Approximating survivable networks with β-metric costs

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Abstract

The Survivable Network Design (SND) problem seeks a minimum-cost subgraph that satisfies prescribed node-connectivity requirements. We consider SND on both directed and undirected complete graphs with *β-metric costs* when $c(xz) \leq \beta[c(xy)+c(yz)]$ for all $x,y,z \in V$, which varies from uniform costs ($\beta = 1/2$) to metric costs ($\beta = 1$).

For the k-Connected Subgraph (k-CS) problem our ratios are: $1+\frac{2\beta}{k(1-\beta)}-\frac{1}{2k-1}$ for undirected graphs, and $1+\frac{4\beta^3}{k(1-3\beta^2)}-\frac{1}{2k-1}$ for directed graphs and $\frac{1}{2} \leq \beta < \frac{1}{\sqrt{3}}$. For undirected graphs this improves the ratios $\frac{\beta}{1-\beta}$ of [2] and $2+\beta\frac{k}{n}$ of [9] for all $k \geq 4$ and $\frac{1}{2}+\frac{1}{2k} \leq \beta \leq \frac{k^2}{(k+1)^2-2}$. We also show that SND admits the approximation ratios $\frac{2\beta}{1-\beta}$ for undirected graphs, and $\frac{4\beta^3}{1-3\beta^2}$ for directed graphs and $1/2 \leq \beta \leq 1/\sqrt{3}$. For two important particular cases of SND, so called Rooted SND and Subset k-CS, our ratios are $\frac{2\beta^3}{1-3\beta^2}$ for directed graphs and $\frac{\beta}{1-\beta}$ for Subset k-CS on undirected graphs.

1 Introduction

1.1 Problems considered

For a graph H, let $\kappa_H(u,v)$ denote the *uv-connectivity of H*, that is, the maximum number of internally-disjoint uv-paths in H. We consider variants the following fundamental problem:

Survivable Network Design (SND)

Instance: A directed/undirected complete graph $G=(V,E)$ with edge-cost $\{c(e) : e \in E\}$, and connectivity requirements $\{r(u,v) : u,v \in V\}$.

Objective: Find a min-cost subgraph H of G satisfying $\kappa_H(u,v) \geq r(u,v)$ for all $u,v \in V$.

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Let $k = \max_{u,v \in V} r(u,v)$ denote the maximum requirement of an SND instance. Important particular cases of SND are:

- k-Connected Subgraph (k-CS), when $r(u,v) = k$ for all $u,v \in V$.
- Subset k-CS, when $r(u,v) = k$ for all $u,v \in T \subseteq V$ and $r(u,v) = 0$ otherwise.
- Rooted SND, when there is a node $s \in V$ so that $r(u,v) > 0$ implies $u = s$.

We consider instances of SND with *β-metric costs*, when *the input graph is complete* and for some $1/2 \leq \beta < 1$ the costs satisfy the *β-triangle inequality* $c(xz) \leq \beta(c(xy) + c(yz))$ for all $x, y, z \in V$. When $\beta = \frac{1}{2}$ the costs are uniform, and we have the "cardinality version" of the problem (in a complete graph). If we allow the case $\beta = 1$, then the costs satisfy the ordinary triangle inequality and we have the metric version of the problem. Many practical instances of the problem may have costs which are between metric and uniform.

1.2 Previous work and our results

The k-CS problem (and thus also SND) with β-metric costs is APX-hard for $k = 2$ and any $\beta > 1/2$ [1]. Approximation ratios and hardness of approximation results for SND and k-CS are summarized in Table 1. In [2] is also given a $(1 + \frac{5(2\beta-1)}{9(1-\beta)})$-approximation algorithm for undirected 3-CS with β-metric costs. For a survey on various min-cost connectivity problems see [10]. For recent work on SND problems see [7, 12, 13]. We mention a recent result [11] that for $k = n/2 + k'$ the approximability of undirected SND is the same as that of directed SND with maximum requirement k'. This is so also for k-CS. However, the reduction in [11] does not preserve metric costs.

We analyze the algorithm of Cheriyan & Thurimella [4] originally suggested for k-CS with $1, \infty$-costs, and show that for β-metric costs it achieves the following ratios:

Theorem 1 *k-CS with β-metric costs admits the following approximation ratios:*

- $1 + \frac{2\beta}{k(1-\beta)} - \frac{1}{2k-1} \leq 1 + \frac{1}{k}\left(\frac{2\beta}{1-\beta} - \frac{1}{2}\right)$ *for undirected graphs.*
- $1 + \frac{4\beta^3}{k(1-3\beta^2)} - \frac{1}{2k-1} \leq 1 + \frac{1}{k}\left(\frac{4\beta^3}{1-3\beta^2} - \frac{1}{2}\right)$ *for directed graphs and* $1/2 \leq \beta \leq 1/\sqrt{3}$.

For undirected graphs, this improves the ratios $\frac{\beta}{1-\beta}$ of [2] and $2 + \beta\frac{k}{n}$ of [9][1] for all $k \geq 4$ and $\frac{1}{2} + \frac{1}{2k} \leq \beta \leq \frac{k^2}{(k+1)^2-2}$.

[1] In [9] is given a $(2 + (k-1)/n)$-approximation algorithm for metric costs; a slight adjustement of the analysis of [9] shows that this algorithm has ratio $2 + \beta\frac{k}{n}$ for β-metric costs.

Costs	**Requirements**	**Approximability**	
		Undirected	*Directed*
general	general	$O(\min\{k^3 \log n, n^2\}$ [7], $\Omega(k^\varepsilon)$ [6]	$\Omega(2^{\log^{1-\varepsilon} n})$ for $k=1$ [8]
general	k-CS	$O(\log \frac{n}{n-k} \log k)$ [12]	$O(\log \frac{n}{n-k} \log k)$ [12]
general	Subset k-CS	$O(\min\{k^2 \log k, n^2\})$ [13], $\Omega(k^\varepsilon)$ [6]	$O(n^2)$
general	Rooted SND	$O(\min\{k^2, n\})$ [13]	$O(n)$
metric	general	$O(\log k)$ [5]	$\Omega(2^{\log^{1-\varepsilon} n})$ for $k=1$ [8]
metric	k-CS	$2+(k-1)/n$ [9]	$2+k/n$ [9]
β-metric	general	–	–
β-metric	k-CS	$2+\beta\frac{k}{n}$ [9], $\frac{\beta}{1-\beta}$ [2], APX-hard [1]	–

Table 1: Approximation ratios and hardness of approximation results for SND and k-CS (recall that in the case of β-metric costs we assume $1/2 \le \beta < 1$).

Graph	**Requirements**	**Approximability**	**Improvement Range**
undirected	general	$\frac{2\beta}{1-\beta}$	$1/2 \le \beta < 1$
undirected	subset k-CS	$\frac{\beta}{1-\beta}$	$1/2 \le \beta < 1$
undirected	k-CS	$1+\frac{2\beta}{k(1-\beta)}-\frac{1}{2k-1}$	$k \ge 4$, $\frac{1}{2}+\frac{3k-2}{2(4k^2-7k+2)} < \beta < \frac{k^2}{(k+1)^2-2}$
directed	general	$\frac{4\beta^3}{1-3\beta^2}$	$\frac{1}{2} \le \beta < \frac{1}{\sqrt{3}}$
directed	subset k-CS	$\frac{2\beta^3}{1-3\beta^2}$	$\frac{1}{2} \le \beta < \frac{1}{\sqrt{3}}$
directed	rooted	$\frac{2\beta^3}{1-3\beta^2}$	$\frac{1}{2} \le \beta < \frac{1}{\sqrt{3}}$
directed	k-CS	$1+\frac{4\beta^3}{k(1-3\beta^2)}-\frac{1}{2k-1}$	$\frac{1}{2} \le \beta < \frac{1}{\sqrt{3}}$

Table 2: Improvement ranges of our results.

For other versions of the problem our results are as follows.

Theorem 2 SND *with β-metric costs admits approximation ratios $\frac{2\beta}{1-\beta}$ for undirected graphs, and $\frac{4\beta^3}{1-3\beta^2}$ for directed graphs with $1/2 \le \beta < 1/\sqrt{3}$. For* Subset k-CS *the ratios are $\frac{\beta}{1-\beta}$ for undirected graphs and $\frac{2\beta^3}{1-3\beta^2}$ for directed graphs with $1/2 \le \beta < 1/\sqrt{3}$; for directed* Rooted SND *the ratio is $\frac{2\beta^3}{1-3\beta^2}$, $1/2 \le \beta < 1/\sqrt{3}$.*

In our proofs, we will often use the following statement:

Lemma 3 ([1, 3]) *Let e, e' be a pair of edges in a complete graph G with β-metric costs.*

(i) *If G is undirected, and if e, e' are adjacent then $c(e) \le \frac{\beta}{1-\beta} c(e')$.*

(ii) *If G is directed, and if $\frac{1}{2} \le \beta \le \frac{1}{\sqrt{3}}$, then $c(e) \le \frac{2\beta^3}{1-3\beta^2} c(e')$.*

1.3 Notation

Given an instance $G = (V, E), c, r$ of SND we use the following notation. Let $V = \{v_1, \ldots, v_n\}$. For undirected graphs, the *requirement r_i of v_i* is the maximum requirement of a pair containing v_i. For directed graphs $r_i^{out} = \max_{v_j \in V} r(v_i, v_j)$ is the *out-requirement of v_i*, and $r_i^{in} = \max_{v_j \in V} r(v_j, v_i)$ is the *in-requirement of v_i*. For an edge set F and a node v, let $\deg_F(v)$ denote the degree of v in F. For directed graphs, let $\deg_F^{in}(v)$ and $\deg_F^{out}(v)$ denote the indegree and the outdegree of v in F.

2 Proof of Theorem 1

Definition 2.1 *An edge set F on node set V is a k-*cover *if for all $v \in V$:*

(i) $\deg_F(v) \geq k$ *if F is undirected.*

(ii) $\deg_F^{in}(v) \geq k$ *and* $\deg_F^{out}(v) \geq k$ *if F is directed.*

Lemma 4 *For both directed and undirected graphs, any k-cover J contains a $(k-1)$-cover F of cost $c(F) \leq \left(1 - \frac{1}{2k-1}\right) c(J)$.*

Proof: The following procedure finds $M \subseteq J$ such that $F = J - M$ is a $(k-1)$-cover and $c(M) \geq c(J)/(2k-1)$. Start with $M = \emptyset$, $F = J$, and all edges in F unmarked, and iteratively do the following, until all edges that remain in F are marked. Among all unmarked edges in F, let $e = uv$ be one of the maximum cost. Remove e from F and add it to M. In the case of undirected graphs, if the degree in F of an endnode of e is exactly $k-1$, mark all edges incident to this endnode. In the case of directed graphs, if $\deg_F^{out}(u) = k-1$ mark all edges leaving u, and if $\deg_F^{in}(v) = k-1$ mark all edges entering v. It is easy to see that at the end $F = J - M$ is a $(k-1)$-cover. At every iteration, at most $2k-1$ edges in F are removed or marked, and each of them is cheaper than the edge e added to M. Hence $c(M) \geq c(J)/(2k-1)$. $\square$

Let $F \subseteq E$ be a minimum-cost $(k-1)$-cover. Such F of minimum-costs can be computed in polynomial time, for both directed and undirected graphs, c.f. [14]. As any feasible solution to k-CS is a k-cover, $c(F) \leq \left(1 - \frac{1}{2k-1}\right) \mathsf{opt}$, by Lemma 4. Now let $I \subseteq E - F$ be an inclusion-minimal augmenting edge set so that $H = (V, F + I)$ is k-connected. It is known that I is a forest in the case of undirected graphs, and $|I| \leq 2n - 1$ in the case of directed graphs, c.f. [4] and [10].

In the case of undirected graphs, since I is a forest, there exists an orientation D of I (namely, D is a directed graph obtained by directing every edge of I) so that the outdegree of every node w.r.t. D is at most 1. Let D_i be the set of edges in D leaving v_i, so either $D_i = \emptyset$ or $|D_i| = 1$ for all i. Let J be an optimal solution, and let J_i be the set of edges in J incident to v_i. As $J_i \geq k$, we have $c(D_i) \leq c(J_i)\frac{\beta}{k(1-\beta)}$, by Lemma 3. Hence

$$c(I) = \sum_{i=1}^{n} c(D_i) \leq \frac{\beta}{k(1-\beta)} \sum_{i=1}^{n} c(J_i) \leq \frac{2\beta}{k(1-\beta)} c(J) = \frac{2\beta}{k(1-\beta)} \cdot \mathsf{opt}\ .$$

Consequently,

$$c(H) = c(F) + c(I) \leq \left(1 - \frac{1}{2k-1}\right) \cdot \mathsf{opt} + \frac{2\beta}{k(1-\beta)} \cdot \mathsf{opt} = \left(1 + \frac{2\beta}{k(1-\beta)} - \frac{1}{2k-1}\right) \cdot \mathsf{opt}\ .$$

In the case of directed graphs, $|I| \leq 2n-1$. As any feasible solution has at least kn edges, we have

$$c(I) \leq \frac{2n-1}{kn} \cdot \frac{2\beta^3}{1-3\beta^2} \cdot \mathsf{opt} \leq \frac{4\beta^3}{k(1-3\beta^2)} \cdot \mathsf{opt}\ .$$

Consequently,

$$c(H) = c(F) + c(I) \leq \left(1 - \frac{1}{2k-1}\right) \cdot \mathsf{opt} + \frac{4\beta^3}{k(1-3\beta^2)} \cdot \mathsf{opt} = \left(1 + \frac{4\beta^3}{k(1-3\beta^2)} - \frac{1}{2k-1}\right) \cdot \mathsf{opt}\ .$$

3 Proof of Theorem 2

Our strategy to prove Theorem 2 is to give an explicit construction of a graph H so that the following holds. In the case of directed graphs, the number of edges in H is at least α times a lower bound on the number of edges in any feasible solution. Using part (ii) from Lemma 3, this immediately implies the ratio $\alpha \cdot \frac{2\beta^3}{1-3\beta^2}$. In the case of undirected graphs, we will show that we can orient the edges of H so that the number of the edges leaving every node v is at most α times the number of edges incident to v in any feasible solution. Using part (i) from Lemma 3, this immediately implies the ratio $\alpha \cdot \frac{\beta}{1-\beta}$. For both directed and undirected graphs, we will have $\alpha = 2$ for SND and $\alpha = 1$ for Subset k-CS. For directed Rooted SND we will also have $\alpha = 1$.

3.1 General SND

For general SND we use the following simple construction.

Lemma 5 *Let $V = \{v_1, \ldots, v_n\}$ be a node set, and for $i = 1, \ldots, n$ let $r_i^{out}, r_i^{in} \leq n-1$ be non-negative integers. Let A_i^{out} be the set of edges from v_i to the first $r^{out}(v_i)$ nodes in $V - \{v_i\}$, and A_i^{in} be the set of edges from the first $r^{out}(v_i)$ nodes in $V - \{v_i\}$ to v_i. Namely:*

$$A_i^{out} = \begin{cases} \{v_i v_j : 1 \leq j \leq r(v_i)\} & \textit{if } \ r^{out}(v_i) < i \\ \{v_i v_j : 1 \leq j \leq r(v_i)+1, j \neq i\} & \textit{otherwise} \end{cases}$$

$$A_i^{in} = \begin{cases} \{v_j v_i : 1 \leq j \leq r(v_i)\} & \textit{if } \ r^{in}(v_i) < i \\ \{v_j v_i : 1 \leq j \leq r(v_i)+1, j \neq i\} & \textit{otherwise} \end{cases}$$

Then for any $i \neq j$, the graph $H_{ij} = (V, A_i^{out} \cup A_j^{in})$ contains at least $\min\{r_i^{out}, r_j^{in}\}$ internally disjoint v_iv_j-paths.

Proof: Note that there is a set C of $\min\{r(v_i), r(v_j)\} - 1$ nodes so that in H_{ij} there is an edge from v_i to every node in C and from every node in C to v_j; furthermore, either $v_iv_j \in H_{ij}$ or v_iv_j there is one more node that can be added to C. The statement follows. $\square$

The algorithm is as follows. In the case of directed graphs, we compute the edge sets A_i^{out} and A_i^{in} as in Lemma 5, and output their union graph H. In the case of undirected graphs, we consider the directed problem on the bi-direction of G with the requirements $r^{in}(v_i) = 0$ for all i, $r^{out}(v_i, v_j) = \max\{r(v_i, v_j), r(v_j, v_i)\}$ for $i > j$ and $r^{out}(v_i, v_j) = 0$ otherwise. Hence we will have $A_i^{in} = \emptyset$ for all i. The graph H is the underlying graph of the union of the sets A_i^{out}. For both directed and undirected graphs we have $\kappa_H(v_i, v_j) \geq \min\{r(v_i), r(v_j)\} \geq r(v_i, v_j)$, hence H is a feasible solution.

To establish the approximation ratio, we will use Lemma 3. Fix some optimal solution J; let J_i^{out} and J_i^{in} be the sets of edges in J leaving and entering v_i, respectively. In the case of directed graphs, note that $|A_i^{out}| = r_i^{out}$ and $|A_i^{in}| = r_i^{in}$ while $|J_i^{out}| \geq r_i^{out}$ and $|J_i^{in}| \geq r_i^{in}$. Hence the number of edges in the constructed solution is $\sum_{i=1}^{n}(r_i^{out} + r_i^{in})$, while any feasible solution has at least half this number of edges. Combined with part (ii) of Lemma 3, this immediately implies the ratio $\frac{4\beta^3}{1-3\beta^2}$.

In the case of undirected graphs, let A_i be the set of undirected edges that corresponding to A_i^{out} in the bi-direction of G. Let J_i be the set of edges incident to v_i in an optimal solution J. Note that $|A_i| = r(v_i)$ and that $|J_i| \geq r_i$ for all i. Hence $c(A_i) \leq \frac{\beta}{1-\beta} c(J_i)$, by part (i) of Lemma 3. Thus

$$c(H) \leq \sum_{i=1}^{n} c(A_i) \leq \frac{\beta}{1-\beta} \sum_{i=1}^{n} c(J_i) \leq \frac{2\beta}{1-\beta} c(J) = \frac{2\beta}{1-\beta} \cdot \mathsf{opt} \ .$$

3.2 Subset k-CS

Recall that Subset k-CS is the case of SND when for some $T \subseteq V$ we have $r(u,v) = k$ for all $u, v \in T$. Let $t = |T|$. For the case $t > k$ we can apply our algorithm for k-CS while ignoring the nodes in $V - T$, thus obtaining ratios as in Theorem 1. We can also obtain the ratios as in Theorem 2. Such an algorithm is described in [2] for undirected graphs, and we extend it to directed graphs. We will use the following statement.

Lemma 6 *For any integers k, n so that, $n \geq k+1$ there exist a directed k-connected graph H on n nodes with exactly kn edges, and such H can be constructed in polynomial time.*

Proof: Let $V = \{v_0, \ldots, v_{n-1}\}$. Let A_i be the set of k edges from v_i to $v_{i+1}, v_{i+2}, \ldots, v_{i+k}$, where the indices are modulo k. Let $A = \bigcup_{i=0}^{n-1} A_i$ and let $H = (V, A)$. Then $|A| = kn$ by the construction; we will show that H is k-connected. A theorem of Whitney states that a directed/undirected graph $H = (V, A)$ is k-connected if, and only if, $uv \in A$ or $\kappa_H(u,v) \geq k$ for all $u, v \in V$. Since the construction is symmetric, it is sufficient to show a set of k internally disjoint paths from v_0 to any node u not adjacent to v_0. Consider the BFS layers L_i with root v_0. We have $L_0 = \{v_0\}$, and there are k nodes in every other layer except of maybe the last one. Namely, $L_1 = \{v_1, \ldots, v_k\}$, $L_2 = \{v_{k+1}, \ldots, v_{2k}\}$, and in general $L_j = \{v_{(j-1)k+1}, \ldots, v_{jk}\}$ (in the last layer the last index is $n-1$). Let $j \geq 1$ and let $u \in L_{j+1}$ be arbitrary, say $u = v_{jk+i}$ for some $1 \leq i \leq k$. Let P'_q be the path $v_0 \to v_q \to v_{k+q} \to \cdots \to \cdots v_{(j-1)k+q}$. Let P_q be the v_0u path obtained by adding to P'_q: the edges $v_{(j-1)k+q} \to v_{jk+q} \to u$ if $q \leq i$, and the edge $v_{(j-1)k+q} \to u$ otherwise; note that the edges we add exist in A, by the definition of A. Now it is easy to see that $P_1, \ldots, P_k$ is a set of k intertnally disjoint v_1u-paths, as required. $\square$

Let $\ell \leq t-1$ be an integer. Let H_ℓ be an ℓ-connected graph on T with the following property. In the case of directed graphs, we require that H_ℓ has ℓn edges. In the case of undirected graphs, we require that H_ℓ has an orientation so that the outdegree of every node is exactly k. By Lemma 6 such graphs exist, and can be constructed in polynomial time; in the undirected case the underlying graph of the graph H as in Lemma 6 has the desired property. If $t \geq k+1$ then our algorithm for k-CS returns any graph H_k as above. The approximation ratio is shown as follows. In the case of undirected graphs, let A_i be the set of edges corresponding to the edges leaving v_i in the above orientation of H_k. For any feasible solution, the degree of every node in T is at least k. The ratio of $\frac{\beta}{1-\beta}$ now immediately follows from part (i) of Lemma 3. In the case of directed graphs, any feasible solution has at least kt edges. The ratio of $\frac{2\beta^3}{1-3\beta^2}$ now immediately follows from part (ii) of Lemma 3.

Our construction for the case $t \leq k$ is a slight extension of this construction. Note that $|V| \geq k+1$, as otherwise the problem has no feasible solution. We choose a set $U \subseteq V - T$ of arbitrary $k-t+1$ nodes, and obtain a graph H by adding all possible edges between H_{t-1} and U. It is easy to see that H is a feasible solution. For the analysis of the approximation ratio, we use the following simple observation.

Lemma 7 *Let J be a feasible solution to a* Subset k-CS *instance. Then:*

(i) *For undirected graphs, every node in T has in J at least $k-t+1$ neighbors in $V-T$.*

(ii) *For directed graphs, J has at least $t(t-1)+2t(k-t+1)$ edges.*

Proof: In undirected J, every node in T has at least k neighbors. At most $t-1$ of these neighbors can lie in T, hence all the other at least $k-t+1$ neighbors are in $V-T$. In directed J, every node has outdegree and indegree at least k. At most $t-1$ edges can enter a node from nodes in T, or leave a node to a node in T. Hence for every $v \in T$, at least $k-t+1$ edges go from v to $V-T$, and at least $k-t+1$ edges go from $V-T$ to v. Thus the number of edges in J is at least $t(t-1)+2t(k-t+1)$, as claimed. $\square$

For undirected graphs, we orient the edges of our solution H as follows. We can orient the edges of H_{t-1} so that the outdegree of every node is k, and we orient the edges between H_{t-1} and U from T to U. In this orientation, the outdegree of every node is exactly $k-t+1$. For directed graphs our solution H has exactly $t(t-1)+2t(k-t+1)$ edges, Thus the ratios $\frac{\beta}{1-\beta}$ for undirected graphs and $\frac{2\beta^3}{1-\beta^2}$ for directed graphs follow from Lemmas 7 and 3.

3.3 Directed Rooted SND

Let $G = (V,E), c, r, s$ be an instance of directed Rooted SND where s is the root. Let $T = \{u \in V : r(s,u) > 0\}$ be the set of nodes with positive in-requirements.

Lemma 8 *Any feasible solution J for directed* Rooted SND *has at least $\sum_{v \in T} r^{in}(v)$ edges if $k \leq |T|$, and at least $\sum_{v \in T} r^{in}(v) + k - |T|$ edges if $k > |T|$.*

Proof: Clearly, $\deg_J^{in}(v) \geq r^{in}(v)$ and $\deg_J^{out}(r) \geq k$. Now consider the edges in J leaving r. At most $|T|$ of these edges can go to nodes in T, hence if $k > |T|$ then there are at least $k - |T|$ edges that go to nodes in $V-T$. The statement follows. $\square$

Now we show that the lower bound in Lemma 8 is achievable. Construct a graph H as follows. Let $U \subseteq V \setminus \{s\}$ be an arbitrary set of $\max\{k, |T|\}$ nodes containing T, so $U = T$ if

$|T| \geq k$. Take an edge from s to every node in U, and for every $v \in T$ take arbitrary $r^{in}(v)-1$ edges entering v from any $r^{in}(v)-1$ nodes in $U \setminus \{v\}$. It is easy to see that H is a feasible solution for the directed Rooted SND instance, and the number of edges in H coincides with the lower bound in Lemma 8. Applying Lemma 3(ii) we obtain $c(H) \leq \frac{2\beta^3}{1-3\beta^2} \cdot \mathsf{opt}$.

The proof of Theorem 2 is complete.

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