2 = I_1 \cup \{e_2\}$ is an $\mathcal{F}(u_2, C)$-cover. We can continue this process until some edge e_q covers $\mathcal{F}_{I_{q-1}}$. Namely, given the edge set $I_{j-1} = \{e_1, \ldots, e_{j-1}\}$ that still does not cover $\mathcal{F}$, C_j is the min-core of $\mathcal{F}_{I_{j-1}}$, and $e_j = u_j v_j$ is the edge in I covering $\hat{C}_j$, where $v_j \in C_j$. Then $C_{j-1} \subset C_j$. As X_j is a min-core of $F_{I_{j-1}}$ and $v_j \in X_j$, it follows that I_{j-1} is an $\mathcal{F}(v_j, C)$-cover and I_j is an $\mathcal{F}(u_j, C)$-cover. The lemma follows. □

Recall that a *directed spider* is a directed tree (arborescence) with at most one node (the root) of outdegree ≥ 2. The following statement is an immediate consequence from [Chuzhoy and Khanna 2008, Theorem 4].

LEMMA 3.4 [CHUZHOY AND KHANNA 2008]. *Let $\mathcal{Q}$ be a set of directed simple paths ending at a set $A = \{a_P : P \in \mathcal{P}\}$ of distinct nodes. There exists $\mathcal{P} \subseteq \mathcal{Q}$ with $\mathcal{P} \geq \lceil 2|\mathcal{Q}|/3 \rceil$ such that the following holds. Every $P \in \mathcal{P}$ has a subpath P' (possibly of length zero) that ends at a_P and has no internal node in A, such that in the (simple) graph J induced by the subpaths $\{P' : P \in \mathcal{P}\}$, every connected component is either a directed spider with at least 2 nodes in A, or is a path in $\mathcal{P}$.*

Now we finish the proof of Theorem 3.1. For every $C \in \mathcal{C}_\mathcal{F}$ fix some inclusion-minimal cover $I_C \subseteq I$ of $\mathcal{F}(C)$. Let $e_1, \ldots, e_q$ be an ordering of I_C as in Lemma 3.3, where $e_j = u_j v_j$ is as in the lemma. Obtain a directed path P_C (see Fig. 2(b)) by orienting every edge $e_j = u_j v_j$ from u_j to v_j and adding for every $j = q, \ldots, 2$ the directed edge $v_j u_{j-1}$, if $v_j \neq u_{j-1}$; hence if $v_j \neq u_{j-1}$ for all j, then the node sequence of P_C is $(u_q, v_q, u_{q-1}, v_{q-1}, \ldots, u_1, v_1)$. Let $a_C = v_1$ and note that $a_C \in C$. Let $\mathcal{Q} = \{P_C : C \in \mathcal{C}_\mathcal{F}\}$. As the min-cores of $\mathcal{F}$ are pairwise disjoint, the path in $\mathcal{Q}$ end at distinct nodes. Hence Lemma 3.4 applies, and thus there exists a subfamily $\mathcal{C} \subseteq \mathcal{C}_\mathcal{F}$ of $\mathcal{F}$-cores of size at least $|\mathcal{C}| \geq \lceil 2|\mathcal{C}_\mathcal{F}|/3 \rceil$, such that the following holds. Every P_C with $C \in \mathcal{C}$ has a subpath P'_C that ends at a_C, such that if $J_1, \ldots, J_h$ are the connected components of the (simple) graph J induced by the subpaths $\{P'_C : C \in \mathcal{C}\}$, then every J_t is either a directed spider with at least 2 nodes in $\{a_C : C \in \mathcal{C}\}$, or is a path in $\mathcal{P}$. For every $t = 1, \ldots, h$ let $\mathcal{C}_t = \{C : v_C \in J_t\}$ and let $S_t = J \cap I$ be the set of those edges $e \in I$ for which the orientation of e is in J_t. From the construction and Lemma 3.3 it follows that S_t is an $\mathcal{F}(\mathcal{C}_t)$-spider-cover. Thus the collection $\mathcal{S} = \{S_1, \ldots, S_h\}$ of edge-sets spider-covers $\mathcal{C}$. Since $|\mathcal{C}| \geq \lceil 2|\mathcal{C}_\mathcal{F}|/3 \rceil$, Theorem 3.1 follows.

We now briefly describe how Theorem 3.1 implies Theorem 1.3; for details see [Nutov 2010a; 2010b]. Note that Definitions 3.1 and 3.2 consider covers only of bisets in $\mathcal{F}$ for which the inner parts are cores, namely, the relevant bifamily is $\{\hat{X} \in \mathcal{F} : X \text{ is an } \mathcal{F}\text{-core}\}$; the latter is uncrossable if $\mathcal{F}$ is, by Lemma 2.2. Any uncrossable simple bifamily satisfies the assumptions of Theorem 3.1, by Lemma 2.2. Thus we have the following.

THEOREM 3.5. *Any cover I of an uncrossable bifamily $\mathcal{F}$ admits a subpartition that spider-covers a subfamily $\mathcal{C} \subseteq \mathcal{C}_\mathcal{F}$ of $\mathcal{F}$-cores of size at least $|\mathcal{C}| \geq \lceil 2|\mathcal{C}_\mathcal{F}|/3 \rceil$.*

For a bifamily $\mathcal{F}$ and an edge set I, let $\nu(I)$ denote the number $|\mathcal{C}_{\mathcal{F}_I}|$ of min-cores of the residual bifamily $\mathcal{F}_I$. Given a partial solution I, the *density* of an edge set $S \subseteq E \setminus I$ is $c(S)/(\nu(I) - \nu(I \cup S))$. The ρ-Greedy Algorithm starts with $I = \emptyset$, and as long as $\nu(I) \geq 1$, it finds and adds to I an edge-set $S \subseteq E \setminus I$ of density at most $\rho \cdot \mathsf{opt}/\nu(I)$, where opt denotes the optimal solution value. It is known that the ρ-Greedy Algorithm, if can be applied, computes a solution I such that $c(I) \leq \rho \ln(\nu(\emptyset) + 1) \cdot \mathsf{opt}$.

One can show that if S is an $\mathcal{F}_I(\mathcal{C})$-spider-cover, then $\nu(I) - \nu(I \cup S) \geq |\mathcal{C}|/3$ (in the particular case of set-families arising from the Steiner Forest problem, we have an improved bound $\nu(I) - \nu(I \cup S) \geq |\mathcal{C}|/2$ [Klein and Ravi 1995]). By an averaging argument, Theorem 3.5 implies that there exists an $\mathcal{F}(\mathcal{C})$-spider-cover of density at most $3 \cdot 3/2 \cdot \mathsf{opt}/\nu(I)$ (for set-families a density of $3 \cdot \mathsf{opt}/\nu(I)$ is achieved, while for set-families arising from the Steiner Forest problem a density of $2 \cdot \mathsf{opt}/\nu(I)$ is achieved). Using this, one can design, under Assumptions 1, 2, a polynomial time algorithm that finds an edge set S (which may not be a spider-cover) of density at most $9 \cdot \mathsf{opt}/\nu(I)$. An additional factor of 2 is invoked since to find such S, we "guess" the center s and compute for every min-core C a 2-approximate $\mathcal{F}(s,C)$-cover; no polynomial time algorithm is known for computing an optimal $\mathcal{F}(s,C)$-cover (for set-families arising from the Steiner Forest problem, one can compute an optimal $\mathcal{F}(s,C)$-cover by a shortest path computation [Klein and Ravi 1995]). We note that in [Nutov 2010b], an algorithm that computes a 2-approximate $\mathcal{F}(s,C)$-cover for an uncrossable *set-family* $\mathcal{F}$ uses as a subroutine a 2-approximation algorithm for finding a minimum node-weight edge-cover of a ring-set-family. Such an algorithm easily follows from the observation that if I is an inclusion-minimal cover of a ring-set-family, then $|\delta_I(v)| \leq 2$ for all $v \in V$. The same statement is true for *ring-bifamilies*, by Lemma 3.3.

4. PROOF OF THEOREM 1.6

The following statement finishes the proof of Proposition 1.5.

CLAIM 4.1. *Suppose that for edge-costs* Bifamily Edge-Cover *with T-uncrossable $\mathcal{F}$ admits a polynomial time algorithm that computes a solution of cost $\leq \rho(\gamma)\cdot\tau(\mathcal{F})$, where $\mathcal{F}$ is the T-uncrossable bifamily from Corollary 1.1, and γ, ρ are as in Proposition 1.5. Then* Rooted Survivable Network *admits a polynomial time algorithm that computes a solution of cost $\leq \mathsf{opt} \cdot \sum_{\ell=1}^{k} \frac{\rho(\ell)}{k-\ell+1}$.*

PROOF. Consider the following sequential algorithm. Start with $J = \emptyset$. At iteration $\ell = 1, \ldots, k$, add to J an augmenting edge-set I_ℓ that increases by 1 the connectivity between pairs in $\mathcal{T}_\ell = \{\{s,t\} : \lambda_J^S(s,t) = r(s,t)-k+\ell-1, s,t \in T\}$. Let T_ℓ be the union of the pairs in $\mathcal{T}_\ell$ and $\mathcal{F}_\ell$ the corresponding T_ℓ-uncrossable bifamily as in Corollary 1.1. Note that $\max\limits_{\{s,t\}\in\mathcal{T}_\ell} \lambda_J^S(s,t) \leq \ell$, hence $c(I_\ell) = O(\ell)\cdot\tau(\mathcal{F}_\ell)$. After iteration ℓ, we have $\lambda_J^S(s,t) \geq r(s,t) - k + \ell$ for all $s,t \in T$. Consequently, after k iterations $\lambda_J^S(s,t) \geq r(s,t)$ holds for all $s,t \in T$, thus the computed solution is feasible. For the approximation ratio, it is sufficient to show that $\tau(\mathcal{F}_\ell) \leq \mathsf{opt}/(k-\ell+1)$. For all $U \in \mathcal{F}_\ell$, any feasible solution H to Rooted Survivable Network has at least $k-\ell+1$ edges covering U, by Menger's Theorem. Thus if x is a characteristic vector of $E(H)$, then $x/(k-\ell+1)$ is a feasible solution for the LP-relaxation for edge-covering $\mathcal{F}_\ell$. The statement follows. □

CLAIM 4.2. *For both edge-costs and node-costs, a ρ-approximation for* Rooted Survivable Network *with requirements in $\{0,k\}$ implies a $\rho\cdot\min\{k,|T|-1\}$-approximation for* Subset k-Connected Subgraph.

PROOF. Choose arbitrary $\min\{k,|T|-1\}$ roots and for each root s compute a ρ-approximation for Rooted Survivable Network with requirements $r(s,t) = k$ for each $t \in T \setminus \{s\}$. Then take the union of the $\min\{k,|T|-1\}$ subgraphs computed. It is known and easy to see that the computed solution is feasible, and its cost is as claimed. □

Except for Survivable Network with node-costs, Theorem 1.6 easily follows from Theorem 1.4, Corollary 1.1, and Claims 4.1 and 4.2. As for Survivable Network with node-costs, it is remarked in [Chuzhoy and Khanna 2009] that a β-approximation for Element-Connectivity Survivable Network implies an $O(k^3\beta\log|T|)$-approximation for Survivable Network. Thus for node-costs, our $O(k\log|T|)$-approximation for Element-Connectivity Survivable Network together with the result of [Chuzhoy and Khanna 2009], implies an $O(k^4\log^2|T|)$-approximation algorithm for Survivable Network. This finishes the proof of Theorem 1.6.

5. CONCLUSIONS, RECENT DEVELOPMENTS, AND OPEN PROBLEMS

In this paper we developed approximation algorithms for Survivable Network problems for both edge-costs and node-costs. Our algorithms are simple and combinatorial, and they achieve much better approximation guarantees than those previously known. For edge-costs, our ratios are $O(k\ln k)$ for Rooted Survivable Network and $O(k^2\ln k)$ for Subset k-Connected Subgraph. These ratios are constants for constant values of k. For node-costs, we give the first non-trivial algorithm for Element-Connectivity Survivable Network, matching (up to constants) the best known ratio for Edge-Connectivity Survivable Network.

We mention some recent developments. [Laekhanukit 2011] showed that for undirected graphs and $|T|\le 2k$, one can solve an instance of Subset k-Connected Subgraph Augmentation by solving $O(\log k)$ instances of Rooted Survivable Network Augmentation. Thus for $|T|\ge 2k$, our $O(k)$-approximation algorithm leads to the ratio $O(k\log^2 k)$ for Subset k-Connected Subgraph. By an additional analysis, he reduced the ratio to $O(k\log k)$ when $|T|\ge k^2$. In [Nutov 2011] it is shown that for both directed and undirected graphs, and edge-costs and node-costs, Subset k-Connected Subgraph Augmentation can be reduced to solving only one instance (or two instances, in the case of directed graphs) of Rooted Survivable Network Augmentation and $\left(\frac{|T|}{|T|-k}\right)^2\cdot O\left(\log\frac{|T|}{|T|-k}\right)$ instances of the Min-Cost k-Flow problem. Still, the main procedure in both algorithms is an algorithm for the rooted case.

Recently, [Panigrahi 2011] suggested a generalization of Survivable Network with node-costs, that also captures several known problems in wireless network design. In [Nutov 2012b], similar ratios to the ones obtained in this paper are shown for this more general problem, and in addition, an $O(\log|\mathcal{C}_{\mathcal{F}}|)$ ratio is obtained for *directed* Bifamily Edge-Cover problem with *intersecting* bifamily $\mathcal{F}$.

A minor open problem is to achieve a ratio of $O(k)$ for Rooted Survivable Network with edge-costs. Such an algorithm would probably rely on the iterative rounding method, while all algorithms in this paper are combinatorial. A more important open problem is to achieve a ratio of $\tilde{O}(k^2)$ for general Survivable Network.

Finally, we note that for all problems considered, a significant obstacle lies in the way of achieving a ratio sublinear in k. As mentioned in [Lando and Nutov 2009], this would imply a sublinear in n ratio for the *directed* variant. Such algorithms are known only for $k=1$, and they are highly non-trivial; see [Charikar et al. 1999] for the rooted case and [Feldman et al. 2012] for the general case. Furthermore, for the directed variant, even for rooted requirements, no ratio better than the trivial $O(n)$ is known even for $k=2$; it is a major open problem to achieve a sublinear ratio for this variant.

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