Small ℓ-edge-covers in k-connected graphs

Zeev Nutov

The Open University of Israel
nutov@openu.ac.il

Abstract. Let $G = (V, E)$ be a k-edge-connected graph with edge-costs $\{c(e) : e \in E\}$ and minimum degree d. We show by a simple and short proof, that for any integer ℓ with $\frac{d}{k} \leq \ell \leq d\left(1 - \frac{1}{k}\right)$, G contains an ℓ-edge cover I such that: $c(I) \leq \frac{\ell}{d} c(E)$ if G is bipartite, or if $\ell|V|$ is even, or if $|E| \geq \frac{d|V|}{2} + \frac{d}{2\ell}$; otherwise, $c(I) \leq \left(\frac{\ell}{d} + \frac{1}{d|V|}\right) c(E)$. The particular case $d = k = \ell + 1$ and unit costs already includes a result of Cheriyan and Thurimella [1], that G contains a $(k-1)$-edge-cover of size $|E| - \lfloor |V|/2 \rfloor$. Using our result, we slightly improve the approximation ratios for the k-Connected Subgraph problem (the node-connectivity version) with uniform and β-metric costs. We then consider the dual problem of finding a spanning subgraph of maximum connectivity k^* with a prescribed number of edges. We give an algorithm that computes a $(k^* - 1)$-connected subgraph, which is tight, since the problem is NP-hard.

1 Introduction

Let $G = (V, E)$ be an undirected graph, possibly with parallel edges. Let $n = |V|$. For $S \subseteq V$ let $\delta(S)$ denote the set of edges in E with exactly one endnode in S. An edge set $I \subseteq E$ is a *d-edge-cover* (of V) if the graph (V, I) has minimum degree $\geq d$. For $x \in \mathbb{R}^E$ and $F \subseteq E$ let $x(F) = \sum_{e \in F} x(e)$. Let $P^f_{cov}(G, d)$ denote the *fractional d-edge-cover polytope* determined by the linear constraints

$$x(\delta(v)) \geq d \qquad v \in V \tag{1}$$

$$1 \geq x_e \geq 0 \qquad e \in E \tag{2}$$

Clearly, for any $1 \leq \ell \leq d - 1$, if $x \in P^f_{cov}(G, d)$ then $\frac{\ell}{d} \cdot x \in P^f_{cov}(G, \ell)$. Let $P_{cov}(G, \ell)$ denote the *integral ℓ-edge-cover polytope*, which is the convex hull of the characteristic vectors of of the ℓ-edge-covers in G. It is known that if G is bipartite then $P^f_{cov}(G, \ell) = P_{cov}(G, \ell)$ (see [5], (31.7) on page 340). This implies the following.

Proposition 1. *Let $G = (V, E)$ be a bipartite graph, let $1 \leq \ell \leq d - 1$, and let $x \in P_{cov}(G, d)$. Then $\frac{\ell}{d} \cdot x \in P_{cov}(G, \ell)$.*

Corollary 1. *Let $G = (V, E)$ be a bipartite graph with edge costs $\{c(e) : e \in E\}$ and minimum degree $\geq d \geq 2$. Then for any $1 \leq \ell \leq d - 1$, G contains an ℓ-edge cover $I \subseteq E$ of cost $c(I) \leq \frac{\ell}{d} c(E)$.*

Cheriyan an Thurimella [1] showed that if G is bipartite and has minimum degree $\geq d$, then G contains a $(d-1)$-edge-cover I such that $|I| \leq |E| - n/2$. Note that this bound follows from Corollary 1 by assuming unit costs, substituting $\ell = d-1$, and observing that $|E| \geq \frac{dn}{2}$. Unfortunately, Corollary 1 does not extend to the general (non-bipartite) case, e.g., if G is a cycle of length 3, $d = 2$, and $\ell = 1$. On the positive side, it is proved in [2] that if G has minimum degree $\geq d$, then G contains a $(d-1)$-edge-cover I of cost $c(I) \leq \frac{2d-2}{2d-1} c(E)$. Let $\zeta(S)$ denote the set of edges in E with at least one endnode in S. It is known that in the general case, $P_{cov}(G,d)$ is determined by adding to the constraints of $P^f_{cov}(G,d)$ the following inequalities (see [5], page 581, Theorem 34.13):

$$x(\zeta(S)) - x(F) \geq \frac{d|S|}{2} - \frac{|F|-1}{2} \qquad S \subseteq V, F \subseteq \delta(S), d|S| - |F| \geq 1 \text{ odd} \quad (3)$$

A graph G is *k-edge-connected* if $|\delta(S)| \geq k$ for all $\emptyset \neq S \subset V$. Cheriyan and Thurimella [1] showed that if G is k-edge-connected, then G contains a $(k-1)$-edge-cover I such that $|I| \leq |E| - \lfloor n/2 \rfloor$. We present an analogue of Proposition 1 and Corollary 1 for general graphs, with simple and short proof, that also implies this bound of [1]. Let $P^f_{con}(G,k)$ denote the *fractional k-edge-connectivity polytope*, determined by

$$\begin{aligned} x(\delta(S)) &\geq k \qquad && \emptyset \neq S \subset V \\ 1 \geq x_e &\geq 0 && e \in E \end{aligned}$$

Note that $P^f_{cov}(G,k) \subseteq P^f_{con}(G,k)$, and that if $x \in P^f_{cov}(G,d)$ then $x(E) \geq \frac{dn}{2}$. The main result of this paper is the following analogue of Proposition 1.

Theorem 1. *Let $G = (V,E)$ be a graph, let $1 \leq \ell \leq d-1$ and $d \geq k$, and let $x \in P^f_{cov}(G,d) \cap P^f_{con}(G,k)$. Then $\mu \cdot x \in P_{cov}(G,\ell)$, where μ is defined as follows.*

(i) *Suppose that $k \geq \max\left\{\frac{d}{\ell}, \frac{d}{d-\ell}\right\}$ (namely, that $\frac{d}{k} \leq \ell \leq d\left(1 - \frac{1}{k}\right)$, which includes the case $k = d$). Then $\mu = \ell/d$ if $\ell|V|$ is even or if $x(E) \geq \frac{d}{2}\left(|V| + \frac{1}{\ell}\right)$; otherwise, $\mu = \frac{\ell|V|+1}{2x(E)} \leq \frac{\ell}{d} + \frac{1}{d|V|}$.*

(ii) *Suppose that $k < \max\left\{\frac{d}{\ell}, \frac{d}{d-\ell}\right\}$.*
   (a) *Suppose that $k < d/\ell$ and that $d \geq 2\ell + 1$. Then $\mu = \frac{2\ell+1}{2d+k}$ if $\ell|V|$ is even or if $x(E) \geq \frac{d}{2}\left(|V| + \frac{1}{\ell}\right)$; otherwise, $\mu = \max\left\{\frac{\ell|V|+1}{2x(E)}, \frac{2\ell+1}{2d+k}\right\}$.*
   (b) *Suppose that $k < \frac{d}{d-\ell}$ and that $d \leq 2\ell$. Then $\mu = \frac{2\ell+1-k}{2d-k}$ if $\ell|V|$ is even or if $x(E) \geq \frac{d}{2}\left(|V| + \frac{1}{\ell}\right)$; otherwise, $\mu = \max\left\{\frac{\ell|V|+1}{2x(E)}, \frac{2\ell+1-k}{2d-k}\right\}$.*

Clearly, the cases of the theorem are exclusive, and it is not hard to verify that they cover all relevant values of d, ℓ, k. To see this, note that if $k < d/\ell$ and $d \leq 2\ell$, then $k \leq \frac{d}{d-\ell}$; hence this case is included in part (iib) of the theorem. Similarly, if $k < \frac{d}{d-\ell}$ and $d \geq 2\ell + 1$, then $k \leq \frac{d}{\ell}$ hence this case is included in part (iia) of the theorem.

Note that for the case $k = 0$ and $\ell = d - 1$ considered in [2], part (iib) of Theorem 1 gives $\mu = \frac{2d-1}{2d}$, which is slightly worse than the bound $\frac{2d-2}{2d-1}$ of [2]. However, the bound of [2] uses a stronger assumption that $x \in P_{cov}(G, d)$, while we assume only that $x \in P^f_{cov}(G, d)$.

Theorem 1 immediately implies the following.

Corollary 2. *Let $G = (V, E)$ be a k-edge-connected graph with edge costs $\{c(e) : e \in E\}$ and let $1 \leq \ell \leq d-1$ and $d \geq k$. Then G contains an ℓ-edge cover $I \subseteq E$ such that $c(I) \leq \mu \cdot c(E)$, where μ is as in Theorem 1.*

Note that the bound $|I| \leq |E| - \lfloor n/2 \rfloor$ of Cheriyan and Thurimella [1] follows from Corollary 2 by assuming unit costs, substituting $d = k = \ell + 1$, and observing that $|E| \geq \frac{kn}{2}$. Indeed, by Corollary 2, $|E| - |I| \geq |E|/k \geq n/2$ if $(k-1)n$ is even or if $|E| \geq \frac{kn}{2} + 1$. Otherwise, k is even, n is odd, $|E| = \frac{kn}{2}$, and then, by Corollary 2, $|E| - |I| \geq \frac{n-1}{kn}|E| = \frac{n-1}{2} = \lfloor n/2 \rfloor$.

We now discuss some applications of Corollaries 1 and 2 for both directed and undirected graphs, for the following classic NP-hard problem. A (simple) directed or undirected graph is *k-connected* if it contains k internally disjoint paths from every node to the other.

> k-Connected Subgraph
> *Instance:* A graph $G' = (V, E')$ with edge costs and an integer k.
> *Objective:* Find a minimum cost k-connected spanning subgraph G of G'.

The case of unit costs is the Minimum Size k-Connected Subgraph problem. Cheriyan and Thurimella [1] suggested and analyzed the following algorithm for the Minimum Size k-Connected Subgraph problem, for both directed and undirected graphs; in the case of a directed graph $G = (V, E)$, we say that $I \subseteq E$ is an ℓ-edge-cover if (V, I) has minimum outdegree and minimum indegree $\geq \ell$.

> **Algorithm 1**
>
> 1. Find a minimum size $(k-1)$-edge cover $I \subseteq E'$.
> 2. Find an inclusion minimal edge set $F \subseteq E' \setminus I$ such that $(V, I \cup F)$ is k-connected.
> 3. Return $I \cup F$.

They showed that this algorithm has approximation ratios

- $1 + \frac{n}{\mathsf{opt}} \leq 1 + \frac{1}{k}$ for directed graphs;
- $1 + \frac{n}{2\mathsf{opt}} \leq 1 + \frac{1}{k}$ for undirected graphs.

Here opt denotes the optimum solution value of a problem instance at hand. Step 1 in the algorithm can be implemented in polynomial time, c.f. [5]. Recently, the performance of this algorithm was also analyzed in [2] for so called *β-metric costs*, when the input graph is complete and for some $1/2 \leq \beta < 1$ the costs satisfy the *β-triangle inequality* $c(uv) \leq \beta[c(ua) + c(av)]$ for all $u, a, v \in V$. When $\beta = 1/2$, the costs are uniform, and we have the min-size version of the problem. If we allow the case $\beta = 1$, then the costs satisfy the ordinary triangle inequality and we have the metric version of the problem. In [2] it is shown that for undirected graphs with β-metric costs the above algorithm has ratio $1 - \frac{1}{2k-1} + \frac{2\beta}{k(1-\beta)}$. We prove the following.

Theorem 2. (i) *For the* Minimum Size k-Connected Subgraph *problem, Algorithm 1 has approximation ratios*

- $1 - \frac{1}{k} + 2n/\mathsf{opt} \le 1 + \frac{n}{\mathsf{opt}}$ *for directed graphs;*
- $1 - \frac{1}{k} + n/\mathsf{opt} \le 1 + \frac{n}{2\mathsf{opt}}$ *for undirected graphs.*

(ii) *In the case of undirected graphs and β-metric costs, Algorithm 1 has approximation ratio* $1 - \frac{1}{k} + \frac{1}{kn} + \frac{2\beta}{k(1-\beta)}$.

(iii) *There exists a polynomial time algorithm that given an instance of the* Minimum Size k-Connected Subgraph *problem returns a $(k-1)$-connected spanning subgraph G of G' with at most* opt *edges.*

Note that in part (i) of Theorem 2 we do *not* improve the worse performance guarantee $1+\frac{1}{k}$ of [1]. However, the ratio $1+\frac{1}{k}$ applies only if $\mathsf{opt} = kn$ in the case of directed graphs and $\mathsf{opt} = kn/2$ in the case of undirected graphs. Otherwise, if opt is larger than these minimum possible values, then both our analysis and that of [1] give better ratios. But the ratios provided by our analysis are smaller, since $2n/\mathsf{opt} - \frac{1}{k} \le n/\mathsf{opt}$ in the case of directed graphs, and $n/\mathsf{opt} - \frac{1}{k} \le n/2\mathsf{opt}$ in the case of undirected graphs. For example, in the case of directed graphs, if $\mathsf{opt} = \frac{3}{2}kn$ then our ratio is $1 + \frac{1}{3k}$, while that of [1] is $1 + \frac{2}{3k}$.

Part (iii) of Theorem 2 can be used to obtain a tight approximation algorithm to the Maximum Connectivity m-Edge Subgraph problem: given a graph G' and an integer m, find a spanning subgraph G of G' with at most m edges and maximum connectivity k^*. We can apply the algorithm in part (iii) to find the maximum integer k for which the algorithm returns a subgraph with at most m edges. Then $k \ge k^* - 1$, hence we obtain a polynomial time algorithm that computes a $(k^* - 1)$-connected spanning subgraph with at most m edges. Note that this is tight, since the problem is NP-hard.

2 Proof of Theorem 1

Let $x \in P^f_{cov}(G, d) \cap P^f_{con}(G, k)$. We need to show that then $\mu \cdot x \in P_{cov}(G, \ell)$, namely, that

$$\mu x(\delta(v)) \ge \ell \qquad v \in V \tag{4}$$

$$\mu(x(\zeta(S)) - x(F)) \ge \frac{\ell|S|}{2} - \frac{|F| - 1}{2} \qquad S \subseteq V, F \subseteq \delta(S), \ell|S| - |F| \ge 1 \text{ odd} \tag{5}$$

$$1 \ge \mu x_e \ge 0 \qquad e \in E \tag{6}$$

Recall that for $1 \le \ell \le d-1$ and $d \ge k$, the parameter μ is defined as follows.

(i) Suppose that $k \ge \max\left\{\frac{d}{\ell}, \frac{d}{d-\ell}\right\}$. Then $\mu = \ell/d$ if ℓn is even or if $x(E) \ge \frac{d}{2}\left(n + \frac{1}{\ell}\right)$; otherwise, $\mu = \frac{\ell n+1}{2x(E)} \le \frac{\ell}{d} + \frac{1}{dn}$.

(ii) Suppose that $k < \max\left\{\frac{d}{\ell}, \frac{d}{d-\ell}\right\}$.

(a) Suppose that $k < d/\ell$ and that $d \geq 2\ell + 1$. Then $\mu = \frac{2\ell+1}{2d+k}$ if ℓn is even or if $x(E) \geq \frac{d}{2}\left(n + \frac{1}{\ell}\right)$; otherwise, $\mu = \max\left\{\frac{\ell n+1}{2x(E)}, \frac{2\ell+1}{2d+k}\right\}$.

(b) Suppose that $k < \frac{d}{d-\ell}$ and that $d \leq 2\ell$. Then $\mu = \frac{2\ell+1-k}{2d-k}$ if ℓn is even or if $x(E) \geq \frac{d}{2}\left(n + \frac{1}{\ell}\right)$; otherwise, $\mu = \max\left\{\frac{\ell n+1}{2x(E)}, \frac{2\ell+1-k}{2d-k}\right\}$.

It is not hard to verify that $\frac{\ell}{d} \leq \mu \leq 1$ for all ℓ, k, d. The following statement is also easily verified.

Lemma 1. *Let $x \in P^f_{cov}(G, d)$. Then for any $\frac{\ell}{d} \leq \mu \leq 1$, (4) and (6) hold.*

We therefore focus on the inequalities in (5). Let $\emptyset \neq S \subseteq V$ and let $F \subseteq \delta(S)$ such that $\ell|S| - |F| \geq 1$ is odd. In the following three lemmas 2, 3, and 4, we prove that (5) holds for certain values of μ, and then deduce Theorem 1 from these lemmas.

Lemma 2. *Let $x \in P^f_{cov}(G, d)$. If $S = V$, then (5) holds for $\mu = \ell/d$ if ℓn is even or if $x(E) \geq \frac{d}{2}\left(n + \frac{1}{\ell}\right)$; otherwise, (5) holds for $\mu = \frac{\ell n+1}{2x(E)} \leq \frac{\ell}{d} + \frac{1}{dn}$.*

Proof. If $S = V$ then $\zeta(S) = E$ and $F = \emptyset$. Then (5) reduces to a void condition if $\ell|V|$ is even, and to the condition $\mu x(E) \geq \frac{\ell n+1}{2}$ otherwise, which holds by the definition of μ. The inequality $\frac{\ell n+1}{2x(E)} \leq \frac{\ell}{d} + \frac{1}{dn}$ is since $x(E) \geq \frac{dn}{2}$. □

Henceforth assume that S is a proper subset of V. Note that then

$$d|S| \leq \sum_{v \in S} x(\delta(v)) = 2x(E(S)) + x(\delta(S)) = 2x(\zeta(S)) - x(\delta(S)) \ .$$

Thus $x(\zeta(S)) \geq \frac{d|S|}{2} + \frac{x(\delta(S))}{2}$. Also note that $x(F) \leq |F|$. Substituting in (5) and rearranging terms, we obtain that it is sufficient to prove the following

$$\mu\left(\frac{d|S|}{2} + \frac{x(\delta(S))}{2} - x(F)\right) \geq \frac{\ell|S|}{2} - \frac{|F| - 1}{2} \ .$$

Finally, multiplying both sides by 2 and rearranging terms we obtain

$$|S|(\mu d - \ell) + (|F| - \mu x(F)) + \mu(x(\delta(S)) - x(F)) \geq 1 \ . \tag{7}$$

Lemma 3. *For $\mu = \ell/d$, (7) holds if $k \geq \frac{d}{\ell}$ and $d \geq 2\ell$, or if $k \geq \frac{d}{d-\ell}$ and $d \leq 2\ell$. Consequently, (7) holds for $\mu = \ell/d$ if $k \geq \max\left\{\frac{d}{\ell}, \frac{d}{d-\ell}\right\}$.*

Proof. Substituting in (7) $\mu = \ell/d$, multiplying both sides by d, and observing that $x(F) \leq |F|$, we obtain that it is sufficient to prove that

$$|F|(d - \ell) + \ell(x(\delta(S)) - x(F)) \geq d \ . \tag{8}$$

If $|F| \geq \frac{d}{d-\ell}$ then (8) holds since $x(\delta(S)) - x(F) \geq 0$. Henceforth assume that $|F| < \frac{d}{d-\ell}$, and let us consider the cases of the lemma. If $d \geq 2\ell$ and $k \geq \frac{d}{\ell}$ then

$$|F|(d-\ell) + \ell(x(\delta(S)) - x(F)) \geq |F|(d-2\ell) + k\ell \geq d$$

In the case $d \leq 2\ell$ and $k \geq \frac{d}{d-\ell}$, since we assume that $|F| < \frac{d}{d-\ell}$, we have

$$|F|(d-2\ell)+k\ell \geq \frac{d}{d-\ell}(d-2\ell)+k\ell = d-\frac{\ell d}{d-\ell}+k\ell = d+\ell\left(k-\frac{d}{d-\ell}\right) \geq d\ .$$

The proof of the lemma is complete. □

Part (i) of Theorem 1 follows from Lemmas 1, 2, and 3, after observing that $x(E) < \frac{d}{2}\left(n+\frac{1}{\ell}\right)$ implies $\frac{\ell n+1}{2x(E)} > \frac{\ell}{d}$.

Now we will use Lemmas 1, 2, and the Lemma 4 to follow, to prove part (ii) of Theorem 1. Before that we observe that in the polyhedral description of $P_{cov}(G,d)$, we may skip the inequalities in (3) with $|S| = 1$, since they are implied by the inequalities in (1); the same applies for inequalities in (5). Say, $S = \{v\}$. Then (3) reduces to $x(\delta(v)) \geq x(F) + \frac{d}{2} - \frac{|F|}{2} + \frac{1}{2}$ for $F \subseteq \delta(v)$, $d - |F| \geq 1$ odd. In particular, $|F| \leq d-1$. However, $x(F) \leq |F|$, hence by (1) we have

$$x(F) + \frac{d}{2} - \frac{|F|}{2} + \frac{1}{2} \leq \frac{|F|+d+1}{2} \leq d \leq x(\delta(v))\ .$$

Lemma 4. *If $|S| \geq 2$, then (7) holds in each one of the following cases:*

(a) $\mu = \frac{2\ell+1}{2d+k}$ *and* $k \geq 2(2\ell+1-d)$.
(b) $\mu = \frac{2\ell+1-k}{2d-k}$ *and* $k \leq 2(2\ell+1-d)$.

Proof. Since $|S| \geq 2$ and $x(F) \leq |F|$, then to prove that (7) holds, it is sufficient to prove that

$$2(\mu d - \ell) + |F|(1-\mu) + \mu(x(\delta(S)) - x(F)) \geq 1\ . \tag{9}$$

If $|F| \geq k$ then the l.h.s. of (9) is at least $2(\mu d-\ell)+k(1-\mu) = \mu(2d-k)+k-2\ell$. Hence if $|F| \geq k$, then (9) holds if $\mu \geq \frac{2\ell+1-k}{2d-k}$.

Suppose that $|F| \leq k$. Then the l.h.s. of (9) is at least

$$2(\mu d - \ell) + |F|(1-\mu) + \mu(k - |F|) = \mu(2d + k - 2|F|) + |F| - 2\ell\ .$$

Hence (9) holds if $\mu \geq \frac{2\ell+1-|F|}{2d+k-2|F|}$. Observe that for any a, b, f with $f \geq 0$ and $b - 2f > 0$, we have $\frac{a-f}{b-2f} \geq \frac{a}{b}$ if $2a \geq b$, and $\frac{a-f}{b-2f} \leq \frac{a}{b}$ if $2a \leq b$. Consequently, we obtain that if $|F| \leq k$, then (9) holds if one of the following holds:

(a) $\mu \geq \frac{2\ell+1}{2d+k}$ and $2(2\ell+1) \leq 2d+k$.
(b) $\mu \geq \frac{2\ell+1-k}{2d-k}$ and $2(2\ell+1) \geq 2d+k$.

The lemma now follows by observing that $\frac{2\ell+1}{2d+k} \geq \frac{2\ell+1-k}{2d-k}$ if $k \geq 2(2\ell+1-d)$. □

Now we prove part (ii) of Theorem 1. In what follows, note that for $d = 2\ell$ and $k = 2$, the values of μ in parts (a) and (b) of Lemma 4 coincide, and that $x(E) \geq \frac{d}{2}\left(n + \frac{1}{\ell}\right)$ implies $\frac{\ell}{d} \geq \frac{\ell n+1}{2x(E)}$.

Suppose that $k < d/\ell$ and that $d \geq 2\ell+1$. Then the condition $k \geq 2(2\ell+1-d)$ in part (a) of Lemma 4 reduces to the void condition $k \geq 0$. The result in this case follows by combining part (a) of Lemma 4 with Lemmas 1 and 2, after observing that if $k < \frac{d}{\ell}$ then $\frac{2\ell+1}{2d+k} > \frac{2\ell+1}{2d+d/\ell} = \frac{\ell}{d}$.

Now suppose that $k < \frac{d}{d-\ell} = \frac{\ell}{d-\ell} + 1$ and that $d \leq 2\ell$. If the condition $k \leq 2(2\ell + 1 - d)$ in part (b) of Lemma 4 holds, then the result follows by combining part (a) of Lemma 4 with Lemmas 1 and 2, after observing that if $k < \frac{d}{d-\ell}$ then $\frac{2\ell+1-k}{2d-k} > \frac{\ell}{d}$. Else, $k > 2(2\ell + 1 - d)$. Denoting $p = d - \ell \geq 1$, we obtain the following inequalities:

$$k < \frac{\ell}{p} + 1 \qquad p \leq \ell \qquad k > 2(\ell + 1 - p)$$

This implies $\frac{\ell}{p} + 1 > 2(\ell + 1 - p)$, which gives $\ell < \frac{2p^2}{2p-1}$. Thus we obtain that $k < \frac{\ell}{p} + 1 < \frac{2p}{2p-1} + 1 = 2 + \frac{1}{2p-1} \leq 3$. Since $k \geq 2(\ell + 1 - p)$, we obtain $\ell - p \leq \frac{k}{2} - 1$. Since $\ell - p \geq 0$, we must have $k = 2$ and $p = \ell$, namely, $k = 2$ and $d = 2\ell$. Then $\mu = \frac{2\ell+1}{2d+k} = \frac{2\ell+1}{4\ell+2} = \frac{1}{2}$. This case is included in case (a) of Lemma 4, and then the result follows by combining part (a) of Lemma 4 with Lemmas 1 and 2. This finishes the proof of part (ii) of Theorem 1.

The proof of Theorem 1 is complete.

3 Proof of Theorem 2

Let I and F denote the set of edges computed by Algorithm 1 at steps 1 and 2, respectively. We prove part (i), starting with the case of directed graphs. For a directed graph G, the corresponding bipartite graph $G' = (V \cup V', E')$ is obtained by adding a copy V' of V and replacing every directed edge $uv \in E$ by the undirected edge uv', where $v' \in V'$ is the copy of v. It is not hard to verify that I is an ℓ-edge-cover in G if, and only if, the set I' of edges that corresponds to I is an ℓ-edge-cover in G'. Thus $|I| \leq \frac{k-1}{k}\mathsf{opt}$, by Corollary 1. On the other hand, by the directed Critical Cycle Theorem of Mader [4] (see [1] for details), the set of edges of G' that corresponds to F' forms a forest in G', hence $|F| \leq 2n - 1$. Consequently, $\frac{|I|+|F|}{\mathsf{opt}} \leq 1 - \frac{1}{k} + \frac{2n-1}{\mathsf{opt}}$.

Let us consider undirected graphs. If $(k-1)n$ is even or if $\mathsf{opt} \geq \frac{kn}{2} + \frac{k}{2(k-1)} \geq \frac{kn}{2} + 1$, then $|I| \leq \frac{k-1}{k}\mathsf{opt}$, by Corollary 2. By the undirected Critical Cycle Theorem of Mader [3] (see [1] for details), F is a forest, hence $|F| \leq n - 1$. Consequently, $\frac{|I|+|F|}{\mathsf{opt}} \leq 1 - \frac{1}{k} + \frac{n-1}{\mathsf{opt}}$. If $(k - 1)n$ is odd and $\mathsf{opt} < \frac{kn}{2} + 1$, then an optimal solution is k-regular and hence $|I| \leq \frac{(k-1)n+1}{2} \leq \left(1 - \frac{1}{k}\right)(\mathsf{opt} + 1)$. Combining we get $\frac{|I|+|F|}{\mathsf{opt}} \leq 1 - \frac{1}{k} + \frac{1-1/k}{\mathsf{opt}} + \frac{n-1}{\mathsf{opt}} < 1 - \frac{1}{k} + \frac{n}{\mathsf{opt}}$.

Now let us consider part (ii), the case of β-metric costs. In [2] it is proved that $c(F) \leq \frac{2\beta}{k(1-\beta)}\mathsf{opt}$. If $(k - 1)n$ is even, or if there exists an optimal solution with at least $\frac{kn}{2} + \frac{k}{2(k-1)} \leq \frac{kn}{2} + 1$ edges, then Corollary 2 gives the bound $c(I) \leq \left(1 - \frac{1}{k}\right) \mathsf{opt}$. Else, Corollary 2 gives the bound $c(I) \leq \left(1 - \frac{1}{k} + \frac{1}{kn}\right) \mathsf{opt}$, and the result follows.

We prove part (iii). We apply Algorithm 1 with k replaced by $k-1$, namely, $I \subseteq E$ is a minimum size $(k-2)$-edge cover and $F \subseteq E \setminus I$ is an inclusion minimal edge set such that $(V, I \cup F)$ is $(k-1)$-connected. Now we use the bounds in Corollary 2. In the case of directed graphs we have $|I| \leq \frac{k-2}{k}\mathsf{opt}$, $|F| \leq 2n - 1 \leq \frac{2}{k}\mathsf{opt}$, and the result follows. In the case of undirected graphs we have $|I| \leq \left(\frac{k-2}{k} + \frac{1}{kn}\right) \mathsf{opt}$ and $|F| \leq n - 1 \leq \left(\frac{2}{k} - \frac{2}{kn}\right) \mathsf{opt}$, and the result follows.

The proof of Theorem 2 is complete.

References

1. J. Cheriyan and R. Thurimella. Approximating minimum-size k-connected spanning subgraphs via matching. *SIAM J. Computing*, 30:528–560, 2000.
2. J. David and Z. Nutov. Approximating survivable networks with β-metric costs. *J. Discrete Algorithms*, 9(2):170–175, 2011.
3. W. Mader. Ecken vom grad n in minimalen n-fach zusammenhängenden graphen. *Archive der Mathematik*, 23:219–224, 1972.
4. W. Mader. Minimal n-fach in minimalen n-fach zusammenhängenden digraphen. *J. Comb. Theory B*, 38:102–117, 1985.
5. A. Schrijver. *Combinatorial Optimization Polyhedra and Efficiency.* Springer, 2004.