A note on edge-covers of (S,T)-crossing families

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Abstract

Let S,T be a partition of a groundset V; $X,Y \subseteq V$ (S,T)-*cross* if $X \cap Y \cap S, T \setminus (X \cup Y) \neq \emptyset$. A set-family $\mathcal{F}$ is (S,T)-crossing if $X \cap Y, X \cup Y \in \mathcal{F}$ for any $X,Y \in \mathcal{F}$ that (S,T)-cross. We give an $H(|S||T|)$-approximation algorithm for the problem of covering an (S,T)-crossing family by a minimum-cost set of (S,T)-edges, where $H(n)=\sum_{i=1}^{n} 1/i$ denotes the nth harmonic number. This generalizes and significantly simplifies one of the main results of [1].

1 Introduction

Let S,T be a partition of a groundset V. For $X \subseteq V$ let $\bar{X} = V \setminus X$ and let $X_S = X \cap S$.

Definition 1.1 *Two sets X,Y S-*intersect *if $X_S \cap Y_S \neq \emptyset$, and* (S,T)-cross *if $X_S \cap Y_S, \bar{X}_T \cap \bar{Y}_T \neq \emptyset$. A set-family $\mathcal{F}$ is* S-intersecting/(S,T)-crossing *if $X \cap Y, X \cup Y \in \mathcal{F}$ for any $X,Y \in \mathcal{F}$ that S-intersect/(S,T)-cross.*

A directed edge-set/graph J covers a set-family $\mathcal{F}$ if for every $X \in \mathcal{F}$ there is an edge in J that goes from X to $\bar{X}$. We consider a variant of the following generic problem.

Set-Family Edge-Cover
Instance: A directed graph $G=(V,E)$ with edge-costs $\{c_e : e \in E\}$ and a set-family $\mathcal{F}$ on V.
Objective: Find a minimum-cost edge-set $J \subseteq E$ that covers $\mathcal{F}$.

Definition 1.2 *A set $C \in \mathcal{F}$ is a* core *of a set-family $\mathcal{F}$ if C contains no set in $\mathcal{F} \setminus \{C\}$. Let $\mathcal{C}(\mathcal{F})$ denote the family of $\mathcal{F}$-cores and let $\nu(\mathcal{F}) = |\mathcal{C}(\mathcal{F})|$.*

In the Set-Family Edge-Cover problem, $\mathcal{F}$ may not be given explicitly, and a polynomial in $n=|V|$ implementation of our algorithms requires that certain queries related to $\mathcal{F}$ can be answered in polynomial time. Given an edge set J on V, the *residual family* $\mathcal{F}^J$ of $\mathcal{F}$ consists of all members of $\mathcal{F}$ that are uncovered by the edges of J. It is easy to see that if an edge covers the intersection or the union of two sets X,Y then it covers one of X,Y. This implies that if $\mathcal{F}$ is T-intersecting or (S,T)-crossing, so is $\mathcal{F}^J$, for any edge-set J. The *co-family* of $\mathcal{F}$ is the set-family $\{\bar{X} : X \in \mathcal{F}\}$ of the complements of the sets in $\mathcal{F}$. It is easy to see that $\mathcal{F}$ is (S,T)-crossing if, and only if, its co-family is (T,S)-crossing, and that J covers $\mathcal{F}$ if, and only if, the reverse edge-set of J covers the co-family of $\mathcal{F}$. We assume that for any edge set J on V and any $(s,t) \in S \times T$ we are able to compute in polynomial time the cores of the set-family $\mathcal{F}(s,t) = \{X \in \mathcal{F}^J : s \in S, t \in T\}$ and also the cores of its co-family, or to determine that $\mathcal{F}(s,t)$ is empty. In specific graph problems, this can be implemented in polynomial time using the Ford-Fulkerson Max-Flow Min-Cut Algorithm. Given an instance of Set-Family Edge-Cover we will assume that the problem admits a feasible solution, namely, that E covers $\mathcal{F}$; otherwise our algorithms can be easily modified to return an error message. In particular, if $E \subseteq S \times T$ then we must have $X_S, \bar{X}_T \neq \emptyset$ for all $X \in \mathcal{F}$.

For an edge-set or a graph J and a set X on V let $\delta_J(X)$ denote the set of edges in J covering X. Let $\tau(\mathcal{F})$ denote the optimal value of an LP-relaxation for covering a set-family $\mathcal{F}$, namely,

$$\begin{array}{rll} \tau(\mathcal{F}) = & \min & \sum_{e \in E} c_e x_e \\ & \text{s.t.} & \sum_{e \in \delta_E(U)} x_e \geq 1 \qquad \forall U \in \mathcal{F} \\ & & x_e \geq 0 \qquad\qquad \forall e \in E \end{array}$$

For $E \subseteq S \times T$ Set-Family Edge-Cover admits a polynomial time algorithm that computes a solution of cost $c(\tau(\mathcal{F}))$ for S-intersecting $\mathcal{F}$ [3], and also in the case of unit costs when $\mathcal{F}$ is (S,T)-crossing and $E = S \times T$ [4]. However, the case of (S,T)-crossing $\mathcal{F}$ and arbitrary costs includes the min-cost k-Connected Subgraph Augmentation problem which is NP-hard, and its approximability is not yet understood, see [2, 6]. In fact, the case of (S,T)-crossing $\mathcal{F}$ includes also the more general "standard version" of the k-(S,T)-Connectivity Augmentation problem considered in [1]. In this problem we are given a k-(S,T)-connected graph G_0 (namely, in G_0 are k edge-disjoint st-paths for every $(s,t) \in S \times T$) and an edge-set E with costs. The goal is to find a minimum-cost edge set $J \subseteq E$ such that the graph $G_0 \cup J$ is $(k+1)$-(S,T)-connected. In the "standard version" of the problem we have $E \subseteq S \times T$. This version already includes the augmentation versions of the k-Connected Subgraph problem and also the more general Subset k-Connected Subgraph problem when every edge with positive cost has its endnodes in the subset, see [6]. Recently, generalizing the algorithm of Fackaroenphol and Laekhanukit [2] for the k-Connected Subgraph problem, Cheriyan and Laekhanukit [1] gave an $H(|S||T|)$-approximation algorithm for the "standard version" of the k-(S,T)-Connectivity Augmentation problem, where $H(n) = \sum_{i=1}^{n}(1/i)$ denote the nth harmonic number. They also observed that it is a particular case of the Set-Family Edge-Cover problem with $E \subseteq S \times T$ and (S,T)-crossing $\mathcal{F}$. In this note, following the generic approach of [6], we give a much simpler proof of a more general result.

Theorem Set-Family Edge-Cover with (S,T)-crossing set-family $\mathcal{F}$ and $E \subseteq S \times T$ admits a polynomial time algorithm that computes a solution of cost at most $\tau(\mathcal{F}) \cdot H(|S||T|)$.

2 Proof of the theorem

Note that if $\mathcal{F}$ is (S,T)-crossing, then each of $\mathcal{F}(s,t)$ and the co-family of $\mathcal{F}(s,t)$ has a unique core.

Lemma 2.1 *An (S,T)-crossing set-family $\mathcal{F}$ has at most $|S||T|$ cores that can be computed in polynomial time.*

Proof: For every ordered pair of nodes $(s,t) \in S \times T$ we compute the core C_{st} of the set-family $\mathcal{F}(s,t) = \{X \in \mathcal{F}^J : s \in S, t \in T\}$, if $\mathcal{F}(s,t)$ is non-empty. Then $\mathcal{C}(\mathcal{F})$ consists from the inclusion-minimal members (cores) of the set-family $\{C_{st} : (s,t) \in S \times T\}$. □

Definition 2.1 *Given a set-family $\mathcal{F}$ and a core $C \in \mathcal{C}(\mathcal{F})$ of $\mathcal{F}$ let $\mathcal{F}(C)$ denote the family of the sets in $\mathcal{F}$ that contain C and contain no other core of $\mathcal{F}$.*

The following statement is immediate from the definition of $\mathcal{F}(C)$, c.f. [5].

Claim 2.2 *Let $\mathcal{F}$ be a set-family. If J edge-covers $\mathcal{F}(C)$ for some $C \in \mathcal{C}(\mathcal{F})$, then every $\mathcal{F}^J$-core contains an $\mathcal{F}$-core in $\mathcal{C}(\mathcal{F}) \setminus \{C\}$ or at least two $\mathcal{F}$-cores. In particular, $\nu(\mathcal{F}^J) \leq \nu(\mathcal{F}) - 1$.*

Lemma 2.3 *Let C_X, C_Y be distinct cores of an (S,T)-crossing set-family $\mathcal{F}$, and let $X \in \mathcal{F}(C_X)$ and $Y \in \mathcal{F}(C_Y)$. Then X, Y do not (S,T)-cross, hence no edge from S to T covers both X, Y.*

Proof: Suppose to the contrary that X, Y (S,T)-cross. Then $X \cap Y \in \mathcal{F}$. Thus $X \cap Y$ contains some $\mathcal{F}$-core C. We cannot have $C \neq C_X$ as $C \subseteq C_X$ and C_X is a core, and we cannot have $C = C_X$ as $C \subseteq C_Y$, $C_X \neq C_Y$, and C_Y is a core. This gives a contradiction. The second statement follows from the observation that an edge from S to T covers two sets X, Y simultaneously if, and only if, it goes from $X_S \cap Y_S$ to $\bar{X}_T \cap \bar{Y}_T$, and hence X, Y (S,T)-cross. □

Lemma 2.4 *Let $C \in \mathcal{C}(\mathcal{F})$ be a core of an (S,T)-crossing set-family $\mathcal{F}$, and let $X, Y \in \mathcal{F}(C)$. If X, Y T-intersect then $X \cap Y, X \cup Y \in \mathcal{F}(C)$.*

Proof: Since $\mathcal{F}$ is (S,T)-crossing, $X \cap Y, X \cup Y \in \mathcal{F}$. We prove that each of $X \cap Y, X \cup Y$ contains C and contains no core distinct from C. Since $X \cap Y \subseteq X \subseteq X \cup Y$ and since $X \in \mathcal{F}(C)$, it follows that $X \cap Y \in \mathcal{F}(C)$ and that $C \subseteq X \cup Y$. It remains to prove that $X \cup Y$ contains no core distinct from C. Suppose to the contrary that $X \cup Y$ contains a core C' distinct from C. It is not hard to verify that then C', X (S,T)-cross or C', Y (S,T)-cross, say the former holds. This implies $C' \cap X \in \mathcal{F}$, but since C' is a core we must have $C' \subseteq X$. This contradicts that $X \in \mathcal{F}(C)$. □

Lemma 2.5 *Let $\mathcal{F}$ be an (S,T)-crossing family and let $C \in \mathcal{C}(\mathcal{F})$. Then the co-family $\mathcal{R}(C) = \{\bar{X} : X \in \mathcal{F}(C)\}$ of $\mathcal{F}(C)$ is T-intersecting, and its cores can be found in polynomial time.*

Proof: Let $\bar{X}, \bar{Y} \in \mathcal{R}(C)$ be the co-sets of $X, Y \in \mathcal{F}(C)$, respectively. Suppose that $\bar{X}, \bar{Y}$ T-intersect. Since $X_S \cap Y_S \supseteq C_S \neq \emptyset$, X, Y (S,T)-cross, and hence $X \cap Y, X \cup Y \in \mathcal{F}(C)$, by Lemma 2.4. The co-sets of $X \cap Y$ and $X \cup Y$ are $\bar{X} \cup \bar{Y}$ and $\bar{X} \cap \bar{Y}$, hence $\bar{X} \cup \bar{Y}, \bar{X} \cap \bar{Y} \in \mathcal{R}(C)$.

This implies that $\mathcal{R}(C)$ is a T-intersecting family. Now we show how to find the cores of $\mathcal{R}(C)$ in polynomial time. For an $\mathcal{F}$-core $C' \neq C$ let $K_{C'} = \{uv : u \in C', v \in \bar{C}'\}$ be the set of all edges from C' to $\bar{C}'$. Let $K = \bigcup_{C' \in \mathcal{C}(\mathcal{F}) \setminus \{C\}} K_{C'}$. We claim that $\mathcal{F}^K = \mathcal{F}(C)$. To see this, note that for any $C' \in \mathcal{C}(\mathcal{F}) \setminus \{C\}$, $K_{C'}$ covers all sets in $\mathcal{F}$ that contain C', and cover no set in $\mathcal{F}(C)$, by Lemma 2.3. Now choose $s \in C_S$, and for every $t \in \bar{C} \cap T$ compute the core C_t of the co-family of $\mathcal{F}^K(s,t)$. The $\mathcal{R}(C)$-cores are the inclusion-minimal members of the family $\{C_t : t \in T\}$. □

Corollary 2.6 Set-family Edge-Cover *with (S,T)-crossing $\mathcal{F}$ and with $E \subseteq S \times T$ admits a polynomial time algorithm that given $C \in \mathcal{C}(\mathcal{F})$ computes an $\mathcal{F}(C)$-cover $J_C \subseteq E$ of cost $c(J_C) = \tau(\mathcal{F}(C))$. Moreover, $\sum_{C \in \mathcal{C}} \tau(\mathcal{F}(C)) \leq \tau(\mathcal{F})$; thus there exists $C \in \mathcal{C}(\mathcal{F})$ with $c(J_C) \leq \tau(\mathcal{F})/\nu(\mathcal{F})$.*

Proof: By Lemma 2.5, the co-family $\mathcal{R}(C)$ of $\mathcal{F}(C)$ is T-intersecting. Thus, after reversing the edges in E, we can apply the primal-dual algorithm of Frank [3] to compute an edge-cover of $\mathcal{R}(C)$ of cost $\tau(\mathcal{R}(C)) = \tau(\mathcal{F}(C))$; J_C is the reverse edge set of this cover. This primal-dual algorithm can be implemented in polynomial time if the cores of $\mathcal{R}(C)$ can be found in polynomial time, which is possible by Lemma 2.5. The second statement of the lemma follows from Lemma 2.3. □

Now we finish the proof of the theorem. The algorithm start with $J = \emptyset$. At iteration i, it finds $C_i \in \mathcal{C}(\mathcal{F}^J)$ and $J_i \subseteq E \setminus J$ with $c(J_i) \leq \tau(\mathcal{F}^J)/\nu(\mathcal{F}^J)$, and adds J_i to J; such J_i exists by Corollary 2.6. At each iteration $\nu(\mathcal{F}^J)$ decreases by 1, by Claim 2.2. Thus at the end of iteration i we have $\nu(\mathcal{F}^J) = \nu(\mathcal{F}) - i$. Consequently, $c(J_i) \leq \tau(\mathcal{F}^J)/\nu(\mathcal{F}^J) \leq \tau(\mathcal{F})/(\nu(\mathcal{F}) - i)$. Thus at the end of the algorithm, $c(J) \leq \sum_i c(J_i) \leq \tau(\mathcal{F}) \sum_i 1/(\nu(\mathcal{F}) - i) = \tau(\mathcal{F}) \cdot H(\nu(\mathcal{F})) \leq \tau(\mathcal{F}) \cdot H(|S||T|)$.

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