On extremal k-outconnected graphs

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Abstract

Let G be a minimally k-connected graph with n nodes and m edges. Mader proved that if $n \geq 3k-2$ then $m \leq k(n-k)$, and for $n \geq 3k-1$ an equality is possible if, and only if, G is the complete bipartite graph $K_{k,n-k}$. Cai proved that if $n \leq 3k-2$ then $m \leq \lfloor (n+k)^2/8 \rfloor$, and listed the cases when this bound is tight.

In this paper we prove a more general theorem, which implies similar results for minimally k-outconnected graphs; a graph is called k-outconnected from r if it contains k internally disjoint paths from r to every other node.

Key-words: minimally k-outconnected graphs, extremal graphs

1 Introduction

Let $G = (V, E)$ be a simple undirected graph with node set V and edge set E, where $|V| = n$ and $|E| = m$. G is k-(*node*) *connected* if there are at least k pairwise internally disjoint paths between any pair of nodes of G. If G is k-connected, then an edge $e \in E$ is called *critical* (*w.r.t. k-connectivity*) if $G - e$ is not k-connected, and G is *minimally k-connected* if all its edges are critical. A fundamental result in the theory of k-connected graphs is the following:

Theorem 1.1 (Mader [5]) *In a k-connected graph G, any cycle of critical edges contains a node which degree in G is k.*

In particular, if G is minimally k-connected, then any cycle of G contains a node of degree k. Thus deleting all the nodes of degree k from a minimally k-connected graph results in a (possibly empty) forest. Based on his theorem, Mader derived several essential properties of minimally k-connected graphs, among them the following one (for a proof in English see [2]):

Theorem 1.2 (Mader [5]) *A minimally k-connected graph on $n \geq 3k-2$ nodes has at most $k(n-k)$ edges, and if $n \geq 3k-1$, then the only case when this bound is tight is $K_{k,n-k}$, where $K_{k,n-k}$ is a complete bipartite graph with k nodes on one side and $n-k$ on the other.*

For $n \leq 3k-2$, Cai [3] proved that $m \leq \lfloor (n+k)^2/8 \rfloor = k(n-k) + \lfloor (3k-n)^2/8 \rfloor$ in a minimally k-connected graph, and listed the cases when this bound is tight.

A graph is called *k-outconnected from r* if it is simple and contains k pairwise internally disjoint paths from r to every other node. Note that if G is k-connected, and $r \in V$, then G is k-outconnected from r; for $k \geq 2$, the inverse is, in general, not true.

In this paper we generalize Theorem 1.2 of Mader and the result of Cai [3], see Theorem 2.1. In particular, we determine the maximum number of edges in a minimally k-outconnected graph, and derive all the cases when this bound is tight. Our proof (of a more general theorem) unifies and simplifies the proofs of Mader [5] and of Cai [3] of their theorems, and makes the description and analysis of the extreme cases easier.

Remark. We note that determining the maximal number of edges in a *directed* graph G which is minimally k-outconnected from r is an easy task. This is since it is easy to show that the indegree of any node $v \neq r$ in G is exactly k; thus G has exactly $k(n-1)$ edges.

Here is some notation used in the paper. All the graphs in the paper are assumed to be undirected and simple. An edge e with endnodes x and y is denoted by $e = xy$; if e has one endnode in A and the other in B, then e is an *$\{A,B\}$-edge*. An *$\{x,y\}$-path* is a simple path with ends x, y. Let $G = (V,E)$ be a graph. For any set of edges and nodes X we denote by $G - X$ (resp., $G + X$) the graph obtained from G by deleting X (resp., adding X), where deletion of a node implies also deletion of all the edges incident to it. Let $\Gamma(X) = \{y \in V - X : xy \in E \text{ for some } x \in X\}$ denote the set of *neighbors* of X in G. For two disjoint subsets X, Y of V let $d(X,Y) = |\{xy \in E : x \in X, y \in Y\}|$; for brevity, $d(X) = d(X, V - X)$ is the *degree* of X. Let G_X denote the subgraph of G induced by X, and $G_X \times G_Y = G_X + G_Y + \{xy : x \in X, y \in Y\}$. For any graph H, let $V(H)$ and $E(H)$ denote the set of nodes and edges of H, respectively.

2 The Theorem

Theorem 2.1 *Let $k \geq 2$ and let $G = (V,E)$ be a simple graph with $|V| \geq k+2$ satisfying:*

(C1) *$F = G_U$ is a forest, where $U = \{x \in V : d(x) \geq k+1\}$.*

(C2) *For every $xy \in E(F)$ there are at most k pairwise internally disjoint $\{x,y\}$-paths.*

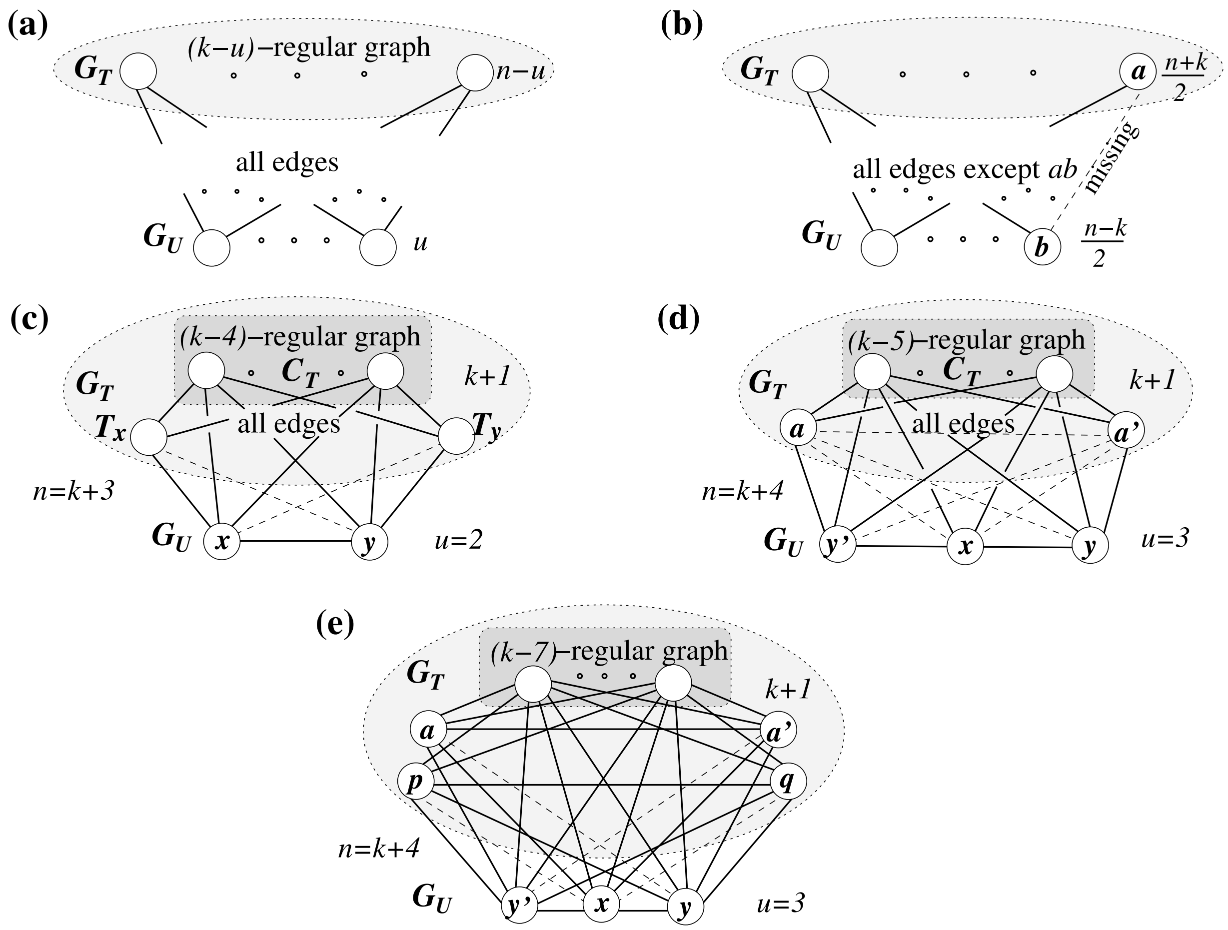

Figure 1: Types of graphs for which the bound of Theorem 2.1 is tight. (Dashed lines show "missing" edges.)

Then $m \leq k(n-k)+\lfloor\gamma^2/8\rfloor$, *where* $\gamma = \max\{0, 3k-n\}$. *Furthermore,* $m = k(n-k)+\lfloor\gamma^2/8\rfloor$ *if and only if (see Fig.* 1*):* $n \geq u+k+1$, $u = |U| \leq k$, $u \in \{k, \frac{n-k}{2}, \frac{n-k\pm1}{2}, \frac{n-k}{2} \pm 1\}$, *and if* $T = V - U$ *then: Either* $E(G_U) = \emptyset$, *and* G *is one of the following graphs:*

(a) $G = G_T \times G_U$, *where* G_T *is* $(k-u)$*-regular (Fig.* 1(a)*).*

(b) $u = \frac{n-k}{2}$, $n \geq k+4$, k *is odd,* $(n+k) = 2 \bmod 4$, *and either:*

1. $G = (G_T \times G_U) - ab$ *where* $a \in T, b \in U$ *and in* G_T*: the degree of every node* $x \neq a$ *is* $k-u$, *and the degree of* a *is* $k-u+1$ *(Fig.* 1(b)*).*
2. $G = G_T \times G_U$ *and there is* $a \in T$ *so that in* G_T*: the degree of every node* $x \neq a$ *is* $k-u$, *and the degree of* a *is* $k-u-1$.

Or $k \geq 4$, $n \in \{k+3, k+4\}$, $u \in \{2,3\}$, F *is a tree, and* G *is one of the following graphs:*

(c) $U = \{x, y\}$ *for some* $x, y \in V$, *and if* $C_T = (\Gamma(x) \cap \Gamma(y)) - \{x, y\}$, $T_x = \Gamma(x) - (C_T + y)$, $T_y = \Gamma(y) - (C_T + x)$, *then* $|C_T| = k-1$, $T_x, T_y \neq \emptyset$, *and* $d(T_x, T_y) = \emptyset$. *Moreover, if* $n = k+3$ *(Fig.* 1(c)*) then* $|T_x| = |T_y| = 1$ *and* G_{C_T} *is* $(k-4)$*-regular, and if* $n = k+4$ *(see Section 5) then* $|T_x| + |T_y| = 3$ *and* k *is odd.*

(d) k *is odd,* $n = k+4$, $U = \{x, y, y'\}$, *and either:*

1. $k \geq 5$, $G_T = G_{C_T} \times (\{a, a'\}, \emptyset)$ *where* $a, a' \in T$ *and* G_{C_T} *is a* $(k-5)$*-regular graph on* $k-1$ *nodes, and* $G = (G_T \times G_U) - \{xa, xa', ay, a'y'\}$ *(Fig.* 1(d)*).*
2. $k \geq 7$, $G_T = G_{C_T} \times (\{a, a', p, q\}, \emptyset)$ *where* $a, a', p, q \in T$ *and* G_{C_T} *is a* $(k-7)$*-regular graph on* $k-3$ *nodes, and* $G = (G_T \times G_U) - \{xp, xq, ay, a'y'\}$ *(Fig.* 1(e)*).*

3 Proof of Theorem 2.1

It is easy to verify that all the graphs listed in Theorem 2.1 obey Conditions (C1),(C2) and have $k(n-k)+\lfloor\gamma^2/8\rfloor$ edges. So, for the extreme cases, we only need to prove that no other case is possible. In the proof of Theorem 2.1 we use the same notation as in the theorem: $U = \{x \in V : d(x) \geq k+1\}$, $u = |U|$, $T = V - U$, $F = G_U$. Also, c is the number of connected components of F, and $f = |E(F)| = u - c$.

3.1 Bounds on $d(U,T)$ and f

Note that if G has all $\{T,U\}$-edges then $d(U,T) = (n-u)\min\{k,u\}$, since $d(v) \le k$ for all $v \in T$. Thus

$$d(U,T) \le (n-u)\min\{k,u\}. \tag{1}$$

A better bound on $d(U,T)$ when $f \ge 1$ is as follows.

Lemma 3.1 *Let G satisfy the conditions of Theorem 2.1 and suppose that $f \ge 1$. Then:*
(i) *$n \ge k+3$ and*

$$d(U,T) \le (n-u)\min\{k,u\} - (n-k-1). \tag{2}$$

(ii) *$f = 1$ if (2) holds with equality.*
(iii) *$f \le 2$ if*

$$d(U,T) = (n-u)\min\{k,u\} - (n-k)\ . \tag{3}$$

The proof of Lemma 3.1 follows. Let us fix some $e = xy \in F$. Condition (C2) and Menger's Theorem imply that there is $C \subseteq V - \{x,y\}$, $|C| = k-1$, such that in $(G-e)-C$ there is no $\{x,y\}$-path. Thus there is a partition $\{S_x, S_y\}$ of $V - C$ such that $x \in S_x$, $y \in S_y$, and $d(S_x, S_y) = 1$. Denote:

$$T_x = T \cap S_x,\ U_x = U \cap S_x,\ T_y = T \cap S_y,\ U_y = U \cap S_y\ .$$

Note that G has no $\{U_x, T_y\}$-edges and no $\{U_y, T_x\}$-edges, hence

$$d(U,T) \le (n-u)\min\{k,u\} - (|U_x||T_y| + |U_y||T_x|)\ . \tag{4}$$

Claim 3.2 *Each one of the sets T_x, U_x, T_y, U_y is nonempty. In particular, $n \ge k+3$ and $|U_x||T_y| + |U_y||T_x| \ge n-k-1$. Thus part* (i) *of Lemma 3.1 holds.*

Proof: Clearly, $U_x, U_y \ne \emptyset$, since $x \in U_x, y \in U_y$. We show that $T_x \ne \emptyset$; the proof for T_y is similar. Suppose to the contrary that $T_x = \emptyset$. Then, $U_x \ne \{x\}$, since otherwise $\Gamma(x) \subseteq C + y$, implying $d(x) \le k$ and contradicting $x \in U$. By condition (C1), $G_U = F$ is a forest. Thus there is a node $z \in U_x - x$, with at most one neighbor in U_x. But then $|\Gamma(z)| \le |C| + 1 = k$, a contradiction. The second statement follows immediately from the first statement. $\square$

It remains to prove parts (ii) and (iii) of Lemma 3.1.

Definition 3.1 *G is* full *(w.r.t. x, y, S_x, S_y) if it has all $\{T,U\}$-edges except the $\{U_x, T_y\}$-edges and the $\{U_y, T_x\}$-edges. G is* semi-full *if G is not full, but can be made full by adding one edge.*

Corollary 3.3 *If equality holds in* (2) *then* G *is full. If* (3) *holds then* G *is full or semi-full.*

Proof: The first statement follows from (4) and Claim 3.2. For the second statement, note that if (3) holds, then (4) implies that either: $|U_x||T_y| + |U_y||T_x| = n - k$ and G is full, or $|U_x||T_y| + |U_y||T_x| = n - k - 1$ and G is semi-full. $\square$

Remark: The case that (3) holds and G is full can occur; e.g., $|T_x| = 1$, $|T_y| = |U_x| = 2$, $|U_y| = n - k - 4$. Then $|U_x||T_y| + |U_y||T_x| = 4 + (n - k - 4) = n - k$.

Denote

$$C_T = \{a \in T : \Gamma(a) = U\},\ C_U = \{b \in U : \Gamma(b) = T\} .$$

Claim 3.4 $C_T \cup C_U \subseteq C$ *and* $C_T \cup C_U = C$ *if* G *is full. Moreover, for any* $e' = x'y' \in F$, $G - (C_T \cup C_U \cup \{e'\})$ *has at most* $|C - (C_T \cup C_U)|$ *pairwise internally disjoint* $\{x', y'\}$*-paths.*

Proof: By Claim 3.2, each one of T_x, U_x, T_y, U_y is nonempty. Since $d(U_x, T_y) = d(U_y, T_x) = 0$, $\Gamma(a) = U$ implies $a \notin T_x \cup T_y$, and $\Gamma(b) = T$ implies $b \notin U_x \cup U_y$. Hence $C_T \cup C_U \subseteq C$. If G is full, then $\Gamma(a) = U$ for every $a \in T \cap C$ and $\Gamma(b) = T$ for every $b \in U \cap C$; hence $C_T \cup C_U = C$. The second statement follows from condition (C2) and Menger's Theorem. $\square$

Corollary 3.5 $f = 1$ *if* G *is full. Thus part* (ii) *of Lemma 3.1 holds.*

Proof: Since G is full, $C_T \cup C_U = C$ by Claim 3.4. Thus C is independent of the choice of $e \in F$. This implies that if $e' = x'y' \in F$, $e' \neq e$, then $x', y' \notin C$, and either $x', y' \in U_x$ or $x', y' \in U_y$. Assume w.l.o.g. that $x', y' \in U_x$. Let $a \in T_x$. Then $ax', ay' \in E$, since G is full. This contradicts that there is no $\{x', y'\}$-path in $G - (C_T \cup C_U \cup \{e'\}) = G - (C \cup \{e'\})$. $\square$

It remains to prove part (iii) of Lemma 3.1.

Claim 3.6 *Assume that:* (3) *holds,* $f \geq 2$, G *is semi-full, and* $G + ab$ *is full for some* $a \in T$ *and* $b \in U$*. Let* $e' = x'y' \in F$, $e' \neq e$*. Then* $|\Gamma(x') \cap \Gamma(y') \cap T| \geq |\Gamma(x) \cap \Gamma(y) \cap T|$, *and an equality holds if, and only if,* $b \in \{x', y'\}$ *and* (*assuming w.l.o.g. that* $x', y' \in S_x \cup C$, *see Fig.* 2(a)) *either:* $T_x = \{a\}$, *or* $a \in C$, $b \notin \{x, y\}$, *and* $|T_x| = 1$.

Proof: Clearly, $C_T \subseteq \Gamma(x) \cap \Gamma(y) \cap T$; similarly, $C_T \subseteq \Gamma(x') \cap \Gamma(y') \cap T$. Let

$$A = (\Gamma(x) \cap \Gamma(y) \cap T) - C_T,\ A' = (\Gamma(x') \cap \Gamma(y') \cap T) - C_T .$$

W.l.o.g. assume that $x', y' \in S_x \cup C$ (see Fig. 2(a)). We need to show that: $|A'| \geq |A|$, and $|A'| = |A|$ if, and only if, $b \in \{x', y'\}$ and either: $T_x = \{a\}$, or $a \in C$, $b \notin \{x, y\}$, $|T_x| = 1$. Note that $A = \{a\}$ if $a \in C$ and $b \notin \{x, y\}$, and $A = \emptyset$ otherwise. If $b \notin \{x', y'\}$, then $A' = T_x \cup \{a\}$ if $a \in C \cup S_x$ and $A' = T_x$ if $a \in S_y$; hence $|A'| \geq |A| + |T_x| > |A|$ in this case.

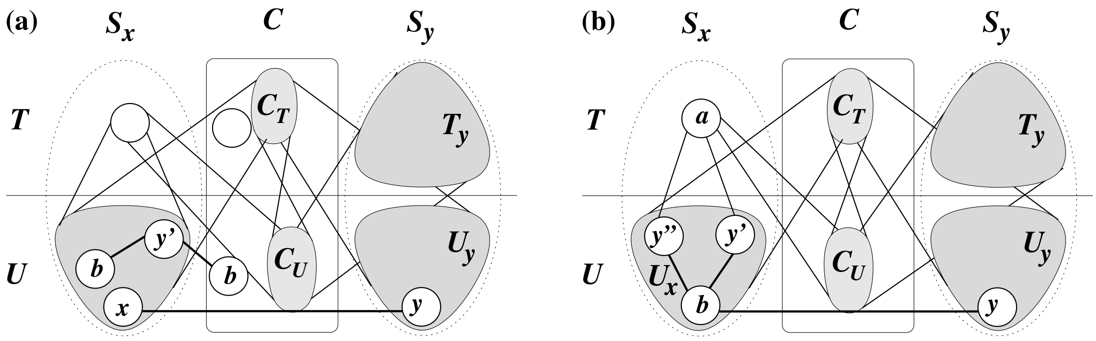

Figure 2: (a) Illustration to Claim 3.6: $|A'| \geq |A|$ holds independently of the location of a, b. $|A'| = |A|$ if, and only if, $b \in \{x', y'\}$ and (assuming that $x', y' \in S_x \cup C$) either: $T_x = \{a\}$, or $a \in C$, $b \notin \{x, y\}$, and $|T_x| = 1$. (b) Illustration to the proof of Corollary 3.7.

If $b \in \{x', y'\}$ then $A' = T_x - \{a\}$; thus $|A'| \geq |A|$ in this case, and if $|A'| = |A|$ then either $T_x = \{a\}$ (so $A = A' = \emptyset$), or $a \in C$, $b \notin \{x, y\}$, $|T_x| = 1$ (so $A = \{a\}$ and $A' = T_x$). $\square$

Corollary 3.7 *Part* (iii) *of Lemma 3.1 holds.*

Proof: Assume (3) holds. We choose $e = xy \in F$ so that $|\Gamma(x) \cap \Gamma(y) \cap T|$ is maximum, and S_x, C, S_y as before. If G is full w.r.t. S_x, C, S_y, then $f = 1$ by Corollary 3.5. Else, G is semi-full, by Corollary 3.3. Then, by Claim 3.6, $|\Gamma(x') \cap \Gamma(y') \cap T| = |\Gamma(x) \cap \Gamma(y) \cap T|$, $b \in \{x', y'\}$ for any $e' = x'y' \in F$, $e' \neq e$, and either $T_x = \{a\}$ or $b \notin \{x, y\}$. Assuming that F has two additional edges $e' = x'y'$ and $e'' = x''y''$, we conclude that e', e'' are incident to b. Since $|\Gamma(x') \cap \Gamma(y') \cap T| = |\Gamma(x) \cap \Gamma(y) \cap T|$, the same argument applies on e' instead of e to conclude that e, e'' are incident to the same node b' (at this point, possibly $b' \neq b$). Similarly, we get that e', e are incident to the same node b''. Consequently, any two edges from e, e', e'' have an endnode in common. Thus e, e', e'' form a star with center $b = b' = b''$ (e, e', e'' is not a cycle since F is a forest, by condition (C1)), say $b = x = x' = x''$; in particular, $T_x = \{a\}$ by Claim 3.6, and $b \in U_x$ (see Fig 2(b)). We obtain a contradiction to Claim 3.4, as then $(by'', y''a, ay')$ is an $\{x', y'\}$-path in $G - (C_T \cup C_U \cup \{e'\})$. $\square$

The proof of Lemma 3.1 is complete.

3.2 The case $u \geq k + 1$

Observe that

$$m = f + \frac{1}{2}[k(n-u) + d(U,T)] = (u - c) + \frac{1}{2}[k(n-u) + d(U,T)] \ . \tag{5}$$

Assume $u \geq k+1$. Then $d(U,T) \leq k(n-u)$ by (1), and thus by (5):

$$m \leq u - c + k(n-u) = kn - u(k-1) - c \leq kn - (k+1)(k-1) - c = k(n-k) - (c-1).$$

This implies $m \leq k(n-k)$, and if equality holds then $u = k+1$, $c = 1$ (thus $f \geq 1$), and $d(U,T) = k(n-u)$. But then, equality holds in (2), so $k = u-1 = f = 1$, by Lemma 3.1(ii).

This completes the proof of Theorem 2.1 for the case $u \geq k+1$.

Henceforth assume that $u \leq k$. Note that then inequality (1) becomes $d(U,T) \leq u(n-u)$. We now split the proof into two cases: the case $f = 0$ and the case $f \geq 1$.

3.3 The case $u \leq k$ and $f = 0$

We will show that if $u \leq k$ and $f = 0$ then $m \leq k(n-k) + \lfloor \gamma^2/8 \rfloor$, and if an equality holds then G must be of type (a) or of type (b). Substituting (1) into (5) we obtain:

$$m \leq \frac{1}{2}\left[k(n-u) + d(U,T)\right] \leq \frac{1}{2}(n-u)(k+u) \equiv g(u) \ . \tag{6}$$

Claim 3.8 *$g(u) \leq k(n-k) + \gamma^2/8$ and thus $m \leq \lfloor g(u) \rfloor \leq k(n-k) + \lfloor \gamma^2/8 \rfloor$. Furthermore, if $m = g(u)$ then:*
(i) *$d(U,T) = u(n-u)$, that is, $\{ab : a \in U, b \in T\} \subseteq E$.*
(ii) *$|E(G_T)| = \frac{1}{2}(n-u)(k-u) = \frac{1}{2}|T|(k-u)$ and thus G_T is $(k-u)$-regular.*

Proof: Let u^* be the maximizer of $g(u)$. As $g'(u) = \frac{1}{2}(n-k-2u)$, $u^* = k$ for $n \geq 3k$ and $u^* = \frac{n-k}{2}$ for $n \leq 3k-1$. The first statement follows since $g(k) = k(n-k)$ and $g(\frac{n-k}{2}) = k(n-k) + \frac{1}{8}(3k-n)^2$. The second statement is obvious. $\square$

Corollary 3.9 *Suppose that $n \geq 3k$. If $m = k(n-k) + \lfloor \gamma^2/8 \rfloor$ then $G = K_{k,n-k}$.*

Proof: In this case $\gamma = 0$, so $m = k(n-k)$. Then we must have $u = k$, and $m = g(u) = g(k)$. By Claim 3.8, the only possible case is $G = K_{k,n-k}$. $\square$

Corollary 3.10 *Assume that $n \leq 3k-1$. Let $u^* = \frac{n-k}{2}$, $\delta = u - u^*$, $\alpha = \lfloor (n+k)/4 \rfloor$, and $i = (n+k) \bmod 4$. If $m = k(n-k) + \lfloor \gamma^2/8 \rfloor$ then $|\delta| \in \{0, 1/2, 1\}$ and:*

$$m = 2\alpha^2 + \alpha i + \left\lfloor \frac{i^2}{8} \right\rfloor = 2\alpha^2 + \alpha i + \left\lfloor \frac{i^2 - 4\delta^2}{8} \right\rfloor = \lfloor g(u) \rfloor \ . \tag{7}$$

Proof: Note that $k(n-k) + \lfloor \gamma^2/8 \rfloor = \lfloor (n+k)^2/8 \rfloor$ for $n \leq 3k-1$, and that $n+k = 4\alpha + i$. Computations show that $\lfloor (n+k)^2/8 \rfloor = 2\alpha^2 + \alpha i + \lfloor i^2/8 \rfloor$ and $g(u) = 2\alpha^2 + \alpha i + (i^2 - 4\delta^2)/8$. Since by (6) $m \leq \lfloor g(u) \rfloor$, if $m = k(n-k) + \lfloor \gamma^2/8 \rfloor$ we must have $\lfloor g(u) \rfloor = \lfloor (n+k)^2/8 \rfloor$, which is equivalent to (7). In particular, we must have $\lfloor (i^2 - 4\delta^2)/8 \rfloor = \lfloor i^2/8 \rfloor$. The only possible cases are $|\delta| \in \{0, 1/2, 1\}$. $\square$

Claim 3.11 *Assume that $n \leq 3k - 1$. If $m = k(n-k) + \lfloor \gamma^2/8 \rfloor$ then either: $\delta = 0$ and G is of type* (a) *or of type* (b)*, or $|\delta| \in \{1/2, 1\}$ and G is of type* (a)*.*

Proof: We consider all the cases when (7) is possible, combining with Claim 3.8.

Suppose that $\delta = 0$. Then $u = u^* = \frac{n-k}{2}$, so both $n - k$ and $n + k$ are even. Hence the cases to consider are $i = 0$ and $i = 2$. If $i = 0$ then $m = g(u)$, and thus G_T is $(k-u)$-regular, by Claim 3.8. Assume now that $i = 2$. Then $u = \frac{n-k}{2}$, $m = \frac{1}{2}[(n-u)^2 - 1]$, $d(U,T) \geq 2m - k(n-u)$. Thus $d(U,T) = u(n-u) - 1$ or $d(U,T) = u(n-u)$. In the former case (type (b1)) there is exactly one pair $\{a, b\}$ such that $a \in T, b \in U$ and $ab \notin E$. This uniquely determines the degrees in G_T: $k-u$ if $x \in T - a$, and the degree of a is $k-u+1$. In the latter case (type (b2)) $ab \in E$ for every $a \in T, b \in U$, and $|E(G_T)| = \frac{1}{2}[(n-u)(k-u)-1]$. This determines the degrees in G_T: $k - u - 1$ for some $a \in T$, and $k - u$ for every $v \in T - a$. In each one of the cases k is odd, as otherwise the number of odd degree nodes in G is even. Thus G is of type (a) or of type (b) in this case.

Suppose that $\delta = \pm 1/2$. Then $i \in \{1, 3\}$, $u = \frac{n-k\pm 1}{2}$ and $m = g(u)$. This implies that G_T is $(k-u)$-regular. Thus G is of type (a) in this case.

Suppose that $\delta = \pm 1$. Then $i = 2$, $u = \frac{n-k}{2} \pm 1$ and $m = g(u)$. This implies that G_T is $(k-u)$-regular. Thus G is of type (a) in this case. $\square$

This completes the proof of the case $u \leq k$ and $f = 0$.

3.4 The case $u \leq k$ and $f \geq 1$

Assume that $u \leq k$ and $f \geq 1$. Using Lemma 3.1, we will show that then $m \leq k(n-k) + \lfloor \gamma^2/8 \rfloor$, and if $m = k(n-k) + \lfloor \gamma^2/8 \rfloor$, then G must be of type (c) or of type (d). By (2), $d(U,T) = u(n-u) - (n-k-1) - j$ for some integer $j \geq 0$. Substituting into (5) we obtain:

$$m \leq g(u) - \frac{1}{2}(n - k - 2u + 2c + j - 1) \equiv h(u) \ . \qquad (8)$$

Claim 3.12 *Assume that $u \leq k$ and that $f \geq 1$. Then $m \leq \lfloor h(u) \rfloor \leq k(n-k) + \lfloor \gamma^2/8 \rfloor$. Furthermore, if $m = k(n-k) + \lfloor \gamma^2/8 \rfloor$, then $c = 1$ and either: $j = 0$, $u = 2$, $n \in \{k+3, k+4\}$, or $j = 1$, $u = 3$, $n = k + 4$.*

Proof: Let u^* be the maximizer of $h(u)$, $2 \leq u \leq k$. As $h'(u) = 1 + \frac{1}{2}(n-k-2u)$, $u^* = k$ for $n \geq 3k-1$ and $u^* = \frac{n-k}{2} + 1$ for $n \leq 3k-2$. Thus

$$h(u) \leq h(k) = k(n-k) - \frac{1}{2}(n-3k+2c+j-1) \quad \text{for } n \geq 3k-1$$

$$h(u) \leq h(\frac{n-k}{2}+1) = k(n-k) + (3k-n)^2/8 - \frac{1}{2}(2c+j-2) \quad \text{for } n \leq 3k-2$$

Thus $\lfloor h(u) \rfloor \leq k(n-k) + \lfloor \gamma^2/8 \rfloor$, and if $m = k(n-k) + \lfloor \gamma^2/8 \rfloor$, then $c = 1$ and $j \in \{0,1\}$; furthermore, by Lemma 3.1 (ii),(iii), $u = 2$ if $j = 0$, and $u \in \{2,3\}$ if $j = 1$.

Assume that $m = k(n-k) + \lfloor \gamma^2/8 \rfloor$. It remains to prove that then either: $j = 0$, $u = 2$, $n \in \{k+3, k+4\}$, or $j = 1$, $u = 3$, $n = k+4$.

Substituting $c = 1$ in (8) and using Claim 3.8 we obtain

$$h(u) = g(u) - \frac{1}{2}(n-k-2u+j+1) \leq k(n-k) + \gamma^2/8 - \frac{1}{2}(n-k-2u+j+1) \ .$$

Therefore, by (8), $n-k-2u+j+1 \leq 1$, so $n \leq k+2u-j$. If $j = 0$ then $u = 2$ and thus $n \in \{k+3, k+4\}$ (recall that $n \geq k+3$, by Lemma 3.1 (i)).

Suppose that $j = 1$. Then $n \leq k+2u-1 \leq k+5$, since $u \in \{2,3\}$, It remains to exclude the case $n = k+3$ and the case $u = 3$, $n = k+5$. These cases are excluded by using the bound (8) $m \leq \lfloor h(u) \rfloor$ and showing via direct computations that $\lfloor h(u) \rfloor < k(n-k) + \lfloor \gamma^2/8 \rfloor$. Substituting $c = j = 1$ and $g(u)$ from (6) into (8) we obtain

$$h(u) = \frac{1}{2}(n-u)(k+u) - \frac{1}{2}(n-k-2u+2) = \frac{1}{2}(n-u+1)(k+u-1) - \frac{1}{2} \ .$$

If $n = k+3$, then $h(2) = h(3) = \frac{1}{2}(k+2)(k+1) - \frac{1}{2}$, so $\lfloor h(2) \rfloor = \lfloor h(3) \rfloor = \frac{1}{2}(k+2)(k+1) - 1$. On the other hand, $k(n-k) + \lfloor \gamma^2/8 \rfloor = \lfloor (n+k)^2/8 \rfloor = \lfloor (2k+3)^2/8 \rfloor$. One can easily verify that $\frac{1}{2}(k+2)(k+1) - 1 < \lfloor (2k+3)^2/8 \rfloor$ for any k.

If $n = k+5$, then $h(3) = \frac{1}{2}(k+3)(k+2) - \frac{1}{2}$, so $\lfloor h(3) \rfloor = \frac{1}{2}(k+3)(k+2) - 1$. On the other hand, $k(n-k) + \lfloor \gamma^2/8 \rfloor = \lfloor (n+k)^2/8 \rfloor = \lfloor (2k+5)^2/8 \rfloor$ for $k \geq 3$. One can easily verify that $\frac{1}{2}(k+3)(k+2) - 1 < \lfloor (2k+5)^2/8 \rfloor$ for any k. For $k = 2$ we have $\lfloor h(3) \rfloor = 9$, while $k(n-k) + \lfloor \gamma^2/8 \rfloor = k(n-k) = 10$. Thus $\lfloor h(3) \rfloor < k(n-k) + \lfloor \gamma^2/8 \rfloor$ if $n = k+5$. $\square$

Claim 3.13 *Assume that $u \leq k$ and that $f \geq 1$. If $m = k(n-k) + \lfloor \gamma^2/8 \rfloor$ and if $d(v) = k+1$ for every $v \in U$ then $d(v) = k$ for every $v \in T$.*

Proof: Otherwise, $m = \frac{1}{2}\sum_{v \in V} d(v) \leq \frac{1}{2}(nk+u-1) \leq \frac{1}{2}k(n+1) - \frac{1}{2}$. On the other hand, $k(n-k) + \lfloor \gamma^2/8 \rfloor = \lfloor (n+k)^2/8 \rfloor$. One can easily verify that $\frac{1}{2}k(n+1) - \frac{1}{2} < \lfloor (n+k)^2/8 \rfloor$ for any $n \geq k+3$. $\square$

Claim 3.14 *Suppose that* $m = k(n-k) + \lfloor \gamma^2/8 \rfloor$. *If* $j = 0$ *then* G *is of type* (c).

Proof: By Claim 3.12 $n \in \{k+3, k+4\}$ and $U = \{x, y\}$. Let us use the notation C, U_x, U_y, T_x, T_y as in the proof of Lemma 3.1. Note that G is full (see Definition 3.1), and thus $C = C_T = \Gamma(x) \cap \Gamma(y)$ is uniquely determined.

If $n = k+3$, then $|T_x| = |T_y| = 1$, so $d(x) = d(y) = k+1$. Thus $d(v) = k$ for every $v \in T$, by Claim 3.13. Consequently, G_{C_T} must be $(k-4)$-regular. Thus G is of type (c).

If $n = k+4$, then $|T_x|+|T_y| = 3$, say $|T_x| = 1$ and $|T_y| = 2$. Then $d(x) = k+1$, $d(y) \leq k+2$. Thus $m = \frac{1}{2}\sum_{v \in V} d(v) \leq \frac{1}{2}[(k+2)^2 - 1]$, and if $m = \frac{1}{2}[(k+2)^2-1]$, then $d(v) = k$ for every $v \in T$ and $d(y) = k+2$. On the other hand, $k(n-k) + \lfloor \gamma^2/8 \rfloor = \lfloor \frac{1}{2}(k+2)^2 \rfloor$. This implies that k is odd and $m = \frac{1}{2}[(k+2)^2 - 1]$, so $d(v) = k$ for every $v \in T$ and $d(y) = k+2$. Thus G is of type (c). $\square$

Claim 3.15 *Suppose that* $m = k(n-k) + \lfloor \gamma^2/8 \rfloor$. *If* $j = 1$ *then* G *is of type* (d).

Proof: By Claim 3.12, $|T| = k+1$ and $|U| = 3$, say $U = \{x, y, y'\}$ and $E(F) = \{xy, xy'\}$. Let $X = \Gamma(x) \cap T$, $Y = \Gamma(y) \cap T$, and $Y' = \Gamma(y') \cap T$. By Claim 3.2, $X \neq T$, $Y \neq T$, and $Y' \neq T$. Hence $|Y| = |Y'| = k$ and $|X| \in \{k, k-1\}$. Furthermore, $|X| \neq k$; otherwise, we show a contradiction to Condition (C2). If $Y = X$ the contradiction is obvious. Else, $|X \cap Y| = k-1$ and $yp \in E$, where $p \in T$ is the unique non-neighbor of x. By a similar argument, $|X \cap Y'| = k-1$ and $y'p \in E$. But then $G - xy$ has k pairwise internally disjoint $\{x, y\}$-paths: $k-1$ paths (xv, vy) for every $v \in X \cap Y$, and another path is $(xy', y'p, py)$.

Henceforth assume that $|Y| = |Y'| = k$ and $|X| = k-1$, so $d(x) = d(y) = d(y') = k+1$. This implies $d(v) = k$ for every $v \in T$, by Claim 3.13. Consequently, k is odd; k cannot be even, as then U is the set of odd degree nodes of G, but any graph has even number of odd degree nodes. Let $T - Y = \{a\}$, $T - Y' = \{a'\}$, and $T - X = \{p, q\}$; note that $p \neq q$, but otherwise a, a', p, q may not be distinct.

We show that if a, a', p, q are not all distinct then G is of type (d1). Consider two cases: the case $a = a'$ and the case $a \neq a'$ and $\{a, a'\} \cap \{p, q\} \neq \emptyset$. The case $a = a'$ is not possible, since then $G - xy$ has k internally disjoint xy-paths, violating Condition (C2) (see Fig. 3(a)): there are $k-2$ paths $\{(xv, vy) : v \in X \cap Y\}$; since $d(a) = k$, $ap \in E$ or $aq \in E$, say $ap \in E$, and this gives two additional paths: $(xy', y'q, qy)$ and (xa, ap, py). Therefore assume that $a \neq a'$ and $\{a, a'\} \cap \{p, q\} \neq \emptyset$, say $a = p$. Then $|X \cap Y| = k-1$, which gives $k-1$ internally disjoint xy-paths. Thus $a' = q$ (see Fig. 3(b)); otherwise the additional path $(xy', y'q, qy)$ violates Condition (C2). It is now easy to see that the only possible case is (d1). Note that $aa' \notin E$; otherwise the additional path $(xy', y'a, aa', a'y)$ gives again a violation of Condition (C2). Consequently, since $d(v) = k$ for every $v \in T$, G must be of type (d1).

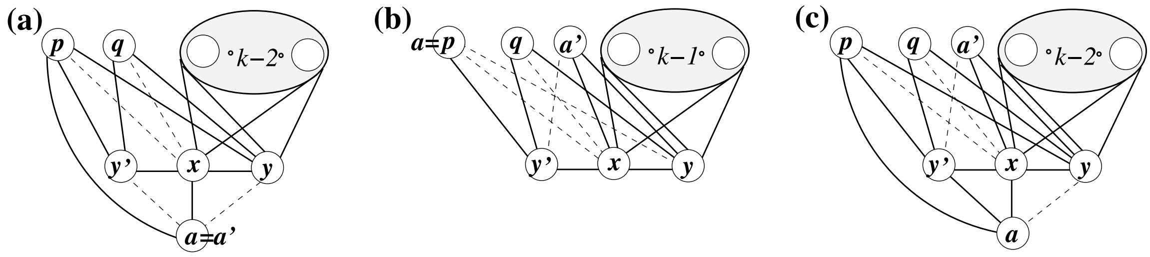

Figure 3: Illustration to the proof of Claim 3.15 (”missing” edges are shown by dashed lines). Condition (C2) is violated in each one of the following cases: (a) $a = a'$; (b) $a = p$ and $a' \neq q$; (c) a, a', p, q are all distinct and $ap \in E$.

If a, a', p, q are all distinct, then it is easy to see that G is of type (d2). The only missing $\{T, U\}$-edges are $xp, xq, ay, a'y'$. Also $ap \notin E$, as otherwise $G - xy$ has k internally disjoint $\{x, y\}$-paths, violating Condition (C2) (see Fig. 3(c)): $k - 2$ paths $\{(xv, vy) : v \in X \cap Y\}$, the path $(xy', y'q, qy)$, and the path (xa, ap, py). By a similar argument, $aq, a'q, a'p \notin E$. Consequently, since $d(v) = k$ for every $v \in T$, G must be of type (d2). □

This completes the proof of the case $f \geq 1$, and thus the proof of Theorem 2.1 is complete.

4 Minimally k-outconnected graphs

Type (b2) graphs are not k-outconnected from any node, since they have a node of degree $k - 1$. All the other extremal graphs listed in Theorem 2.1 are k-outconnected from r for any $r \in G_U$ if $U \neq \emptyset$. This can be easily seen for graphs in Fig 1. For type (c) graphs with $n = k + 4$ the proof is given in Section 5. If $U = \emptyset$ then $0 = u = \frac{n-k}{2} - 1$ must hold, and thus G is a k-regular graph on $n = k + 2$ nodes. This implies that k is even, and G is obtained from a complete graph by deleting a perfect matching. Consequently, if $U = \emptyset$ then G is k-connected, and thus is k-outconnected from any $r \in V$.

Observe that for graphs with $f = 0$ the minimality property is straightforward, while for other extreme graphs follows by an easy analysis. It is also not hard to extract from the extremal graphs listed in Theorem 2.1 the k-connected ones.

Conditions (C1),(C2) of Theorem 2.1 are satisfied for minimally k-connected and minimally k-outconnected graphs. For k-connected graphs, condition (C1) holds by Mader's Theorem 1.1, while validity of condition (C2) was also proved in [5]. For k-outconnected graphs, validity of condition (C2) can be deduced from [4, Lemma 23], while validity of condition (C1) follows from the following analogue of Theorem 1.1.

Theorem 4.1 ([4], Theorem 21) *In a graph G which is k-outconnected from r, any cycle of critical edges (w.r.t. k-outconnectivity from r) contains a node $v \neq r$ which degree in G is k.*

The following theorem summarizes our results for k-outconnected graphs.

Theorem 4.2 *Let G be a graph which is minimally k-outconnected from r. Then:*

(1) $m \leq k(n-k) + \lfloor \gamma^2/8 \rfloor$, *and if $n \geq 3k$ then $m = k(n-k)$ if and only if $G = K_{k,n-k}$.*

(2) $m = k(n-k) + \lfloor \gamma^2/8 \rfloor$ *if and only if G is one of the graphs listed in Theorem* 2.1 *that is not a type* (b2) *graph.*

We will now consider some other questions related to minimally k-outconnected graphs. Mader in [6] showed that any minimally k-connected graph has at least $\frac{(k-1)n+2}{2k-1}$ nodes of degree k. This result can also be extended to minimally k-outconnected graphs.

Lemma 4.3 *Let G be a graph and let $T = \{v \in V : d(v) \leq k\}$. If $F = E(G-T)$ is a forest then $|T| \geq \frac{(k-1)n+2}{2k-1}$.*

Proof: Let $U = V - T$, and let c be the number of connected components in F. If F is empty, then $|T| = n$. Otherwise, $c \geq 1$. Observe that

$$k|T| \geq d(U,T) = \sum_{x \in U} d(x) - 2|F| \geq (k+1)|U| - 2(|U| - c) = (k-1)|U| + 2c. \qquad (9)$$

Substituting $|U| = n - |T|$ in (9) gives $k|T| \geq (k-1)(n-|T|) + 2c$, and the result follows. □

Combining Theorem 4.1 and Lemma 4.3, we obtain:

Theorem 4.4 *Any minimally k-outconnected graph has at least $\frac{(k-1)n+2}{2k-1}$ nodes of degree k.*

Finally, let us consider the following question: what is the least number $\mu = \mu(G)$ of edges which deletion from a minimally k-connected graph G results in a minimally k-outconnected graph. For a node r of a k-connected graph $G = (V, E)$, let F_r denote the set of all the edges in E that are *not* critical w.r.t. k-outconnectivity from r. Note that $\mu \leq \min_r |F_r|$. Clearly, if $k = 1$, then $F_r = \emptyset$ for any $r \in V$. In [1] it was shown that if $k \in \{2, 3\}$, then for any node r of G with $d(r) = k$ holds $F_r = \emptyset$. It is not known whether for an arbitrary k, any minimally k-connected graph has a node r with $F_r = \emptyset$. If this is so, then the family of minimally k-outconnected graphs contains the family of minimally k-connected graphs. Our purpose now is to show that any minimally k connected graph has a node r such that $|F_r|$ (and thus also μ) is relatively small. For an arbitrary k-connected graph G, let F denote the set of all the edges of G that are critical w.r.t. k-connectivity and not incident to a node of degree k. Note that for any node $r \in V$, $F_r \subseteq F$.

Lemma 4.5 *Any k-connected graph G has a node r such that $|F_r| \leq \left\lfloor \frac{|F|(k-1)}{n} \right\rfloor$.*

Proof: For $e = xy \in F$, let $V_e = \{r \in V : e \in F_r\}$. By Menger's Theorem, $e \in F_r$ if and only if there is a set $C \subset V - \{x, y\}$ of size $k-1$ such that $G - e - C$ is disconnected, and r belongs to any such set C. Thus, $|V_e| \leq k-1$ for any $e \in F$. Note that

$$\sum_{r \in V} |F_r| = |\{(r,e) : r \in V, e \in F_r\}| = |\{(e,r) : e \in F, r \in V_e\}| = \sum_{e \in F} |V_e| \leq |F|(k-1).$$

Thus there is a node r for which $|F_r| \leq \frac{|F|(k-1)}{n}$ holds. □

Corollary 4.6 *Let G be a minimally k-connected graph. Then G has a node r such that*

$$|F_r| \leq \left\lfloor \frac{(kn-2k-1)(k-1)}{n(2k-1)} \right\rfloor \leq \left\lfloor \frac{k-1}{2} \right\rfloor.$$

In particular, G can be made minimally k-outconnected by deleting at most $\left\lfloor \frac{k-1}{2} \right\rfloor$ edges.

5 Graphs of type (c) with $n = k + 4$

Here we give a more detailed description of type (c) graphs with $k+4$ nodes, and show that they are all k-outconnected from r for any $r \in U = \{x, y\}$. We will use here the notation of Theorem 2.1. W.l.o.g., let us assume $T_y = \{a, b\}$, thus $|T_x| = 1$.

There are three cases, see Fig. 4. In all these cases, $\Gamma(x) = V - \{a, b\}$, $\Gamma(y) = V - T_x$, $\Gamma(T_x) = V - \{y, a, b\}$.

The first case (type (c1)) in Fig. 4(a) is when the edge ab is not in the graph. Then G_{C_T} must be a $(k-5)$-regular graph and each one of a, b is connected by an edge to every node in C_T.

Another two types (c2)and (c3) arise when $ab \in E$. Then each one of a, b is connected to all the nodes in C_T except of exactly one. (c2) is the case when these nodes are different (α and β in Fig. 4(b)), and then their degree in G_{C_T} is exactly $k-4$. (c3) is the case when these nodes coincide, so it is one node, and then its degree in G_{C_T} is exactly $k-3$. The degrees of any other node in G_{C_T} is $k-5$.

Let us show that in each one of the cases G is k-outconnected from y. That is, in G there are k pairwise internally disjoint paths between y and any other node v. This can be easily

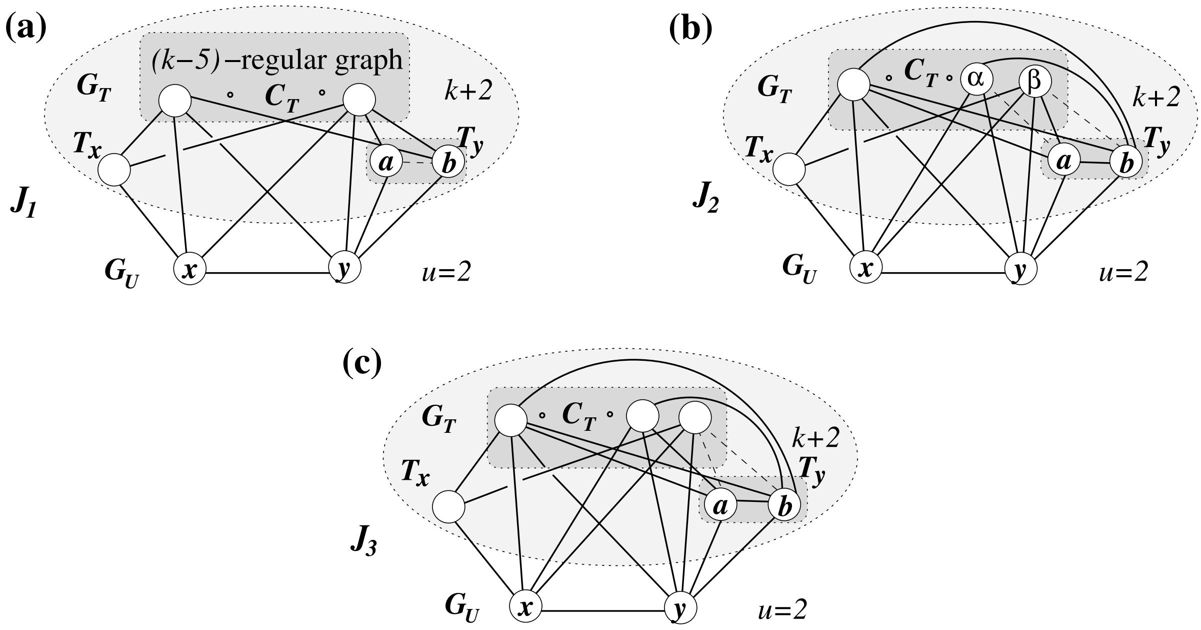

Figure 4: Type (c) graphs with $n = k + 4$ (dashed lines show some "missing" edges.).

seen for $v \notin C_T$: then either $yv \in E$ (if $v \in \{x, a, b\}$) and $|\Gamma(y) \cap \Gamma(v)| = k - 1$, or $yv \in E$ (if $v = T_x$) and $|\Gamma(y) \cap \Gamma(v)| = k$. Now, assume $v \in C_T$. Then $yv \in E$, and, as can be easily checked, $|\Gamma(y) \cap \Gamma(v)| = k - 2$. This gives $k - 1$ pairwise internally disjoint paths. An additional $\{y, v\}$-path that is internally disjoint to the above $k-1$ paths is (y, w, T_x, v), where $w \in C_T - \Gamma(v)$ arbitrary. Thus, any type (c) graph on $n = k + 4$ nodes is k-outconnected from y, where y is as above.

The proof that any type (c) graph is k-outconnected from x is similar.

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