The k-Connected subgraph Problem

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2.1 Introduction

Two paths in a graph are **internally disjoint** if no internal node of one of the paths belongs to the other. A graph is k**-connected** if it contains k pairwise internally disjoint paths from every node to any other node. We survey approximation algorithms for the k-Connected Subgraph problem, formally defined as follows.

k-Connected Subgraph
Input: A directed/undirected graph $\hat{G} = (V, \hat{E})$ with edge costs $\{c_e : e \in \hat{E}\}$ and a positive integer k.
Output: A minimum cost k-connected spanning subgraph of $\hat{G}$.

In the related k-Edge-Connected Subgraph problem the paths are required to be only edge disjoint. For undirected graphs, both problems are NP-hard for $k = 2$ (the case $k = 1$ is the Minimum Spanning Tree problem), even when all edges in $\hat{G}$ have unit costs. This is since any feasible solution with $|V|$ edges is a Hamiltonian cycle. For directed graphs the problem is NP-hard already for $k = 1$, by a similar reduction.

We may assume that the input graph $\hat{G}$ is complete, by assigning infinite costs to "forbidden" edges. Under this assumption, we consider four types of edge costs:

- $\mathbf{\{0, 1\}}$**-costs:** Here we are given a graph G, and the goal is to find a minimum size augmenting edge set J of new edges (any edge is allowed) such that $G \cup J$ is k-connected.

- $\mathbf{\{1, \infty\}}$**-costs:** Here we seek a k-connected spanning subgraph of $\hat{G}$ with minimum number of edges.

- **Metric Costs:** The costs satisfy the triangle inequality $c_{uv} \leq c_{uw} + c_{wv}$ for all $u, w, v \in V$.

- **General Costs:** The costs are arbitrary non-negative integers or ∞.

In this survey we consider only the *node-connectivity* version, when the paths are internally disjoint. We survey only *approximation algorithms* (see [13, 15] for polynomially solvable cases and [23, 28] for previous surveys), with the currently best known approximation ratios, as summarized in Table 2, where for comparison we also included results for edge-connectivity.

	Node-Connectivity		**Edge-Connectivity**	
Costs	*Undirected*	*Directed*	*Undirected*	*Directed*
$\{0,1\}$	$\min\{2, 1+\frac{k^2}{2\mathsf{opt}}\}$ [14, 21]	in P [14]	in P [42]	in P [11]
$\{1,\infty\}$	$1-\frac{1}{k}+\frac{n}{\mathsf{opt}} \leq 1+\frac{1}{k}$ [3] ([36])	$1-\frac{1}{k}+\frac{2n}{\mathsf{opt}} \leq 1+\frac{1}{k}$ [3] ([36])	$1+\frac{1}{2k}+O\left(\frac{1}{k^2}\right)$ [19] $1+\Omega\left(\frac{1}{k}\right)$ [20]	$1+\frac{1}{k}$ [30] $1+\Omega\left(\frac{1}{k}\right)$ [20]
metric	$2+(k-1)/n$ [26]	$2+k/n$ [26]	2 [25]	2 [25]
general	$O\left(\ln\frac{n}{n-k}\cdot\ln k\right)$ [37] 6 if $n \geq k^3$ [4] ([18])	$O\left(\ln\frac{n}{n-k}\cdot\ln k\right)$ [37]	2 [25]	2 [25]

Table 1.1: Known approximability status of k-Connected Subgraph problems. Here and everywhere, references in parenthesis give a simplified proof and/or a slight improvement needed to achieve the approximation ratio stated.

We mention some additional results not appearing in Table 1. The best known ratios for k-Connected Subgraph with small values of k are: $\lceil (k+1)/2 \rceil$ if $k \leq 8$ for undirected graphs and $k+1$ if $k \leq 6$ for directed graphs [1, 6, 26]. For $\{0,1\}$-costs the complexity status of the problem is not known for undirected graphs, but for any constant k an optimal solution can be computed in polynomial time [22]. When $\hat{G}$ contains a spanning $(k-1)$-connected subgraph of cost 0 the $\{0,1\}$-costs case can be solved in polynomial time for any k [40]. In the case of $\{1,\infty\}$-costs, directed 1-Connected Subgraph admits ratio 3/2 [41], and undirected k-Edge-Connected Subgraph admits ratio 4/3 for $k=2$ [39] and ratio $1+\frac{2}{k+1}$ for $3 \leq k \leq 6$ [3]. In the case of metric costs both 2-Connected Subgraph and 2-Edge-Connected Subgraph admit ratio 3/2 [17]. In the case of $\{0,1,\infty\}$-costs when $\hat{G}$ contains a spanning tree of cost 0, 2-Edge-Connected Subgraph admits ratio 3/2 [29].

We will assume that $n \geq k+2$ and that the input graph $\hat{G}$ is simple. This is justified as follows. If $n \leq k+1$ then an easy argument shows that an optimal solution is obtained by taking the cheapest $k-n$ parallel edges from any node to the other; thus this case can be solved in polynomial time. If $n \geq k+1$ then it is known that any k-connected graph has a simple k-connected spanning subgraph. Thus in this case for any set of parallel edges of the input graph $\hat{G}$ we can keep only the cheapest one. The case $k = n-2$ can also be solved in polynomial time. It is not hard to see that a graph G is $(n-2)$-connected if and only if each node has degree (indegree and outdegree, in the case of directed graphs) at least $n-2$. The problem of finding the cheapest subgraph that satisfies some prescribed degree bounds can be solved in polynomial time (c.f. [38]). This gives a polynomial time algorithm for $k = n-2$.

Here is some notation used. An edge from u to v is denoted by uv. A uv-**path** is a path from u to v. For arbitrary sets A, B of nodes and edges (or graphs) $A \setminus B$ is the set (or graph) obtained by deleting B from A, where deletion of a node implies also deletion of all the edges incident to it; similarly, $A \cup B$ is the set (graph) obtained by adding B to A. For real values $\{x_u : u \in U\}$ let $x(U) = \sum_{u \in U} x_u$ and $\max(U) = \max_{u \in U} x_u$. We denote $n = |V|$ and assume that $n \geq k+2$ (otherwise the problem is trivial).

Organization. In the next section 2.2 we give some reductions between directed and undirected versions of our problem and related problems. In Section 2.3 we describe the algorithm of [3] for the $\{1, \infty\}$-costs case. In Section 2.4 we give a Biset-LP formulation of the problem, and some properties of relevant biset functions. In Section 2.5 we consider metric costs. In Section 2.6 we describe a 6-approximation algorithm of [4] for undirected graphs with $n = \Omega(k^3)$. In Section 2.7 we discuss the relation between k-Connected Subgraph and the augmentation version of the problem when the input graph $\hat{G}$ has a $(k-1)$-connected spanning subgraph of cost zero, and in Section 2.8 we describe an $O\left(\ln \frac{n}{n-k}\right)$-approximation algorithm for the augmentation problem. We conclude in Section 2.9 with some open problems in the field.

2.2 Relation between directed and undirected problems

We mention several relevant related problems. Let $\kappa_G(s,t)$ denote the maximum number of internally disjoint st-paths in a graph G. We say that G is k-**inconnected to** s if $\kappa_G(v,s) \geq k$ for all $v \in V \setminus \{s\}$. The corresponding min-cost connectivity problem is k-Inconnected Subgraph. In the related edge-connectivity problem k-Edge-Inconnected Subgraph the paths are required only to be edge disjoint. For directed graphs, these problems can be solved in polynomial time, see [7, 8] for the edge-connectivity case, [16] for the node connectivity case, and [12, 13] for various generalizations. The following reduction shows that undirected problems are not much harder to approximate than the directed ones.

Lemma 2.1 *For the four problems k-Connected Subgraph, k-Inconnected Subgraph, k-Edge-Connected Subgraph, and k-Edge-Inconnected Subgraph, ratio ρ for directed graphs implies ratio 2ρ for undirected graphs.*

Proof: Given an undirected instance $\mathcal{I} = (\hat{G}, c, k)$ of one of the four problems in the lemma, obtain a "bidirected" instance $\mathcal{I}' = (\hat{G}', c', k)$ by replacing every edge uv of $\hat{G}$ by the two opposite directed edges uv, vu each of the same cost as uv. Then compute a ρ approximate solution J' to $\mathcal{I}'$ and output its underlying graph J. It is easy to see that J is a feasible solution for $\mathcal{I}$. Furthermore, if G is an arbitrary subgraph of $\hat{G}$, and G' is the corresponding bidirected subgraph of $\hat{G}'$, then $c'(G') = 2c(G)$, and G is a feasible solution for $\mathcal{I}$ if G' is a feasible solution for $\mathcal{I}'$. Thus $\mathsf{opt}' \leq 2\mathsf{opt}$, where opt and opt' denote the optimal solution value of $\mathcal{I}$ and $\mathcal{I}'$, respectively. Consequently, $c(J) \leq c'(J') \leq \rho \cdot \mathsf{opt}' \leq 2\rho \cdot \mathsf{opt}$. □

Here is a typical application of Lemma 2.1. Since the directed k-Inconnected Subgraph problem can be solved in polynomial time [16], we get from Lemma 2.1 ratio 2 for undirected k-Inconnected Subgraph. Another immediate application is as follows. Since a directed graph $G = (V, E)$ is k-edge-connected iff both G and the reverse graph of G are k-edge-inconnected to s for some $s \in V$, we get ratio 2 for the directed k-Edge-Connected Subgraph problem. From Lemma 2.1 we also get ratio 2 for the undirected k-Edge-Connected Subgraph problem, since an undirected graph G is k-edge-connected if and only if G is k-edge-inconnected to some node s. This method does not work directly for the k-Connected Subgraph problem, since a graph which is k-inconnected to s may not be k-connected. However, many algorithms for k-Connected Subgraph use an extension of this method.

The following reduction shows that for k-Connected Subgraph instances with "high" values of k, the undirected variant cannot be much easier than the directed one.

Theorem 2.1 (Lando & Nutov [31]) *Let $\rho_{\mathsf{dir}}(k, n)$ and $\rho_{\mathsf{und}}(k, n)$ denote the best possible approximation* *ratio for the directed and undirected* k-Connected Subgraph *problem on graphs on* n *nodes, respectively. Then* $\rho_{\mathsf{dir}}(k,n) \leq \rho_{\mathsf{und}}(k+n,2n)$.

For even n and $k \geq n/2+1$ the inequality in Theorem 2.1 can be written as $\rho_{\mathsf{dir}}(k-n/2,n/2) \leq \rho_{\mathsf{und}}(k,n)$. Combining with Lemma 2.1 gives $\rho_{\mathsf{dir}}(k-n/2,n/2) \leq \rho_{\mathsf{und}}(k,n) \leq 2\rho_{\mathsf{dir}}(k,n)$. Loosely speaking, this means that for "high" values of k, say $k = n - o(n)$, the approximability of directed and undirected variants of the k-Connected Subgraph problem is the same, up to a constant factor.

In the rest of this section we prove Theorem 2.1. An ordered pair (S,T) of disjoint subsets of V is called a **setpair**. For $s,t \in V$ we say that (S,T) an st**-setpair** if $s \in S$ and $t \in T$. Let $d_G(S,T)$ denote the number of edges in G that go from S to T. Then the node connectivity version of Menger's Theorem can be formulated as follows.

Lemma 2.2 *Let* $G=(V,E)$ *be a (directed or undirected) graph. Then for any* $s,t \in V$

$$\kappa_G(s,t) = \min\{d_G(S,T) + |V| - (|S|+|T|) : (S,T) \text{ is an } st\text{-setpair in } G\}$$

Furthermore, if $st \notin E$ *then the minimum is attained for an* st*-setpair* (S,T) *with* $d_G(S,T)=0$.

Using Lemma 2.2, it is not hard to prove the following known statement.

Lemma 2.3 *A graph* $G=(V,E)$ *with* $|V| \geq k+1$ *is* k*-connected if and only if* $\kappa_G(s,t) \geq k$ *for all* $s,t \in V$ *with* $st \notin E$.

Let $G=(V,E)$ be a directed graph. The **bipartite graph of** G has node set $V' \cup V''$ where V', V'' are copies of V, and edge set $\{u'v'' : uv \in E\}$, where for $v \in V$ we denote by v' and v'' the copies of v in V' and V'', respectively. The **padded graph of** G is obtained by adding to the bipartite graph of G a **padding edge set** of cliques on each of V' and V' and the matching $M = \{v'v'' : v \in V\}$. Given an instance $\hat{G}=(V,\hat{E}),c,k$ of directed k-Connected Subgraph, obtain an instance $\hat{H},c',k+n$, of undirected k-Connected Subgraph where $\hat{H}$ is the padded graph of $\hat{G}$, and the costs are: $c'(u'v'')=c(uv)$ if $uv \in \hat{E}$ and the padding edges have cost 0. Note that if G is a spanning subgraph of $\hat{G}$ then the padded graph H of G is a spanning subgraph of $\hat{H}$, and G and H have the same cost. Also note that the set of non-adjacent node pairs in H is $\{\{s',t''\} : s,t \in V, st \notin E\}$. Thus the following lemma combined with Lemma 2.3 finishes the proof of Theorem 2.1 (for the proof see [34]).

Lemma 2.4 ([31], ([34])) *If H is the padded graph of a directed graph $G = (V, E)$ on n nodes then $\kappa_H(s', t'') = \kappa_G(s, t) + n$ for all $s, t \in V$. Thus G is k-connected if and only if H is $(k + n)$-connected.*

2.3 Min-Size k-Connected Subgraph ($\{1,\infty\}$-costs)

Here we consider the $\{1,\infty\}$-costs k-Connected Subgraph problem, often called the Min-Size k-Connected Subgraph problem. We present a slight improvement due to [36] of the ratio $1+\frac{1}{k}$ of Cheriyan and Thurimella [3].

Theorem 2.2 (Cheriyan & Thurimella [3] ([36])) Min-Size k-Connected Subgraph *admits the following approximation ratios: $1 - \frac{1}{k} + \frac{n}{\mathsf{opt}} \leq 1 + \frac{1}{k}$ for undirected graphs and $1 - \frac{1}{k} + \frac{2n}{\mathsf{opt}} \leq 1 + \frac{1}{k}$ for directed graphs.*

An edge e of a k-connected graph G is said to be **critical** if $G \setminus \{e\}$ is not k-connected. One of the most important theorems in theory of k-connected graphs is the following.

Theorem 2.3 (Mader's Undirected Critical Cycle Theorem [32])
In a k-connected undirected graph G, any cycle of critical edges has a node v with $d_G(v) = k$.

In [33] Mader also formulated and proved a similar theorem for directed graphs. An even length sequence of directed edges $C = (v_1v_2, v_3v_2, v_3v_4, \ldots, v_{2q-1}v_{2q}, v_1v_{2q})$ of a directed graph G is called an **alternating cycle**; the nodes $v_1, v_3, \ldots, v_{2q-1}$ are C**-out nodes**, and $v_2, v_4, \ldots, v_{2q}$ are C**-in nodes**.

Theorem 2.4 (Mader's Directed Critical Cycle Theorem [33])
In a k-connected directed graph G, any alternating cycle C of critical edges contains a C-in node whose indegree in G is k, or a C-out node whose outdegree in G is k.

Definition 2.1 *A graph $G = (V, E)$ is an* **ℓ-edge-cover** *if it has minimum degree $\geq \ell$ if G is undirected, and minimum outdegree $\geq \ell$ and minimum indegree $\geq \ell$ if G is directed.*

It is not hard to see that a directed graph G has no alternating cycle iff its bipartite graph G' is a forest. From Theorems 2.3 and 2.4 it is easy to deduce the following corollary, also due to Mader [32, 33].

Corollary 2.1 *Let $G = (V, I \cup F)$ be a k-connected graph such that I is a $(k-1)$-edge-cover and all the edges in F are critical. If G is an undirected graph then (V, F) is a forest, and if G is a directed graph then the bipartite graph of (V, F) is a forest.*

The following algorithm from [3] achieves the desired ratio for both directed and undirected graph.

Algorithm 1: Min-Size k-Connected Subgraph$(\hat{G}, k)$

1 find a minimum size $(k-1)$-edge-cover $I \subseteq \hat{E}$

2 find an inclusionwise minimal edge set $F \subseteq \hat{E} \setminus I$ such that $(V, I \cup F)$ is k-connected

3 return $I \cup F$

The problem of finding the cheapest ℓ-edge-cover can be solved in polynomial time (c.f. [38]), and it is also not hard to see that Step 2 of the algorithm can be implemented in polynomial time (see [3] for details). To show the approximation ratio we will use the following theorem, to be proved later.

Theorem 2.5 (Nutov [36]) *Let $G = (V, E)$ be an undirected graph with edge costs $\{c_e : e \in E\}$ and minimum degree $\geq k$ and let $I \subseteq E$ be a minimum cost ℓ-edge cover in G, $1 \leq \ell \leq k-1$. If G is bipartite then $c(F) \leq \frac{\ell}{k} \cdot c(E)$. If G is k-edge-connected then $c(I) \leq \frac{\ell+1/n}{k} \cdot c(E)$ and $c(I) \leq \frac{\ell}{k} \cdot c(E)$ if $\ell|V|$ is even.*

Let us consider directed graphs. Let I' and F' be the "bipartite" edge sets in the bipartite graph G' of G that corresponds to I and F, respectively. Then I is an ℓ-edge-cover if and only if I' is an ℓ-edge-cover. Thus $|I| = |I'| \leq \frac{k-1}{k}\mathsf{opt}$, by Theorem 2.5. On the other hand, by Corollary 2.1, F' is a forest, hence $|F| = |F'| \leq 2n-1$. Consequently, $\frac{|I|+|F|}{\mathsf{opt}} \leq 1 - \frac{1}{k} + \frac{2n-1}{\mathsf{opt}}$.

Let us consider undirected graphs. If $(k-1)n$ is even or if $\mathsf{opt} \geq \frac{kn}{2} + \frac{k}{2(k-1)}$, then $|I| \leq \frac{k-1}{k}\mathsf{opt}$, by Theorem 2.5. By Corollary 2.1, F is a forest, hence $|F| \leq n-1$. Consequently, $\frac{|I|+|F|}{\mathsf{opt}} \leq 1 - \frac{1}{k} + \frac{n-1}{\mathsf{opt}}$. If $(k-1)n$ is odd and $\mathsf{opt} < \frac{kn}{2} + 1$, then an optimal solution is k-regular and hence $|I| \leq \frac{(k-1)n+1}{2} \leq \left(1 - \frac{1}{k}\right)(\mathsf{opt} + 1)$. Combining we get $\frac{|I|+|F|}{\mathsf{opt}} \leq 1 - \frac{1}{k} + \frac{1-1/k}{\mathsf{opt}} + \frac{n-1}{\mathsf{opt}} < 1 - \frac{1}{k} + \frac{n}{\mathsf{opt}}$.

Before proving Theorem 2.5, let us give two additional applications of Theorem 2.5.

Max-Connectivity m-Subgraph: Here we seek a maximum connectivity k^* spanning subgraph G of $\hat{G}$ with at most m edges. Note that if we apply Algorithm 1 with k replaced by $k-1$, then from Theorem 2.5 we get:

$|I| \leq \frac{k-2}{k}\mathsf{opt}$ and $|F| \leq \frac{2}{k}\mathsf{opt}$ for directed graphs, and $|I| \leq \frac{k-2+1/n}{k}\mathsf{opt}$ and $|F| \leq \frac{2-2/n}{k}\mathsf{opt}$ for undirected graphs. Thus the algorithm returns a $(k-1)$-connected spanning subgraph G with at most opt edges. We can apply this algorithm to find the maximum integer k for which the algorithm returns a subgraph with at most m edges. Then $k \geq k^*-1$, hence we obtain a (k^*-1)-connected spanning subgraph with at most m edges. Note that this is tight, since the problem is NP-hard.

β-metric costs: Here for some $\frac{1}{2} \leq \beta < 1$ the costs satisfy the *β-triangle inequality* $c_{uv} \leq \beta(c_{uw}+c_{wv})$ for all $u,w,v \in V$. When $\beta = 1/2$ we have the min-size version of the problem. If we allow $\beta = 1$, then we get metric costs. In [5] it is proved that $c(F) \leq \frac{2\beta}{k(1-\beta)}\mathsf{opt}$. If $(k-1)n$ is even, or if there exists an optimal solution with at least $\frac{kn}{2}+\frac{k}{2(k-1)} \leq \frac{kn}{2}+1$ edges, then Theorem 2.5 gives the bound $c(I) \leq \left(1-\frac{1}{k}\right)\mathsf{opt}$; else, $c(I) \leq \left(1-\frac{1}{k}+\frac{1}{kn}\right)\mathsf{opt}$. Consequently, we get ratio $1-\frac{1}{k}+\frac{1}{kn}+\frac{2\beta}{k(1-\beta)}$ for this version of the problem.

In the rest of this section we prove Theorem 2.5. Let For $A \subseteq V$ let $\delta(A)$ denote the set of edges in E with exactly one endnode in A. Also, let $\zeta(A)$ denote the set of edges in E with at least one endnode in A. For $x \in \mathbb{R}^E$, the **ℓ-edge-cover polytope** $P_{\mathsf{cov}}(G,\ell)$ is defined by the constraints

$$\begin{array}{rcll} 0 \leq x_e & \leq & 1 & e \in E \\ x(\delta(v)) & \geq & \ell & v \in V \\ x(\zeta(A)) - x(F)) & \geq & (\ell|A|-|F|+1)/2 & A \subseteq V, F \subseteq \delta(A), \ell|A|-|F| \text{ odd} \end{array}$$

The **fractional ℓ-edge-cover polytope** $P^f_{\mathsf{cov}}(G,\ell)$ is defined by the first two sets of constraints. It is known that if G is bipartite then $P^{\mathsf{f}}_{\mathsf{cov}}(G,\ell) = P_{\mathsf{cov}}(G,\ell)$ (see [38], (31.7) on page 340). The **fractional k-edge-connectivity polytope** $P^{\mathsf{f}}_{\mathsf{con}}(G,k)$ is defined by the constraints

$$\begin{array}{rcll} 0 \leq x_e & \leq & 1 & e \in E \\ x(\delta(A)) & \geq & k & A \subseteq V \end{array}$$

We will prove the following statement that implies Theorem 2.5.

Lemma 2.5 *Let $G=(V,E)$ be an undirected graph with costs $\{c_e : e \in E\}$ and minimum degree $\geq k \geq 2$ and let $1 \leq \ell \leq k-1$. If G is bipartite and $x \in P^{\mathsf{f}}_{\mathsf{cov}}(G,k)$ then $\frac{\ell}{k}x \in P^{\mathsf{f}}_{\mathsf{cov}}(G,\ell)$. If G is k-edge-connected and $x \in P^{\mathsf{f}}_{\mathsf{con}}(G,k)$ then $\frac{\ell+1/n}{k}x \in P_{\mathsf{cov}}(G,\ell)$ and $\frac{\ell}{k}x \in P_{\mathsf{cov}}(G,\ell)$ if ℓn is even.*

Proof: Clearly, if $x \in P^{\mathsf{f}}_{\mathsf{cov}}(G,k)$ then $\frac{\ell}{k} \in P^{\mathsf{f}}_{\mathsf{cov}}(G,\ell)$. If G is bipartite then $P^{\mathsf{f}}_{\mathsf{cov}}(G,\ell) = P_{\mathsf{cov}}(G,\ell)$ and thus $\frac{\ell}{k}x \in P^{\mathsf{f}}_{\mathsf{cov}}(G,\ell)$. Assume that G is k-edge-connected and let $x \in P^{\mathsf{f}}_{\mathsf{con}}(G,k)$. We show that $\mu \cdot x \in P_{\mathsf{cov}}(G,\ell)$, where $\mu = \frac{\ell}{k}$ if ℓn is even and $\mu = \frac{\ell+1/n}{k}$ otherwise. Note that $\frac{\ell}{k}x$ satisfies the first two sets of constraints in the definition of $P_{\mathsf{cov}}(G,\ell)$. Thus we only need to prove that

$$\mu(x(\zeta(A)) - x(F)) \geq (\ell|A| - |F| + 1)/2 \qquad A \subseteq V, F \subseteq \delta(A), \ell|A| - |F| \text{ odd}$$

Note that $x(\zeta(A)) = \frac{1}{2}(\sum_{v \in A} x(\delta(v)) + x(\delta(A))) \geq \frac{1}{2}(k|A| + x(\delta(A))$. Substituting, multiplying by 2, and rearranging terms, we obtain that it is sufficient to prove that

$$|A|(\mu k - \ell) + (|F| - \mu x(F)) + \mu(x(\delta(A)) - x(F)) \geq 1 \qquad A \subseteq V, F \subseteq \delta(A), \ell|A| - |F| \text{ odd}$$

If $A = V$ then $F = \delta(A) = \emptyset$. Then the above condition is void if $\ell|V|$ is even, and it reduces to the condition $n(\mu k - \ell) \geq 1$ otherwise, which holds as equality for $\mu = \frac{\ell+1/n}{k}$.

Suppose that A is a proper subset of V. Then substituting $\mu = \frac{\ell}{k}$, multiplying both sides by k, and observing that $x(F) \leq |F|$, we obtain that it is sufficient to prove that

$$|F|(k-\ell) - \ell x(F) + \ell x(\delta(A)) \geq k \ .$$

If $|F| \geq \frac{k}{k-\ell}$ then this is so since $x(\delta(A)) \geq x(F)$, If $k \geq 2\ell$ then this is so since $|F| \geq x(F)$ and $x(\delta(A)) \geq k$. The remaining case is $|F| < \frac{k}{k-\ell}$ and $k < 2\ell$. Then since $|F| \geq x(F)$ and $x(\delta(A)) \geq k$

$$|F|(k-\ell) - \ell x(F) + \ell x(\delta(A)) \geq |F|(k-2\ell) + k\ell \geq \frac{k}{k-\ell}(k-2\ell) + k\ell = k - \frac{k\ell}{k-\ell} + k\ell \geq k \ .$$

This concludes the proof of the lemma. □

2.4 Biset functions

In Lemma 2.2 we formulated Mengers's Theorem in terms of setpairs. It would be more convenient to consider instead of a setpair (A,B) the pair of sets $(A, V \setminus B)$ called a "biset".

Definition 2.2 *An ordered pair $\mathbb{A} = (A, A^+)$ of subsets of V with $A \subseteq A^+$ is called a* **biset***; A is the* **inner part** *and A^+ is the* **outer part** *of $\mathbb{A}$, and $\partial\mathbb{A} = A^+ \setminus A$ is the* **boundary** *of $\mathbb{A}$. The* **co-set** *of $\mathbb{A}$ is*

$A^* = V \setminus A^+$; the **co-biset** *of $\mathbb{A}$ is $\mathbb{A}^* = (A^*, V \setminus A)$. We say that $\mathbb{A}$ is* **void** *if $A = \emptyset$,* **co-void** *if $A^+ = V$, and $\mathbb{A}$ is* **proper** *otherwise. Let $\mathcal{V}$ denote the family of bisets over V.*

A **biset function** assigns to every $\mathbb{A} \in \mathcal{V}$ a real number; in our context, it will always be an integer.

Definition 2.3 *An* **edge covers a biset** *$\mathbb{A}$ if it goes from A to A^*. For a biset $\mathbb{A}$ and an edge-set/graph J let $\delta_J(\mathbb{A})$ denote the set of edges in J covering $\mathbb{A}$ and let $d_J(\mathbb{A}) = |\delta_J(\mathbb{A})|$. The* **residual function** *of a biset function f w.r.t. a partial f-cover J is defined by $f^J(\mathbb{A}) = f(\mathbb{A}) - d_J(\mathbb{A})$ for all $\mathbb{A} \in \mathcal{V}$. We say that an edge set/graph J covers a biset function f, or that J is an f-**cover**, if $d_J(\mathbb{A}) \geq f(\mathbb{A})$ for all $\mathbb{A} \in \mathcal{V}$.*

We say that $\mathbb{A}$ is an st-**biset** if $s \in A$ and $t \in A^*$. In biset terms, Menger's Theorem (Lemma 2.2) is

$$\kappa_G(s,t) = \min\{|\partial \mathbb{A}| + d_G(\mathbb{A}) : \mathbb{A} \text{ is an } st\text{-biset}\} \ .$$

Thus $\kappa_G(s,t) \geq k$ iff $d_G(\mathbb{A}) \geq k - |\partial \mathbb{A}|$ for every st-biset $\mathbb{A}$. Consequently, G is k-connected iff G covers the k-**connectivity biset function** $f_{k\text{-}\mathsf{CS}}$ defined by

$$f_{k\text{-}\mathsf{CS}}(\mathbb{A}) = \begin{cases} k - |\partial \mathbb{A}| & \text{if } \mathbb{A} \text{ is proper} \\ 0 & \text{otherwise} \end{cases}$$

We thus will often consider the following generic problem:

<u>Biset-Function Edge-Cover</u>

Input: A graph $\hat{G} = (V, \hat{E})$ with edge costs $\{c_e : e \in \hat{E}\}$ and a biset function f on V.

Output: A minimum cost edge-set $E \subseteq \hat{E}$ that covers f.

Here f may not be given explicitly, and an efficient implementation of algorithms requires that certain queries related to f can be answered in time polynomial in n. We will not consider implementation details. In the applications discussed here, relevant polynomial time oracles are available via min-cut computations. In particular, we have a polynomial time separation oracle for the LP-relaxation due to Frank and Jordán [14]:

$$\textbf{(Biset-LP)} \qquad \begin{array}{rlll} \tau(f) = & \min & c \cdot x & \\ & \text{s.t.} & x(\delta_{\hat{E}}(\mathbb{A})) \geq f(\mathbb{A}) & \forall \mathbb{A} \in \mathcal{V} \\ & & 0 \leq x_e \leq 1 & \forall e \in E \end{array}$$

This LP is particularly useful if the biset function f has good uncrossing/supermodularity properties. To state these properties, we need to define the intersection and the union of bisets.

Definition 2.4 *The* **intersection** *and the* **union** *of two bisets* $\mathbb{A}, \mathbb{B}$ *are defined by* $\mathbb{A} \cap \mathbb{B} = (A \cap B, A^+ \cap B^+)$ *and* $\mathbb{A} \cup \mathbb{B} = (A \cup B, A^+ \cup B^+)$. *The biset* $\mathbb{A} \setminus \mathbb{B}$ *is defined by* $\mathbb{A} \setminus \mathbb{B} = (A \setminus B^+, A^+ \setminus B)$. *We say that* $\mathbb{B}$ **contains** $\mathbb{A}$ *and write* $\mathbb{A} \subseteq \mathbb{B}$ *if* $A \subseteq B$ *and* $A^+ \subseteq B^+$. *We say that* $\mathbb{A}, \mathbb{B}$ **intersect** *if* $A \cap B \neq \emptyset$, *and* **cross** *if* $A \cap B \neq \emptyset$ *and* $A^+ \cup B^+ \neq V$.

The following properties of bisets are easy to verify.

Fact 2.1 *For any bisets* $\mathbb{A}, \mathbb{B}$ *the following holds. If a directed/undirected edge* e *covers one of* $\mathbb{A} \cap \mathbb{B}, \mathbb{A} \cup \mathbb{B}$ *then* e *covers one of* $\mathbb{A}, \mathbb{B}$*; if* e *is an undirected edge, then if* e *covers one of* $\mathbb{A} \setminus \mathbb{B}, \mathbb{B} \setminus \mathbb{A}$, *then* e *covers one of* $\mathbb{A}, \mathbb{B}$. *Furthermore* $|\partial\mathbb{A}| + |\partial\mathbb{B}| = |\partial(\mathbb{A} \cap \mathbb{B})| + |\partial(\mathbb{A} \cup \mathbb{B})| = |\partial(\mathbb{A} \setminus B)| + |\partial(\mathbb{B} \setminus \mathbb{A})|$.

For a biset function f and bisets $\mathbb{A}, \mathbb{B}$ the **supermodular inequality** is

$$f(\mathbb{A} \cap \mathbb{B}) + f(\mathbb{A} \cup \mathbb{B}) \geq f(\mathbb{A}) + f(\mathbb{B}) \ .$$

We say that a biset function f is **supermodular** if the supermodular inequality holds for all $\mathbb{A}, \mathbb{B} \in \mathcal{V}$, and **modular** if the supermodular inequality holds as equality for all $\mathbb{A}, \mathbb{B} \in \mathcal{V}$. f is **symmetric** if $f(\mathbb{A}) = f(\mathbb{A}^*)$ for all $\mathbb{A} \in \mathcal{V}$. From Fact 2.1 one can deduce the following.

- For any (directed or undirected) graph G the function $-d_G(\cdot)$ is supermodular.
- The function $|\partial(\cdot)|$ is modular, and for any $R \subseteq V$ the function $|A \cap R|$ is modular.

Two additional important types of biset functions are given in the following definition.

Definition 2.5 *A biset functin* f *is* **intersecting/crossing supermodular** *if the supermodular inequality holds whenever* $\mathbb{A}, \mathbb{B}$ *intersect/cross.*

Biset-Function Edge-Cover with intersecting supermodular f admits a polynomial time algorithm that for directed graphs computes an f-cover of cost $\tau(f)$ [12]; for undirected graphs the cost is at most $2\tau(f)$, by the "bidirection" reduction from Lemma 2.1. Intersecting supermodular functions usually arise by zeroing a supermodular function on void bisets, while crossing supermodular functions arise by zeroing a supermodular function on non-proper bisets. For example, the k-connectivity function $f_{k\text{-CS}}$ is crossing supermodular, since it is obtained by zeroing the modular function $k - |\partial\mathbb{A}|$ on non-proper bisets.

2.5 Metric costs

For metric costs, Khuller and Raghavachari [24] obtained ratio $2+\frac{2(k-1)}{n}$ for undirected graphs. We survey a result of [26], which gives an improved ratio for undirected graphs, and a similar ratio for directed graphs.

Theorem 2.6 (Kortsarz & Nutov [26]) k-Connected Subgraph *with metric costs admits the following approximation ratios:* $(2+\frac{k-1}{n})$ *for undirected graphs and* $(2+\frac{k}{n})$ *for directed graphs.*

Let $R \subseteq V$ with $|R| \geq k$. A **k-fan from v to R** is a set of k distinct paths (possibly of length 0) from v to R such that any two of them have only the node v in common. Let us say that a graph G is **k-fan-connected to** R if G has a k-fan from any $v \in V$ to R. In a similar way we define a **k-fan from R to v** and say that G is **k-fan-connected from** R if the paths go from R to v. Note that the problem of finding a minimum cost spanning subgraph that is k-fan-connected to R is equivalent to the k-Inconnected Subgraph problem in the graph obtained from $\hat{G}$ by adding a new node s and edges of cost 0 from every $v \in R$ to s. Let the **k-fan-connectivity function** $f_{k,R}$ be obtained by zeroing the modular function $k-|\partial\mathbb{A}|-|A\cap R|$ on void bisets. Then $f_{k,R}$ is intersecting supermodular. By Menger's Theorem G is k-fan-connected to R iff $d_G(\mathbb{A}) \geq f_{k,R}(\mathbb{A})$ for all $\mathbb{A} \in \mathcal{V}$. Note that if $\mathbb{A}$ is co-void then $|\partial\mathbb{A}|+|A\cap R| \geq |R| \geq k$, hence $f_{k,R}(\mathbb{A}) \leq 0$. Summarizing, we have:

Lemma 2.6 *If* $|R| \geq k$ *then* $f_{k,R}$ *is intersecting supermodular and* $f_{k\text{-}\mathsf{CS}}(\mathbb{A})-|A\cap R| \leq f_{k,R}(\mathbb{A}) \leq f_{k\text{-}\mathsf{CS}}(\mathbb{A})$ *for all* $\mathbb{A} \in \mathcal{V}$*. Thus if an edge set* J *covers* $f_{k,R}$ *then* $A\cap R \neq \emptyset$ *whenever* $f^J_{k\text{-}\mathsf{CS}}(\mathbb{A})>0$.

An undirected edge set (or a graph) S is a **star** if all its edges are incident to the same node, called the **center** of the star. In the case of directed graphs, S should be either an **outstar** – all edges in S leave the center, or an **instar** – all edges in S enter the center. An **ℓ-star** is a star with ℓ leaves. For a star S let $\max(S)$ denote the maximum cost of an edge in S. The algorithm for undirected graphs is given in Algorithm 2.

Note that by Lemma 2.6 an edge set F as in step 3 exist. Let e_S denote the maximum cost edge of S, so $c(e_S)=\max(S)$. The approximation ratio follows from the following lemma, which shows that $c(F) \leq \frac{k-1}{n} \cdot \mathsf{opt}$.

Algorithm 2: Undirected Metric k-$\mathsf{Connected\ Subgraph}(\hat{G}, c, k)$

find a $(k-1)$-star (R, S) in $\hat{G}$ for which $c(S) + (k-2)\max(S)$ is minimal

find a 2-approximate k-fan-connected to R spanning subgraph (V, I) of $\hat{G}$

find an inclusionwise minimal edge set F on R such that $(V, I \cup F)$ is k-connected

return $I \cup F$

Lemma 2.7 (i) $c(F) \leq c(S) + (k-2)\max(S)$.

(ii) *There exists a $(k-1)$-star S such that* $c(S) + (k-2)\max(S) \leq \frac{k-1}{n} \cdot \mathsf{opt}$.

Proof: We prove (i). By Corollary 2.1 F is a forest. Let z be the center of S and for $v \in R$ denote $w_v = c_{zv}$ (where $w_z = 0$). Note that $w(R) + (k-2)\max_{v \in R} w_v = c(S) + (k-2)\max(S)$. Since the costs are metric, $c(F) \leq \sum_{uv \in F}(w_u + w_v) = \sum_{v \in R} d_F(v) w_v$. However, it is easy to see that for any non-negative weights $\{w_v : v \in R\}$, the maximum of $\sum_{v \in R} d_F(v) w_v$ is attained when F is a star on R centered at the maximum weight node. Thus $\sum_{v \in R} d_F(v) w_v \leq w(R) + (k-2)\max_{v \in R} w_v \leq c(S) + (k-2)\max(S)$.

We prove (ii). Let $p = \mathsf{opt}/n$. By an averaging argument, there exists a k-star S' of cost $c(S') \leq 2p$. Obtain a star S from S' by deleting the maximum cost edge of S'. Then $c(S) + c(e_S) \leq c(S') \leq 2p$ and $c(e_S) \leq c(S')/2 \leq p$. Consequently, $c(S) + (k-2)c(e_S) = c(S) + c(e_S) + (k-3)c(e_S) \leq (k-1)p$. $\square$

The algorithm for directed graphs is as follows.

Algorithm 3: Directed Metric k-$\mathsf{Connected\ Subgraph}(\hat{G}, c, k)$

find a $(k-1)$-outstar (R^{out}, S^{out}) and a $(k-1)$-instar (R^{in}, S^{in}) with common center such that $c(S^{out}) + c(S^{in}) + (k-1)(\max(S^{out}) + \max(S^{in}))$ is minimal

find a 2-approximate subgraph (V, I) that is k-fan-connected to R^{out} and k-fan-connected from R^{in}

find an inclusionwise minimal edge set F from R^{out} to R^{in} such that $(V, I \cup F)$ is k-connected

return $I \cup F$

The following directed counterpart of Lemmas 2.7 (the proof is omitted) implies that $c(F) \leq \frac{k}{n}\mathsf{opt}$.

Lemma 2.8 (i) $c(F) \leq c(S^{out}) + c(S^{in}) + (k-1)(\max(S^{out}) + \max(S^{in}))$.

(ii) *There exists a $(k-1)$-outstar (R^{out}, S^{out}) and a $(k-1)$-instar (R^{in}, S^{in}) with common center such that $c(S^{out}) + c(S^{in}) + (k-1)\left(\max(S^{out}) + \max(S^{in})\right) \leq \frac{k}{n} \cdot \mathsf{opt}$.*

2.6 Ratio 6 for undirected graphs with $n = \Omega\left(k^3\right)$

This section is devoted for the proof of the following result.

Theorem 2.7 (Cheriyan & Végh [4] ([18])) *Undirected* Biset-Function Edge-Cover *with symmetric crossing supermodular f admits ratio 6 provided that $n \geq \gamma(2\gamma+1)^2 + 2\gamma + 1$, where $\gamma = \gamma_f = \max\limits_{f(\mathbb{A})>0} |\partial\mathbb{A}|$.*

In the k-Connected Subgraph problem the function $f_{k\text{-CS}}$ has $\gamma = k-1$. This implies that undirected k-Connected Subgraph admits ratio 6 provided that $n \geq (k-1)(2k-1)^2 + 2k - 1$. Better bounds can be obtained under stronger assumptions on f, e.g., for $f = f_{k\text{-CS}}$ [18] gives the bound $n \geq k(k-1)(k-1.5) + k$.

In the rest of this section we prove Theorem 2.7. We need some definitions. Let us say that bisets $\mathbb{A}, \mathbb{B}$ **co-cross** if $\mathbb{A} \setminus \mathbb{B}$ and $\mathbb{B} \setminus \mathbb{A}$ are both non-void, and that $\mathbb{A}, \mathbb{B}$ **mesh** if they do not cross nor co-cross. One can verify that $\mathbb{A}, \mathbb{B}$ mesh iff one of the following holds: $A \subseteq \partial\mathbb{B}$, or $A^* \subseteq \partial\mathbb{B}$, or $B \subseteq \partial\mathbb{A}$, or $B^* \subseteq \partial\mathbb{A}$.

A biset function f is **positively intersecting supermodular** if the supermodular inequality holds whenever $\mathbb{A}, \mathbb{B}$ intersect and $f(\mathbb{A}) > 0$ and $f(\mathbb{B}) > 0$. f is **positively skew-supermodular** if the supermodular inequality or the **co-supermodular inequality** $f(\mathbb{A} \setminus \mathbb{B}) + f(\mathbb{B} \setminus \mathbb{A}) \geq f(\mathbb{A}) + f(\mathbb{B})$ holds whenever $f(\mathbb{A}) > 0$ and $f(\mathbb{B}) > 0$. Each of the the corresponding Biset-Function Edge-Cover problems, where f is positively intersecting supermodular, or when f is positively skew-supermodular, admits ratio 2 [16, 10].

The idea of the proof is to cover sequentially three functions dominated by f; one intersecting supermodular, the second positively intersecting supermodular, and the last positively skew-supermodular.

Lemma 2.9 *Let f be a symmetric crossing supermodular biset function. If $\mathbb{A}, \mathbb{B}$ are non-meshing bisets, then the supermodular or the co-supermodular inequality holds for $\mathbb{A}, \mathbb{B}$ and f.*

Proof: If $\mathbb{A}, \mathbb{B}$ cross then the supermodular inequality holds for $\mathbb{A}, \mathbb{B}$. Assume that $\mathbb{A}, \mathbb{B}$ co-cross. Then $\mathbb{A}$ and $\mathbb{B}^*$ cross, and thus the supermodular inequality holds for $\mathbb{A}, \mathbb{B}^*$ and f. Note that (i) $\mathbb{A} \setminus \mathbb{B} = \mathbb{A} \cap \mathbb{B}^*$; (ii) $\mathbb{A} \cup \mathbb{B}^*$ is the co-biset of $\mathbb{B} \setminus \mathbb{A}$, hence $f(\mathbb{A} \cup \mathbb{B}^*) = f(\mathbb{B} \setminus \mathbb{A})$, by the symmetry of f. Thus we get $f(\mathbb{A} \setminus \mathbb{B}) + f(\mathbb{B} \setminus \mathbb{A}) = f(\mathbb{A} \cap \mathbb{B}^*) + f(\mathbb{A} \cup \mathbb{B}^*) \geq f(\mathbb{A}) + f(\mathbb{B}^*) = f(\mathbb{A}) + f(\mathbb{B})$. □

From Lemma 2.9 we obtain a sufficient condition for a symmetric crossing supermodular biset function to be positively skew-supermodular.

Corollary 2.2 *Let f be a symmetric crossing supermodular biset function. If $f(\mathbb{C}) \leq 0$ holds for every biset $\mathbb{C}$ with $|C| \leq \gamma$ then f is positively skew-supermodular.*

Proof: Let $\mathbb{A}, \mathbb{B}$ be bisets with $f(\mathbb{A}) > 0$ and $f(\mathbb{B}) > 0$. We claim that $\mathbb{A}, \mathbb{B}$ do not mesh, and thus by Lemma 2.9 the supermodular or the co-supermodular inequality holds for $\mathbb{A}, \mathbb{B}$ and f, as required.

Suppose to the contrary that $\mathbb{A}, \mathbb{B}$ mesh. Then $A \subseteq \partial\mathbb{B}$, or $A^* \subseteq \partial\mathbb{B}$, or $B \subseteq \partial\mathbb{A}$, or $B^* \subseteq \partial\mathbb{A}$. If $A \subseteq \partial\mathbb{B}$ holds, then $|A| \leq |\partial\mathbb{B}| \leq \gamma$, and thus $f(\mathbb{A}) \leq 0$. If $A^* \subseteq \partial\mathbb{B}$ holds, then $|A^*| \leq |\partial\mathbb{B}| \leq \gamma$, and thus by the symmetry of f we get $f(\mathbb{A}) = f(\mathbb{A}^*) \leq 0$. In both case we obtain a contradiction to the assumption $f(\mathbb{A}) > 0$. The contradiction for the cases $B \subseteq \partial\mathbb{A}$, or $B^* \subseteq \partial\mathbb{A}$ is obtained in a similar way. □

In what follows, for a biset function f on V and $R \subseteq V$ let f_R be defined by

$$f_R(\mathbb{A}) = \begin{cases} f(\mathbb{A}) & \text{if } A \cap R = \emptyset \\ 0 & \text{otherwise} \end{cases}$$

Lemma 2.10 *If f is crossing supermodular and $|R| \geq 2\gamma+1$ then f_R is positively intersecting supermodular.*

Proof: Let $\mathbb{A}, \mathbb{B}$ be intersecting bisets with $f_R(\mathbb{A}), f_R(\mathbb{B}) > 0$. Then $A \cap R = B \cap R = \emptyset$, and thus $(A \cap B) \cap R = (A \cup B) \cap R = \emptyset$. Consequently, $f_R(\mathbb{A}) = f(\mathbb{A})$, $f_R(\mathbb{B}) = f(\mathbb{B})$, $f_R(\mathbb{A} \cap \mathbb{B}) = f(\mathbb{A} \cap \mathbb{B})$, and $f_R(\mathbb{A} \cup \mathbb{B}) = f(\mathbb{A} \cup \mathbb{B})$. Furthermore, $\mathbb{A}, \mathbb{B}$ cross, since $|R| \geq 2\gamma + 1$. Thus since f is crossing supermodular

$$f_R(\mathbb{A}) + f_R(\mathbb{B}) = f(\mathbb{A}) + f(\mathbb{B}) \leq f(\mathbb{A} \cap \mathbb{B}) + f(\mathbb{A} \cup \mathbb{B}) = f_R(\mathbb{A} \cap \mathbb{B}) + f_R(\mathbb{A} \cup \mathbb{B}) \ .$$

Consequently, the supermodular inequality holds for $\mathbb{A}, \mathbb{B}$ and f_R whenever $f_R(\mathbb{A}), f_R(\mathbb{B}) > 0$. □

For a biset function f and an integer p let us denote $U(f,p) = \bigcup\{A : f(\mathbb{A}) > 0, |A| \leq p\}$. Note that:

- $f(\mathbb{A}) \leq 0$ whenever $|A| \leq p$ and $A \setminus U(f,p) \neq \emptyset$.
- If I covers f_R then $f^I(\mathbb{A}) \leq 0$ whenever $A \cap R = \emptyset$, namely, $R \cap A \neq \emptyset$ whenever $f^I(\mathbb{A}) > 0$.

This implies that if $R \subseteq V \setminus U(f,p)$ and if I covers f_R then $f^I(\mathbb{A}) \leq 0$ whenever $|A| \leq p$. Combining with Lemma 2.10(ii) we get:

Corollary 2.3 *Let f be symmetric crossing supermodular, let $R \subseteq V \setminus U(f,\gamma)$, and let I be an f_R-cover. Then f^I is positively skew-supermodular.*

Corollary 2.3 gives a "cheap" method to "convert" a symmetric crossing supermodular function f into a positively skew-supermodular function: just find $R \subseteq V \setminus U(f,\gamma)$ with $|R| = 2\gamma + 1$ and compute a 2-approximate cover I of f_R – the residual function f^I of f will be positively crossing supermodular. The difficulty is that such R may not exist, e.g., if $f = f_{k\text{-CS}}$ then $U(f,\gamma) = V$. We thus find some "cheap" edge set I' such that $|U(g,\gamma)| \leq n - (2\gamma+1)$ will hold for the residual function $g = f^{I'}$. The next lemma shows that such I' can be a cover of the function f_S for arbitrary $S \subseteq V$, provided that n is not too small.

Lemma 2.11 *Let g be a symmetric crossing supermodular biset function and let $\mathcal{F} = \{\mathbb{A} : g(\mathbb{A}) > 0, |A| \leq p\}$. Let $S \subseteq V$ such that $S \cap A \neq \emptyset$ for all $\mathbb{A} \in \mathcal{F}$. Then $|U(g,p)| \leq (2\gamma+1)|S|p$.*

Proof: Let $\mathcal{F}' = \{\mathbb{A}_1, \ldots, \mathbb{A}_\ell\}$ be a minimum size subfamily of $\mathcal{F}$ with $\bigcup\limits_{\mathbb{A} \in \mathcal{F}} A = \bigcup\limits_{\mathbb{A} \in \mathcal{F}'} A$. It is sufficient to show that $|\mathcal{F}'| \leq (2\gamma+1)|S|$. By the minimality of $|\mathcal{F}'|$, for every $\mathbb{A}_i \in \mathcal{F}'$ there is $v_i \in A_i$ such that $v_i \notin A_j$ for every $j \neq i$. For every i, let $\mathbb{C}_i$ be an inclusionwise minimal member of the family $\{\mathbb{A} \in \mathcal{F} : \mathbb{A} \subseteq \mathbb{A}_i, v_i \in A\}$. By Corollary 2.2 $\mathcal{F}$ has the following property: $\mathbb{A} \setminus \mathbb{B} \in \mathcal{F}$ or $\mathbb{B} \setminus \mathbb{A} \in \mathcal{F}$ whenever none of $\mathbb{A} \setminus \mathbb{B}, \mathbb{B} \setminus \mathbb{A}$ is void. Thus the minimality of $\mathbb{C}_i$ implies that one of the following holds for any $i \neq j$:

- $v_i \in \partial\mathbb{C}_j$ or $v_j \in \partial\mathbb{C}_i$.
- $\mathbb{C}_i = \mathbb{C}_i \setminus \mathbb{C}_j$ or $\mathbb{C}_j = \mathbb{C}_j \setminus \mathbb{C}_i$.

Construct an auxiliary directed graph $\mathcal{J}$ on node set $\mathcal{C} = \{\mathbb{C}_1, \mathbb{C}_2, \ldots, \mathbb{C}_\ell\}$. Add an arc $\mathbb{C}_i\mathbb{C}_j$ if $v_i \in \partial\mathbb{C}_j$. The in-degree in $\mathcal{J}$ of a node $\mathbb{C}_i$ is at most $|\partial\mathbb{C}_i| \leq \gamma$. Thus every subgraph of the underlying graph of $\mathcal{J}$ has a node of degree $\leq 2\gamma$. A graph is d-degenerate if every subgraph of it has a node of degree $\leq d$. It is known that any d-degenerate graph is $(d+1)$-colorable. Hence $\mathcal{J}$ is $(2\gamma+1)$-colorable, so its node set can be partitioned into $2\gamma+1$ independent sets. The bisets in each independent set are inner part disjoint, hence their number is at most $|S|$. Consequently, $\ell \leq (2\gamma+1)|S|$, as claimed. □

Let I' be a cover of f_S where $|S| = 2\gamma+1$ and let $g = f^{I'}$ be the residual function of f w.r.t. I'. Note that $A \cap S \neq \emptyset$ whenever $g(\mathbb{A}) > 0$, thus Lemma 2.11 gives the bound $|U(g,\gamma)| \leq \gamma(2\gamma+1)^2$. Thus if $n \geq \gamma(2\gamma+1)^2 + 2\gamma + 1$ then there *exists* $R \subseteq V \setminus U(g,\gamma)$ with $|R| = 2\gamma+1$.

However, no polynomial time algorithm for *finding* R as above is known. This can be resolved as follows. The iterative rounding algorithm of [10], when applied on an *arbitrary* biset function h, either returns a 2-approximate cover J of h, or a **failure certificate**: a pair $\mathbb{A}, \mathbb{B}$ of bisets with $h(\mathbb{A}) > 0$ and $h(\mathbb{B}) > 0$ for which both the supermodular and the co-supermodular inequality does not hold. In the case when $h = g^I$, where g is symmetric supormodular and I is a cover of g_R with $|R| \geq 2\gamma + 1$, this can happen only if $\mathbb{A}, \mathbb{B}$ mesh (by Corollary 2.2) and $A \cap R, B \cap R$ are both non-empty (since $A \cap R \neq \emptyset$ whenever $g^I(\mathbb{A}) > 0$). Since g^I is symmetric, then by interchanging the roles of $\mathbb{A}, \mathbb{A}^*, \mathbb{B}, \mathbb{B}^*$, we can assume that our failure certificate $\mathbb{A}, \mathbb{B}$ satisfies $A \subseteq \partial\mathbb{B}$. Then $A \subseteq U(g, \gamma)$ and $A \cap R \neq \emptyset$, and we can apply the same procedure by choosing a different R from a smaller range set that excludes A. Formally, the algorithm is:

Algorithm 4: Crossing Supermodular Edge-Cover($\hat{G}, c, f$) (Assume $n \geq \gamma(2\gamma+1)^2 + 2\gamma + 1$)

```
choose S ⊆ V with |S| = 2γ + 1 and find a 2-approximate cover I' ⊆ Ê of f_S
g ← f^{I'}, Ê ← Ê \ I', U ← ∅, J ← NIL
while J = NIL do
    choose R ⊆ V \ U with |R| = 2γ + 1 and find a 2-approximate cover I ⊆ Ê of g_R
    apply the algorithm of [10] on g^I and Ê \ I
        - if the algorithm returns a failure certificate A, B with A ⊆ ∂B then do U ← U ∪ A
        - else, J ← a 2-approximate cover of g^I computed by the algorithm of [10]
return I' ∪ I ∪ J
```

Clearly, the algorithm computes a feasible solution. Ratio 6 follows from the fact that the solution is a union of 3 edge sets such that each of them is a 2-approximate cover of a function dominated by f.

2.7 Biset families and k-connectivity augmentation problems

Any $\{0,1\}$-valued biset function f bijectively corresponds to the **biset family** $\mathcal{F} = \{\mathbb{A} \in \mathcal{V} : f(\mathbb{A}) = 1\}$. We thus use for biset families the same terminology and notation as for biset functions, e.g., we say that J covers $\mathcal{F}$ if $d_J(\mathbb{A}) \geq 1$ for all $\mathbb{A} \in \mathcal{F}$, in the Biset-Family Edge-Cover problem we seek a minimum cost edge-set $E \subseteq \hat{E}$ that covers $\mathcal{F}$, and $\tau(\mathcal{F})$ is the optimal value of the Biset-LP for covering $\mathcal{F}$.

Let k-Connectivity Augmentation be the restriction of $(k+1)$-Connected Subgraph to instances when the input graph contains a k-connected spanning subgraph G of cost 0.

k-Connectivity Augmentation
Input: An integer k, a k-connected graph $G=(V,E)$, and an edge set $\hat{E}$ with costs $\{c_e : e \in \hat{E}\}$.
Output: A min-cost augmenting edge set $J \subseteq \hat{E}$ such that $G \cup J$ is $(k+1)$-connected.

Given an instance of k-Connectivity Augmentation let us say that $\mathbb{A} \in \mathcal{V}$ is a **tight biset** if $d_G(\mathbb{A})=0$, $|\partial\mathbb{A}|=k$, and $\mathbb{A}$ is proper. By Lemma 2.2, k-Connectivity Augmentation is equivalent to the Biset-Family Edge-Cover problem with $\mathcal{F}=\{\mathbb{A} \in \mathcal{V} : d_G(\mathbb{A})=0, |\partial\mathbb{A}|=k, \mathbb{A} \text{ is proper}\}$ being the family of tight bisets.

Suppose that k-Connectivity Augmentation admits ratio $\rho(k)$, and consider the following algorithm:

Algorithm 5: Sequential Augmentation$(\hat{G}, c, k)$

```
E ← ∅
for ℓ = 0 to ℓ = k − 1 do
    find a ρ(ℓ)-approximate solution J to ℓ-Connectivity Augmentation instance G = (V,E), Ê
    E ← E ∪ J, Ê ← Ê \ J
return E
```

Clearly, the algorithm computes a feasible solution for k-Connected Subgraph and has ratio $\sum_{\ell=0}^{k-1}\rho(\ell)$. One can show a better ratio if the ratio $\rho(\ell)$ is w.r.t. the Biset-LP. Let us say that Biset-Function Edge-Cover **admits LP-ratio** ρ if there exists a polynomial time algorithm that computes an f-cover of cost at most $\rho \cdot \tau(f)$. A similar terminology is used for k-Connected Subgraph and k-Connectivity Augmentation instances, meaning that the ratio ρ is w.r.t. the Biset-LP for these problems.

Lemma 2.12 *If k-Connectivity Augmentation admits LP-ratio $\rho(k)$ then k-Connected Subgraph admits LP-ratio* $\displaystyle\sum_{\ell=0}^{k-1}\frac{\rho(\ell)}{k-\ell}$; *thus if $\rho(\ell)$ is a non-decreasing function of ℓ, then the ratio is bounded by $\rho(k-1)H(k)$.*

Proof: At iteration ℓ of Algorithm 5, we cover the family $\mathcal{F}_\ell=\{\mathbb{A} \in \mathcal{V} : d_G(\mathbb{A})=0, |\partial\mathbb{A}|=\ell, \mathbb{A} \text{ is proper}\}$ by an edge set J_ℓ of cost $c(J_\ell) \leq \rho(\ell)\tau(\mathcal{F}_\ell)$. Let x be a feasible solution to the Biset-LP for $f_{k\text{-CS}}$. Then $x(\delta(\mathbb{A})) \geq k-|\partial\mathbb{A}|=k-\ell$ for every $\mathbb{A} \in \mathcal{F}$, hence $\frac{x}{k-\ell}$ is a feasible solution to the Biset-LP for covering $\mathcal{F}_\ell$. Thus $\tau(\mathcal{F}_\ell) \le \frac{1}{k-\ell}\tau(f_{k\text{-CS}})$. Consequently, $c(J_\ell) \le \rho(\ell)\tau(\mathcal{F}_\ell) \le \rho(\ell)\cdot\frac{1}{k-\ell}\tau(f_{k\text{-CS}})$, and thus $c(E) \le \sum_{\ell=0}^{\ell=k-1} c(J_\ell) \le \tau(f_{k\text{-CS}})\sum_{\ell=0}^{\ell=k-1}\frac{\rho(\ell)}{k-\ell}$, as claimed. □

We now establish some properties of tight bisets that will be used later.

Definition 2.6 *A biset family $\mathcal{F}$ is* **intersecting/crossing** *if $\mathbb{A}\cap\mathbb{B}, \mathbb{A}\cap\mathbb{B} \in \mathcal{F}$ whenever $\mathbb{A},\mathbb{B}$ intersect/cross. A crossing family is p-***crossing** *if whenever $\mathbb{A},\mathbb{B}\in\mathcal{F}$ intersect and $|A\cup B| \le n-p-1$ holds, $\mathbb{A}$ and $\mathbb{B}$ cross.*

For a biset family $\mathcal{F}$ the **co-family** of $\mathcal{F}$ is defined to be $\mathcal{F}^* = \{\mathbb{A}^* : \mathbb{A}\in\mathcal{F}\}$. We will use the following property of the family of tight bisets.

Lemma 2.13 *Let $\mathcal{F} = \{\mathbb{A}\in\mathcal{V} : d_G(\mathbb{A}) = 0, |\partial\mathbb{A}| = k, \mathbb{A} \text{ is proper}\}$ be the family of tight bisets of a k-connected (directed or undirected) graph G. Then $\mathcal{F}$ and $\mathcal{F}^*$ are both k-crossing.*

Proof: We will show that $\mathcal{F}$ is k-crossing; the proof for $\mathcal{F}^*$ is similar. Let $\mathbb{A},\mathbb{B}\in\mathcal{F}$. Note that by Fact 2.1 $d_G(\mathbb{A}\cap\mathbb{B}) = d_G(\mathbb{A}\cup\mathbb{B}) = 0$.

If $\mathbb{A},\mathbb{B}$ cross, then none of $\mathbb{A}\cap\mathbb{B}, \mathbb{A}\cup\mathbb{B}$ is void or co-void, thus $|\partial(\mathbb{A}\cap\mathbb{B})| \ge k$ and $|\partial(\mathbb{A}\cup\mathbb{B})| \ge k$, since G is k-connected. Thus by Fact 2.1 $k+k = |\partial\mathbb{A}|+|\partial\mathbb{B}| = |\partial(\mathbb{A}\cap\mathbb{B})|+|\partial(\mathbb{A}\cup\mathbb{B})| \ge k+k$. Hence $|\partial(\mathbb{A}\cap\mathbb{B})| = k$ and $|\partial(\mathbb{A}\cup\mathbb{B})| = k$, which implies $\mathbb{A}\cap\mathbb{B}, \mathbb{A}\cup\mathbb{B}\in\mathcal{F}$. Thus $\mathcal{F}$ is a crossing family.

Now let $\mathbb{A},\mathbb{B}$ intersect and satisfy $|A\cup B| \le n-k-1$. Note that $\mathbb{A}\cap\mathbb{B}$ is not co-void, and thus $|\partial(\mathbb{A}\cap\mathbb{B})| \ge k$, since G is k-connected. If $\mathbb{A},\mathbb{B}$ do not cross, then $|\partial(\mathbb{A}\cup\mathbb{B})| = n-|A\cup B| \ge k+1$, and thus $|\partial(\mathbb{A}\cap\mathbb{B})| = |\partial\mathbb{A}|+|\partial\mathbb{B}|-|\partial(\mathbb{A}\cup\mathbb{B})| \le k-1$, contradicting that G is k-connected. □

2.8 LP-ratio $O\left(\ln\frac{n}{n-k}\right)$ for k-Connectivity Augmentation

Let q be a parameter eventually set to $q = \lceil\frac{n-k}{2}\rceil$ and let $\mu = \left\lfloor\frac{n}{q+1}\right\rfloor \le \frac{2n}{n-k}$. We survey the following result:

Theorem 2.8 (Nutov [37]) *Directed k-Connectivity Augmentation admits LP-ratio $H(\mu)+2$.*

Note that combining Theorem 2.8 with Lemma 2.12 gives ratio $H(k)(H(\mu)+2) = O\left(\ln k\cdot\ln\frac{n}{n-k}\right)$ for directed k-Connected Subgraph, and via Lemma 2.1 also for undirected k-Connected Subgraph.

Recall that the k-Connectivity Augmentation problem is equivalent to covering the family $\mathcal{F} = \{\mathbb{A} : d_G(\mathbb{A}) = 0, |\partial\mathbb{A}| = k, \mathbb{A} \text{ is proper}\}$ of tight bisets, and that by Lemma 2.13 both $\mathcal{F}$ and $\mathcal{F}^*$ are k-crossing. In the rest of this section we will prove the following general theorem, that implies Theorem 2.8.

Theorem 2.9 (Nutov [37]) Biset-Family Edge-Cover *such that $\mathcal{F}$ and $\mathcal{F}^*$ are both k-crossing admits LP-ratio $O(\ln\mu)$. Moreover, if $|\partial\mathbb{A}| \geq k$ for all $\mathbb{A} \in \mathcal{F}$ then the problem admits LP-ratio $H(\mu)+2$.*

Since any crossing family $\mathcal{F}$ is 2γ-crossing, where $\gamma = \max\limits_{\mathbb{A}\in\mathcal{F}} |\partial\mathbb{A}|$, we get that Biset-Family Edge-Cover with arbitrary crossing $\mathcal{F}$ admits LP-ratio $O\left(\ln\frac{n}{n-2\gamma}\right)$.

In the rest of this section we prove Theorem 2.9. We need the following definition from [27]:

Definition 2.7 *The inclusionwise minimal members of a biset family $\mathcal{F}$ are called $\mathcal{F}$-**cores**, or simply **cores**, if $\mathcal{F}$ is clear from the context. Let $\mathcal{C}(\mathcal{F})$ denote the family of $\mathcal{F}$-cores. For an $\mathcal{F}$-core $\mathbb{C} \in \mathcal{C}(\mathcal{F})$, the **halo-family** $\mathcal{F}(\mathbb{C})$ of $\mathbb{C}$ is the family of those members of $\mathcal{F}$ that contain $\mathbb{C}$ and contain no $\mathcal{F}$-core distinct from $\mathbb{C}$.*

The following lemma summarizes the properties of halo-families that we need.

Lemma 2.14 *For any crossing biset family $\mathcal{F}$ the following holds.*

(i) *For any $\mathcal{F}$-core $\mathbb{C}$, $\mathcal{F}(\mathbb{C})$ is a crossing family and $\mathcal{F}(\mathbb{C})^* = \{\mathbb{A}^* : \mathbb{A} \in \mathcal{F}(\mathbb{C})\}$ is an intersecting family.*

(ii) *Bisets $\mathbb{A}_1, \mathbb{A}_2$ that belong to distinct halo-families do not cross; thus no edge can cover both $\mathbb{A}_1$ and $\mathbb{A}_2$.*

(iii) *For any $\mathcal{F}$-core $\mathbb{C}$, if J is an inclusion minimal edge set that covers $\mathcal{F}(\mathbb{C})$ then $\mathcal{C}(\mathcal{F}^J) = \mathcal{C}(\mathcal{F}) \setminus \{\mathbb{C}\}$.*

Proof: We prove (i). Let $\mathbb{A}, \mathbb{B} \in \mathcal{F}(\mathbb{C})$ cross. Then $\mathbb{A}\cap\mathbb{B}, \mathbb{A}\cup\mathbb{B} \in \mathcal{F}$. Since $\mathbb{A}\cap\mathbb{B} \subseteq \mathbb{A} \subseteq \mathbb{A}\cup\mathbb{B}$ and $\mathbb{A} \in \mathcal{F}(\mathbb{C})$, $\mathbb{A}\cap\mathbb{B} \in \mathcal{F}(\mathbb{C})$ and $\mathbb{C} \subseteq \mathbb{A}\cup\mathbb{B}$. We claim that $\mathbb{A}\cup\mathbb{B}$ contains no core $\mathbb{C}'$ distinct from $\mathbb{C}$. Otherwise, since none of $\mathbb{A}, \mathbb{B}$ can contain $\mathbb{C}'$, we must have that $\mathbb{C}', \mathbb{A}$ cross or $\mathbb{C}', \mathbb{B}$ cross, so $\mathbb{C}' \cap \mathbb{A} \in \mathcal{F}$ or $\mathbb{C}' \cap \mathbb{B} \in \mathcal{F}$; this contradicts that $\mathbb{C}'$ is a core. Thus $\mathcal{F}(\mathbb{C})$ is a crossing family. We prove that $\mathcal{F}(\mathbb{C})^*$ is an intersecting family. Let $\mathbb{A}, \mathbb{B} \in \mathcal{F}(\mathbb{C})^*$ intersect. Then $\mathbb{C} \subseteq \mathbb{A}^* \cap \mathbb{B}^*$ so $\mathbb{A}, \mathbb{B}$ cross. Thus since $\mathcal{F}(\mathbb{C})$ is a crossing family, we get that $\mathbb{A}\cap\mathbb{B}, \mathbb{A}\cup\mathbb{B} \in \mathcal{F}(\mathbb{C})^*$.

We prove (ii). Let $\mathbb{A}_1 \in \mathcal{F}(\mathbb{C}_1)$ and $\mathbb{A}_2 \in \mathcal{F}(\mathbb{C}_2)$ cross, for $\mathbb{C}_1, \mathbb{C}_2 \in \mathcal{C}(\mathcal{F})$. Then $\mathbb{A}_1 \cap \mathbb{A}_2 \in \mathcal{F}$, so $\mathbb{A}_1 \cap \mathbb{A}_2$ contains some $\mathcal{F}$-core $\mathbb{C}$. We have $\mathbb{C} = \mathbb{C}_1$ since $\mathbb{C} \subseteq \mathbb{A}_1$ and $\mathbb{C} = \mathbb{C}_2$ since $\mathbb{C} \subseteq \mathbb{A}_2$, hence $\mathbb{C}_1 = \mathbb{C}_2$.

Part (iii) follows from part (ii), since every $e \in J$ covers some biset in $\mathcal{F}(\mathbb{C})$ (by the minimality of J) and thus by (ii) cannot cover a core distinct from $\mathbb{C}$. $\square$

We need the following theorem for the proof of Theorem 2.9.

Theorem 2.10 (Fakcharoenphol & Laekhanukit [9] ([2, 37])) *Directed* Biset-Family Edge-Cover *with crossing $\mathcal{F}$ admits a polynomial time algorithm that given $\mathcal{C} \subseteq \mathcal{C}(\mathcal{F})$ computes $J \subseteq \hat{E}$ such that $\mathcal{C}(\mathcal{F}^J) = \mathcal{C}(\mathcal{F}) \setminus \mathcal{C}$ and $c(J) \leq H(|\mathcal{C}|) \cdot \tau(\mathcal{F})$. In particular,* Biset-Family Edge-Cover *with crossing $\mathcal{F}$ admits ratio $H(|\mathcal{C}(\mathcal{F})|)$.*

Proof: Consider the following algorithm. Start with a partial solution $J = \emptyset$. While $|\mathcal{C} \cap \mathcal{C}(\mathcal{F}^J)| \geq 1$ continue with iterations. At iteration i, compute for each $\mathbb{C} \in \mathcal{C} \cap \mathcal{C}(\mathcal{F}^J)$ an optimal inclusion minimal edge cover $J_{\mathbb{C}}$ of the family $\mathcal{F}^J(\mathbb{C})$ (the halo-family of $\mathbb{C}$ in $\mathcal{F}^J$); then add to J a minimum cost edge set J_i among the edge sets $\left\{J_{\mathbb{C}} : \mathbb{C} \in \mathcal{C} \cap \mathcal{C}\left(\mathcal{F}^J\right)\right\}$. By part (i) of Lemma 2.14, each $J_{\mathbb{C}}$ can be computed in polynomial time and $c(J_{\mathbb{C}}) = \tau(\mathcal{F}^J(\mathbb{C}))$. By part (ii), $\sum_{\mathbb{C} \in \mathcal{C}(\mathcal{F})} c(J_{\mathbb{C}}) \leq \tau(\mathcal{F})$. Thus there is $\mathbb{C} \in \mathcal{C}$ such that $c(J_{\mathbb{C}}) \leq \tau(\mathcal{F})/|\mathcal{C}|$. By part (iii), at iteration i we have $|\mathcal{C}(\mathcal{F}^J) \cap \mathcal{C}| \leq |\mathcal{C}| - i + 1$. Thus $c(J_i) \leq \tau(\mathcal{F})/(|\mathcal{C}| - i + 1)$ at iteration i, and the statement follows. $\square$

Definition 2.8 *A biset family $\mathcal{F}$ is* **intersection closed** *if $\mathbb{A} \cap \mathbb{B} \in \mathcal{F}$ for any intersecting $\mathbb{A}, \mathbb{B} \in \mathcal{F}$. An intersection closed $\mathcal{F}$ is* q**-semi-intersecting** *if $|A| \leq q$ for every $\mathbb{A} \in \mathcal{F}$, and if $\mathbb{A} \cup \mathbb{B} \in \mathcal{F}$ for any intersecting $\mathbb{A}, \mathbb{B} \in \mathcal{F}$ with $|A \cup B| \leq q$. The* q**-truncated family** *of a biset family $\mathcal{F}$ is $\mathcal{F}_{\leq q} := \{\mathbb{A} \in \mathcal{F} : |A| \leq q\}$.*

We obtain a q-semi-intersecting family from a k-crossing family as follows.

Lemma 2.15 *If $\mathcal{F}$ is k-crossing and $q \leq \frac{n-k}{2}$ then the q-truncated family $\mathcal{F}_{\leq q}$ of $\mathcal{F}$ is q-semi-intersecting.*

Proof: Let $\mathbb{A}, \mathbb{B} \in \mathcal{F}_{\leq q}$ intersect. Then $|A \cup B| \leq |A| + |B| - 1 \leq 2q - 1 \leq n - k - 1$. Thus $\mathbb{A} \cap \mathbb{B}, \mathbb{A} \cup \mathbb{B} \in \mathcal{F}$, since $\mathcal{F}$ is k-crossing. Hence $\mathbb{A} \cap \mathbb{B} \in \mathcal{F}_{\leq q}$ (since $|A \cap B| \leq |A| \leq q$), and $\mathbb{A} \cup \mathbb{B} \in \mathcal{F}_{\leq q}$ if $|A \cup B| \leq q$. $\square$

Let $\nu(\mathcal{F})$ denote the maximum number of pairwise inner part disjoint bisets in $\mathcal{F}$.

Theorem 2.11 (Nutov [37]) *Directed* Biset-Family Edge-Cover *with q-semi-intersecting $\mathcal{F}$ admits a polynomial time algorithm that computes an edge-set $J \subseteq \hat{E}$ such that $\nu(\mathcal{F}^J) \leq \mu$ and $c(J) \leq \tau(\mathcal{F})$.*

Theorem 2.11 will be proved later. From Theorem 2.11 and Lemma 2.15 we have the following.

Corollary 2.4 *Directed* Biset-Family Edge-Cover *with k-crossing $\mathcal{F}$ admits a polynomial time algorithm that for $q=\lceil\frac{n-k}{2}\rceil$ computes $J \subseteq E$ such that $\nu(\mathcal{F}^J_{\leq q}) \leq \mu$ and $c(J) \leq \tau(\mathcal{F}_{\leq q})$.*

A slightly weaker version of the following statement was proved in [35].

Lemma 2.16 ([35]) *If both $\mathcal{F}$ and $\mathcal{F}^*$ are k-crossing then $|\mathcal{C}(\mathcal{F})| \leq \nu(\mathcal{F}_{\leq q})+\nu(\mathcal{F}^*_{\leq q})+\mu^2 H(\mu)$.*

Proof: We show that if $\mathcal{C} \subseteq \mathcal{F}$ has no two bisets that cross then $|\mathcal{C}| \leq \nu(\mathcal{F}_{\leq q})+\nu(\mathcal{F}^*_{\leq q})+\mu^2 H(\mu)$. Let $\mathcal{B}=\{\mathbb{A}\in\mathcal{C} : |A|,|A^*| \geq q+1\}$. Clearly, $|\mathcal{C}| \leq |\mathcal{C}_{\leq q}|+|\mathcal{C}^*_{\leq q}|+|\mathcal{B}|$. Note that $|\mathcal{C}_{\leq q}| \leq \nu(\mathcal{F}_{\leq q})$, since $\mathcal{F}_{\leq q}$ is intersection closed, by Lemma 2.15. Similarly, $|\mathcal{C}^*_{\leq q}| \leq \nu(\mathcal{F}^*_{\leq q})$. To see that $|\mathcal{B}| \leq \mu^2 H(\mu)$, note that:

(i) $\Delta(\mathcal{B}) \leq \mu$, where $\Delta(\mathcal{B})$ is the maximum degree in the hypergraph $\mathcal{B}^{in}$ formed by the inner parts of the bisets in $\mathcal{B}$. This is so since no two bisets in $\mathcal{B}$ cross, and thus for any $v \in V$ the sets in the family $\{A^* : \mathbb{A}\in\mathcal{B}, v\in A\}$ of $\mathcal{B}^*$ are pairwise disjoint; hence their number is at most $\nu(\mathcal{B}^*) \leq \left\lfloor\frac{n}{q+1}\right\rfloor=\mu$.

(ii) The hypergraph $\mathcal{B}^{in}$ has a hitting-set U of size $|U| \leq \mu H(\Delta(\mathcal{B})) \leq \mu H(\mu)$. This is so since this hypergraph has a fractional hitting-set h of value μ defined by $h(v)=\frac{1}{q+1}$ for all $v\in V$.

Since $|\mathcal{B}| \leq |U|\cdot\Delta(\mathcal{B})$ for any hitting-set U of $\mathcal{B}^{in}$, the bound $|\mathcal{B}| \leq \mu^2 H(\mu)$ follows. □

The algorithm is as follows.

Algorithm 6: DIRECTED-COVER1$(\mathcal{F},\hat{G},c)$ ($\mathcal{F},\mathcal{F}^*$ are both k-crossing)

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1 **compute** $J_1 \subseteq E$ with $\nu(\mathcal{F}^{J_1}_{\leq q}) \leq \mu$ and $c(J_1) \leq \tau(\mathcal{F}_{\leq q})$ using the algorithm from **Corollary 2.4**;
compute a similar edge set $J^*_1 \subseteq E$ for the family $\mathcal{F}^*_{\leq q}$.

2 **compute** $J_2 \subseteq E$ covering $\mathcal{F}^{J_1\cup J^*_1}$ using the algorithm from **Theorem 2.10**.

3 **return** $J=J_1\cup J^*_1\cup J_2$.

By Lemma 2.16, $\left|\mathcal{C}\left(\mathcal{F}^{J_1\cup J^*_1}\right)\right|=O(\mu^2\ln\mu)$ and thus $c(J_2)=\tau(\mathcal{F})O(\ln\mu)$. Consequently, the cost of the solution computed is bounded by $\tau(\mathcal{F})(c(J_1)+c(J^*_1)+c(J_2)) \leq \tau(\mathcal{F})(1+1+O(\ln\mu))=O(\ln\mu)$.

In the case when $|\partial\mathbb{A}| \geq k$ for all $\mathbb{A}\in\mathcal{F}$ we get a slightly better ratio $H(\mu)+2$ by the following observation.

Lemma 2.17 *Let $\mathcal{F}$ be a biset family such that $\mathcal{F}^*$ is k-crossing and $|\partial\mathbb{A}| \geq k$ for all $\mathbb{A} \in \mathcal{F}$. Let $q = \lceil \frac{n-k}{2} \rceil$. If $\mathcal{F}_{\leq q} = \emptyset$ then $\mathcal{F}^*$ is an intersecting family;*

Proof: If $|A \cup B| \leq n-k-1$ then $\mathbb{A} \cap \mathbb{B}, \mathbb{A} \cup \mathbb{B} \in \mathcal{F}^*$, since $\mathcal{F}^*$ is k-regular. Assume that $|A \cup B| \geq n-k$. Then $\max\{|A|,|B|\} \geq \frac{n-k}{2}$, say $|A| \geq \frac{n-k}{2}$. This implies $|A^*| \leq n-k-|A| \leq \frac{n-k}{2}$, so $\mathbb{A}^* \in \mathcal{F}_{\leq q}$, contradicting that $\mathcal{F}_{\leq q} = \emptyset$. □

Relying on Corollary 2.4, Theorem 2.10, and Lemma 2.17, it is easy to see that Algorithm 7 computes a feasible solution of cost at most $\tau(\mathcal{F})(H(\mu)+2)$.

Algorithm 7: DIRECTED-COVER2($\mathcal{F}, \hat{G}, c$) ($\mathcal{F}$ and $\mathcal{F}^*$ are both k-crossing, $|\partial\mathbb{A}| \geq k$ for all $\mathbb{A} \in \mathcal{F}$)

---

compute $J_1 \subseteq \hat{E}$ with $\nu(\mathcal{F}^{J_1}_{\leq q}) \leq \mu$ and $c(J_1) \leq \tau(\mathcal{F}_{\leq q})$ using the algorithm from **Corollary 2.4**.

compute $J_2 \subseteq \hat{E}$ covering $\mathcal{F}^{J_1}_{\leq q}$ with $c(J_2) \leq H(\mu)\tau\left(\mathcal{F}^{J_1}\right)$ using the algorithm from **Theorem 2.10**.

compute $J_3 \subseteq \hat{E}$ covering $\mathcal{F}^{J_1 \cup J_2}$ with $c(J_3) \leq \tau\left(\mathcal{F}^{J_1 \cup J_2}\right)$ using the algorithm from **Lemma 2.17**.

return $J_1 \cup J_2 \cup J_3$.

In the rest of this section we prove Theorem 2.11. Consider the dual program of the Biset-LP for covering $\mathcal{F}$ and the following primal-dual algorithm for covering $\mathcal{F}$.

$$\max\left\{\sum_{\mathbb{A} \in \mathcal{V}} y_{\mathbb{A}} : \sum_{\delta(\mathbb{A}) \ni e} y_{\mathbb{A}} \leq c_e \ \forall e \in E,\ y_{\mathbb{A}} \geq 0 \ \forall \mathbb{A} \in \mathcal{V}\right\} .$$

Algorithm 8: q-SEMI-INTERSECTING FAMILY EDGE-COVER($\mathcal{F}, G, c$)

$J \leftarrow \emptyset$, $y \leftarrow 0$, $\mathcal{L} \leftarrow \emptyset$.

while $\nu(\mathcal{F}^J) \geq 1$ **do**

  add some $\mathbb{C} \in \mathcal{C}(\mathcal{F}^J)$ to $\mathcal{L}$

  raise $y_{\mathbb{C}}$ until the dual constraint of some $e \in \delta_{\hat{E} \setminus J}(\mathbb{C})$ becomes tight and add e to J

Let $e_1, \ldots, e_j$ be the order in which the edges were added to J

for $i = j$ **downto** 1 **do**

  if $J \setminus \{e_i\}$ covers the family $\mathcal{F}' = \{\mathbb{A} \in \mathcal{F} : \mathbb{A} \subseteq \mathbb{B} \text{ for some } \mathbb{B} \in \mathcal{L}\}$ then do $J \leftarrow J \setminus \{e_i\}$

return J

Let I denote the set of edges in J right before the reverse-delete phase (steps 5,6,7). Note that I covers $\mathcal{F}$, but in the reverse-delete phase we care to cover just the subfamily $\mathcal{F}'$ of $\mathcal{F}$. In fact, the algorithm coincides with a standard primal-dual algorithm for covering the biset family $\mathcal{F}'$. We will show that $\mathcal{F}'$ is an intersecting biset family and conclude that $c(J) = \tau(\mathcal{F}') \leq \tau(\mathcal{F})$. In what follows, let $\mathcal{M}$ denote the family of inclusionwise maximal members of $\mathcal{L}$, and for an $\mathcal{F}^J$-core $\mathbb{C}_i$ let $\mathcal{M}_i$ denote the family of bisets in $\mathcal{M}$ that intersect with $\mathbb{C}_i$, and $\mathbb{B}_i$ the union of $\mathbb{C}_i$ and the bisets in $\mathcal{M}_i$.

Note that each family $\mathcal{M}_i$ is non-empty, since $\mathbb{C}_i$ is covered by some edge $e \in I \setminus J$, and since any edge $e \in I$ covers some $\mathbb{A} \in \mathcal{L}$. Let us say that a biset family $\mathcal{L}$ is **laminar** if for any $\mathbb{A}, \mathbb{B} \in \mathcal{L}$ that intersect $\mathbb{A} \subseteq \mathbb{B}$ or $\mathbb{B} \subseteq \mathbb{A}$ holds. In the following lemma we establish some properties of the families $\mathcal{L}$ and $\mathcal{F}'$.

Lemma 2.18 *At the end of the algorithm the following holds:*

(i) *$\mathcal{L}$ is a laminar biset family and $\mathcal{F}'$ is an intersecting biset family.*

(ii) *For any $\mathbb{A} \in \mathcal{M}$ there is a unique edge $e_{\mathbb{A}}$ in I that covers $\mathbb{A}$, and $e_{\mathbb{A}} \in J$. Furthermore, if $\mathbb{A}$ and an $\mathcal{F}^J$-core $\mathbb{C}$ intersect, then $\delta_J(\mathbb{A} \cap \mathbb{C}) = \{e_{\mathbb{A}}\}$.*

Proof: We prove (i). Let $\mathbb{A}_1, \mathbb{A}_2 \in \mathcal{L}$ intersect where $\mathbb{A}_1$ was added to $\mathcal{L}$ before $\mathbb{A}_2$. When $\mathbb{A}_1$ was added to $\mathcal{L}$, we had $\mathbb{A}_1 \in \mathcal{C}(\mathcal{F}^J)$ and $\mathbb{A}_2 \in \mathcal{F}^J$. Thus $\mathbb{A}_1 \cap \mathbb{A}_2 = \mathbb{A}_1$ (namely, $\mathbb{A}_1 \subseteq \mathbb{A}_2$) by the minimality of $\mathbb{A}_1$ and since $\mathcal{F}$ (and thus also $\mathcal{F}^J$) is intersection closed. This implies that $\mathcal{L}$ is laminar. We show that $\mathcal{F}'$ is an intersecting biset family. Let $\mathbb{A}_1, \mathbb{A}_2 \in \mathcal{F}'$ intersect. Then, since $\mathcal{L}$ is laminar, $\mathbb{A}_1 \cup \mathbb{A}_2 \subseteq \mathbb{B}$ for some $\mathbb{B} \in \mathcal{L}$. Thus $\mathbb{A}_1 \cup \mathbb{A}_2 \in \mathcal{F}$, since $|A_1 \cup A_2| \leq |B| \leq q$ and since $\mathcal{F}$ is q-semi-intersecting. This implies $\mathbb{A}_1 \cup \mathbb{A}_2 \in \mathcal{F}'$, and clearly $\mathbb{A}_1 \cap \mathbb{A}_2 \in \mathcal{F}'$ since $\mathbb{A}_1 \cap \mathbb{A}_2 \subseteq \mathbb{B}$ and since $\mathcal{F}$ is intersection closed.

We prove (ii). Let $e_{\mathbb{A}}$ be the edge that was added to J at step 4 of the algorithm after $\mathbb{A}$ was added to $\mathcal{L}$ at step 3 (the first edge that covered $\mathbb{A}$). After $\mathbb{A}$ was added to $\mathcal{L}$, no biset that intersects with $\mathbb{A}$ was added to $\mathcal{L}$, since $\mathbb{A} \in \mathcal{M}$ and since $\mathcal{L}$ is laminar. Thus edges added to J after $e_{\mathbb{A}}$ do not cover $\mathbb{A}$, since their tails are in $V \setminus A$. Consequently, $e_{\mathbb{A}}$ is the unique edge in I that covers $\mathbb{A}$, and thus $e_{\mathbb{A}} \in J$. Now suppose that $\mathbb{A}$ and an $\mathcal{F}^J$-core $\mathbb{C}$ intersect. Then $\mathbb{A} \cap \mathbb{C} \in \mathcal{F}'$, since $\mathcal{F}$ is intersection closed and since $\mathbb{A} \cap \mathbb{C} \subseteq \mathbb{A}$. Thus $\delta_J(\mathbb{A} \cap \mathbb{C}) \neq \emptyset$. Let $e \in \delta_J(\mathbb{A} \cap \mathbb{C})$. Then e covers $\mathbb{A}$, since e covers $\mathbb{A}$ or $\mathbb{C}$ by Fact 2.1, but e cannot cover C since $e \in J$ and J does not cover $\mathbb{C}$. Thus $e = e_{\mathbb{A}}$ for any $e \in \delta_J(\mathbb{A} \cap \mathbb{C})$, namely, $\delta_J(\mathbb{A} \cap \mathbb{C}) = \{e_{\mathbb{A}}\}$. $\square$

Lemma 2.19 *If $\mathcal{F}$ is q-semi-intersecting then at the end of the algorithm the following holds:*

(i) $|\delta_J(\mathbb{A})| = 1$ *for any* $\mathbb{A} \in \mathcal{L}$.

(ii) *The sets B_i are pairwise disjoint and each of them has size $\geq q+1$.*

Proof: For part (i), let $\mathbb{A} \in \mathcal{L}$ and suppose to the contrary that there are $e_1, e_2 \in \delta_J(\mathbb{A})$ with $e_1 \neq e_2$. For $i = 1, 2$ let $\mathbb{A}_i$ be some biset in $\mathcal{F}'$ that became uncovered when e_i was considered for deletion at step 7. Note that $\delta_J(\mathbb{A}_i) = \{e_i\}$ and that $\mathbb{A} \subseteq \mathbb{A}_i$, since the edges in J were considered for deletion in the reverse order. Thus $\mathbb{A} \subseteq \mathbb{A}_1 \cap \mathbb{A}_2$, and by Lemma 2.18(i) $\mathbb{A}_1 \cup \mathbb{A}_2 \in \mathcal{F}'$. Consequently, there is $e \in \delta_J(\mathbb{A}_1 \cup \mathbb{A}_2)$, hence $e \in \delta_J(\mathbb{A}_1)$ or $e \in \delta_J(\mathbb{A}_2)$, by Fact 2.1. Thus $e = e_1$ or $e = e_2$. Since the tail of each of e_1, e_2 is in $A \subseteq A_1 \cap A_2$, so is the tail of e. The head of e is in $A_1^* \cap A_2^*$. This gives the contradiction $e \in \delta_J(\mathbb{A}_1) \cap \delta_J(\mathbb{A}_2)$.

We prove part (ii). Let $\mathbb{C}_i, \mathbb{C}_j$ be distinct $\mathcal{F}^J$-cores. Note that no two bisets in $\mathcal{M}$ intersect (since $\mathcal{L}$ is laminar) and that $C_i \cap C_j = \emptyset$ (since $\mathcal{F}$ is intersection closed). Thus to prove that $B_i \cap B_j = \emptyset$ it is sufficient to prove that $\mathcal{M}_i \cap \mathcal{M}_j = \emptyset$. Suppose to the contrary that there is $\mathbb{A} \in \mathcal{M}_i \cap \mathcal{M}_j$. By Lemma 2.18(ii), the tail of $e_{\mathbb{A}}$ is both in $A \cap C_i$ and $A \cap C_j$. This contradicts $C_i \cap C_j = \emptyset$. We prove that $|B_i| \geq q+1$. Note that $|B_i| \leq q$ implies $\mathbb{B}_i \in \mathcal{F}$, since $\mathcal{F}$ is q-semi-intersecting. Thus to prove that $|B_i| \geq q+1$ it is sufficient to prove that $\delta_I(\mathbb{B}_i) = \emptyset$, since this implies $\mathbb{B}_i \notin \mathcal{F}$ (as I covers $\mathcal{F}$). Suppose to the contrary that there is $e \in \delta_I(\mathbb{B}_i)$. Then there is a biset $\mathbb{A} \in \mathcal{M}$ whose inner part contains the tail of e, and we must have $\mathbb{A} \in \mathcal{M}_i$, by the definition of $\mathbb{B}_i$ and since no two bisets in $\mathcal{M}$ intersect. As e covers the biset $\mathbb{B}_i$ that contains $\mathbb{A}$, e covers $\mathbb{A}$, and thus $e = e_{\mathbb{A}}$ and $\delta_J(\mathbb{A} \cap \mathbb{C}_i) = \{e_{\mathbb{A}}\}$, by Lemma 2.18(ii). The edge $e_{\mathbb{A}}$ has its tail in C_i and covers the biset $\mathbb{B}_i$ that contains $\mathbb{C}_i$. Consequently, $e_{\mathbb{A}}$ covers $\mathbb{C}_i$, contradicting that $\mathbb{C}_i \in \mathcal{F}^J$. $\square$

Lemma 2.19(ii) implies $\nu(\mathcal{F}^J) \leq \lfloor n/(q+1) \rfloor$. To see that $c(J) = \tau(\mathcal{F}')$ let $x \in \{0,1\}^F$ be the characteristic vector of J and y the dual solution produced by the algorithm. It is easy to see that x and y are feasible solutions for the primal and dual LPs, respectively, and that the Primal Complementary Slackness Conditions hold for x and y. The Dual Complementary Slackness Conditions are: $|\delta_J(\mathbb{A})| = 1$ whenever $y_{\mathbb{A}} > 0$, and they hold by Lemma 2.19(i), since $\{\mathbb{A} : y_{\mathbb{A}} > 0\} \subseteq \mathcal{L}$.

This concludes the proof of Theorem 2.11, and thus the proof of Theorem 2.9 is also complete.

2.9 Open problems

In this section we list some open problems in the field.

k-Connected Subgraph **with** $\{0,1\}$ **costs.** Can this problem be solved in polynomial time? A polynomial time algorithm is known for the augmentation version of increasing the connectivity by 1.

Min-Size k-Connected Subgraph **(**$\{1,\infty\}$**-costs).** The best known ratios for this problem are $1-\frac{1}{k}+\frac{n}{\mathsf{opt}} \leq 1+\frac{1}{k}$ for undirected graphs and $1-\frac{1}{k}+\frac{2n}{\mathsf{opt}} \leq 1+\frac{1}{k}$ for directed graphs. In both cases, the ratio is better than $1+\frac{1}{k}$, unless an optimal solution is a k-regular graph. Can the ratio $1+\frac{1}{k}$ be improved? Can it be improved for $k=2$? Such improvements are known in the undirected edge-connectivity case.

k-Connected Subgraph **with general costs.** The best known ratios are: $O\left(\ln k \ln \frac{n}{n-k}\right)$, for undirected graphs with $n=\Omega(k^3)$ ratio 6 is achievable, and for the augmentation version of increasing the connectivity by 1 ratio $O\left(\ln \frac{n}{n-k}\right)$ is known. One main open question here is whether the later problem admits a constant ratio. Another open question is whether the ratio $O\left(\ln \frac{n}{n-k}\right)$ extends to the general version of the problem.

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