Approximating Minimum-Power k-Connectivity

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Abstract. The Minimum-Power k-Connected Subgraph (MPkCS) problem seeks a power (range) assignment to the nodes of a given wireless network such that the resulting communication (sub)network is k-connected and the total power is minimum. We give a new very simple approximation algorithm for this problem that significantly improves the previously best known approximation ratios. Specifically, the approximation ratios of our algorithm are:
- 3 (improving $(3 + 2/3)$) for $k = 2$;
- 4 (improving $(5 + 2/3)$) for $k = 3$;
- $k+3$ for $k \in \{4, 5\}$ and $k+5$ for $k \in \{6, 7\}$ (improving $k+2\lceil(k+1)/2\rceil$);
- $3(k - 1)$ (improving $3k$) for any constant k.
Our results are based on a $(k + 1)$-approximation algorithm (improving the ratio $k + 4$) for the problem of finding a Min-Power k-Inconnected Subgraph, which is of independent interest.

1 Introduction

1.1 Preliminaries

Wireless networks are studied extensively due to their wide applications. The power consumption of a station determines its transmission range, and thus also the stations it can send messages to; the power typically increases at least quadratically in the transmission range. Assigning power levels to the stations (nodes) determines the resulting communication network. Conversely, given a communication network, the cost required at v only depends on the furthest node that is reached directly by v. This is in contrast with wired networks, in which every pair of stations that need to communicate directly incurs a cost.

In network design problems one seeks to design a "cheap" communication (sub)network that satisfies some prescribed properties. An important network property is fault-tolerance, often measured by node-connectivity of the network. Node-connectivity is much more central here than edge-connectivity, as it models stations failures. Such problems were vastly studied; see [3, 1, 4, 9, 21, 17, 12, 11] for only a small sample of papers in this area. We consider the Min-Power k-Connected Subgraph (MPkCS) problem which is the power variant of the classic Min-Cost k-Connected Subgraph (MCkCS) problem. We give an approximation algorithm for MPkCS that significantly improves the previously best known ones.

Definition 1. *Let $H = (V, I)$ be a graph with edge-costs $\{c(e) : e \in I\}$. For $v \in V$, the* power $p(v) = p_H(v)$ of *v in H (w.r.t. c) is the maximum cost of an edge in I leaving v, i.e., $p(v) = p_I(v) = \max_{vu \in I} c(vu)$. The power of the graph is the sum of the powers of its nodes.*

Note that $p(H)$ differs from the cost $c(H) = \sum_{e \in I} c(e)$ of H even for unit costs; for unit costs, if H is undirected, then $c(H) = |I|$ and (if H has no isolated nodes) $p(H) = |V|$. For example, if I is a perfect matching on V then $p(H) = 2c(H)$. If H is a clique then $p(H)$ is roughly $c(H)/\sqrt{|I|/2}$. For directed graphs, the ratio of the cost over the power can be equal to the maximum outdegree, e.g., for stars with unit costs. The following statement, parts of which appeared in various papers, c.f., [9, 11], shows that these are the extremal cases for general edge costs.

Proposition 1. *$c(H)/\sqrt{|I|/2} \leq p(H) \leq 2c(H)$ for any undirected graph $H = (V, I)$, and if H is a forest then $c(H) \leq p(H) \leq 2c(H)$. For any directed graph D holds: $c(D)/\Delta(D) \leq p(D) \leq c(D)$, where $\Delta(D)$ is the maximum outdegree of a node in D.*

Minimum-power problems are usually harder than their minimum-cost versions. The Minimum-Power Spanning Tree problem is APX-hard. The problem of finding minimum-cost k pairwise edge-disjoint paths is in P (this is the Minimum-Cost k-Flow problem, c.f., [22]) while both directed and undirected minimum-power variants are unlikely to have even a polylogarithmic approximation [9, 17]. Another example is finding an arborescence rooted at s, that is, a subgraph that contains an sv-path for every node v. The minimum-cost case is in P (c.f., [22]), while the minimum-power variant is at least as hard as the Set-Cover problem. For more examples see [1, 21].

A *network* is a (possibly directed) graph with edge costs. For a graph $H = (V, I)$ and $X \subseteq V$, let $d_I(X) = d_H(X)$ denote the degree of X in H, that is the number of edges from X to $V - X$. All the graphs in the paper are assumed to be simple, and, unless stated otherwise, undirected.

A graph $H = (V, I)$ is *k-connected* if it contains k internally-disjoint uv-paths for all $u, v \in V$. We consider the min-power variant of the extensively studied classic Min-Cost k-Connected Subgraph (MCkCS) problem.

Minimum-Power k-Connected Subgraph (MPkCS):
Instance: A graph $G = (V, E)$ with edge costs $\{c(e) : e \in E)$, and an integer k.
Objective: Find a minimum-power k-connected spanning subgraph H of G.

1.2 Previous and related work

We now introduce some additional related problems, that will also play an important role later. The first problem is the min-power variant of the Min-Cost k-Flow problem (with unit node capacities).

Min-Power k Disjoint Paths (MPkDP)
Instance: A graph $G = (V, E)$, edge-costs $\{c(e) : e \in E\}$, $u, v \in V$, an integer k.
Objective: Find a min-power subgraph H of G with k internally-disjoint uv-paths.

A (possibly directed) graph $H = (V, I)$ is *k-inconnected to s* if it contains k internally-disjoint vs-paths for all $v \in V - s$.

Min-Power k-Inconnected Subgraph (MPkIS)
Instance: A graph $G = (V, E)$, edge-costs $\{c(e) : e \in E\}$, $s \in V$, an integer k.
Objective: Find a min-power k-inconnected to s spanning subgraph H of G.

Min-Power k-Edge-Cover (MPkEC)
Instance: A graph $G = (V, E)$, edge-costs $\{c(e) : e \in E\}$, an integer k.
Objective: Find a min-power edge set $I \subseteq E$ so that $d_I(v) \geq k$ for all $v \in V$.

It is easy to see (c.f., [11, 9]) that the simplest heuristic for MPkEC that for every node $v \in V$ takes the k cheapest edges incident to v is a $(k + 1)$-approximation algorithm for MPkEC. In [12] the approximation ratio $O(\log n)$ was derived. For $k = 1$ a 3/2-approximation algorithm is given in [13].

It turns out that approximating MPkCS is closely related to approximating MCkCS and MPkEC as shows the following observation from [9], which first part was implicitly observed independently in [11].

Theorem 1 ([9, 11]).
(i) *An α-approximation for* MCkCS *and a β-approximation for* MP$(k-1)$EC *implies a $(2\alpha + \beta)$-approximation for* MPkCS*.*
(ii) *A ρ-approximation for* MPkCS *implies a $(2\rho+1)$-approximation for* MCkCS*.*

One can combine various values of α, β with Theorem 1(i) to get approximation algorithms for MPkCS. As was mentioned, currently $\beta = \min\{k, O(\log n)\}$ [9], and $\beta = 3/2$ for $k = 2$ [13] (note that here β is the ratio for MP$(k-1)$EC and not for MPkEC). The best known values for α are: $\alpha = \lceil (k+1)/2 \rceil$ for $2 \leq k \leq 7$ (see [2] for $k = 2, 3$, [6] for $k = 4, 5$, and [14] for $k = 6, 7$), $\alpha = k$ for other small values of k [14], and $\alpha = O\left(\log k \cdot \log \frac{n}{n-k}\right)$ otherwise [20]. Thus for undirected MPkCS the following ratios follow: $3k$ for any k, $k + 2\lceil (k+1)/2 \rceil$ for $2 \leq k \leq 7$, $O(\log n)$ unless $k = n - o(n)$, and $O(\log^2 n)$ if $k = n - o(n)$.

Improvements over the above ratios for MPkCS are known only for $k \leq 5$: $(2k - 1/3)$ for $k \in \{2, 3\}$ [13], and 9 for $k = 4$ [11].

For further results on other min-power connectivity problems, among them problems on directed graphs see [9, 21, 17]. For results on min-cost k-connectivity problems see [2, 6, 14, 5, 15, 7, 20, 18]; see also a recent survey in [16] on various min-cost connectivity problems.

1.3 Results

The previously best known ratio for undirected MPkIS was $\min\{k+4, O(\log n)\}$ [17]. We improve the ratio $k + 4$ for $k = O(\log n)$ as follows:

Theorem 2. *Undirected* MPkIS *admits a $(k + 1)$-approximation algorithm.*

Combining Theorem 2 with a direct analysis of the algorithms in [2, 6, 14] for MCkCS, we obtain the following result:

Theorem 3. *Suppose that* MPkIS *admits a* γ*-approximation algorithm and that* MPkDP *admits a* θ*-approximation algorithm. Then* MPkCS *admits the following approximation ratios:* $\gamma + \theta(k-2)$ *for any constant* k *and* $\gamma + \theta(\lfloor k/2 \rfloor - 1)$ *for* $k \leq 7$*. In particular, for* $k \leq 7$ *the ratios are:* γ *for* $k \in \{2,3\}$*,* $\gamma + \theta$ *for* $k \in \{4,5\}$*, and* $\gamma + 2\theta$ *for* $k \in \{6,7\}$*.*

As MPkDP admits a 2-approximation algorithm (c.f., [9, 17]), then by combining Theorems 2 and 3 we obtain:

Theorem 4. MPkCS *admits the following approximation ratios:* $k+1$ *for* $k \in \{2,3\}$*,* $k+3$ *for* $k \in \{4,5\}$*,* $k+5$ *for* $k \in \{6,7\}$*, and* $3(k-1)$ *for any constant* k*.*

Theorem 4 significantly improves the previously best known ratios for MPkCS with $2 \leq k \leq 7$, as summarized in the following table:

k	Prior art	This paper
1	$(5/3+\varepsilon)$ [1]	–
2	$(3+2/3)$ [13]	3
3	$(5+2/3)$ [13]	4
4	9 [11]	7
5	11 [11, 9]	8
6	14 [11, 9]	11
7	15 [11, 9]	12
constant k	$3k$ [11, 9]	$3k-3$

Table 1. Approximation ratios for MPkCS.

Theorems 2 and 3 are proved in Sections 2 and 3, respectively.

2 Algorithm for MPkIS (Proof of Theorem 2)

A *bi-direction* of an undirected network H is a directed network obtained by replacing every edge $e = uv$ of H by two opposite directed edges uv, vu each having the same cost as e. Clearly, if D is a bi-direction of H, then $p(H) = p(D)$. The *underlying network* of a directed network D is a network H obtained from D by ignoring the directions (but keeping costs) of the edges, and then keeping one (arbitrary) edge from every maximal set of parallel edges, if non-empty. If H is the underlying network of a directed star D with unit costs, then $p(H) = (\Delta(D)+1)p(D)$. The following statement shows that this is the extremal case for general costs.

Lemma 1. $p(H) \leq (\Delta(D)+1)p(D)$ *for the underlying network* H *of a directed network* D*.*

Proof. By induction on the number m of edges in D. For $m = 1$ the statement is obvious. Assume that the statement is true for digraphs with at most $m-1$

edges. Let $v \in V$ be a node in D of maximum power $c_{\max}$. Let D' be obtained from D by removing the edges leaving v, and let H' be the underlying graph of D'. Clearly, $p(D') = p(D) - c_{\max}$ and $p(H') \geq p(H) - (\Delta(D)+1)c_{\max}$. Combining with the induction hypothesis gives:

$$\begin{aligned} p(H) &\leq p(H') + (\Delta(D)+1)c_{\max} \\ &\leq (\Delta(D)+1)(p(D)+c_{\max}) \\ &= (\Delta(D)+1)p(D) \ . \end{aligned}$$

We need several results from [17].

Theorem 5 ([17]). *Directed* MPkIS *can be solved in polynomial time.*

Definition 2. *An edge e of a k-inconnected to s graph J is* critical *if $J - e$ is not k-inconnected to s. A graph is minimally k-inconnected to s if all its edges are critical.*

Theorem 6 ([17]). *Let uv' and uv'' be two distinct critical edges of a k-inconnected to s directed graph J. Then $d_J(u) = k$. In particular, $d_J(u) = k$ for every node $u \neq s$ if J is minimally k-inconnected to s.*

The $(k+1)$-approximation algorithm for MPkIS is as follows:

1. Let D be the bi-direction of G.
2. Compute a min-power k-inconnected to s spanning subgraph J of D.
3. Return the underlying graph H of J.

Step 2 can be implemented in polynomial time using the algorithm of [17] (Theorem 5). We now show that the approximation ratio of the algorithm is $k+1$. Let H^* be an optimal solution to MPkIS instance (so $p(H^*) = \mathsf{opt}$) and let J^* be the bi-direction of H^*. Let H and J be as in the algorithm. Combining Theorem 6 with Lemma 1 we get:

$$\begin{aligned} p(H) &\leq (\Delta(J)+1)p(J) \\ &\leq (k+1)p(J) \\ &\leq (k+1)p(J^*) \\ &\leq (k+1)p(H^*) \\ &= (k+1)\mathsf{opt} \ . \end{aligned}$$

The proof of Theorem 2 is complete.

3 Algorithm for MPkCS (Proof of Theorem 3)

We need the following summary of several statements from [2, 6, 14].

Lemma 2 ([2, 6, 14]). *Let $H = (V, I)$ be k-inconnected to s graph with $d_H(s) = k$. Then one can find in polynomial time a set F of at most $k-2$ new edges on the neighbors of s in H so that $H + F$ is k-connected. Furthermore, $|F| \leq \lfloor k/2 \rfloor - 1$ for $k \leq 7$.*

Halin [10] proved that any minimally k-connected graph has a node of degree k. A stronger statement was proved by Mader [19]:

Theorem 7 ([19]). *A minimally k-connected graph contains at least $\frac{(k-1)n+2}{2k-1}$ nodes of degree k.*

This motivates the following auxiliary problem, which min-cost variant is the basis for the algorithms in [2, 6, 14].

Restricted MPkIS
Instance: A graph $G = (V, E)$, edge costs $\{c(e) : e \in E\}$, $s \in V$, an integer k.
Objective: Find a min-power k-inconnected to s spanning subgraph H of G with $d_H(s) = k$.

Lemma 3. *If* MPkIS *admits a γ-approximation algorithm then* Restricted MPkIS *admits a γ-approximation algorithm for any constant k.*

Proof. The algorithm for Restricted MPkIS is derived from the algorithm for MPkIS by "guessing" the k edges incident to s in some optimal solution for Restricted MPkIS. For any subset $K \subseteq E$ of k edges incident to s, we remove the other edges incident to s, and compute a γ-approximate solution H_K to MPkIS (or declare that the resulting graph is not k-inconnected to s). Then, among the subgraphs H_K computed, we output one H of the minimum power. The running time is $\binom{n}{k} = O(n^k)$ times the running time of the γ-approximation algorithm for MPkIS, hence polynomial for any constant k.

Remark: In [6], it was shown that the min-cost version of directed Restricted MPkIS is solvable in polynomial time; this was done by using the algorithm of [8] for the min-cost version of directed MPkIS and penalty methods. Although MPkIS is solvable in polynomial time [17], it seems that the penalty method used in [6] does not work for directed Restricted MPkIS.

We now finish the proof of Theorem 3. The algorithm is as follows:

1. For every $s \in V$, compute a γ-approximate solution H_s to Restricted MPkIS with G, s.
   Among the subgraphs H_s computed, let H be one of the minimum power.
2. Compute an edge set F as in Lemma 2.
3. For every $uv \in F$ compute a 2-approximate solution for MPkDP in $G, \{u, v\}$.
4. Return $H + \bigcup\{F_{uv} : uv \in F\}$.

The fact that the returned graph is k-connected was already established in [2, 14], and easily follows from the definition of F. For any constant k, Step 1 can be implemented in polynomial time, by Lemma 3. All the other steps can be implemented in polynomial time for any k. Thus the running time is polynomial for any constant k, as claimed.

We prove the approximation ratio. Note that a k-connected graph is also k-inconnected to s for every $s \in V$. Let H^* be some optimal solution to MPkCS;

clearly, we may assume that H^* is minimally k-connected. From Theorem 7 it follows that there is a node $s \in V$ so that the degree of s in H is k. Thus for the graph H computed at Step 1 we have $p(H) \leq \gamma p(H^*) = \gamma \mathsf{opt}$. Also, H^* contains k internally disjoint uv for all $u, v \in V$. Thus $F_{uv} \leq \theta \mathsf{opt}$ for all $uv \in F$. Consequently,

$$\begin{aligned} p\left(H + \bigcup\{F_{uv} : uv \in F\}\right) &\leq p(H) + \sum_{uv \in F} p(F_{uv}) \\ &\leq \gamma \mathsf{opt} + \theta |F| \mathsf{opt} \\ &= (\gamma + \theta|F|)\mathsf{opt} \ . \end{aligned}$$

Substituting the sizes of F from Lemma 2 we obtain the following. For any k we have $|F| \leq k-2$, and thus in this case the approximation ratio is $\gamma + \theta|F| = \gamma + \theta(k-2)$. For $k \leq 7$ we have $|F| \leq \lfloor k/2 \rfloor - 1$, and thus in this case the approximation ratio is $\gamma + \theta|F| = \gamma + \theta(\lfloor k/2 \rfloor - 1)$. Substituting the specific values of k, we obtain the last statement of the Theorem.

The proof of Theorem 3 is complete.

4 Open problems

The main open problem in the context of this paper is to determine whether the *undirected* MPkDP is in P or is NP-hard (the directed MPkDP is in P, c.f., [9]). If MPkDP is in P, then we can substitute $\theta = 1$ in Theorem 3 and obtain the following ratios for MPkCS: $2k-1$ (instead of $3k-3$) for any constant k, and $k + \lfloor k/2 \rfloor$ (improving $k - 1 + 2\lfloor k/2 \rfloor$) for $k \leq 7$.

We note that we do not know the answer even to the following "easier" question. Let MPkDP Augmentation be the restriction of MPkDP to instances where $E_0 = \{e \in E : c(e) = 0\}$ contains $k-1$ pairwise internally disjoint paths. We do not know if (undirected) MPkDP Augmentation is in P, but we conjecture this is so. A polynomial algorithm for MPkDP Augmentation can be used to improve the ratios for MPkCS for $k = 4, 5$: from 7 to 6 for $k = 4$ and from 8 to 7 for $k = 5$. This is since in [2, 6] it is shown that if H is k-inconnected to s and $d_H(s) = k$ then H is $(\lceil k/2 \rceil + 1)$-connected. Thus for $k = 4, 5$, H is $k-1$-connected, and, by Lemma 2, H contains two nodes u, v so that increasing the connectivity between them by one results in a k-connected graph.

Except directed MPkDP and MPkIS that are in P, there is still a large gap between upper and lower bounds of approximation for many other min-power node connectivity problems, for both directed and undirected graphs, see [21, 17, 12].

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