

Approximating minimum-cost connectivity problems via uncrossable bifamilies

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We give approximation algorithms for the **Generalized Steiner Network (GSN)** problem. The input consists of a graph $G = (V, E)$ with edge/node-costs, a node subset $S \subseteq V$, and connectivity requirements $\{r(s, t) : s, t \in T \subseteq V\}$. The goal is to find a minimum cost subgraph H of G that for all $s, t \in T$ contains $r(s, t)$ pairwise edge-disjoint st -paths such that no two of them have a node in $S - \{s, t\}$ in common. Three extensively studied particular cases are: **Edge-GSN** ($S = \emptyset$), **Node-GSN** ($S = V$), and **Element-GSN** ($r(s, t) = 0$ whenever $s \in S$ or $t \in S$). Let $k = \max_{s, t \in T} r(s, t)$. In **Rooted GSN** there is $s \in T$ so that $r(u, t) = 0$ for all $u \neq s$, and in the **Subset k -Connected Subgraph** problem $r(s, t) = k$ for all $s, t \in T$.

For edge-costs, our ratios are $O(k \log k)$ for **Rooted GSN** and $O(k^2 \log k)$ for **Subset k -Connected Subgraph**. This improves the previous ratio $O(k^2 \log n)$, and for bounded values of k settles the approximability of these problems to a constant.

For node-cost, our ratios are:

- $O(k \log |T|)$ for **Element-GSN**, matching the best known ratio for **Edge-GSN**.
- $O(k^2 \log |T|)$ for **Rooted GSN** and $O(k^3 \log |T|)$ for **Subset k -Connected Subgraph**, improving the ratio $O(k^8 \log^2 |T|)$.
- $O(k^4 \log^2 |T|)$ for **GSN**; this is the first non-trivial approximation algorithm for the problem.

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1. INTRODUCTION

In network design connectivity problems the goal is to find a low cost subgraph that satisfies prescribed connectivity requirements. When only connectedness is required between certain pairs of nodes, classic examples are: **Shortest Path**, **Minimum Spanning Tree**, **Minimum Steiner Tree/Forest**, and others. The corresponding examples with high connectivity requirements are: **Min-Cost k -Flow**, **k -Edge/Node-Connected Spanning Subgraph**, **Steiner Network**, and others.

For an edge set I on node set V let $V(I) = \bigcup_{uv \in I} \{u, v\}$ denote the set of end-nodes of the edges in I . Given node-costs $\{c(v) : v \in V\}$, let $c(I) = c(V(I))$ be

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the *node-cost* of I . For a subset S of nodes in a graph H , let $\lambda_H^S(s, t)$ denote the S -*connectivity* between s and t in H , namely, the maximum number of pairwise edge-disjoint st -paths in H so that no two of them have a node in $S - \{s, t\}$ in common. We consider the following fundamental problem on undirected graphs, that includes as a special case the problems mentioned above.

Generalized Steiner Network (GSN)

Instance: A graph $G = (V, E)$ with edge/node-costs, $S \subseteq V$, and S -connectivity requirements $\{r(s, t) : s, t \in T \subseteq V\}$.

Objective: Find a minimum cost subgraph H of G so that $\lambda_H^S(s, t) \geq r(s, t)$ for all $s, t \in T$.

Extensively studied particular cases of GSN are: **Edge-GSN** ($S = \emptyset$), **Node-GSN** ($S = V$), and **Element-GSN** ($r(s, t) = 0$ whenever $s \in S$ or $t \in S$). Edge-GSN is often called **Steiner Network** in the literature, c.f. [Jain 2001], and various variants of GSN are also referred to as the **Survivable Network Design Problem (SNDP)** in the literature, c.f. [Goemans et al. 1994; Ravi and Williamson 1997]. Element-GSN is essentially the edge-connectivity version of the problem on hypergraphs, studied in the 90s by Frank, Benczur, and many others; see e.g. [Nutov 2009b] and the references therein. We note that GSN can be reduced to Node-GSN by elementary constructions. Thus all our results for Node-GSN extend to GSN, and we simply write GSN to mean Node-GSN. In **Rooted GSN** there is $s \in T$ so that $r(u, t) = 0$ for all $u \neq s$ and in **Subset k -Connected Subgraph** $r(s, t) = k$ for all $s, t \in T$. The latter problem generalizes the **k -Connected Subgraph** problem; see [Nutov 2009a; Fackharoenphol and Laekhanukit 2008] and the references therein. We refer the reader to a survey [Kortsarz and Nutov 2007], and here mention some literature relevant to this paper. Let $k = \max_{s, t \in T} r(s, t)$. The first approximation algorithms for the problem appeared in the 90s for the **Steiner Forest** problem – the case $k = 1$. Agrawal, Klein, & Ravi [Agrawal et al. 1995] gave a 2-approximation for edge-costs (see also [Goemans and Williamson 1995; Goemans et al. 1994] for a more general result and a simpler analysis), and [Klein and Ravi 1995] gave an $O(\log n)$ -approximation for node-costs. The latter ratio is essentially (up to constants) the best possible, as the node-costs version is **Set-Cover** hard [Klein and Ravi 1995].

For $k \geq 2$, a line of research initiated by Frank, Goemans and Williamson, and others, was to study a more general setting of edge-covering the “set-function” arising from the GSN variant. For example, **Edge-GSN** can be formulated as a **Set-Function Edge-Cover** problem as follows. An edge e covers a set X if it has exactly one endnode in X . Let $\delta_H(X)$ denote the set of edges in a graph H that cover X . By Menger’s Theorem a subgraph H of G is a feasible solution to a GSN instance if, and only if $|\delta_H(X)| \geq f(X)$ for all $X \subseteq V$, where $f(X) = \max\{r(s, t) : |X \cap \{s, t\}| = 1\}$. This set-function f is *weakly supermodular*, namely,

$$f(X) + f(Y) \leq \max\{f(X \cap Y) + f(X \cup Y), f(X \setminus Y) + f(Y \setminus X)\} \text{ for all } X, Y \subseteq V.$$

A set-family \mathcal{F} is *uncrossable* if for any $X, Y \in \mathcal{F}$ we have $X \cap Y, X \cup Y \in \mathcal{F}$ or $X \setminus Y, Y \setminus X \in \mathcal{F}$. It is known (c.f. [Goemans et al. 1994]) that the problem of edge-covering a weakly supermodular set-function f can be decomposed into $f_{\max} = \max_{X \subseteq V} f(X)$ problems of edge-covering an uncrossable set-family.

The seminal paper [Jain 2001], and numerous papers preceding it, considered Edge-GSN with edge-costs, and developed novel tools for approximating minimum cost edge-covers of several types of set functions and families. [Jain 2001] gave a 2-approximation algorithm for edge-covering a weakly-supermodular set-function using the iterative rounding method. Earlier, [Goemans et al. 1994] gave a combinatorial (primal-dual/local-ratio) 2-approximation algorithm for the special case of uncrossable set-families. The 2-approximation of [Jain 2001] for Edge-GSN was extended to Element-GSN by Fleisher, Jain, and Williamson [Fleischer et al. 2006] and by Cheriyan, Vempala, and Vetta [Cheriyan et al. 2006].

Recently, progress has also been made for node-costs. In [Nutov 2010] was developed an $O(\log |V|)$ -approximation algorithm for edge-covering an uncrossable set-family by a minimum *node-cost* edge set. This algorithm generalizes the algorithm of [Klein and Ravi 1995] for GSN with $k = 1$, and for node-costs implies an $O(k \log |T|)$ -approximation algorithm for Edge-GSN, and also for Node-GSN with $k \leq 2$. In [Nutov 2010] it is also proved that for large values of k , even the simplest version of Edge-GSN with node-costs when $r(s, t) \neq 0$ for only one pair s, t , is at least as hard to approximate as the Densest k -Subgraph problem.

We survey some results for GSN with edge-costs. A hardness result of [Kortsarz et al. 2004] suggests that Subset k -Connected Subgraph is unlikely to admit a polylogarithmic approximation; this is so even when the input graph is complete and the costs are in $\{0, 1\}$ [Nutov 2009b]. Chakraborty, Chuzhoy, and Khanna [Chakraborty et al. 2008] extended this to $\Omega(k^\varepsilon)$ -hardness for any $k \geq k_0$, where k_0 and $\varepsilon > 0$ are universal constants. [Lando and Nutov 2009] proved that for $k = n/2 + k'$ the approximability of the undirected Node-GSN variant is the same (up to a factor of 2) as that of the directed one with maximum requirement k' . This is so also for Rooted Node-GSN. The directed variant of Rooted GSN includes as a special case, when $k' = 1$, the Directed Steiner Tree problem. The latter is not known to admit a polylogarithmic approximation, but admits an $O(n^\varepsilon)$ -approximation scheme [Charikar et al. 1999]; for $k' = 2$ no sublinear approximation for the directed rooted variant is known. On the positive side, the best known ratios for GSN problems were: $O(k^3 \log n)$ for Node-GSN [Chuzhoy and Khanna 2009], $O(k^2 \log n)$ for Subset k -Connected Subgraph by [Chuzhoy and Khanna 2008], and $O(k^2 \log n)$ for Rooted Node-GSN [Chuzhoy and Khanna 2008] and [Nutov 2009e]. GSN also admits an $O(\log k)$ -approximation for metric edge-costs [Cheriyan and Vetta 2007]. For node-costs, non-trivial approximation ratios were known only for Rooted GSN; $O(k^8 \log^2 n)$ by [Chuzhoy and Khanna 2008]. We note that in [Nutov 2009a] the author announced an $O(k^4 \log^2 n)$ -approximation algorithm for Rooted Node-GSN with node-costs, but a full proof of this result was not published, since the much better result presented in this paper was found.

As was mentioned, Edge-GSN can be formulated as a Set-Function Edge-Cover problem with weakly supermodular set function f . For other GSN problems a similar formulation can be given in terms of *setpairs* instead of sets [Frank and Jordán 1995; Fleischer et al. 2006; Cheriyan and Vempala 2001]. Following [Frank 2009], we will use the following equivalent formulation.

DEFINITION 1.1. *An ordered pair $\hat{X} = (X, X^+)$ of subsets of a groundset V is called a biset if $X \subseteq X^+$; X is the inner part and X^+ is the outer part of \hat{X} . Let*

$\Gamma(\hat{X}) = X^+ \setminus X$. The intersection and the union of bisets \hat{X}, \hat{Y} is naturally defined by $\hat{X} \cap \hat{Y} = (X \cap Y, X^+ \cap Y^+)$ and $\hat{X} \cup \hat{Y} = (X \cup Y, X^+ \cup Y^+)$. The biset $\hat{X} \setminus \hat{Y}$ is defined by $\hat{X} \setminus \hat{Y} = (X \setminus Y^+, X^+ \setminus Y) = \hat{X} \cap (V \setminus Y^+, V \setminus Y^-)$.

An edge e covers a biset \hat{X} if it covers both sets X, X^+ , namely, if it has one endnode in X and the other in $V \setminus X^+$. For an edge-set or a graph H and a biset \hat{X} on a node set V let $\delta_H(\hat{X})$ denote the set of edges in H covering \hat{X} . Given an instance of GSN let

$$r(\hat{X}) = \max\{r(s, t) : |X \cap \{s, t\}| = |X^+ \cap \{s, t\}| = 1\}.$$

By the S -connectivity version of Menger's Theorem, a subgraph H of G is a feasible solution to a GSN instance if, and only if $|\delta_H(\hat{X})| \geq f(\hat{X})$ for all bisets \hat{X} on V , where here f is a *biset-function* defined by

$$f(\hat{X}) = r(\hat{X}) - |\Gamma(\hat{X})| \text{ if } \Gamma(\hat{X}) \subseteq S$$

and $f(\hat{X}) = 0$ otherwise.

We study biset-families arising from Rooted GSN and Element-GSN instances. For all applications considered in this paper it suffices to consider biset-families \mathcal{F} that are:

- *bijective* – $X = Y$ implies $X^+ = Y^+$ for any $\hat{X}, \hat{Y} \in \mathcal{F}$.
- *monotone* – $X \subseteq Y$ implies $X^+ \subseteq Y^+$ for any $\hat{X}, \hat{Y} \in \mathcal{F}$.

A biset-family \mathcal{F} is called a *bifamily* if it is bijective, monotone, and $X, V \setminus X^+$ are both nonempty for every $\hat{X} \in \mathcal{F}$.

DEFINITION 1.2. *Given a bifamily \mathcal{F} on V and a set $T \subseteq V$ of terminals, we say that $\hat{X}, \hat{Y} \in \mathcal{F}$ are T -dependent if $X \cap T \subseteq \Gamma(\hat{Y})$ or if $Y \cap T \subseteq \Gamma(\hat{X})$, and \hat{X}, \hat{Y} are T -independent otherwise. We say that \mathcal{F} is T -uncrossable if $X \cap T, T \setminus X^+ \neq \emptyset$ for all $\hat{X} \in \mathcal{F}$, and if $\hat{X} \cap \hat{Y}, \hat{X} \cup \hat{Y} \in \mathcal{F}$ or $\hat{X} \setminus \hat{Y}, \hat{Y} \setminus \hat{X} \in \mathcal{F}$ for any T -independent $\hat{X}, \hat{Y} \in \mathcal{F}$. We say that \mathcal{F} is uncrossable if it is V -uncrossable and any $\hat{X}, \hat{Y} \in \mathcal{F}$ are V -independent (equivalently, a bifamily \mathcal{F} is uncrossable if $\hat{X} \cap \hat{Y}, \hat{X} \cup \hat{Y} \in \mathcal{F}$ or $\hat{X} \setminus \hat{Y}, \hat{Y} \setminus \hat{X} \in \mathcal{F}$ for any $\hat{X}, \hat{Y} \in \mathcal{F}$).*

Let GSN Augmentation be the restriction of GSN to instances where G contains a subgraph J of cost 0 such that $\lambda_J^S(s, t) \geq \max\{r(s, t) - 1, 0\}$ for all $s, t \in T$. Namely, we seek to increase the connectivity by 1 between certain pairs. Formally:

GSN Augmentation:

Instance: A graph $G = (V, E \cup J)$ with edge/node-costs, $S \subseteq V$, and a set \mathcal{T} of node pairs from a set $T \subseteq V$ of terminals.

Objective: Find a minimum cost edge-set $I \subseteq E$ so that $\lambda_{J \cup I}^S(s, t) \geq \lambda_J^S(s, t) + 1$ for all $\{s, t\} \in \mathcal{T}$.

To avoid considering “mixed” cuts that contain both nodes and edges, we may assume that $st \notin E_J$ for all $\{s, t\} \in \mathcal{T}$. One way to achieve this is to subdivide every edge $st \in E_J$ with $\{s, t\} \in \mathcal{T}$ by a new node (of cost 0, in the case of node-costs), and to add all these new nodes to S .

For $X \subseteq V$, let X^+ be the union of X and the set of neighbors of X in J . Let us say that a biset $\hat{X} = (X, X^+)$ is *tight* if there exists $\{s, t\} \in \mathcal{T}$ such that

$|X \cap \{s, t\}| = |X^+ \cap \{s, t\}| = 1$, $\Gamma(\hat{X}) \subseteq S$, and $|\Gamma(\hat{X})| = \lambda_j^S(s, t)$. By Menger's Theorem, I is a feasible solution to GSN Augmentation if, and only if, I covers the family \mathcal{F} of tight bisets; see [Kortsarz and Nutov 2007]. It is easy to see that \mathcal{F} is a bifamily. This bifamily is uncrossable for Element-GSN, by [Fleischer et al. 2006; Cheriyan et al. 2006]. In the case of Rooted GSN, it is sufficient to cover the bifamily $\{\hat{X} \in \mathcal{F} : s \notin X^+\}$. This bifamily is T -uncrossable for Rooted GSN by [Nutov 2009e]. We therefore consider the following generic problem which includes GSN Augmentation problems.

Bifamily Edge-Cover

Instance: A graph $G = (V, E)$ with edge/node-costs and a bifamily \mathcal{F} on V .

Objective: Find a minimum cost edge-cover $I \subseteq E$ of \mathcal{F} .

A polynomial time implementation of our algorithms requires that certain queries related to \mathcal{F} can be answered in polynomial time. We need some definitions to describe these queries.

Given an edge set I on V (I is a partial edge-cover of \mathcal{F}), the *residual bifamily* \mathcal{F}_I of \mathcal{F} (w.r.t. I) consists of all members of \mathcal{F} that are uncovered by the edges of I . It is easy to verify that if \mathcal{F} is T -uncrossable, so is \mathcal{F}_I , for any I , c.f. [Fleischer et al. 2006] for the particular case of uncrossable bifamilies.

DEFINITION 1.3. *A set $C \in \{X : \hat{X} \in \mathcal{F}\}$ is a core of a bifamily \mathcal{F} , or C is an \mathcal{F} -core, if C does not contain two distinct inclusion-minimal members of the set-family $\{X : \hat{X} \in \mathcal{F}\}$. An inclusion-minimal (inclusion-maximal) core is a min-core (max-core). Let $\mathcal{C}_{\mathcal{F}}$ ($\mathcal{M}_{\mathcal{F}}$) denote the set-family of min-cores (max-cores) of \mathcal{F} .*

ASSUMPTION 1. *Given the inner part X of a biset $\hat{X} \in \mathcal{F}$, the outer part X^+ of \hat{X} can be computed in polynomial time.*

ASSUMPTION 2. *For any edge set I on V , the families $\mathcal{C}_{\mathcal{F}_I}$ of min-cores and $\mathcal{M}_{\mathcal{F}_I}$ of max-cores of \mathcal{F}_I can be computed in polynomial time.*

Using standard max-flow min-cut methods, it is easy to see that Assumptions 1, 2 hold for the family \mathcal{F} of tight bisets, c.f. [Nutov 2009e; 2009d]. Summarizing, we have the following.

CLAIM 1.1. *Element-GSN Augmentation is a particular case of Bifamily Edge-Cover with uncrossable \mathcal{F} , and Rooted GSN Augmentation is a particular case of Bifamily Edge-Cover with T -uncrossable \mathcal{F} . Furthermore, in both cases, Assumptions 1 and 2 hold for \mathcal{F} .*

Our first result is the following decomposition, which is obtained by an improved analysis of the algorithm from [Nutov 2009e].

THEOREM 1.2. *There exists a polynomial time algorithm that given a T -uncrossable bifamily \mathcal{F} sequentially finds $4\gamma + \lceil \lg(\lfloor \gamma/2 \rfloor + 1) \rceil = O(\gamma)$ uncrossable sub-bifamilies of \mathcal{F} such that the union of their edge-covers is an edge-cover of \mathcal{F} , where $\gamma = \gamma(\mathcal{F}, T) = \max_{\hat{X}, \hat{Y} \in \mathcal{F}} |\Gamma(\hat{X}) \cap Y \cap T| \geq 1$. In particular, if Bifamily Edge-Cover with uncrossable \mathcal{F} admits a ρ -approximation algorithm, then Bifamily Edge-Cover with T -uncrossable \mathcal{F} admits an $O(\rho\gamma)$ -approximation algorithm.*

Let $\tau(\mathcal{F})$ denote the optimal value of a standard LP-relaxation for Bifamily Edge-Cover, namely

$$\tau(\mathcal{F}) = \min \left\{ \sum_{e \in E} c_e x_e : \sum_{e \in \delta_E(\hat{U})} x_e \geq 1 \forall \hat{U} \in \mathcal{F}, x_e \geq 0 \forall e \in E \right\}.$$

For edge-costs, Bifamily Edge-Cover with uncrossable \mathcal{F} admits a polynomial time algorithm that computes an edge-cover of \mathcal{F} of cost $\leq 2\tau(\mathcal{F})$. In [Fleischer et al. 2006; Cheriyan et al. 2006] such an algorithm uses the iterative rounding method, and applies for a more general setpair-function edge-cover problem. A combinatorial algorithm that relies on Assumptions 1, 2 only can be found in [Nutov 2009e].

For node-costs, Bifamily Edge-Cover with uncrossable \mathcal{F} includes the Set-Cover problem, and thus is $\Omega(\log n)$ -hard to approximate. The only approximation algorithm known was for set-families; [Nutov 2010] gives a $6H(|\mathcal{C}_{\mathcal{F}}|)$ -approximation algorithm, where $H(n)$ denotes the n th Harmonic number. In this paper we prove the following generalization:

THEOREM 1.3. *For node-costs, Bifamily Edge-Cover with uncrossable \mathcal{F} admits a $9H(|\mathcal{C}_{\mathcal{F}}|)$ -approximation algorithm.*

From Theorems 1.2 and 1.3 we obtain the main result of this paper.

THEOREM 1.4. *Bifamily Edge-Cover with T -uncrossable bifamily \mathcal{F} admits the following approximation algorithms.*

- For edge-costs, an algorithm that computes a solution of cost $O(\gamma) \cdot \tau(\mathcal{F})$.
- For node-costs, an $O(\gamma \log |T|)$ -approximation algorithm.

We now consider some applications of Theorem 1.4. For Rooted GSN, two different $O(k^2 \log n)$ -approximation algorithms were suggested independently in [Chuzhoy and Khanna 2008] and [Nutov 2009e]. A particularly elegant and simple approach was suggested recently by [Chuzhoy and Khanna 2009]. They showed that Rooted GSN can be decomposed into p instances of Element-GSN, where $p = p(|T|, k)$ is the minimum number of subsets T_1, \dots, T_p of T , so that for every pair (t, Q) with $Q \subset T$, $|Q| = k$, $t \in V \setminus Q$, there exists T_i with $t \in T_i$ and $Q \cap T_i = \emptyset$. Chuzhoy & Khanna proved that $p = O(k^2 \log |T|)$. A factor of $\log |T|$ is unavoidable here even for $k = 1$. However, for $k \leq 2$ GSN admits a constant ratio [Ravi and Williamson 1997; Fleischer et al. 2006; Cheriyan et al. 2006]. Hence it seems reasonable that GSN with edge-costs admits an approximation ratio that depends on k only. This was proved recently in [Nutov 2009d] for the special case when the input graph G is complete and the costs are in $\{0, 1\}$. Here we prove this for Rooted GSN and Subset k -Connected Subgraph with arbitrary costs, by deducing it from Theorem 1.4.

The following statement will be proved later in Section 4.

PROPOSITION 1.5. *For edge-costs, if Bifamily Edge-Cover with T -uncrossable \mathcal{F} admits a polynomial time algorithm that computes a solution of cost $\leq \rho(\gamma) \cdot \tau(\mathcal{F})$, where ρ is a monotone non-decreasing function, then Rooted GSN admits a polynomial time algorithm that computes a solution of cost $\leq \text{opt} \cdot \sum_{\ell=1}^k \frac{\rho(\ell)}{k-\ell+1} \leq \rho(k) \cdot H(k) \cdot \text{opt}$, where opt denotes the optimal solution cost for Rooted GSN.*

Combining Theorem 1.4 with Proposition 1.5 we prove in Section 4 the following result.

THEOREM 1.6. *GSN problems admit the following approximation ratios:*

- For edge-costs, $O(k \log k)$ for Rooted GSN and $O(k^2 \log k)$ for Subset k -Connected Subgraph.
- For node-costs, $O(k \log |T|)$ for Element-GSN, $O(k^2 \log |T|)$ for Rooted GSN, $O(k^3 \log |T|)$ for Subset k -Connected Subgraph, and $O(k^4 \log^2 |T|)$ for GSN.

For bounded values of k , this settles the approximability of Rooted GSN and of Subset k -Connected Subgraph with edge-costs to a constant, and of Element-GSN, Rooted GSN, and Subset k -Connected Subgraph with node-costs to $O(\log |T|)$.

Theorems 1.2, 1.3, and 1.6, are proved in Sections 2, 3, and 4, respectively.

2. PROOF OF THEOREM 1.2

Let us say that a bifamily \mathcal{F} is *simple* if the inner part of every member of \mathcal{F} is a core. Note that the sub-bifamily $\{\hat{X} \in \mathcal{F} : X \text{ is an } \mathcal{F}\text{-core}\}$ of any bifamily \mathcal{F} is always simple, and we will prove that it is T -uncrossable if \mathcal{F} is. To prove Theorem 1.2, we will show how to decompose any simple T -uncrossable bifamily into a small number of uncrossable bifamilies, see Lemma 2.3 to follow. We start with the following useful property of min-cores.

LEMMA 2.1. *Let $C \in \mathcal{C}_{\mathcal{F}}$ be a min-core of a T -uncrossable bifamily \mathcal{F} . If \hat{C} and $\hat{X} \in \mathcal{F}$ are T -independent, then $C \subseteq X$ or $C \cap X^+ = \emptyset$. In particular, the min-cores of \mathcal{F} are pairwise disjoint on T .*

PROOF. Since \hat{C} and \hat{X} are T -independent, $\hat{C} \cap \hat{X} \in \mathcal{F}$ or $\hat{C} \setminus \hat{X} \in \mathcal{F}$. Thus one of the sets $C \cap X$ or $C \setminus X^+$ is an inner part of a biset in \mathcal{F} . If the statement in the lemma does not hold, then these sets are strictly contained in C . This contradicts the minimality of C . \square

For a bifamily \mathcal{F} on V and $C \subseteq V$ let

$$\mathcal{F}(C) = \{\hat{X} \in \mathcal{F} : X \supseteq C, X \text{ is an } \mathcal{F}\text{-core}\}.$$

Let us say that a bifamily \mathcal{F} is a *ring-bifamily* if $\hat{X} \cap \hat{Y}, \hat{X} \cup \hat{Y} \in \mathcal{F}$ for any $\hat{X}, \hat{Y} \in \mathcal{F}$ and if \mathcal{F} has a unique min-core. Note that any ring-bifamily also has a unique max-core. We will be repeatedly use the following property of cores.

LEMMA 2.2. *Let \mathcal{F} be a T -uncrossable bifamily and let X, Y be \mathcal{F} -cores.*

- (i) *If X, Y contain the same min-core $C \in \mathcal{C}_{\mathcal{F}}$ (namely, if $\hat{X}, \hat{Y} \in \mathcal{F}(C)$) then \hat{X}, \hat{Y} are T -independent and $\hat{X} \cap \hat{Y}, \hat{X} \cup \hat{Y} \in \mathcal{F}(C)$.*
- (ii) *If X, Y contains distinct min-cores $C_X, C_Y \in \mathcal{C}_{\mathcal{F}}$, respectively, and if \hat{X}, \hat{Y} are T -independent, then $\hat{X} \setminus \hat{Y} \in \mathcal{F}(C_X)$ and $\hat{Y} \setminus \hat{X} \in \mathcal{F}(C_Y)$.*

Consequently, the bifamily $\{\hat{X} \in \mathcal{F} : X \text{ is an } \mathcal{F}\text{-core}\}$ is also T -uncrossable, and for every min-core $C \in \mathcal{C}_{\mathcal{F}}$ of \mathcal{F} the following holds: $\mathcal{F}(C)$ is a ring bifamily, there is a unique max-core M containing C , and $\mathcal{F}(C) = \{\hat{X} \in \mathcal{F} : X \subseteq M\}$.

PROOF. We prove (i). Let $\hat{X}, \hat{Y} \in \mathcal{F}(C)$. As $X \cap Y \cap T \supseteq C \cap T \neq \emptyset$, \hat{X}, \hat{Y} are T -independent. We cannot have $\hat{X} \setminus \hat{Y}, \hat{Y} \setminus \hat{X} \in \mathcal{F}$, as then X will contain two disjoint sets $C, X \setminus Y^+$ that are inner parts of bisets in \mathcal{F} , contradicting that X is a core. Hence $\hat{X} \cap \hat{Y}, \hat{X} \cup \hat{Y} \in \mathcal{F}$. Clearly, $X \cap Y$ is a core. It remains to show that $X \cup Y$ is a core. Otherwise, there is a min-core $C' \subseteq X \cup Y$, $C' \neq C$. Then $X \cap T$ or $Y \cap T$ contains a node from $C' \cap T$, say $X \cap C' \cap T \neq \emptyset$. Thus \hat{X}, \hat{C}' are T -independent, and hence $C' \subseteq X$, by Lemma 2.1. But then X contains two distinct cores C, C' . This contradicts that X is a core.

We prove (ii). If (ii) does not hold, then $\hat{X} \cap \hat{Y}, \hat{X} \cup \hat{Y} \in \mathcal{F}$, hence $X \cap Y$ contains some $C \in \mathcal{C}_{\mathcal{F}}$. If $C \neq C_X$ then X contains two distinct min-cores C, C_X , and if $C \neq C_Y$ then Y contains two distinct min-cores C, C_Y . In both cases we obtain a contradiction.

The last statement of the lemma easily follows from (i) and (ii). \square

Later, we will prove the following decomposition lemma.

LEMMA 2.3. *Let \mathcal{F} be a T -uncrossable bifamily with $|C \cap T| \geq q$ for all $C \in \mathcal{C}_{\mathcal{F}}$. If $q \geq \gamma + 1$ then any $\hat{X}, \hat{Y} \in \mathcal{F}$ are T -independent and thus \mathcal{F} is uncrossable. If \mathcal{F} is simple, then $\mathcal{C}_{\mathcal{F}}$ can be partitioned into at most $2\lceil \gamma/q \rceil + 1$ parts such that for every part \mathcal{C} , the bifamily $\bigcup_{C \in \mathcal{C}} \mathcal{F}(C)$ is uncrossable. Furthermore, given the families $\mathcal{C}_{\mathcal{F}}$ and $\mathcal{M}_{\mathcal{F}}$ of min-cores and max-cores of \mathcal{F} , such a partition can be found in polynomial time.*

The following lemma enables us to estimate a progress made towards covering \mathcal{F} , if we cover the sub-bifamily $\{\hat{X} \in \mathcal{F} : X \text{ is an } \mathcal{F}\text{-core}\}$ of \mathcal{F} .

LEMMA 2.4. *Suppose that the min-cores of a bifamily \mathcal{F} are pairwise disjoint on a subset T of V , and that $|C \cap T| \geq q$ for all $C \in \mathcal{C}_{\mathcal{F}}$. If an edge-set I covers the bifamily $\{\hat{X} \in \mathcal{F} : X \text{ is an } \mathcal{F}\text{-core}\}$ then $|C' \cap T| \geq 2q$ for every min-core C' of \mathcal{F}_I .*

PROOF. Let C' be an \mathcal{F}_I -core. As I covers $\{\hat{X} \in \mathcal{F} : X \text{ is an } \mathcal{F}\text{-core}\}$, C' contains at least two min-cores of \mathcal{F} . These two min-cores have no node of T in common, and each of them contains at least q nodes from T , by the assumption. The statement follows. \square

Now we deduce Theorem 1.2 from Lemmas 2.1, 2.3, and 2.4. The algorithm starts with $I = \emptyset$, and while $q = \min\{|C \cap T| : C \in \mathcal{C}_{\mathcal{F}_I}\} \leq \lceil (\gamma + 1)/2 \rceil = \lceil \gamma/2 \rceil + 1$, it adds to I a cover of the bifamily $\{\hat{X} \in \mathcal{F}_I : X \text{ is an } \mathcal{F}_I\text{-core}\}$. Then, at the last iteration, the algorithm adds to I a cover of the entire residual bifamily \mathcal{F}_I . Note that at the beginning of an iteration, the bifamily $\{\hat{X} \in \mathcal{F}_I : X \text{ is an } \mathcal{F}_I\text{-core}\}$ is simple and T -uncrossable, by Lemma 2.1, and thus can be partitioned into at most $2\lceil \gamma/q \rceil + 1$ uncrossable bifamilies, by Lemma 2.3. Initially, $q \geq 1$, and q is at least doubled during each iteration, by Lemmas 2.1 and 2.4. At the beginning of the last iteration, we have $q \geq \gamma + 1$, and then the residual bifamily \mathcal{F}_I is uncrossable, by Lemma 2.3. Consequently, the total number of uncrossable bifamilies we cover is bounded by

$$1 + \sum_{p=0}^{\lceil \lg(\lceil \gamma/2 \rceil + 1) \rceil} (2\lceil \gamma/2^p \rceil + 1) \leq 2 + \lceil \lg(\lceil \gamma/2 \rceil + 1) \rceil + 2\gamma \sum_{p=0}^{\lceil \lg(\lceil \gamma/2 \rceil + 1) \rceil} (1/2)^p$$

$$\begin{aligned}
 &\leq 2 + \lceil \lg(\lfloor \gamma/2 \rfloor + 1) \rceil + 2\gamma \frac{1 - 1/2^{\lceil \lg(\lfloor \gamma/2 \rfloor + 1) + 1}}{1 - 1/2} \\
 &\leq 2 + \lceil \lg(\lfloor \gamma/2 \rfloor + 1) \rceil + 4\gamma \left(1 - \frac{1}{2(\lfloor \gamma/2 \rfloor + 1)} \right) \\
 &\leq 4\gamma + \lceil \lg(\lfloor \gamma/2 \rfloor + 1) \rceil = O(\gamma).
 \end{aligned}$$

In the rest of this section we prove Lemma 2.3. Let \mathcal{F} be a T -uncrossable bifamily on V . If $q \geq \gamma + 1$ then \mathcal{F} is uncrossable, since any $\hat{X}, \hat{Y} \in \mathcal{F}$ are T -independent; otherwise, there are $\hat{X}, \hat{Y} \in \mathcal{F}$ such that $|X \cap \Gamma(\hat{Y}) \cap T| \geq \gamma + 1$, contradicting the definition of γ .

Now assume that \mathcal{F} is simple. Let $\mathcal{C}_{\mathcal{F}} = \{C_1, \dots, C_\nu\}$ be the family of min-cores of \mathcal{F} and let M_i be the (unique, by Lemma 2.2) max-core containing C_i .

DEFINITION 2.1. *We say that $C_i, C_j \in \mathcal{C}_{\mathcal{F}}$ are strongly T -independent if both \hat{M}_i, \hat{C}_j are T -independent and \hat{M}_j, \hat{C}_i are T -independent.*

Note that $\mathcal{F}(C_i) = \{\hat{X} \in \mathcal{F} : X \subseteq M_i\}$ for every i , by Lemma 2.2. Hence if C_i, C_j are strongly T -independent then any $\hat{X}, \hat{Y} \in \mathcal{F}$ with $X \subseteq M_i$ and $Y \subseteq M_j$ are T -independent, by the monotonicity of \mathcal{F} . Thus we have the following.

COROLLARY 2.5. *Let $C_i, C_j \in \mathcal{C}_{\mathcal{F}}$ be strongly T -independent. Then for any $\hat{X}, \hat{Y} \in \mathcal{F}$ with $X \subseteq M_i$ and $Y \subseteq M_j$ we have $\hat{X} \cap \hat{Y}, \hat{X} \cup \hat{Y} \in \mathcal{F}$ if $i = j$, and $\hat{X} \setminus \hat{Y}, \hat{Y} \setminus \hat{X} \in \mathcal{F}$ if $i \neq j$. Thus for any $\mathcal{C} \subseteq \mathcal{C}_{\mathcal{F}}$, if the members of \mathcal{C} are pairwise strongly T -independent, then the bifamily $\bigcup_{C \in \mathcal{C}} \mathcal{F}(C) = \{\hat{X} \in \mathcal{F} : X \subseteq M_i, C_i \in \mathcal{C}\}$ is uncrossable.*

Consequently, the following lemma finishes the proof of Lemma 2.3.

LEMMA 2.6. *If $|C_i \cap T| \geq q$ for all i , then $\mathcal{C}_{\mathcal{F}}$ admits a partition into at most $2\lfloor \gamma/q \rfloor + 1$ parts such that the members of each part are pairwise strongly T -independent, and given the families $\mathcal{C}_{\mathcal{F}}, \mathcal{M}_{\mathcal{F}}$ such a partition can be found in polynomial time.*

PROOF. Construct an auxiliary directed graph \mathcal{J} as follows. The node set of \mathcal{J} is $\mathcal{C}_{\mathcal{F}}$. Add an arc $C_i C_j$ if $T \cap C_i \subseteq \Gamma(\hat{M}_j)$. The indegree of every node in \mathcal{J} is at most $\lfloor \gamma/q \rfloor$, by Lemma 2.2(i). This implies that every subgraph of the underlying graph of \mathcal{J} has a node of degree $\leq 2\lfloor \gamma/q \rfloor$. A graph is d -degenerate if every subgraph of it has a node of degree $\leq d$. It is known that any d -degenerate graph can be colored in polynomial time with $(d + 1)$ colors. Hence \mathcal{J} is $(2\lfloor \gamma/q \rfloor + 1)$ -colorable, and such coloring can be computed in polynomial time. Consequently, $\mathcal{C}_{\mathcal{F}}$ can be partitioned in polynomial time into $2\lfloor \gamma/q \rfloor + 1$ independent sets, as required. \square

The proof of Lemma 2.3, and thus also of Theorem 1.2, is complete.

3. PROOF OF THEOREM 1.3

The proof of Theorem 1.3 is an extension of the proof in [Nutov 2010] for the case of uncrossable set-families. We present a full proof only of an analogue of the main theorem from [Nutov 2010], since it slightly differs from the proof for set-families, while the other parts of the proof of Theorem 1.3 are essentially identical to the ones in [Nutov 2010].

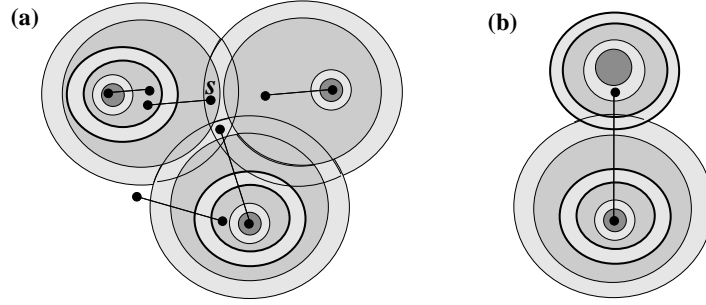


Fig. 1. Illustration of an $\mathcal{F}(s, \mathcal{C})$ spider-cover with $|\mathcal{C}| = 3$. The inner parts of the bisets are shown by darker gray circles, min-cores in \mathcal{C} are shown by darkest gray circles.

We start by extending the concept “spider-cover” introduced in [Nutov 2010] from set-families to bifamilies. For $s \in V$ and $C \in \mathcal{C}_{\mathcal{F}}$ let

$$\mathcal{F}(s, C) = \{\hat{X} \in \mathcal{F}(C) : s \notin X^+\}.$$

Note that by Lemma 2.2(i), $\mathcal{F}(C)$, and thus also $\mathcal{F}(s, C)$, is a ring-bifamily if \mathcal{F} is an uncrossable (or even a T -uncrossable) bifamily.

DEFINITION 3.1. *Let \mathcal{F} be a bifamily on V and let $\mathcal{C} \subseteq \mathcal{C}_{\mathcal{F}}$. We say that an edge-set S on V is an $\mathcal{F}(s, \mathcal{C})$ -spider-cover (for illustration see Fig. 1) if $s \in V(S)$ and if S can be partitioned into $\mathcal{F}(s, C_i)$ -covers $\{P_i : C_i \in \mathcal{C}\}$ such that the node sets $\{V(P_i) \setminus \{s\} : C_i \in \mathcal{C}\}$ are pairwise disjoint.*

We say that S is an $\mathcal{F}(\mathcal{C})$ -spider-cover (or simply a spider-cover, if \mathcal{C} is clear from the context) if the following holds:

- If $|\mathcal{C}| \geq 2$ then there exists $s \in V$ (a center of the spider cover) such that S is an $\mathcal{F}(s, \mathcal{C})$ -spider-cover.
- If $|\mathcal{C}| = 1$, say $\mathcal{C} = \{C_i\}$, then S covers $\mathcal{F}(C_i)$.

Equivalently, for $|\mathcal{C}| \geq 2$, an $\mathcal{F}(\mathcal{C})$ -spider-cover S with center $s \in V(S)$ is a union of $\mathcal{F}(s, C_i)$ -covers $\{P_i : C_i \in \mathcal{C}\}$ so that only s can be a common end-node of two of them. Note that there might be $C_i \in \mathcal{C}$ so that P_i does not cover C_i . This may happen if $|\mathcal{C}| \geq 2$ and $s \in C_i^+$ for some $C_i \in \mathcal{C}$; then $\mathcal{F}(s, C_i) = \emptyset$ and $P_i = \emptyset$ is an $\mathcal{F}(s, C_i)$ -cover, although no edge in P_i covers \hat{C}_i itself.

DEFINITION 3.2. *Let \mathcal{F} be a bifamily on V and let $\mathcal{C} \subseteq \mathcal{C}_{\mathcal{F}}$. We say that a collection $\mathcal{S} = \{S_1, \dots, S_q\}$ of edge-sets spider-covers \mathcal{C} if the following holds:*

- The node-sets $V(S_1), \dots, V(S_q)$ are pairwise disjoint.
- \mathcal{C} admits a partition $\Pi = \{\mathcal{C}_1, \dots, \mathcal{C}_q\}$ such that each S_t is an $\mathcal{F}(\mathcal{C}_t)$ -spider-cover.

In [Nutov 2010] it is proved that any cover I of an uncrossable bifamily \mathcal{F} admits a subpartition that spider-covers the entire family $\mathcal{C}_{\mathcal{F}}$ of \mathcal{F} -cores. In the case of biset families the situation is more involved, and we prove the following.

THEOREM 3.1. *Any cover I of an uncrossable bifamily \mathcal{F} admits a subpartition that spider-covers a subfamily $\mathcal{C} \subseteq \mathcal{C}_{\mathcal{F}}$ of \mathcal{F} -cores of size at least $|\mathcal{C}| \geq \lceil 2|\mathcal{C}_{\mathcal{F}}|/3 \rceil$.*

We briefly describe how Theorem 3.1 implies Theorem 1.3; for details see [Nutov 2010]. For a bifamily \mathcal{F} and an edge set I , let $\nu(I)$ denote the number $|\mathcal{C}_{\mathcal{F}_I}|$ of min-cores of the residual bifamily \mathcal{F}_I . Given a partial solution I , the *density* of an edge set $S \subseteq E - I$ is $c(S)/(\nu(I) - \nu(I \cup S))$. The ρ -Greedy Algorithm starts with $I = \emptyset$, and as long as $\nu(I) \geq 1$, it finds and adds to I an edge-set $S \subseteq E - I$ of density at most $\rho \cdot \text{opt}/\nu(I)$, where opt denotes the optimal solution value. It is known that the ρ -Greedy Algorithm, if can be applied, computes a solution I such that $c(I) \leq \rho H(\nu(\emptyset)) \cdot \text{opt}$.

One can show that if S is an $\mathcal{F}_I(\mathcal{C})$ -spider-cover, then $\nu(I) - \nu(I \cup S) \geq |\mathcal{C}|/3$ (in the particular case of set-families arising from the Steiner Forest problem, we have an improved bound $\nu(I) - \nu(I \cup S) \geq |\mathcal{C}|/2$ [Klein and Ravi 1995]). By an averaging argument, Theorem 3.1 implies that there exists an $\mathcal{F}(\mathcal{C})$ -spider-cover of density at most $3 \cdot 3/2 \cdot \text{opt}/\nu(I)$ (for set-families a density of $3 \cdot \text{opt}/\nu(I)$ is achieved, while for set-families arising from the Steiner Forest problem a density of $2 \cdot \text{opt}/\nu(I)$ is achieved). Using this, one can design a polynomial time algorithm that finds an edge set S (may not be a spider-cover) of density at most $9 \cdot \text{opt}/\nu(I)$. An additional factor of 2 is invoked since to find such S , we “guess” the center s and compute for every min-core C a 2-approximate $\mathcal{F}(s, C)$ -cover; no polynomial time algorithm is known for computing an optimal $\mathcal{F}(s, C)$ -cover (for set-families arising from the Steiner Forest problem, one can compute an optimal $\mathcal{F}(s, C)$ -cover by a shortest path computation [Klein and Ravi 1995]).

In what follows we prove Theorem 3.1. Let \mathcal{F} be an *uncrossable bifamily* and let I be an *inclusion minimal \mathcal{F} -cover*.

A set-family \mathcal{L} is laminar if for any distinct sets $X, Y \in \mathcal{L}$ either $X \subset Y$, or $Y \subset X$, or $X \cap Y = \emptyset$. The following definition extends this to bifamilies.

DEFINITION 3.3. *A bifamily \mathcal{L} is laminar if for any $\hat{X}, \hat{Y} \in \mathcal{L}$ either $X \subset Y$, or $Y \subset X$, or $X \cap Y^+ = \emptyset$ (note that the latter implies $X \cap Y = \emptyset$).*

DEFINITION 3.4. *We say that an edge set I is a fit-cover of a bifamily $\mathcal{L} \subseteq \mathcal{F}$, or that \mathcal{L} is a fit-bifamily for I , if $|\mathcal{L}| = |I|$ and for every $e \in I$ there is a fit-set $\hat{X}_e \in \mathcal{L}$ so that $\delta_I(\hat{X}_e) = \{e\}$; namely, e is the unique edge in I that covers \hat{X}_e .*

By the minimality of I , for every $e \in I$ there exists $\hat{X}_e \in \mathcal{F}$ such that e is the unique edge in I that covers \hat{X}_e . Thus there exists $\mathcal{L} \subseteq \mathcal{F}$ so that I is a fit-cover of \mathcal{L} . Previous work (c.f. [Nutov 2009e]) shows that there exists a fit-bifamily \mathcal{L} that is laminar.

Let $\mathcal{L} \subseteq \mathcal{F}$ be a laminar fit-bifamily for a minimal \mathcal{F} -cover I . To simplify the exposition, we will make the following two assumptions (used also in [Nutov 2010]) about the bifamilies \mathcal{F} and \mathcal{L} .

ASSUMPTION A: We may assume that the bifamily \mathcal{F} is simple, namely, it would be sufficient to prove Theorem 3.1 for simple bifamilies. This is because Definitions 3.1 and 3.2 consider covers only of bisets in \mathcal{F} for which the inner parts are cores. Thus we may replace \mathcal{F} by the relevant bifamily $\{\hat{X} \in \mathcal{F} : X \text{ is an } \mathcal{F}\text{-core}\}$; the latter is uncrossable if \mathcal{F} is, by Lemma 2.2.

ASSUMPTION B: We may assume that the inclusion-minimal members of the set-family $\{X : \hat{X} \in \mathcal{L}\}$ are the min-cores of \mathcal{F} , namely, that $\mathcal{C}_{\mathcal{L}} = \mathcal{C}_{\mathcal{F}}$. Otherwise

(assuming \mathcal{F} is simple, by Assumption A), apply the following transformation, to obtain $V', \mathcal{F}', I', \mathcal{L}'$. For every $C \in \mathcal{C}_{\mathcal{F}}$ do the following:

V' – add to V the new node v_C ;

\mathcal{F}' – replace every $\hat{X} \in \mathcal{F}(C)$ by the biset $(X \cup \{v_C\}, X^+ \cup \{v_C\})$ and add the biset $(\{v_C\}, \{v_C\})$ to \mathcal{F} ;

I' – add to I an edge $u_C v_C$ where $u_C \in C$ arbitrary;

\mathcal{L}' – replace every $\hat{X} \in \mathcal{L}(C)$ by the biset $(X \cup \{v_C\}, X^+ \cup \{v_C\})$ and add the biset $(\{v_C\}, \{v_C\})$ to \mathcal{L} .

It is not hard to verify that:

- The new bifamily \mathcal{F}' is simple and uncrossable if \mathcal{F} is.
- I covers \mathcal{F} if, and only if, I' covers \mathcal{F}' .
- \mathcal{L}' is a laminar fit-bifamily for I' , where $(\{v_C\}, \{v_C\})$ is the fit-biset for $u_C v_C$.
- For any $s \in V$ (so $s \neq v_C$ for every $C \in \mathcal{C}_{\mathcal{F}}$) and $\mathcal{C} \subseteq \mathcal{C}_{\mathcal{F}}$ the following holds:
 $S \subseteq I$ is an $\mathcal{F}(s, \mathcal{C})$ -spider-cover if, and only if, $S' = S \cup \{u_C v_C : C \in \mathcal{C}\}$ is an $\mathcal{F}'(s, \mathcal{C}')$ -spider-cover, where $\mathcal{C}' = \{\{v_C\} : C \in \mathcal{C}\}$.

Thus proving Theorem 3.1 for \mathcal{F}', I' implies Theorem 3.1 for \mathcal{F}, I , provided every spider-cover in the decomposition derived for \mathcal{F}', I' has a choice of the center that belongs to V (namely, not in $\{v_C : C \in \mathcal{C}_{\mathcal{F}}\}$). More generally, relying on the property that there exists a laminar fit-bifamily $\mathcal{L}' \subseteq \mathcal{F}'$ for I' so that $\mathcal{C}_{\mathcal{L}'} = \mathcal{C}_{\mathcal{F}'}$, we will construct a spider-cover decomposition for I' , so that every spider-cover in the decomposition has a center that does not belong to a min-core.

Assume that Assumptions A, B are valid, namely, that \mathcal{F} is simple and that $\mathcal{C}_{\mathcal{L}} = \mathcal{C}_{\mathcal{F}}$. To derive our decomposition, we will study the maximal members of \mathcal{L} .

DEFINITION 3.5. For every $C_i \in \mathcal{C}_{\mathcal{F}}$ define (see Fig. 2):

- L_i is the maximal set in $\{X : \hat{X} \in \mathcal{L}(C_i)\}$ (L_i is an \mathcal{F} -core, by Assumptions A, B).
- $e_i = s_i v_i$ is the unique edge in I covering \hat{L}_i , where $v_i \in L_i$ and $s_i \in V \setminus L_i^+$.
- P_i is the set of edges in I with at least one end-node in L_i .

The following statement gives some properties of the sets L_i, P_i in Definition 3.5.

LEMMA 3.2.

- (i) $L_i \cap L_j^+, L_j \cap L_i^+ = \emptyset$ for any $j \neq i$; thus the sets L_i are pairwise disjoint.
- (ii) The edge-sets P_i partition I , and $V(P_i) \cap V(P_j) \subseteq V \setminus (L_i \cup L_j)$ for any $i \neq j$.
- (iii) If $\hat{X} \in \mathcal{F}$ and $X \subseteq L_i$ then P_i covers \hat{X} and no edge in $I \setminus P_i$ covers \hat{X} .

PROOF. Part (i) follows from the laminarity of \mathcal{L} and the maximality of L_i . Part (ii) follows from (i) and the fact that \mathcal{L} is a fit-bifamily for I . Part (iii) follows from (ii) and the observation that if an edge e covers a biset which inner part is contained in L_i , then e has at least one end-node in L_i . \square

LEMMA 3.3.

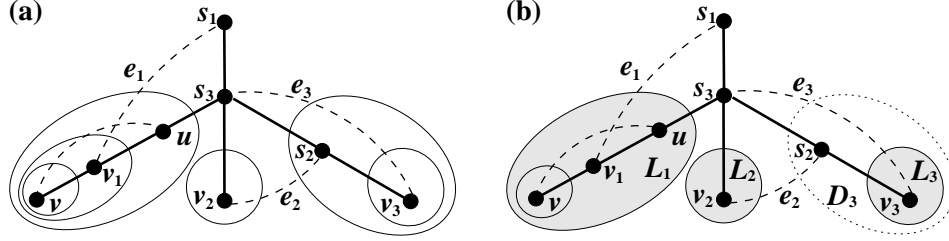


Fig. 2. Illustration to Definition 3.5. (a) An instance of Rooted GSN Augmentation; the graph J is a tree shown by solid lines, and we want to increase the connectivity from 1 to 2 between each of v, v_1, v_2 and s_1 . The cores of the biset family $\{\hat{X} : \hat{X} \text{ is tight, } s \notin X^+\}$ are shown by ellipses; for every core X , X^+ is the union of X and the neighbor of X in J . The set of dashed edges covers the bifamily corresponding to cores; other tight bisets, e.g. $(\{v_2, s_2, v_3\}, \{v_2, s_2, v_3, s_3\})$, may not be covered. (b) The bifamily \mathcal{L} , the sets L_i , and the edges $s_i v_i$ as in Definition 3.5. The biset \hat{D}_3 , whose inner parts is shown by the dotted ellipse, is in $\mathcal{F}(C_3)$, but it is covered by e_2 and not by $P_3 = \{e_3\}$.

DEFINITION 3.6. For $\mathcal{C} \subseteq \mathcal{C}_{\mathcal{F}}$ let $S_{\mathcal{C}} = \bigcup_{C_i \in \mathcal{C}} P_i$. We say that a partition Π of $\mathcal{C}_{\mathcal{F}}$ is a good partition if the node sets $\{V(S_{\mathcal{C}}) : \mathcal{C} \in \Pi\}$ are pairwise disjoint and if for every $\mathcal{C} \in \Pi$ the following holds:

- If $|\mathcal{C}| \geq 2$ then there is $C_j \in \mathcal{C}$ such that P_i is an $\mathcal{F}(s_j, C_i)$ -spider-cover for every $C_i \in \mathcal{C}$.
- If $|\mathcal{C}| = 1$, say $\mathcal{C} = \{C_i\}$, then P_i covers $\mathcal{F}(C_i)$.

From Lemma 3.2(ii) we have the following (see Definitions 3.1 and 3.2).

PROPOSITION 3.4. If Π is a good partition then $\{S_{\mathcal{C}} : \mathcal{C} \in \Pi\}$ is a spider-cover decomposition of I .

To prove Theorem 3.1 we will show that $\mathcal{C}_{\mathcal{F}}$ admits a good partition Π .

DEFINITION 3.7. Let Π^* denote the partition of $\mathcal{C}_{\mathcal{F}}$ by the equivalence classes of the relation $\{(C_i, C_j) : s_i = s_j\}$ on $\mathcal{C}_{\mathcal{F}}$.

The following statement stems from Lemma 3.2(ii) and the definition of Π^* .

CLAIM 3.5. Let Π be a partition of $\mathcal{C}_{\mathcal{F}}$. Then the node sets $\{V(S_{\mathcal{C}}) : \mathcal{C} \in \Pi\}$ are pairwise disjoint if, and only if, Π^* is a refinement of Π .

DEFINITION 3.8. For $C_i \in \mathcal{C}_{\mathcal{F}}$ let us use the notation

$$\mathcal{D}(C_i) = \{\hat{X} \in \mathcal{F}(C_i) : \hat{X} \text{ is not covered by } P_i\}.$$

We say that a min-core $C_i \in \mathcal{C}_{\mathcal{F}}$ is dangerous if $\mathcal{D}(C_i) \neq \emptyset$, namely, if P_i does not cover $\mathcal{F}(C_i)$. (For illustration see the biset \hat{D}_3 in Fig. 2(b).)

To get some intuition, we prove the following.

CLAIM 3.6. If there are no dangerous min-cores then Π^* is a good partition.

PROOF. The node sets $\{V(S_{\mathcal{C}}) : \mathcal{C} \in \Pi^*\}$ are pairwise disjoint, by Claim 3.5. Let $\mathcal{C} \in \Pi^*$ and let s be the node such that $s = s_i$ for all $C_i \in \mathcal{C}$. If $|\mathcal{C}| \geq 2$ then for every $C_i \in \mathcal{C}$, P_i covers $\mathcal{F}(C_i)$, since C_i is not dangerous; in particular, P_i is

an $\mathcal{F}(s, C_i)$ -cover. If $|\mathcal{C}| = 1$, say $\mathcal{C} = \{C_i\}$, then P_i covers $\mathcal{F}(C_i)$, since C_i is not dangerous. The statement follows. \square

Note that the proof of Claim 3.6 heavily relies on the assumption that there are no dangerous min-cores. In fact, with some additional minor effort, we can show a stronger statement. In what follows, let

$$\mathcal{A} = \{C_i \in \mathcal{C}_{\mathcal{F}} : C_i \text{ is danderous and } |\delta_I(s_i)| = 1\}.$$

We can show that if $\mathcal{A} = \emptyset$, namely, if $|\delta_I(s_i)| \geq 2$ for every dangerous min-core C_i , then Π^* is still a good partition. However, in the case when dangerous min-cores C_i for which $|\delta_I(s_i)| = 1$ exist, our proof is more involved. We will show that in this case we can obtain a good partition by grouping members of \mathcal{A} together or joining them to other parts. For that, we need to derive some properties of dangerous min-cores.

The following statement is easily verified; see [Nutov 2009e].

FACT 3.7. *Let \hat{X}, \hat{Y} be arbitrary bisets. If an edge e covers one of the bisets $\hat{X} \cap \hat{Y}, \hat{X} \cup \hat{Y}, \hat{X} \setminus \hat{Y}, \hat{Y} \setminus \hat{X}$ then e covers \hat{X} or \hat{Y} . Furthermore, if e covers $\hat{X} \setminus \hat{Y}$ and does not cover \hat{X} , then e has one endnode in $Y \cap X^+$ and the other in $X \setminus Y^+$.*

LEMMA 3.8. *Let C_i be a dangerous min-core. Then:*

- (i) $\mathcal{D}(C_i)$ is a ring-bifamily and thus has a unique min-core, denoted by D_i .
- (ii) $\hat{D}_i \cap \hat{L}_i$ is a fit-set for e_i ; hence $v_i \in D_i$ and $s_i \in D_i^+$.
- (iii) $\hat{D}_i \setminus \hat{L}_j = \hat{D}_i$, namely, $D_i \cap L_j^+, D_i^+ \cap L_j = \emptyset$, for any $j \neq i$.
- (iv) If e covers \hat{D}_i then $e = e_j$ for some $j \neq i$ and $s_j \in D_i$; furthermore, if C_j is also a dangerous min-core then $s_j \in D_j^+$.

PROOF. We prove (i). Let $\hat{X}, \hat{Y} \in \mathcal{D}(C_i)$. Then $\hat{X} \cap \hat{Y}, \hat{X} \cup \hat{Y} \in \mathcal{F}(C_i)$, by Lemma 2.2(ii). It remains to prove that none of $\hat{X} \cap \hat{Y}, \hat{X} \cup \hat{Y}$ is covered by P_i . Otherwise, by Fact 3.7, P_i covers one of \hat{X}, \hat{Y} , contradicting that $\hat{X}, \hat{Y} \in \mathcal{D}(C_i)$.

We prove (ii). By Lemma 2.2(ii), $\hat{D}_i \cap \hat{L}_i \in \mathcal{F}(C_i)$. Consider an arbitrary edge $e \in I$ that covers $\hat{D}_i \cap \hat{L}_i$. Then $e \in P_i$, since e has one endnode in $D_i \cap L_i \subseteq L_i$. Thus e does not cover \hat{D}_i , hence e covers \hat{L}_i , by Fact 3.7. Consequently, $e = e_i$.

We prove (iii). By Lemma 2.2(ii), $\hat{D}_i \setminus \hat{L}_j \in \mathcal{F}(C_i)$. Consider an arbitrary edge $e \in I$ that covers $\hat{D}_i \setminus \hat{L}_j$. By Fact 3.7, e covers \hat{D}_i or \hat{L}_j . If e covers \hat{D}_i then $e \notin P_i$, and if e covers \hat{L}_j then $e = e_j \notin P_i$. In both cases, $e \notin P_i$. We conclude that $\hat{D}_i \setminus \hat{L}_j$ is not covered by P_i ; thus $\hat{D}_i - \hat{L}_j \in \mathcal{D}(C_i)$. Consequently, the inner part of $\hat{D}_i \setminus \hat{L}_j$ must be D_i , by the minimality of D_i . Hence $\hat{D}_i \setminus \hat{L}_j = \hat{D}_i$, by the bijectiveness of \mathcal{F} .

We prove (iv). As $e \notin P_i$, we must have $e \in P_j$ for some $j \neq i$, by Lemma 3.2(ii). By (iii), $D_i \cap L_j^+ = \emptyset$. Thus one endnode of e is in L_j and the other in $D_i \subseteq V \setminus L_j^+$. Consequently, e covers L_j ; thus $e = e_j$ and $s_j \in D_i$. If C_j is dangerous, then $s_j \in D_j^+$ by (ii). \square

LEMMA 3.9. *If $D_i \cap D_j^+ \neq \emptyset$ for $i \neq j$ then the following holds: $s_i \in D_j \cap D_i^+$, $v_i \in D_i \setminus D_j^+$, and e_i covers D_j . Consequently, also $D_j \cap D_i^+ \neq \emptyset$, and thus the same holds with the roles of i, j exchanged.*

PROOF. By Lemma 2.2(ii), $\hat{D}_i \setminus \hat{D}_j \in \mathcal{F}(C_i)$. As $D_i \cap D_j^+ \neq \emptyset$, the inner part $D_i \setminus D_j^+$ of $\hat{D}_i \setminus \hat{D}_j$ is strictly contained in D_i . Thus by the minimality of D_i , there is an edge $e \in P_i$ that covers $\hat{D}_i \setminus \hat{D}_j$. As $e \in P_i$, e does not cover \hat{D}_i . By Fact 3.7, e covers \hat{D}_j , and has one endnode $u \in D_j \cap D_i^+$ and the other $v \in D_i \setminus D_j^+$. Thus all we need to prove is that $e = e_i$ and $u = s_i$. As $e \in P_i$, e has one endnode in L_i . By Lemma 3.8(ii) with roles of i, j exchanged, $D_j \cap L_i^+, D_j^+ \cap L_i = \emptyset$, hence the endnode of e in L_i must be v while $u \notin L_i^+$. Consequently, e covers L_i and hence $e = e_i$ and $u = s_i$, as claimed. \square

COROLLARY 3.10. *The relation $\mathcal{R} = \{(C_i, C_j) : D_i \cap D_j^+ \neq \emptyset\}$ on \mathcal{A} (or on arbitrary subset of dangerous min-cores) is an equivalence.*

PROOF. Reflexivity is obvious. Symmetry is by Lemma 3.9. We prove transitivity. Let D_i, D_p, D_j be distinct and suppose that $D_i \cap D_p^+, D_p \cap D_j^+ \neq \emptyset$. Then by Lemma 3.9, $s_p \in D_i \cap D_p^+$ and $s_p \in D_j \cap D_p^+$, hence $s_p \in D_i \cap D_j \subseteq D_i \cap D_j^+$. \square

LEMMA 3.11. *If C_i is a dangerous min-core then P_i is an $\mathcal{F}(s, C_i)$ -cover for any $s \in D_i^+$.*

PROOF. Let $\hat{X} \in \mathcal{F}(s, C_i)$. By Lemma 2.2(ii), $\hat{X} \cap \hat{D}_i \in \mathcal{F}(C_i)$. Note that $X \cap D_i$ is strictly contained in D_i , as we cannot have $X \supseteq D_i$ since then we would have $\hat{X} \notin \mathcal{F}(s, C_i)$. Hence, by the minimality of D_i , there is $e \in P_i$ covering $\hat{X} \cap \hat{D}_i$. By Fact 3.7 and since $e \in P_i$, e covers \hat{X} . \square

Now we can describe the desired good partition Π of $\mathcal{C}_{\mathcal{F}}$. Let Π' be the subpartition of \mathcal{A} into equivalence classes of size at least 2 of the relation \mathcal{R} from Corollary 3.10. Let \mathcal{C}' be the union of the parts of Π' , and note that we may have $\mathcal{A} - \mathcal{C}' \neq \emptyset$ because the singleton classes of \mathcal{R} are not included in Π' . Note that by Lemmas 3.9 and 3.11 we have:

CLAIM 3.12. *Let $\mathcal{C} \in \Pi'$ and let $C_j \in \mathcal{C}$. Then P_i is an $\mathcal{F}(s_j, C_i)$ -spider-cover for every $C_i \in \mathcal{C}$.*

Let Π'' be a partition of $\mathcal{C}'' = \mathcal{C}_{\mathcal{F}} \setminus \mathcal{C}'$ defined as follows. First, partition $\mathcal{C}'' \setminus \mathcal{A}$ by stars of the graph formed by the edges e_i (a star might consist of a single edge); namely, the parts are the equivalence classes of the relation $\{(C_i, C_j) : s_i = s_j\}$ on $\mathcal{C}'' \setminus \mathcal{A}$. Second, join every $C_i \in \mathcal{A} \cap \mathcal{C}''$ to some part of $\mathcal{C}'' \setminus \mathcal{A}$ as follows. By Lemma 3.8(iv) and the definition of \mathcal{R} and Π' , there exists $C_j \in \mathcal{C}'' \setminus \mathcal{A}$ such that e_j covers D_i ; we join C_i to the part containing C_j . Note that indeed $C_j \notin \mathcal{A}$, as otherwise C_i, C_j would belong to the same part of Π' , by Lemma 3.8(iv) and the definition of \mathcal{R} .

CLAIM 3.13. *Let $\mathcal{C} \in \Pi''$ with $|\mathcal{C}| \geq 2$, and let $C_j \in \mathcal{C} \cap (\mathcal{C}'' \setminus \mathcal{A})$ (such C_j exists, by the construction). Then P_i is an $\mathcal{F}(s_j, C_i)$ -spider-cover for every $C_i \in \mathcal{C}$.*

PROOF. Suppose that C_i is dangerous, as otherwise P_i covers $\mathcal{F}(C_i)$ and the statement is obvious. If $\{C_i\} \notin \mathcal{A}$, then $s_i \in D_i^+$ since C_i is dangerous, and $s_i = s_j$ by the construction; hence $s_j \in D_i^+$. If $\{C_i\} \in \mathcal{A}$, then there is an edge $e_{j'}$ with $s_{j'} = s_j$ that covers \hat{D}_i , by the construction. Then $s_j \in D_i$, by Lemma 3.8(iv). In both cases we have $s_j \in D_i^+$, and hence the claim follows from Lemma 3.11. \square

We claim that the partition $\Pi = \Pi' \cup \Pi''$ of $\mathcal{C}_{\mathcal{F}}$ obtained is a good partition, hence by Proposition 3.4, $\{S_{\mathcal{C}} : \mathcal{C} \in \Pi\}$ is a spider-cover decomposition of I . Note that Π^* is a refinement of Π , hence the node sets $\{V(S_{\mathcal{C}}) : \mathcal{C} \in \Pi\}$ are pairwise disjoint, by Claim 3.5. To finish the proof of Theorem 3.1, it is sufficient to prove:

LEMMA 3.14. *For every $\mathcal{C} \in \Pi$ the following holds:*

- If $|\mathcal{C}| \geq 2$ then there is $C_j \in \mathcal{C}$ such that P_i is an $\mathcal{F}(s_j, C_i)$ -spider-cover for every $C_i \in \mathcal{C}$.
- If $|\mathcal{C}| = 1$, say $\mathcal{C} = \{C_i\}$, then P_i covers $\mathcal{F}(C_i)$.

PROOF. Suppose that $|\mathcal{C}| \geq 2$. Then, by the construction, either $\mathcal{C} \in \Pi'$, or $\mathcal{C} \in \Pi''$. If $\mathcal{C} \in \Pi'$, then the statement of the Lemma follows from Claim 3.12. If $\mathcal{C} \in \Pi''$, then the statement of the Lemma follows from Claim 3.13.

If $|\mathcal{C}| = 1$, say $\mathcal{C} = \{C_i\}$, then C_i is not dangerous, by the construction. Hence P_i covers $\mathcal{F}(C_i)$, as required. \square

The proof of Theorem 3.1 is now complete.

The rest of the proof of Theorem 1.3, namely, deducing Theorem 1.3 from Theorem 3.1, is a slight modification of the one in [Nutov 2010], except the following minor change. To achieve the ratio $6H(|\mathcal{C}_{\mathcal{F}}|)$, the algorithm in [Nutov 2010] uses as a subroutine a 2-approximation algorithm for finding a minimum node-weight edge-cover of a ring *set-family*. Such an algorithm easily follows from the observation that if F is an inclusion-minimal cover of a ring set-family, then $|\delta_F(v)| \leq 2$ for all $v \in V$. The same statement is true for ring-bifamilies, as shown by the following statement.

LEMMA 3.15. *Let F be an inclusion-minimal cover of a ring-bifamily \mathcal{F} . Then $|\delta_F(v)| \leq 2$ for all $v \in V$.*

PROOF. Note that any ring bifamily is uncrossable; hence there exists a laminar fit-bifamily $\mathcal{L} \subseteq \mathcal{F}$ for F . Suppose to the contrary that there are three edges $vx, vy, vz \in F$, and let $\hat{X}, \hat{Y}, \hat{Z} \in \mathcal{L}$ be their fit-bisets, respectively. Since $\{\hat{X}, \hat{Y}, \hat{Z}\}$ is a laminar ring bifamily, we have w.l.o.g. that $X \subset Y \subset Z$. If $v \in Y$ then vz covers Y , and if $v \notin Y$ then vx covers Y , by the monotonicity of \mathcal{F} . In both cases this contradicts that $\delta_F(Y) = \{vy\}$. \square

4. PROOF OF THEOREM 1.6

The following statement finishes the proof of Proposition 1.5.

CLAIM 4.1. *Suppose that for edge-costs Bifamily Edge-Cover with T -uncrossable \mathcal{F} admits a polynomial time algorithm that computes a solution of cost $\leq \rho(\gamma) \cdot \tau(\mathcal{F})$, where \mathcal{F} is the T -uncrossable bifamily from Claim 1.1, and γ, ρ are as in Proposition 1.5. Then Rooted GSN admits a polynomial time algorithm that computes a solution of cost $\leq \text{opt} \cdot \sum_{\ell=1}^k \frac{\rho(\ell)}{k-\ell+1}$.*

PROOF. Consider the following sequential algorithm. Start with $J = \emptyset$. At iteration $\ell = 1, \dots, k$, add to J an augmenting edge-set I_{ℓ} that increases by 1 the connectivity between pairs in $\mathcal{T}_{\ell} = \{\{s, t\} : \lambda_J^S(s, t) = r(s, t) - k + \ell - 1, s, t \in T\}$. Let T_{ℓ} be the union of the pairs in \mathcal{T}_{ℓ} and \mathcal{F}_{ℓ} the corresponding T_{ℓ} -uncrossable bifamily

as in Claim 1.1. Note that $\max_{\{s,t\} \in \mathcal{T}_\ell} \lambda_J^S(s,t) \leq \ell$, hence $c(I_\ell) = O(\ell) \cdot \tau(\mathcal{F}_\ell)$. After iteration ℓ , we have $\lambda_J^S(s,t) \geq r(s,t) - k + \ell$ for all $s, t \in T$. Consequently, after k iterations $\lambda_J^S(s,t) \geq r(s,t)$ holds for all $s, t \in T$, thus the computed solution is feasible. For the approximation ratio, it is sufficient to show that $\tau(\mathcal{F}_\ell) \leq \text{opt}/(k - \ell + 1)$. For all $U \in \mathcal{F}_\ell$, any feasible solution H to Rooted GSN has at least $k - \ell + 1$ edges covering U , by Menger's Theorem. Thus if x is a characteristic vector of $E(H)$, then $x/(k - \ell + 1)$ is a feasible solution for the LP-relaxation for edge-covering \mathcal{F}_ℓ . The statement follows. \square

CLAIM 4.2. *For both edge and node-costs, a ρ -approximation for Rooted GSN with requirements in $\{0, k\}$ implies a $\rho \cdot \min\{k, |T| - 1\}$ -approximation for Subset k -Connected Subgraph.*

PROOF. Choose arbitrary $\min\{k, |T| - 1\}$ roots and for each root s compute a ρ -approximation for Rooted GSN with requirements $r(s, t) = k$ for each $t \in T - \{s\}$. Then take the union of the $\min\{k, |T| - 1\}$ subgraphs computed. It is known and easy to see that the computed solution is feasible, and its cost is as claimed. \square

Except for GSN with node-costs, Theorem 1.6 easily follows from Theorem 1.4 and Claims 1.1, 4.1, and 4.2. As for GSN with node-costs, it is remarked in [Chuzhoy and Khanna 2009] that a β -approximation for Element-GSN implies an $O(k^3 \beta \log |T|)$ -approximation for GSN. Thus for node-costs, our $O(k \log |T|)$ -approximation for Element-GSN together with the result of [Chuzhoy and Khanna 2009], implies an $O(k^4 \log^2 |T|)$ -approximation algorithm for GSN. This finishes the proof of Theorem 1.6.

5. CONCLUSIONS AND OPEN PROBLEMS

In this paper we developed approximation algorithm for GSN problems for both edge and node-costs. Our algorithms are simple and combinatorial, and they achieve much better approximation guarantees than those previously known ones. For edge-costs, our ratios are $O(k \ln k)$ for Rooted GSN and $O(k^2 \ln k)$ for Subset k -Connected Subgraph. These ratios are constants for bounded values of k . For node-costs, we gave the first non-trivial algorithm for Element-GSN, matching the best known ratio for Edge-GSN.

An open problem is to achieve an $O(k)$ ratio for Rooted GSN with edge-costs. Such an algorithm would probably rely on the iterative rounding method, while all algorithms in this paper are combinatorial. Another open problem is to achieve a ratio of $\tilde{O}(k^2)$ for general GSN.

Finally, we note that for all problems considered, a significant obstacle lies in the way of achieving a ratio sublinear in k . As was mentioned by [Lando and Nutov 2009], this would imply a sublinear in n ratio for the *directed* variant. Such algorithms are known only for $k = 1$, and they are highly non-trivial; see [Charikar et al. 1999] for the rooted case and [Feldman et al. 2009] for the general case. Furthermore, for the directed variant, even for rooted requirements, no ratio better than the trivial $O(n)$ is known even for $k = 2$.

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