

Prize-Collecting Steiner Network Problems^{*}

MohammadTaghi Hajiaghayi¹, Rohit Khandekar²,
Guy Kortsarz³ **, and Zeev Nutov⁴

¹ AT&T Research Lab Research. Hajiaghayi@research.att.com

² IBM T.J.Watson Research Center. rohitk@us.ibm.com

³ Rutgers University, Camden. guyk@camden.rutgers.edu

⁴ The Open University of Israel nutov@openu.ac.il

Abstract. In the **Steiner Network** problem we are given a graph G with edge-costs and connectivity requirements r_{uv} between node pairs u, v . The goal is to find a minimum-cost subgraph H of G that contains r_{uv} edge-disjoint paths for all $u, v \in V$. In **Prize-Collecting Steiner Network** problems we do not need to satisfy all requirements, but are given a *penalty function* for violating the connectivity requirements, and the goal is to find a subgraph H that minimizes the cost plus the penalty. The case when $r_{uv} \in \{0, 1\}$ is the classic **Prize-Collecting Steiner Forest** problem.

In this paper we present a novel linear programming relaxation for the **Prize-Collecting Steiner Network** problem, and by rounding it, obtain the first constant-factor approximation algorithm for submodular and monotone non-decreasing penalty functions. In particular, our setting includes all-or-nothing penalty functions, which charge the penalty even if the connectivity requirement is slightly violated; this resolves an open question posed in [SSW07]. We further generalize our results for element-connectivity and node-connectivity.

1 Introduction

Prize-collecting Steiner problems are well-known network design problems with several applications in expanding telecommunications networks (see for example [JMP00,SCRS00]), cost sharing, and Lagrangian relaxation techniques (see e.g. [JV01,CRW01]). A general form of these problems is the **Prize-Collecting Steiner Forest** problem¹: given a network (graph) $G = (V, E)$, a set of source-sink pairs $\mathcal{P} = \{\{s_1, t_1\}, \{s_2, t_2\}, \dots, \{s_k, t_k\}\}$, a non-negative cost function $c : E \rightarrow \mathfrak{R}_+$, and a non-negative penalty function $\pi : \mathcal{P} \rightarrow \mathfrak{R}_+$, our goal is a minimum-cost way of installing (buying) a set of links (edges) and paying the penalty for those pairs which are not connected via installed links. When all penalties are ∞ , the problem is the classic APX-hard **Steiner Forest** problem, for

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¹ In the literature, this problem is also called “prize-collecting generalized Steiner tree”.

which the best known approximation ratio is $2 - \frac{2}{n}$ (n is the number of nodes of the graph) due to Agrawal, Klein, and Ravi [AKR95] (see also [GW95] for a more general result and a simpler analysis). The case of **Prize-Collecting Steiner Forest** problem when all sinks are identical is the classic **Prize-Collecting Steiner Tree** problem. Bienstock, Goemans, Simchi-Levi, and Williamson [BGSLW93] first considered this problem (based on a problem earlier proposed by Balas [Bal89]) and gave for it a 3-approximation algorithm. The current best ratio for this problem is 1.992 by Archer, Bateni, Hajiaghayi, and Karloff [ABHK09], improving upon a primal-dual $(2 - \frac{1}{n-1})$ -approximation algorithm of Goemans and Williamson [GW95]. When in addition all penalties are ∞ , the problem is the classic **Steiner Tree** problem, which is known to be APX-hard [BP89] and for which the best approximation ratio is 1.55 [RZ05]. Very recently, Byrka et al. [BGRS10] have announced an improved approximation algorithm for the Steiner tree problem.

The general form of the **Prize-Collecting Steiner Forest** problem first has been formulated by Hajiaghayi and Jain [HJ06]. They showed how by a primal-dual algorithm to a novel integer programming formulation of the problem with doubly-exponential variables, we can obtain a 3-approximation algorithm for the problem. In addition, they show that the factor 3 in the analysis of their algorithm is tight. However they show how a direct randomized LP-rounding algorithm with approximation factor 2.54 can be obtained for this problem. Their approach has been generalized by Sharma, Swamy, and Williamson [SSW07] for network design problems where violated arbitrary 0-1 connectivity constraints are allowed in exchange for a very general penalty function. The work of Hajiaghayi and Jain has also motivated a game-theoretic version of the problem considered by Gupta et al. [GKL⁺07].

In this paper, we consider a much more general high-connectivity version of **Prize-Collecting Steiner Forest**, called **Prize-Collecting Steiner Network**, in which we are also given connectivity requirements r_{uv} for pairs of nodes u and v and a penalty function in case we do not satisfy all r_{uv} . Our goal is to find a minimum way of constructing a network (graph) in which we connect u and v with $r'_{uv} \leq r_{uv}$ edge-disjoint paths and paying a penalty for all violated connectivity between source-sink pairs. This problem can arise in real-world network design, in which a typical client not only might want to connect to the network but also might want to connect via a few disjoint paths (e.g., to have a higher bandwidth or redundant connections in case of edge failures) and a penalty might be charged if we cannot satisfy its connectivity requirement. When all penalties are ∞ , the problem is the classic **Steiner Network** problem. Improving on a long line of earlier research that applied primal-dual methods, Jain [Jai01] obtained a 2-approximation algorithm for **Steiner Network** using the iterative rounding method. This algorithm was generalized to so called “element-connectivity” by Fleischer, Jain, and Williamson [FJW01] and by Cheriyan, Vempala, and Vetta [CVV06]. Recently, some results were obtained for the node-connectivity version; the currently best known ratios for the node-connectivity case are $O(R^3 \log n)$ for general requirements [CK09] and $O(R^2)$ for rooted requirements [Nut09],

where $R = \max_{u,v \in V} r_{uv}$ is the maximum requirement. See also the survey by Kortsarz and Nutov [KN07] for various min-cost connectivity problems.

Hajiaghayi and Nasri [HN10] generalize the iterative rounding approach of Jain to Prize-Collecting Steiner Network when there is a separate non-increasing marginal penalty function for each pair u, v whose r_{uv} -connectivity requirement is not satisfied. They obtain an iterative rounding 3-approximation algorithm for this case. For the special case when penalty functions are linear in the violation of the connectivity requirements, Nagarajan, Sharma, and Williamson [NSW08] using Jain's iterative rounding algorithm as a black box give a 2.54-factor approximation algorithm. They also generalize the 0-1 requirements of Prize-Collecting Steiner Forest problem introduced by Sharma, Swamy, and Williamson [SSW07] to include general connectivity requirements. Assuming the monotone submodular penalty function of Sharma et al. is generalized to a multiset function that can be decomposed into functions in the same type as that of Sharma et al., they give an $O(\log R)$ -approximation algorithm (recall that R is the maximum connectivity requirement). In this algorithm, they assume that we can use each edge possibly many times (without bound). They raise the question whether we can obtain a constant ratio without all these assumptions, when penalty is a submodular multi-set function of the set of disconnected pairs? More importantly they pose as an open problem to design a good approximation algorithm for the all-or-nothing version of penalty functions: penalty functions which charge the penalty even if the connectivity requirement is slightly violated. In this paper, we answer affirmatively all these open problems by proving the first constant factor 2.54-approximation algorithm which is based on a novel LP formulation of the problem. We further generalize our results for element-connectivity and node-connectivity. In fact, for all types of connectivities, we prove a very general result (see Theorem 1) stating that if Steiner Network (the version without penalties) admits an LP-based ρ -approximation algorithm, then the corresponding prize-collecting version admits a $(\rho + 1)$ -approximation algorithm.

1.1 Problems we consider

In this section, we define formally the terms used in the paper. For a subset S of nodes in a graph H , let $\lambda_H^S(u, v)$ denote the S -connectivity between u and v in H , namely, the maximum number of edge-disjoint uv -paths in H so that no two of them have a node in $S - \{u, v\}$ in common. In the Generalized Steiner-Network (GSN) problem we are given a graph $G = (V, E)$ with edge-costs $\{c_e \geq 0 \mid e \in E\}$, a node subset $S \subseteq V$, a collection $\{u_1, v_1\}, \dots, \{u_k, v_k\}$ of node pairs from V , and S -connectivity requirements r_1, \dots, r_k . The goal is to find a minimum cost subgraph H of G so that $\lambda_H^S(u_i, v_i) \geq r_i$ for all i . Extensively studied particular cases of GSN are: the Steiner Network problem, called also Edge-Connectivity GSN ($S = \emptyset$), Node-Connectivity GSN ($S = V$), and Element-Connectivity GSN ($S \cap \{u_i, v_i\} = \emptyset$ for all i). The case of *rooted requirements* is when there is a "root" s that belongs to all pairs $\{u_i, v_i\}$. We consider the following "prize-collecting" version of GSN.

All-or-Nothing Prize Collecting Generalized Steiner Network (PC-GSN):

Instance: A graph $G = (V, E)$ with edge-costs $\{c_e \geq 0 \mid e \in E\}$, $S \subseteq V$, a collection $\{u_1, v_1\}, \dots, \{u_k, v_k\}$ of node pairs from V , S -connectivity requirements $r_1, \dots, r_k > 0$, and a penalty function $\pi : 2^{\{1, \dots, k\}} \rightarrow \mathfrak{R}_+$.

Objective: Find a subgraph H of G that minimizes the *value*

$$\text{val}(H) = c(H) + \pi(\text{unsat}(H))$$

of H , where $\text{unsat}(H) = \{i \mid \lambda_H^S(u_i, v_i) < r_i\}$ is the set of requirements *not* satisfied by H .

We will assume that the penalty function π is given by an evaluation oracle. We will also assume that π is *submodular*, namely, that $\pi(A) + \pi(B) \geq \pi(A \cap B) + \pi(A \cup B)$ for all A, B and that it is *monotone non-decreasing*, namely, $\pi(A) \leq \pi(B)$ for all A, B with $A \subseteq B$. As was mentioned, approximating the edge-connectivity variant of PC-GSN was posed as the main open problem by Nagarajan, Sharma, and Williamson [NSW08]. We resolve this open problem for the submodular function $\text{val}(H)$ considered here.

We next define the second problem we consider.

Generalized Steiner Network with Generalized Penalties (GSN-GP):

Instance: A graph $G = (V, E)$ with edge-costs $\{c_e \geq 0 \mid e \in E\}$, $S \subseteq V$, a collection $\{u_1, v_1\}, \dots, \{u_k, v_k\}$ of node pairs from V , and non-increasing penalty functions $p_1, \dots, p_k : \{0, 1, \dots, n-1\} \rightarrow \mathfrak{R}_+$.

Objective: Find a subgraph H of G that minimizes the *value*

$$\text{val}'(H) = c(H) + \sum_{i=1}^k p_i(\lambda_H^S(u_i, v_i)).$$

The above problem captures general penalty functions of the S -connectivity $\lambda^S(u_i, v_i)$ for given pairs $\{u_i, v_i\}$. It is natural to assume that the penalty functions are non-increasing, i.e., we pay less in the objective function if the achieved connectivity is more. This problem was posed as an open question by Nagarajan et al. [NSW08]. In this paper, we use the convention that $p_i(n) = 0$ for all i .

We need some definitions to introduce our results. A pair $T = \{T', T''\}$ of subsets of V is called a *setpair* (of V) if $T' \cap T'' = \emptyset$. Let $K = \{1, \dots, k\}$. Let $T = \{T', T''\}$ be a setpair of V . We denote by $\delta(T)$ the set of edges in E between T' and T'' . For $i \in K$ we use $T \odot (i, S)$ to denote that $|T' \cap \{u_i, v_i\}| = 1$, $|T'' \cap \{u_i, v_i\}| = 1$ and $V \setminus (T' \cup T'') \subseteq S$. While in the case of edge-connectivity a “cut” consists of edges only, in the case of S -connectivity a cut that separates

between u and v is “mixed”, meaning it may contain both edges in the graph and nodes from S . Note that if $T \odot (i, S)$ then $\delta(T) \cup (V \setminus (T' \cup T''))$ is such a mixed cut that separates between u_i and v_i . Intuitively, Menger’s Theorem for S -connectivity (c.f. [KN07]) states that the S -connectivity between u_i and v_i equals the minimum size of such a mixed cut. Formally, for a node pair u_i, v_i of a graph $H = (V, E)$ and $S \subseteq V$ we have:

$$\lambda_H^S(u_i, v_i) = \min_{T \odot (i, S)} (|\delta(T)| + |V \setminus (T' \cup T'')|) = \min_{T \odot (i, S)} (|\delta(T)| + |V| - (|T'| + |T''|))$$

Hence if $\lambda_H^S(u_i, v_i) \geq r_i$ for a graph $H = (V, E)$, then for any setpair T with $T \odot (i, S)$ we must have $|\delta(T)| \geq r_i(T)$, where $r_i(T) = \max\{r_i + |T'| + |T''| - |V|, 0\}$. Consequently, a standard “cut-type” LP-relaxation of the GSN problem is as follows (c.f. [KN07]):

$$\min \left\{ \sum_{e \in E} c_e x_e \mid \sum_{e \in \delta(T)} x_e \geq r_i(T) \ \forall T \odot (i, S), \ \forall i \in K, \ x_e \in [0, 1] \ \forall e \right\}. \quad (1)$$

1.2 Our results

We introduce a novel LP relaxation of the problem which is shown to be better, in terms of the integrality gap, than a “natural” LP relaxation considered in [NSW08]. Using our LP relaxation, we prove the following main result.

Theorem 1. *Suppose that there exists a polynomial time algorithm that computes an integral solution to LP (1) of cost at most ρ times the optimal value of LP (1) for any subset of node pairs. Then PC-GSN admits a $(1 - e^{-1/\rho})^{-1}$ -approximation algorithm, provided that the penalty function π is submodular and monotone non-decreasing.*

Note that since $1 - \frac{1}{\rho} < e^{-\frac{1}{\rho}} < 1 - \frac{1}{\rho+1}$ holds for $\rho \geq 1$, we have $\rho < (1 - e^{-1/\rho})^{-1} < \rho + 1$.

Let $R = \max_i r_i$ denote the maximum requirement. The best known values of ρ are as follows: 2 for Edge-GSN [Jai01], 2 for Element-GSN [FJW01, CVV06], $O(R^3 \log |V|)$ for Node-GSN [CK09], and $O(R^2)$ for Node-GSN with rooted requirements [Nut09]. Substituting these values in Theorem 1, we obtain:

Corollary 1. *PC-GSN problems admit the following approximation ratios provided that the penalty function π is submodular and monotone non-decreasing: 2.54 for edge- and element-connectivity, $O(R^3 \log |V|)$ for node-connectivity, and $O(R^2)$ for node-connectivity with rooted requirements.*

Our results for GSN-GP follow from Corollary 1.

Corollary 2. *GSN-GP problems admit the following approximation ratios: 2.54 for edge- and element-connectivity, $O(R^3 \log |V|)$ for node-connectivity, and $O(R^2)$ for node-connectivity with rooted requirements. Here $R = \max_{1 \leq i \leq k} \min\{\lambda \geq 0 \mid p_i(\lambda) = 0\}$.*

Proof. We present an approximation ratio preserving reduction from the GSN-GP problem to the corresponding PC-GSN problem. Given an instance of the GSN-GP problem, we create an instance of the PC-GSN problem as follows. The PC-GSN instance inherits the graph G , its edge-costs, and the set S . Let (u_i, v_i) be a pair in GSN-GP and let $R_i = \min\{\lambda \geq 0 \mid p_i(\lambda) = 0\}$. We introduce R_i copies of this pair, $\{(u_i^1, v_i^1), \dots, (u_i^{R_i}, v_i^{R_i})\}$, to the set of pairs in the PC-GSN instance. We set the edge-connectivity requirement of a pair (u_i^t, v_i^t) to be t for $1 \leq t \leq R_i$. We also set the penalty function for singleton sets as follows $\pi(\{(u_i^t, v_i^t)\}) = p_i(t-1) - p_i(t)$ for all $1 \leq t \leq R_i$. Finally, we extend this function π to a set of pairs P by *linearity*, i.e., $\pi(P) = \sum_{p \in P} \pi(\{p\})$. Note that such a function π is clearly submodular and monotone non-decreasing.

It is sufficient to show that for any subgraph H of G , its value in the GSN-GP instance equals its value in the PC-GSN instance, i.e., $\text{val}(H) = \text{val}'(H)$; then we can use the algorithm from Corollary 1 to complete the proof. Fix a pair (u_i, v_i) in the GSN-GP instance. Let $\lambda_H^S(u_i, v_i) = t_i$. Thus the contribution of pair (u_i, v_i) to the objective function $\text{val}(H)$ of the GSN-GP instance is $p_i(t_i)$. On the other hand, since π is linear, the total contribution of pairs $\{(u_i^1, v_i^1), \dots, (u_i^{R_i}, v_i^{R_i})\}$ to the objective function $\text{val}'(H)$ of the PC-GSN instance is $\sum_{t=t_i+1}^{R_i} \pi(\{(u_i^t, v_i^t)\}) = \sum_{t=t_i+1}^{R_i} (p_i(t-1) - p_i(t)) = p_i(t_i)$. Note that the pairs (u_i^t, v_i^t) for $1 \leq t \leq t_i$ do not incur any penalty. Summing up over all pairs, we conclude that $\text{val}(H) = \text{val}'(H)$, as claimed. \blacksquare

2 A new LP relaxation

We use the following LP-relaxation for the PC-GSN problem. We introduce variables x_e for $e \in E$ ($x_e = 1$ if $e \in H$), $f_{i,e}$ for $i \in K$ and $e \in E$ ($f_{i,e} = 1$ if $i \notin \text{unsat}(H)$ and e appears on a chosen set of r_i S -disjoint $\{u_i, v_i\}$ -paths in H), and z_I for $I \subseteq K$ ($z_I = 1$ if $I = \text{unsat}(H)$).

$$\begin{aligned}
\text{Minimize} \quad & \sum_{e \in E} c_e x_e + \sum_{I \subseteq K} \pi(I) z_I \\
\text{Subject to} \quad & \sum_{e \in \delta(T)} f_{i,e} \geq (1 - \sum_{I: i \in I} z_I) r_i(T) \quad \forall i \quad \forall T \odot (i, S) \\
& f_{i,e} \leq 1 - \sum_{I: i \in I} z_I \quad \forall i \quad \forall e \\
& x_e \geq f_{i,e} \quad \forall i \quad \forall e \\
& \sum_{I \subseteq K} z_I = 1 \\
& x_e, f_{i,e}, z_I \in [0, 1] \quad \forall i \quad \forall e \quad \forall I
\end{aligned} \tag{2}$$

We first prove that (2) is a valid LP-relaxation of the PC-GSN problem.

Lemma 1. *The optimal value of LP (2) is at most the optimal solution value to the PC-GSN problem. Moreover, if π is monotone non-decreasing, the optimum solution value to the PC-GSN problem is at most the value of the optimum integral solution of LP (2).*

Proof. Given a feasible solution H to the PC-GSN problem define a feasible solution to LP (2) as follows. Let $x_e = 1$ if $e \in H$ and $x_e = 0$ otherwise. Let $z_I = 1$ if $I = \text{unsat}(H)$ and $z_I = 0$ otherwise. For each $i \in \text{unsat}(H)$ set $f_{i,e} = 0$ for all $e \in E$, while for $i \notin \text{unsat}(H)$ the variables $f_{i,e}$ take values as follows: fix a set of r_i pairwise S -disjoint $\{u_i, v_i\}$ -paths, and let $f_{i,e} = 1$ if e belongs to one of these paths and $f_{i,e} = 0$ otherwise. The defined solution is feasible for LP (2): the first set of constraints are satisfied by Menger's Theorem for S -connectivity, while the remaining constraints are satisfied by the above definition of variables. It is also easy to see that the above solution has value exactly $\text{val}(H)$.

If π is monotone non-decreasing, we prove that for any *integral* solution $\{x_e, f_{i,e}, z_I\}$ to (2), the graph H with edge-set $\{e \in E \mid x_e = 1\}$ has $\text{val}(H)$ at most the value of the solution $\{x_e, f_{i,e}, z_I\}$. To see this, first note that there is a unique set $I \subseteq K$ with $z_I = 1$, since the variables z_I are integral and $\sum_{I \subseteq K} z_I = 1$. Now consider an index $i \notin I$. Since $\sum_{I: i \in I} z_I = 0$, we have $\sum_{e \in \delta(T)} x_e \geq \sum_{e \in \delta(T)} f_{i,e} \geq r_i(T)$ for all $T \odot (i, S)$. This implies that $i \notin \text{unsat}(H)$, by Menger's Theorem for S -connectivity. Consequently, $\text{unsat}(H) \subseteq I$, hence $\pi(\text{unsat}(H)) \leq \pi(I)$ by the monotonicity of π . Thus $\text{val}(H) = c(H) + \pi(\text{unsat}(H)) \leq \sum_{e \in E} c_e x_e + \sum_{I \subseteq K} \pi(I) z_I$ and the lemma follows. \blacksquare

2.1 Why does a “natural” LP relaxation not work?

One may be tempted to consider a natural LP without using the flow variables $f_{i,e}$, namely, the LP obtained from LP (2) by replacing the the first three sets of constraints by the set of constraints

$$\sum_{e \in \delta(T)} x_e \geq (1 - \sum_{I: i \in I} z_I) r_i(T)$$

for all i and $T \odot (i, S)$. Here is an example demonstrating that the integrality gap of this LP can be as large as $R = \max_i r_i$ even for edge-connectivity. Let G consist of $R - 1$ edge-disjoint paths between two nodes s and t . All the edges have cost 0. There is only one pair $\{u_1, v_1\} = \{s, t\}$ that has requirement $r_1 = R$ and penalty $\pi(\{1\}) = 1$. Let $\pi(\emptyset) = 0$. Clearly, π is submodular and monotone non-decreasing. We have $S = \emptyset$. No integral solution can satisfy the requirement r_1 , hence an optimal integral solution pays the penalty $\pi(\{1\})$ and has value 1. A feasible fractional solution (without the flow variables) sets $x_e = 1$ for all e , and sets $z_{\{1\}} = 1/R$, $z_\emptyset = 1 - 1/R$. The new set of constraints is satisfied since $\sum_{e \in \delta(T)} x_e \geq (1 - 1/R) \cdot R = (1 - z_{\{1\}}) r_1(T)$ for any $\{s, t\}$ -cut T . Thus the optimal LP-value is at most $1/R$, giving a gap of at least R .

With flow variables, however, we have an upper bound $f_{1,e} \leq 1 - z_{\{1\}}$. Since there is an $\{s, t\}$ -cut T with $|\delta(T)| = R - 1$, we cannot satisfy the constraints $\sum_{e \in \delta(T)} f_{1,e} \geq (1 - z_{\{1\}}) r_1(T)$ and $f_{1,e} \leq 1 - z_{\{1\}}$ simultaneously unless we set $z_{\{1\}} = 1$. Thus in this case, our LP (2) with flow variables has the same optimal value of as the integral optimum.

2.2 Some technical results regarding LP (2)

We will prove the following two statements that together imply Theorem 1.

Lemma 2. *Any basic feasible solution to (2) has a polynomial number of non-zero variables. Furthermore, an optimal basic solution to (2) (the non-zero entries) can be computed in polynomial time.*

Lemma 3. *Suppose that there exists a polynomial time algorithm that computes an integral solution to LP (1) of cost at most ρ times the optimal value of LP (1) for any subset of node pairs. Then there exists a polynomial time algorithm that given a feasible solution to (2) computes as a solution to PC-GSN a subgraph H of G so that $\text{val}(H) = c(H) + \pi(\text{unsat}(H))$ is at most $(1 - e^{-1/\rho})^{-1}$ times the value of this solution, assuming π is submodular and monotone non-decreasing.*

Before proving these lemmas, we prove some useful results. The following statement can be deduced from a theorem of Edmonds for polymatroids (c.f. [KV02, Chapter 14.2]), as the dual LP $D(\gamma)$ in the lemma seeks to optimize a linear function over a polymatroid. We provide a direct proof for completeness of exposition.

Lemma 4. *Let $\gamma \in [0, 1]^k$ be a vector. Consider a primal LP*

$$P(\gamma) := \min \left\{ \sum_{I \subseteq K} \pi(I) z_I \mid \sum_{I: i \in I} z_I \geq \gamma_i \quad \forall i \in K, z_I \geq 0 \quad \forall I \subseteq K \right\}$$

and its dual LP

$$D(\gamma) := \max \left\{ \sum_{i \in K} \gamma_i y_i \mid \sum_{i \in I} y_i \leq \pi(I) \quad \forall I \subseteq K, y_i \geq 0 \quad \forall i \in K \right\}.$$

Let σ be a permutation of K such that $\gamma_{\sigma(1)} \leq \gamma_{\sigma(2)} \leq \dots \leq \gamma_{\sigma(k)}$. Let us also use the notation that $\gamma_{\sigma(0)} = 0$. The optimum solutions to $P(\gamma)$ and $D(\gamma)$ respectively are given by

$$z_I = \begin{cases} \gamma_{\sigma(i)} - \gamma_{\sigma(i-1)}, & \text{for } I = \{\sigma(i), \dots, \sigma(k)\}, i \in K \\ 0, & \text{otherwise;} \end{cases}$$

and

$$y_{\sigma(i)} = \pi(\{\sigma(i), \dots, \sigma(k)\}) - \pi(\{\sigma(i+1), \dots, \sigma(k)\}), \text{ for } i \in K.$$

Proof. To simplify the notation, we assume without loss of generality that $\gamma_1 \leq \gamma_2 \leq \dots \leq \gamma_k$, i.e., that σ is the identity permutation.

We argue that the above defined $\{z_I\}$ and $\{y_i\}$ form feasible solutions to the primal and dual LPs respectively. Note that $z_I \geq 0$ for all I and $\sum_{I: i \in I} z_I = \sum_{j=1}^i (\gamma_j - \gamma_{j-1}) = \gamma_i$ for all i . Since π is monotone non-decreasing, the above

defined y_i satisfy $y_i \geq 0$ for all i . Now fix $I \subseteq K$. Let $I = \{i_1, \dots, i_p\}$ where $i_1 < \dots < i_p$. Therefore

$$\begin{aligned} \sum_{i \in I} y_i &= \sum_{j=1}^p y_{i_j} = \sum_{j=1}^p [\pi(\{i_j, \dots, k\}) - \pi(\{i_j + 1, \dots, k\})] \\ &\leq \sum_{j=1}^p [\pi(\{i_j, i_{j+1}, \dots, i_p\}) - \pi(\{i_{j+1}, i_{j+2}, \dots, i_p\})] \\ &= \pi(\{i_1, \dots, i_p\}) = \pi(I). \end{aligned}$$

The above inequality holds because of the submodularity of π . Next observe that the solutions $\{z_I\}$ and $\{y_i\}$ satisfy

$$\begin{aligned} \sum_I \pi(I) z_I &= \sum_{i=1}^k \pi(\{i, \dots, k\}) \cdot (\gamma_i - \gamma_{i-1}) \\ &= \sum_{i=1}^k \gamma_i \cdot (\pi(\{i, \dots, k\}) - \pi(\{i+1, \dots, k\})) = \sum_{i=1}^k \gamma_i \cdot y_i. \end{aligned}$$

Thus from weak LP duality, they in fact form optimum solutions to primal and dual LPs respectively. \blacksquare

Recall that a sub-gradient of a convex function $g : \mathfrak{R}^k \rightarrow \mathfrak{R}$ at a point $\gamma \in \mathfrak{R}^k$ is a vector $d \in \mathfrak{R}^k$ such that for any $\gamma' \in \mathfrak{R}^k$, we have $g(\gamma') - g(\gamma) \geq d \cdot (\gamma' - \gamma)$. For a differentiable convex function g , the sub-gradient corresponds to gradient ∇g . The function $P(\gamma)$ defined in Lemma 4 is essentially Lovasz's continuous extension of the submodular function π . The fact that P is convex and its subgradient can be computed efficiently is given in [Fuj05]. We provide a full proof for completeness of exposition.

Lemma 5. *The function $P(\gamma)$ in Lemma 4 is convex and given $\gamma \in [0, 1]^k$, both $P(\gamma)$ and its sub-gradient $\nabla P(\gamma)$ can be computed in polynomial time.*

Proof. We first prove that P is convex. Fix $\gamma_1, \gamma_2 \in [0, 1]^k$ and $\alpha \in [0, 1]$. To show that P is convex, we will show $P(\alpha\gamma_1 + (1-\alpha)\gamma_2) \leq \alpha P(\gamma_1) + (1-\alpha)P(\gamma_2)$. Let $\{z_I^1\}$ and $\{z_I^2\}$ be the optimum solutions of the primal LP defining P for γ_1 and γ_2 respectively. Note that the solution $\{\alpha z_I^1 + (1-\alpha)z_I^2\}$ is feasible for this LP for $\gamma = \alpha\gamma_1 + (1-\alpha)\gamma_2$. Thus the optimum solution has value not greater than the value of this solution which is $\alpha P(\gamma_1) + (1-\alpha)P(\gamma_2)$.

From Lemma 4, it is clear that given $\gamma \in [0, 1]^k$, the value $P(\gamma)$ can be computed in polynomial time. Lemma 4 also implies that the optimum dual solution $y^* = (y_1^*, \dots, y_k^*) \in \mathfrak{R}_+^k$ can be computed in polynomial time. We now argue that y^* is a sub-gradient of P at γ . Fix any $\gamma' \in \mathfrak{R}^k$. First note that, from LP duality, $P(\gamma) = y^* \cdot \gamma$. Thus we have

$$P(\gamma) + y^* \cdot (\gamma' - \gamma) = y^* \cdot \gamma + y^* \cdot (\gamma' - \gamma) = y^* \cdot \gamma' \leq P(\gamma').$$

The last inequality holds from weak LP duality since y^* is a feasible solution for the dual LP $D(\gamma')$ as well. The lemma follows. \blacksquare

3 Proof of Lemma 3

We now describe how to round LP (2) solutions to obtain a $(\rho+1)$ -approximation for PC-GSN. Later we show how to improve it to $(1 - e^{-1/\rho})^{-1}$. Let $\{x_e^*, f_{i,e}^*, z_I^*\}$ be a feasible solution to LP (2). Let $\alpha \in (0, 1)$ be a parameter to be fixed later. We partition the requirements into two classes: we call a requirement $i \in K$ *good* if $\sum_{I:i \in I} z_I^* \leq \alpha$ and *bad* otherwise. Let K_g denote the set of good requirements. The following statement shows how to satisfy the good requirements.

Lemma 6. *There exists a polynomial time algorithm that computes a subgraph H of G of cost $c(H) \leq \frac{\rho}{1-\alpha} \cdot \sum_e c_e x_e^*$ that satisfies all good requirements.*

Proof. Consider the LP-relaxation (1) of the GSN problem with good requirements only, with K replaced by K_g ; namely, we seek a minimum cost subgraph H of G that satisfies the set K_g of good requirements. We claim that $x_e^{**} = \min\{1, x_e^*/(1-\alpha)\}$ for each $e \in E$ is a feasible solution to LP (1). Thus the optimum value of LP (1) is at most $\sum_{e \in E} c_e x_e^{**}$. Consequently, using the algorithm that computes an integral solution to LP (1) of cost at most ρ times the optimal value of LP (1), we can construct a subgraph H that satisfies all good requirements and has cost at most $c(H) \leq \rho \sum_{e \in E} c_e x_e^{**} \leq \frac{\rho}{1-\alpha} \sum_e c_e x_e^*$, as desired.

We now show that $\{x_e^{**}\}$ is a feasible solution to LP (1), namely, that $\sum_{e \in \delta(T)} x_e^{**} \geq r_i(T)$ for any $i \in K_g$ and any $T \odot (i, S)$. Let $i \in K_g$ and let $\zeta_i = 1 - \sum_{I:i \in I} z_I^*$. Note that $\zeta_i \geq 1 - \alpha$, by the definition of K_g . By the second and the third sets of constraints in LP (2), for every $e \in E$ we have $\min\{\zeta_i, x_e^*\} \geq f_{i,e}^*$. Thus we obtain: $x_e^{**} = \min\left\{1, \frac{x_e^*}{1-\alpha}\right\} = \frac{1}{\zeta_i} \min\left\{\zeta_i, \frac{\zeta_i}{1-\alpha} x_e^*\right\} \geq \frac{1}{\zeta_i} \min\{\zeta_i, x_e^*\} \geq \frac{f_{i,e}^*}{\zeta_i} = \frac{f_{i,e}^*}{1 - \sum_{I:i \in I} z_I^*}$. Consequently, combining with the first set of constraints in LP (2), for any $T \odot (i, S)$ we obtain that $\sum_{e \in \delta(T)} x_e^{**} \geq \frac{\sum_{e \in \delta(T)} f_{i,e}^*}{1 - \sum_{I:i \in I} z_I^*} \geq r_i(T)$. ■

Let H be as in Lemma 6, and recall that $\text{unsat}(H)$ denotes the set of requirements not satisfied by H . Clearly each requirement $i \in \text{unsat}(H)$ is bad. The following lemma bounds the total penalty we pay for $\text{unsat}(H)$.

Lemma 7. $\pi(\text{unsat}(H)) \leq \frac{1}{\alpha} \cdot \sum_I \pi(I) z_I^*$.

Proof. Define $\gamma \in [0, 1]^k$ as follows: $\gamma_i = 1$ if $i \in \text{unsat}(H)$ and 0 otherwise. Now consider LP $P(\gamma)$ defined in Lemma 4. Since each $i \in \text{unsat}(H)$ is bad, from the definition of bad requirements, it is clear that $\{z_I^*/\alpha\}$ is a feasible solution to LP $P(\gamma)$. Furthermore, from Lemma 4, the solution $\{z_I\}$ defined as $z_I = 1$ if $I = \text{unsat}(H)$ and 0 otherwise is the optimum solution to $P(\gamma)$. The cost of this solution, $\pi(\text{unsat}(H))$, is therefore at most the cost of the feasible solution $\{z_I^*/\alpha\}$ which is $\frac{1}{\alpha} \cdot \sum_I \pi(I) z_I^*$. The lemma thus follows. ■

Combining Lemmas 6 and 7, we obtain $\max\{\frac{\rho}{1-\alpha}, \frac{1}{\alpha}\}$ -approximation. If we substitute $\alpha = 1/(\rho+1)$, we obtain a $(\rho+1)$ -approximation for PC-GSN.

Improving the approximation to $(1 - e^{-1/\rho})^{-1}$. We use a technique introduced by Goemans as follows. We pick α uniformly at random from the interval $(0, \beta]$ where $\beta = 1 - e^{-1/\rho}$. From Lemmas 6 and 7, the expected cost of the solution is at most

$$\mathbb{E}_\alpha \left[\frac{\rho}{1-\alpha} \right] \cdot \sum_{e \in E} c_e x_e^* + \mathbb{E}_\alpha [\pi(\text{unsat}(H))]. \quad (3)$$

To complete the proof of $\frac{1}{\beta}$ -approximation, we now argue that the above expectation is at most $\frac{1}{\beta} \cdot \sum_{e \in E} (c_e x_e^* + \sum_I \pi(I) z_I^*)$.

Since $\mathbb{E}_\alpha \left[\frac{\rho}{1-\alpha} \right] = \frac{1}{\beta}$, the first term in (3) is at most $\frac{1}{\beta} \cdot \sum_{e \in E} c_e x_e^*$. Since $\text{unsat}(H) \subseteq \{i \mid \sum_{I:i \in I} z_I^* \geq \alpha\}$ and since π is monotone non-decreasing, the second term in (3) is at most $\mathbb{E}_\alpha [\pi(\{i \mid \sum_{I:i \in I} z_I^* \geq \alpha\})]$. Lemma 8 bounds this quantity as follows. The ideas used here are also presented in Sharma et al. [SSW07].

Lemma 8. *We have*

$$\mathbb{E}_\alpha \left[\pi \left(\{i \mid \sum_{I:i \in I} z_I^* \geq \alpha\} \right) \right] \leq \frac{1}{\beta} \cdot \sum_I \pi(I) z_I^*. \quad (4)$$

Proof. Let $\gamma_i = \sum_{I:i \in I} z_I^*$ for all $i \in K$. Let us, without loss of generality, order the elements $i \in K$ such that $\gamma_1 \leq \gamma_2 \leq \dots \leq \gamma_k$. We also use the notation $\gamma_0 = 0$. Note that $\{z_I^*\}$ forms a feasible solution to the primal LP $P(\gamma)$ given in Lemma 4. Therefore, from Lemma 4, its objective value is at least that of the optimum solution:

$$\sum_I \pi(I) z_I^* \geq \sum_{i=1}^k [(\gamma_i - \gamma_{i-1}) \cdot \pi(\{i, \dots, k\})]. \quad (5)$$

We now observe that the LHS of (4) can be expressed as follows. Since α is picked uniformly at random from $(0, \beta]$, we have that for all $1 \leq i \leq k$, with probability at most $\frac{\gamma_i - \gamma_{i-1}}{\beta}$, the random variable α lies in the interval $(\gamma_{i-1}, \gamma_i]$. When this event happens, we get that $\{i' \mid \sum_{I:i' \in I} z_I^* \geq \alpha\} = \{i' \mid \gamma_{i'} \geq \alpha\} = \{i, \dots, k\}$. Thus the expectation in LHS of (4) is at most

$$\sum_{i=1}^k \left[\frac{\gamma_i - \gamma_{i-1}}{\beta} \cdot \pi(\{i, \dots, k\}) \right]. \quad (6)$$

From expressions (5) and (6), the lemma follows. \blacksquare

Thus the proof of $(1 - e^{-1/\rho})^{-1}$ -approximation is complete. It is worth mentioning so far in this section we obtain a solution with a bound on its expected cost. However, the choice of α can be simply derandomized by trying out all the breakpoints where a good demand pair becomes a bad one (plus 0 and β).

4 Proof of Lemma 2

We next show that even if LP (2) has exponential number of variables and constraints, the following lemma holds.

Lemma 9. *Any basic feasible solution to LP (2) has a polynomial number of non-zero variables.*

Proof. Fix a basic feasible solution $\{x_e^*, f_{i,e}^*, z_i^*\}$ to (2). For $i \in K$, let

$$\gamma_i = 1 - \frac{\min_{T:T \odot i} \sum_{e \in \delta(T)} f_{i,e}^*}{r_i} \quad \text{and} \quad \gamma'_i = 1 - \max_e f_{i,e}^* .$$

Now fix the values of variables $\{x_e, f_{i,e}\}$ to $\{x_e^*, f_{i,e}^*\}$ and project the LP (2) onto variables $\{z_I\}$ as follows.

$$\sum_{e \in E} c_e x_e^* + \min \left\{ \sum_{I \subseteq K} \pi(I) z_I \mid \sum_{I \subseteq K} z_I = 1, \gamma_i \leq \sum_{I: i \in I} z_I \leq \gamma'_i \forall i \in K, z_I \geq 0 \forall I \subseteq K \right\}. \quad (7)$$

Since $\{x_e^*, f_{i,e}^*, z_i^*\}$ is a basic feasible solution to (2), it cannot be written as a convex combination of two distinct feasible solutions to (2). Thus we get that $\{z_I^*\}$ cannot be written as a convex combination of two distinct feasible solutions to (7), and hence it forms a basic feasible solution to (7). Since there are $1 + 2|K|$ non-trivial constraints in (7), at most $1 + 2|K|$ variables z_I can be non-zero in any basic feasible solution of (7). Thus the lemma follows. ■

We prove that LP (2) can be solved in polynomial time. Introduce variables $\gamma \in [0, 1]^k$ and obtain the following program (the function P is as in Lemma 4).

$$\begin{aligned} \text{Minimize} \quad & \sum_{e \in E} c_e x_e + P(\gamma) \\ \text{Subject to} \quad & \sum_{e \in \delta(T)} f_{i,e} \geq (1 - \gamma_i) r_i(T) \quad \forall i \quad \forall T \odot (i, S) \\ & f_{i,e} \leq 1 - \gamma_i \quad \forall i \quad \forall e \\ & x_e \geq f_{i,e} \quad \forall i \quad \forall e \\ & x_e, f_{i,e}, \gamma_i \in [0, 1] \quad \forall i \quad \forall e \end{aligned} \quad (8)$$

It is clear that solving (8) is enough to solve (2). Now note that this is a convex program since P is a convex function. To solve (8), we convert its objective function into a constraint $\sum_{e \in E} c_e x_e + P(\gamma) \leq \text{OPT}$ where OPT is the target objective value and thus reduce it to a feasibility problem. Now to find a feasible solution using an ellipsoid algorithm, we need to show a polynomial time separation oracle. The separation oracle for the first set of constraints can be reduced to a minimum u - v cut problem using standard techniques. The separation oracle for the remaining constraints is trivial.

The separation oracle for the objective function is as follows. Given a point $(x, \gamma) = \{x_e, \gamma_i\}$ that satisfies $\sum_{e \in E} c_e x_e + P(\gamma) > \text{OPT}$, we compute a sub-gradient of the function $\sum_{e \in E} c_e x_e + P(\gamma)$ w.r.t. variables $\{x_e, \gamma_i\}$. The sub-gradient of $\sum_{e \in E} c_e x_e$ w.r.t. x is simply the cost vector c . The sub-gradient of $P(\gamma)$ w.r.t. γ is computed using Lemma 5, denote it by $y \in \mathbb{R}_+^k$. From the definition of sub-gradient, we have that the sub-gradient (c, y) to the objective function at point (x, γ) satisfies

$$\left(\sum_{e \in E} c_e x'_e + P(\gamma') \right) - \left(\sum_{e \in E} c_e x_e + P(\gamma) \right) \geq (c, y) \cdot ((x', \gamma') - (x, \gamma)).$$

Now fix any feasible solution (x^*, γ^*) , i.e., the one that satisfies $\sum_{e \in E} c_e x_e^* + P(\gamma^*) \leq \text{OPT}$. Substituting $(x', \gamma') = (x^*, \gamma^*)$ in the above equation we get,

$$\begin{aligned} 0 &= \text{OPT} - \text{OPT} > \left(\sum_{e \in E} c_e x_e^* + P(\gamma^*) \right) - \left(\sum_{e \in E} c_e x_e + P(\gamma) \right) \\ &\geq (c, y) \cdot (x^*, \gamma^*) - (c, y) \cdot (x, \gamma). \end{aligned}$$

Thus (c, y) defines a separating hyperplane between the point (x, γ) and any point (x^*, γ^*) that satisfies $\sum_{e \in E} c_e x_e^* + P(\gamma^*) \leq \text{OPT}$. Hence we have a polynomial time separation oracle for the objective function as well.

Thus we can solve (8) using the ellipsoid algorithm. The proof of Lemma 2 is hence complete.

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