# Approximating rooted connectivity augmentation problems

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**Abstract**

A graph is called *$\ell$-connected from $U$ to $r$* if there are $\ell$ internally disjoint paths from every node $u \in U$ to $r$. The *Rooted Subset Connectivity Augmentation Problem* (*RSCAP*) is as follows: given a graph $G = (V + r, E)$, a node subset $U \subseteq V$, and an integer $k$, find a smallest set $F$ of new edges such that $G + F$ is $k$-connected from $U$ to $r$. In this paper we consider mainly a restricted version of RSCAP in which the input graph $G$ is already $(k-1)$-connected from $U$ to $r$. For this version we give an $O(\ln |U|)$-approximation algorithm, and show that the problem cannot achieve a better approximation guarantee than the Set Cover Problem (SCP) on $|U|$ elements and with $|V| - |U|$ sets. For the general version of RSCAP we give an $O(\ln k \ln |U|)$-approximation algorithm.

For $U = V$ we get the *Rooted Connectivity Augmentation Problem* (*RCAP*). For directed graphs RCAP is polynomially solvable, but for undirected graphs its complexity status is not known: no polynomial algorithm is known, and it is also not known to be NP-hard. For undirected graphs with the input graph $G$ being $(k-1)$-connected from $V$ to $r$, we give an algorithm that computes a solution of size at most $opt + \min\{opt, k\}/2$, where $opt$ denotes the optimal solution size.

**Key words:** rooted connectivity, augmentation problems, hardness of approximation, approximation algorithms.

# 1 Introduction and preliminaries

A graph is called *$\ell$-connected from $U$ to $r$* if there are $\ell$ internally disjoint paths from every node in $U$ to $r$. In this paper we consider the following problem (for motivation see [4, 6, 2]):

*Rooted Subset Connectivity Augmentation Problem* (RSCAP):
Input: A graph $G = (V + r, E)$, a node subset $U \subseteq V$, and an integer $k$.
Output: A minimum size set $F$ of new edges such that $G + F$ is $k$-connected from $U$ to $r$.

For $G$ being $k_0$-connected from $U$ to $r$ and $k - k_0$ bounded by a polynomial in $n = |V|$, we give an $O(\ln|U|\ln(k - k_0 + 1))$-approximation algorithm for both a directed and an undirected RSCAP. On the other hand, we show that even for $k_0 = k - 1$, RSCAP cannot have a better approximation ratio than the Set Cover Problem (SCP) on $|U|$ elements and with $|V| - |U|$ sets.

For $U = V$ we get the *Rooted Connectivity Augmentation Problem* (RCAP). A generalization of the RCAP when one seeks an augmenting edge set of minimum weight is polynomially solvable for directed graphs [6], but is NP-hard for undirected graphs. However, the complexity status of an undirected RCAP (where every new edge has weight 1) is not known: no polynomial algorithm is known, and it is also not known to be NP-hard. The problem admits an easy 2-approximation algorithm. We give an algorithm that computes a solution of size at most $opt + \min\{opt, k\}/2$, where $opt$ denotes the optimal solution size.

**Remark** A more common definition of "$k$-connected to $r$" assumes that the graph is simple; in particular, $k \leq |V|$. In the corresponding version of RSCAP, the edges should be added while preserving simplicity; we refer to this version as a simple RSCAP. Our main results are valid for this version as well, except the $O(\ln|U|\ln(k-k_0+1))$-approximation algorithm for RSCAP.

RCAP is related to the extensively studied *Vertex-Connectivity Augmentation Problem*: given a graph $G$ and an integer $k$, find a smallest set $F$ of new edges so that $G+F$ is $k$-(node) connected. For directed graphs, Frank and Jordán [5] showed a polynomial algorithm. For undirected graphs, the complexity status of this problem is a long standing open question, and the following algorithms were obtained. A 2-approximation algorithm follows from [5]. For the case of $G$ being $(k-1)$-connected, Jordán [9, 10] gave an algorithm that computes a solution which size exceeds the optimum by at most $(k-1)/2$ edges. Jordán and Jackson [7] gave an algorithm that computes a solution with an additive gap being roughly $k(k-k_0)/2$ ($k_0$ is the initial connectivity of $G$); in [8] they gave an algorithm that for any fixed $k$ computes an optimal solution in polynomial time. We note that the techniques used in this paper were applied in [11] to give a simple and fast version of Jordán's algorithm [9, 10].

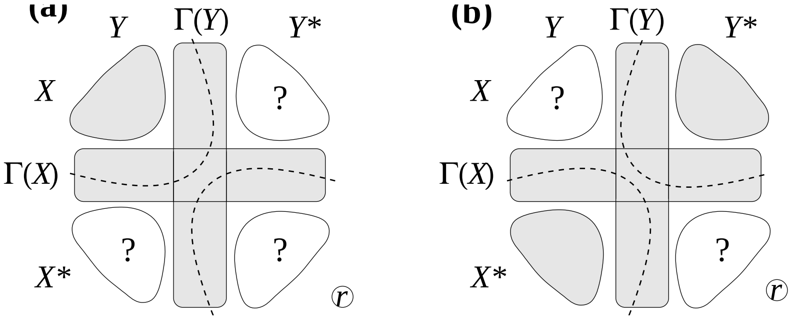


Figure 1: Illustration to (1),(2) and (3),(4).

This paper is organized as follows. In the rest of the section we give some notation and preliminary statements used in the paper. In Section 2 we will show that the hardness of approximation of RSCAP is the same as of the SCP, and give an almost matching approximation algorithm. In Section 3 we give our algorithm for the undirected RCAP, and discuss some applications and related results.

Here is some notation and preliminary statements used in the paper. An edge from $u$ to $v$ is denoted by $uv$. Given a graph, we call new edges that can be added to the graph *links*, to distinguish them from the existing edges. Let $opt = opt_k(G, U)$ denote the optimal solution size of an instance at hand. For an arbitrary edge set $F$ and a set $X$ let $\deg_F(X)$ denote the *degree* of $X$ with respect to $F$ (that is, the number of edges in $F$ from $X$ to $V - X$).

Let $G = (V + r, E)$ be a graph. For $X \subseteq V$ we denote by $\Gamma_G(X) = \Gamma(X)$ the set $\{v \in V - X : uv \in E \text{ for some } u \in X\}$ of *neighbors* of $X$ in $V$, and let $X^* = V - (X + \Gamma(X))$. It is easy to see that for any $X, Y \subseteq V$ the following holds (see Figure 1), where (1),(2) are valid for both directed and undirected graphs, and (3),(4) are valid for undirected graphs only.

$$\Gamma(X \cap Y) \subseteq (\Gamma(X) \cap Y) \cup (\Gamma(Y) \cap X) \cup (\Gamma(X) \cap \Gamma(Y)) \tag{1}$$

$$\Gamma(X \cup Y) \subseteq (\Gamma(X) \cap Y^*) \cup (\Gamma(Y) \cap X^*) \cup (\Gamma(X) \cap \Gamma(Y)) \tag{2}$$

$$\Gamma(X \cap Y^*) \subseteq (\Gamma(X) \cap Y^*) \cup (\Gamma(Y) \cap X) \cup (\Gamma(X) \cap \Gamma(Y)) \tag{3}$$

$$\Gamma(Y \cap X^*) \subseteq (\Gamma(X) \cap Y) \cup (\Gamma(Y) \cap X^*) \cup (\Gamma(X) \cap \Gamma(Y)) \tag{4}$$

Let $d_r(X)$ denote the number of edges going from $X$ to $r$, and define $g(X) = d_r(X) + |\Gamma(X)|$. Then (1),(2) and (3),(4) imply the following inequalities, where (5) is valid for both directed and undirected graphs, and (6) is valid for undirected graphs only.

$$g(X) + g(Y) \geq g(X \cap Y) + g(X \cup Y) \tag{5}$$

$$g(X) + g(Y) \geq g(X \cap Y^*) + g(Y \cap X^*) + d_r(X - Y^*) + d_r(Y - X^*) \tag{6}$$

Moreover, by counting the contribution to the sides of (5) and (6) of the nodes in $\Gamma(X), \Gamma(Y)$ and of the edges incident to $r$ we get:

**Proposition 1.1** *If equality holds in (5) then equality holds in each one of (1) and (2), and if (for an undirected graph) equality holds in (6) then* $d_r(X - Y^*) = d_r(Y - X^*) = 0$ *(in particular,* $d_r(X \cap Y) = 0$*) and equality holds in each one of (3) and (4).*

We say that $X$ is *$\ell$-tight* (or simply that $X$ is tight, if $\ell$ is understood) if $g(X) = \ell$. The following fact, which applies for both directed and undirected graphs, stems from Menger's Theorem.

**Fact 1.2** *$G$ is $\ell$-connected from $U$ to $r$ if and only if $g(X) \geq \ell$ for all $X \subseteq V$ with $X \cap U \neq \emptyset$.*

**Lemma 1.3** *Let $G$ be $\ell$-connected from $U$ to $r$, and let $X, Y$ be $\ell$-tight sets such that $X \cap Y \cap U \neq \emptyset$. Then $X \cap Y$ and $X \cup Y$ are both $\ell$-tight, and each one of (1),(2) holds with equality.*

**Proof:** By inequality (5) and Fact 1.2 we have

$$2\ell = g(X) + g(Y) \geq g(X \cap Y) + g(X \cup Y) \geq \ell + \ell = 2\ell.$$

Hence equality holds everywhere, and by Proposition 1.1 the claim follows. □

# 2 Rooted Subset Connectivity Augmentation

**Theorem 2.1** *For the restriction of a directed RSCAP to instances in which $G$ is $(k-1)$-connected from $U$ to $r$, there exists:*

*(i) An $O(\ln |U|)$-approximation algorithm, and*

*(ii) For any $k \geq 1$, a polynomial time reduction from the Set Cover Problem (SCP) on universe $U$ with $|V| - |U|$ sets such that there is a solution of size $\tau$ to SCP if and only if there is a solution of size $\tau$ to RSCAP.*

Part (ii) of the theorem says that if one finds an algorithm with an approximation guarantee better than $O(\ln |U|)$ for the above restricted version of the RSCAP, then one can get an approximation guarantee better than $O(\ln |U|)$ for the SCP on groundset $U$; the latter is possible only if P=NP, see [13].

To prove the theorem, we will use the following well-known formulation of the SCP; in this formulation, $J$ is the incidence graph of sets and elements, where $A$ is the family of sets and $B$ is the universe.

Input: A bipartite graph $J = (A + B, I)$ without isolated nodes.
Output: A minimum size subset $D \subseteq A$ such that $\Gamma_J(D) = B$.

The proof of Theorem 2.1 follows, starting with part (i). Given an instance of the directed RSCAP with the input graph $G$ being $\ell$-connected from $U$ to $r$, $\ell < k$ arbitrary, we construct an instance $J = (A + B, I)$ of the SCP as follows: $B$ is the family of the inclusion minimal sets among the $\ell$-tight sets intersecting $U$, $A = V$, and for $a \in A, b \in B$ we have $ab \in I$ if, and only if, the subset of $V$ corresponding to $b$ contains $a$. Note that $|B| \leq |U|$, by Lemma 1.3. The above construction is polynomial, since for every node $s \in U$ we can compute the unique set in $B$ containing $s$ (or determine that such does not exist) in polynomial time using max-flow techniques. Specifically, one can view $G$ as a max-flow network with source $s$ and sink $r$ where the nodes in $V - s$ and all edges have unit capacity. Apply a standard conversion of node capacities to edge capacities: replace every node $v \in V - s$ by the two nodes $v^+, v^-$ connected by the edge $v^+v^-$ having the same capacity as $v$, and redirect the heads of the edges entering $v$ to $v^+$ and the tails of the edges leaving $v$ to $v^-$. In the resulting network, compute a maximum $(s, r)$-flow. If its value is $\ell$, then in the corresponding residual network the set of nodes $\{v \in V : v^+, v^- \text{ are both reachable from } s\}$ is the minimal $\ell$-tight set containing $s$; otherwise, such set does not exist.

Let $\tau^*$ be the optimal value of the following LP-relaxation for the obtained instance of the SCP:

$$\tau^* = \min \left\{ \sum_{a \in A} x_a : \sum_{a \in \Gamma_J(b)} x_a \geq 1 \quad \forall b \in B, \quad x_a \geq 0 \quad \forall a \in A \right\}.$$

By a well-known result of Lovász [12], the greedy algorithm (which repeatedly removes from $J$ the node of maximum degree in $A$ and all its neighbors, until $B$ becomes empty) computes a feasible solution $D \subseteq A$ to the SCP of size at most $H(|B|)\tau^*$, where $H(j)$ denotes the $j$th harmonic number. By Fact 1.2, $G + \{vr : v \in D\}$ is $(\ell + 1)$-connected from $U$ to $r$. We claim that $|D| \leq \frac{1}{k-\ell} H(|U|) opt_k$. Let $F$ be a link set such that $G + F$ is $k$-connected from $U$ to $r$, and let $x$ be the vector on $A = V$ defined by $x_v = \frac{1}{k-\ell}\deg_F(v)$. Since $\deg_F(X) \geq k - \ell$ for any $\ell$-tight set $X$ of $G$, $x$ is a feasible solution to the above LP-relaxation. Thus $|D| \leq H(|B|)\tau^* \leq H(|B|) \sum_{v \in V} x_v = H(|U|) \frac{1}{k-\ell} |F|$. Consequently, the algorithm finds a link set that augments $G$ to be $(\ell + 1)$-connected from $U$ to $r$ of size at most $\frac{1}{k-\ell} H(|U|) opt_k$. Thus we have proved the following statement, which for $k_0 = k - 1$ implies part (i) of Theorem 2.1:

**Corollary 2.2** *There exists an $H(|U|)H(k - k_0)$-approximation algorithm for the directed RSCAP with the input graph $G$ being $k_0$-connected from $U$ to $r$ and $k - k_0$ bounded by a polynomial in $n$.*

To prove part (ii) of Theorem 2.1, we will show that given an instance $J = (A + B, I)$ of the SCP, one can construct in polynomial time an instance $G = (V + r, E)$ of the directed RSCAP with $k_0 = k - 1$, $V = A + B$, and $U = B$, and such that:
(a) For any solution $F'$ for the RSCAP there exists a solution $F$ with $|F| = |F'|$ such that every edge in $F$ connects some node in $V - U = A$ to $r$.
(b) $D \subseteq A$ is a solution to the SCP on $J$ if, and only if, $F = \{vr : v \in D\}$ is a solution to the RSCAP on $G$.

Note that by Fact 1.2, replacing any edge $xy$ in a directed graph which is $k$-connected from $U$ to $r$ by a new edge $xr$ results again in a graph that is $k$-connected from $U$ to $r$. This implies that for any feasible solution $F'$ for a directed RSCAP there always exists a feasible solution $F$ with $|F| = |F'|$ such that $r$ is the head of all the edges in $F$.

Given an instance $J = (A + B, I)$ for the SCP, we construct an instance $G = (V + r, E)$ for a directed RSCAP by directing the edges in $J$ from $B$ to $A$, adding a new node $r$ and $k - 1$ edges from each node in $B$ to $r$, and setting $U = B$. Then $G$ is $(k - 1)$-connected from $U$ to $r$, and by Fact 1.2, (b) holds. Now let $F'$ be a set of links incident to $r$ such that $G + F'$ is $k$-connected from $U$ to $r$. If there is $ur \in F'$ with $u \in U$, then $\Gamma_G(u) \neq \emptyset$ (since $J$ has no isolated nodes), and for any $a \in \Gamma_G(u)$ the graph $G + F$ where $F = F' - ur + ar$ is $k$-connected from $U$ to $r$. Thus for the obtained instance of the RSCAP (a) holds. This finishes the proof of Theorem 2.1.

Let us now show similar results for the undirected RSCAP. Using standard constructions, it is easy to prove that a $\rho$-approximation algorithm for the directed RSCAP implies a $2\rho$-approximation algorithm for the undirected RSCAP. In particular, by Corollary 2.2, there exists a $2H(|U|)H(k - k_0)$-approximation algorithm for the undirected RSCAP. However, the construction used to prove part (ii) of Theorem 2.1 does not work for the undirected RSCAP. In fact, for $k = 1$ the undirected RSCAP is polynomially solvable: one just needs to add a tree on the connected components that intersect $U + r$. We describe a slightly more complicated construction, which transfers an instance $J = (A + B, I)$ of SCP to an instance of the undirected RSCAP with $k \geq |B| + 1$ arbitrary, and $k_0 = k - 1$, such that if one finds a solution of size $\tau$ to the corresponding instance of the RSCAP then one can find a solution of size at most $2\tau$ to the SCP.

Given an instance $J = (A + B, I)$ for the SCP, we convert it into a new instance $\mathcal{J} = (\mathcal{A} + B, \mathcal{I})$ for the SCP by replacing every node $a \in A$ by a set $\{a_1, \ldots, a_{k_0}\}$ of $k_0$ copies of $a$ and every edge $ab \in I$ by the edge set $\{a_1 b, \ldots, a_{k_0} b\}$, where $k_0 \geq |B|$ arbitrary. For every $D \subseteq A$ let $\mathcal{D}$ denote an arbitrary set obtained by choosing for every $a \in D$ at least one copy of $a$. Clearly, $D$ is a solution to the SCP on $J$ if, and only if, $\mathcal{D}$ is a solution to the SCP on

$\mathcal{J}$, that is if, and only if, $\mathcal{D} \subseteq \mathcal{A}$ and $\mathcal{D}$ intersects the set $\Gamma_{\mathcal{J}}(b) + b$ for every $b \in B$. Let $\Gamma^2(b) = \Gamma_J(\Gamma_J(b)) - b$. Construct an instance $G$ for an undirected RSCAP by adding to $\mathcal{J}$ a new node $r$ and $k_0 - |\Gamma^2(b)| \geq 1$ edges from each node $b \in B$ to $r$, and setting $U = B$. Note that in the resulting graph $G$ there is an edge $rb$ for every $b \in B$. It is not hard to see that $G$ is $k_0$-connected from $U$ to $r$; for every $b \in U$ there are $|\Gamma^2(b)|$ internally disjoint paths between $b$ and $r$ of length 3 each, and there are additional $k_0 - |\Gamma^2(b)|$ edges between $b$ and $r$. Let $F$ be a solution for the obtained instance of the RSCAP, and let $Z$ be set of endnodes of the links in $|F|$. Then $Z$ intersects every minimal tight set. By the construction, for every $b \in B$ the set $\Gamma_G(b) + b$ is $k_0$-tight, and it is not hard to see that it is the minimal $k_0$-tight set containing $b$. This implies that there exists a set $\mathcal{D} \subseteq \mathcal{A}$ with $|\mathcal{D}| \leq |Z| \leq 2|F|$ such that $\mathcal{D}$ also intersects every minimal tight set. By the above, the corresponding set $D \subseteq A$ is a solution of size at most $2|F|$ to the SCP on $J$, as required.

**Remark** Our hardness results for RSCAP easily extend to the simple RSCAP by adding to the graph $G$ constructed a clique on $k_0$ nodes, redirecting every edge entering $r$ to the nodes of the clique so that no parallel edges arise, and connecting every node of the clique to $r$. Another (simpler but less compact) possibility is to subdivide every edge entering $r$ in the graph $G$ constructed.

# 3 Undirected Rooted Connectivity Augmentation

In the rest of the paper we consider an undirected RCAP with $k_0 = k - 1$; that is, we will assume that $G$ is $(k-1)$-connected (from $V$) to $r$, and "tight" means $(k-1)$-tight. By Lemma 1.3, the (inclusion) minimal tight sets are pairwise disjoint, and let $\nu = \nu(G)$ denote their number. For $T \subseteq V$, the $T$-*components* are the connected components of $G - T$, and the $T$-components not containing $r$ are the *sides* of $T$. Let $b(T)$ be the number of $T$-components. If $|T| = k-1$ and $b(T) \geq 3$ then $T$ is a *shredder*. Let $b(G) = \max\{b(T) : T \subset V, |T| = k-1\}$. If $G+F$ is $k$-connected to $r$ then $|F| \geq \nu(G)/2$ (since $\deg_F(X) \geq 1$ for every tight set $X \subseteq V$) and $|F| \geq b(G) - 1$ (since for any $T \subseteq V$ with $|T| = k-1$, $F$ must induce a connected graph on the $T$-components). Thus

$$opt(G) \geq \max\{\lceil \nu(G)/2 \rceil, b(G) - 1\}.$$

For $k - 1 = 0$ any tree on the components of $G$ is an augmenting link set of size $b(G) - 1$. For $k - 1 = 1$ it is possible to compute an optimal solution using the lower bound $\max\{\lceil \nu'(G)/2 \rceil, b(G) - 1\}$, where $\nu'(G) = \nu(G) + 1$ if there is a tight set that contains all the minimal tight sets, and $\nu'(G) = \nu(G)$ otherwise. For $k - 1 \geq 2$ we prove the following theorem:

**Theorem 3.1** *There is a polynomial algorithm that given a graph $G$ which is $(k-1)$-connected to $r$ finds a link set $F$ of size at most $opt + \min\{opt, k\}/2$ such that $G + F$ is $k$-connected to $r$. Furthermore, $|F| \leq \max\{\lceil \nu(G)/2 \rceil + \lceil k/2 \rceil - 1, b(G) - 1\}$.*

**Remark** Note that the lower bound $opt(G) \geq \max\{\lceil \nu(G)/2 \rceil, b(G) - 1\}$ is not sufficient to get for some $\varepsilon > 0$ an approximation ratio $2 - \varepsilon$; see Example 2 at the end of Section 3.1.

Two disjoint subsets $X, Y$ of $V$ are *adjacent* if there is an edge with one end in $X$ and the other end in $Y$. The following lemma summarizes some simple properties of tight sets used in the rest of the paper.

**Lemma 3.2** *Let $X, Y$ be two tight sets in $G$.*

*(i) If $X \cap Y \neq \emptyset$ then $X \cap Y, X \cup Y$ are both tight and each one of (1),(2) holds with equality.*

*(ii) If the sets $X \cap Y^*, Y \cap X^*$ are nonempty, then they are both tight and nonadjacent, $d_r(X - Y^*) = d_r(Y - X^*) = 0$ (in particular, $d_r(X \cap Y) = 0$), and each one of (3),(4) holds with equality.*

*(iii) If $X, Y$ are disjoint and $|X| \leq |Y|$ then exactly one of the following holds:*

*(a) $X \cap Y^*, Y \cap X^*$ are nonempty and thus (ii) holds, or*

*(b) $X \subseteq \Gamma(Y)$.*

**Proof:** Part (i) follows from Lemma 1.3, while the proof of part (ii) is similar to that of (i) using inequality (6) instead of (5). We prove part (iii). Assume that at least one of $X \cap Y^*, Y \cap X^*$ is empty. We claim that then $X \cap Y^* = \emptyset$, which implies $X \subseteq \Gamma(Y)$ since $X, Y$ are disjoint. If not, then $Y \subseteq \Gamma(X)$, $|Y| - |X \cap \Gamma(Y)| \geq 1$, and we get a contradiction:

$$g(X \cap Y^*) \leq g(X) - |\Gamma(X) \cap Y| + |X \cap \Gamma(Y)| = g(X) - (|Y| - |X \cap \Gamma(Y)|) \leq g(X) - 1 = k - 2.$$

$\square$

**Remark** The following statement shows that Theorem 3.1 is valid for the simple RCAP.

**Proposition 3.3** *Let $G = (V + r, E)$ be $(k-1)$-connected to $r$ with $|V| \geq k$, and let $G + F$ be $k$-connected to $r$. Then there exists a polynomial time algorithm that finds a link set $F'$ with $|F'| \leq |F|$ such that $G + F'$ is $k$-connected to $r$ and has no pair of parallel edges so that at least one of them belongs to $F'$. In particular, if $G$ is simple, so is $G + F'$.*

**Proof:** We can assume that $|\Gamma_{G+F}(r)| \geq k$, as otherwise by Menger's Theorem there cannot be $k$ internally disjoint paths from $v \in V - \Gamma_{G+F}(r)$ to $r$ (such $v$ exists since $|V| \geq k$). Suppose that there is $f \in F$ so that there is $e \in E + F$ parallel to $f$. If $G + F - f$ is also $k$-connected to $r$ we can simply delete $f$; it is easy to see that this is the case if $f$ is not incident to $r$. Assume therefore that $f = ur$ for some $u \in V$. Then $H = (G + F) - f$ has one minimal tight set $C$, and one maximal tight set $S$. Clearly, $u \in C$. Note that $V - (S + \Gamma_H(S))$ contains a neighbor of $r$, say $v$, since $|\Gamma_H(r)| = |\Gamma_G(r)| \geq k$ and since $S$ is tight in $H$. Let $f' = uv$. Then $H + f'$ is $k$-connected to $r$, and $H$ contains no edge parallel to $f'$. By repeatedly replacing every link $f \in F$ as above by an appropriate link $f'$, we obtain a link set $F'$ as required. $\square$

Given an instance $(G = (V + r, E), k)$ of an undirected simple RCAP so that $G$ is $(k-1)$-connected to $r$, we compute an edge set $F$ as in Theorem 3.1, and then replace $F$ by $F'$ as in Proposition 3.3.

## 3.1 Independent families

**Definition 3.1** *A family $\mathcal{F}$ of pairwise disjoint tight sets is* independent *if there exists a partition $\Pi$ of $\mathcal{F}$ and a family $\mathcal{S}(\mathcal{F}) = \{S_\mathcal{P} : \mathcal{P} \in \Pi\}$ of pairwise disjoint tight sets such that:*

*(i) For every $\mathcal{P} \in \Pi$ holds: $\cup\{S : S \in \mathcal{P}\} \subseteq S_\mathcal{P}$, if $|\mathcal{P}| \neq 2$ then equality holds, and if $|\mathcal{P}| \geq 3$ then $\mathcal{P}$ consists of some sides of a shredder.*

*(ii) For any disjoint $X, Y \in \mathcal{F} \cup \mathcal{S}(\mathcal{F})$ holds: $X - \Gamma(Y), Y - \Gamma(X)$ are both nonempty if, and only if, $X, Y$ belong to the same part in $\Pi$.*

*If in addition to (i) and (ii), for any part $\mathcal{P} = \{S_i, S_j\} \in \Pi$ we have that any tight set that is intersecting with $S_\mathcal{P}$ is contained in one of $S_i, S_j$ then $\mathcal{F}$ is* strongly independent.

It is not hard to verify that any subfamily of an independent family is also independent. We call an independent family *trivial* if the corresponding partition is trivial, that is if $\Pi = \{\mathcal{F}\}$. Let $\beta(G)$ denote the maximum cardinality of an independent family in $G$. Note that even trivial independent families strictly generalize shredders. Indeed, any subfamily of sides of a shredder forms a trivial independent family; thus $\beta(G) \geq b(G) - 1$. However, even trivial independent families with two sets might not correspond to a shredder, see Example 1 below. To prove Theorem 3.1 we use the following new lower bound:

**Lemma 3.4** *Let $\mathcal{F}$ be a nonempty independent family in a $(k-1)$-connected to $r$ graph $G$. Then $opt(G) \geq |\mathcal{F}| \geq \beta(G)$.*

**Proof:** The statement will be proved if we can show that for any link $e = xx'$, $G^e = G + e$ contains an independent family $\mathcal{F}^e$ with $|\mathcal{F}^e| \geq |\mathcal{F}| - 1$. Let $\Pi$ and $\mathcal{S}(\mathcal{F})$ be as in the definition of an independent family. Let $Q = \cup\{S : S \in \mathcal{S}(F)\}$. If $x, x' \notin Q$ then $\mathcal{F}$ remains independent in $G^e$. If $x' \notin Q$ and $x \in S_{\mathcal{P}}$ for some $P \in \Pi$, then $\mathcal{F}^e = \mathcal{F} - \{X\}$ is as required, where $X \in \mathcal{P}$ so that $x \in X$ if $|\mathcal{P}| \neq 2$ and $x \notin Y$ if $\mathcal{P} = \{X, Y\}$.

Assume henceforth that $x \in S = S_{\mathcal{P}}$ and $x' \in S' = S_{\mathcal{P}'}$ for some $\mathcal{P}, \mathcal{P}' \in \Pi$. If $\mathcal{P} = \mathcal{P}'$ then there are $X, X' \in \mathcal{P}$ so that $x, x' \in X \cup X'$. Let $Z = S_{\mathcal{P}}$ if $|\mathcal{P}| = 2$ and $Z = X \cup X'$ otherwise. Then $\mathcal{F}^e = \mathcal{F} + \{Z\} - \{X, X'\}$ is as required (with $\mathcal{S}(\mathcal{F}^e) = \mathcal{S}(\mathcal{F})$).

Assume that $\mathcal{P} \neq \mathcal{P}'$. It is easy to see that if $|\mathcal{P}|, |\mathcal{P}'| \neq 2$ then in $G^e$, $\mathcal{F} - \{X\}$ is independent or $\mathcal{F} - \{X'\}$ is independent, where $x \in X \in \mathcal{P}$ and $x' \in X' \in \mathcal{P}'$.

Assume therefore that $|\mathcal{P}| = 2$ or $|\mathcal{P}'| = 2$, say $|\mathcal{P}'| = \{X', Y'\}$ and $x' \notin Y'$. Consider the case $|\mathcal{P}| \neq 2$. Let $X \in \mathcal{P}$ so that $x \in X$. If $x' \in \Gamma(X)$ then $\mathcal{F}^e = \mathcal{F} - \{X'\}$ is as required. Otherwise, one can verify that $\mathcal{F}^e = \mathcal{F} - \{X\}$ is as required.

The remaining case is when $x \in S = S_{\mathcal{P}}$ and $x' \in S' = S_{\mathcal{P}'}$ for some $\mathcal{P} \neq \mathcal{P}' \in \Pi$, and $\mathcal{P} = \{X, Y\}$, $\mathcal{P}' = \{X', Y'\}$. W.l.o.g. assume that $x \notin Y$ and $x' \notin Y'$. Then $Y, Y'$ are tight in $G^e$. We claim that: $X, S$ are both tight in $G^e$ or $X', S'$ are both tight in $G^e$, say w.l.o.g the former holds. Then $\mathcal{F}^e = \mathcal{F} - \{X'\}$ is as required, with $\mathcal{S}(\mathcal{F}^e) = \mathcal{S}(F) - \{S'\} + \{Y'\}$. W.l.o.g. we will show that if $X', S'$ are not tight in $G^e$ then $X, S$ are both tight in $G^e$. Since $S'$ is not tight in $G^e$

$$x \notin \Gamma(S'). \tag{7}$$

Thus $S' \subseteq \Gamma(S)$, by condition (ii) in the definition of an independent family. This implies that $S$ is tight in $G^e$. It remains to show that $X$ is tight in $G^e$. Suppose to the contrary that this is not so. Then we must have $x \in X$ and $x' \notin \Gamma(X)$. Thus $X \subseteq \Gamma(X')$, by condition (ii) in the definition of an independent family. This implies $x \in \Gamma(X')$. Consequently, $x \in \Gamma(S')$, contradicting (7). $\square$

Let $\mathcal{R}$ be the following relation on tight sets: $(X, Y) \in \mathcal{R}$ if $X - \Gamma(Y)$ and $Y - \Gamma(X)$ are both nonempty. Given a family $\mathcal{F}$ of tight sets, let $\mathcal{R}(\mathcal{F})$ denote the restriction of $\mathcal{R}$ to $\mathcal{F}$. Clearly, $\mathcal{R}(\mathcal{F})$ is symmetric and reflexive, and, if $\mathcal{F}$ is independent, then $\mathcal{R}(\mathcal{F})$ is an equivalence, with the corresponding partition into equivalence classes $\Pi$ as in the definition. Note that condition (ii) in the definition of an independent family implies $S' \subseteq \Gamma(S'')$ or $S'' \subseteq \Gamma(S')$ for any distinct $S', S'' \in \mathcal{S}(\mathcal{F})$. But Lemma 3.2(iii) implies a stronger statement:

**Proposition 3.5** *If $|S''| \leq |S'|$ for distinct $S', S'' \in \mathcal{S}(\mathcal{F})$ then $S'' \subseteq \Gamma(S')$.*

If $\mathcal{F}$ is a trivial independent family, then $|\mathcal{F}|$ can be as large as $n-k+1$. However, as Proposition 3.6 below implies, a nontrivial independent family has at most $k-1$ sets; Examples 2,3 below show that this bound is tight. For a family $\mathcal{F}$ of sets, let $||\mathcal{F}||$ denote the cardinality of the union of the sets in $\mathcal{F}$.

**Proposition 3.6** *Let $\mathcal{F}$ be a nontrivial independent family, and let $S'=S_{\mathcal{P}'}$ be the largest set in $\mathcal{S}(\mathcal{F})$. Then $|\mathcal{P}'|+||\Pi-\mathcal{P}'||\leq k-1$.*

**Proof:** We need the following claim:
*Claim: Let $Y$ be an $\ell$-tight set, and suppose that there is a node $v\in Y$ such that there are $\ell$ internally disjoint paths from $r$ to $v$. Then for any set $X\subseteq V$ disjoint to $Y$ holds:*

$$d_r(X)+|\Gamma(X)-(Y\cup\Gamma(Y))|\geq|X\cap\Gamma(Y)|.$$

*Proof:* Consider a set of $\ell$ internally disjoint paths from $r$ to $v$. Then $|X\cap\Gamma(Y)|$ of them contain a node from $X$. In each of these $|X\cap\Gamma(Y)|$ paths, pick the first node whose successor is in $X$. Such a node is either $r$ or in $\Gamma(X)-(Y\cup\Gamma(Y))$, so it contributes 1 to the left side of the inequality. □

Note that $|S_{\mathcal{P}}|\geq||\mathcal{P}||$ and $|S_{\mathcal{P}}|=||\mathcal{P}||$ if $|\mathcal{P}|\neq 2$ for any $\mathcal{P}\in\Pi$. Let $S''=S_{\mathcal{P}''}$ be the second largest set in $\mathcal{S}(\mathcal{F})$. Then $|\Gamma(S'')\cap S'|\geq|\mathcal{P}'|$ (by condition (ii) in Definition 3.1) and $S''\cap\Gamma(S')=S''$ (by Proposition 3.5). Applying the claim above on $S'=Y$ and $S''=X$ gives:

$$\begin{aligned}k-1=g(S'') &= d_r(S'')+|\Gamma(S'')-(S'\cup\Gamma(S'))|+|\Gamma(S'')\cap S'|+|\Gamma(S'')\cap\Gamma(S'))|\geq\\ &\geq |S''|+|\mathcal{P}'|+||\Pi-\mathcal{P}'-\mathcal{P}''||=|\mathcal{P}'|+||\Pi-\mathcal{P}'||.\end{aligned}$$

□

**Examples:**
**1.** Let $u,v$ be two nodes of a cycle, where $r\neq u,v$ is arbitrary, and $k-1=2$. Then $\mathcal{F}=\{\{u\},\{v\}\}$ is an independent family. If $u,v$ are adjacent, then $\mathcal{F}$ is nontrivial and strongly independent. Assume that $u,v$ are nonadjacent. Then $\mathcal{F}$ is trivial and not strongly independent. To consider some modifications, assume that none of $u,v$ is incident to $r$. Let $P$ be the path between $u$ and $v$, let $u'$ be the neighbor of $u$ not belonging to $P$ and define $v'$ in the same way. Let $G$ be the graph obtained by connecting each of $u',v'$ to all the internal nodes of $P$. If $P$ has at least two internal nodes, then $\mathcal{F}$ is strongly independent in $G$. Otherwise ($P$ has one internal node) $\mathcal{F}$ is not strongly independent, but the family $\{\{u,u'\},\{v,v'\}\}$ is strongly independent; $\mathcal{F}$ becomes strongly independent if we add to $G$ the links $ru',rv'$. Note that there are no shredders in the graphs considered.

**2.** Let $G = K_k$ be a complete graph on $k$ nodes. Then $\lceil \nu(G)/2 \rceil = \lceil (k-1)/2 \rceil$ and $b(G) - 1 = 0$, but $opt(G) = \beta(G) = k - 1$. Here any family of pairwise disjoint tight sets forms a trivial independent family. Let us replace one node of $G$ by a clique of size at least $k-1$, connecting the edges of $K_k$ to distinct nodes of the clique. The new graph contains a nontrivial independent family of size $k-1$.
**3.** Let $G = K_{k-1,k-1}$ be a complete bipartite graph with $k-1$ nodes on each side and parts $R, S$ where $r \in R$. Then $\lceil \nu(G)/2 \rceil = \lceil (2k-3)/2 \rceil = k-1$ and $\beta(G) = b(G) + 1 = k - 1$. Indeed, $b(G) = b(S) = k-2$, and $\beta(G) = |R| - 1 + 1 = k - 1$ since $R - r + s$ is an independent family for any $s \in S$ (so there are $k-1$ distinct nontrivial independent families of size $\beta(G) = k-1$ in $G$). Also, $opt(G) = k - 2 + \lceil (k-1)/2 \rceil$. An optimal augmenting link set is obtained by connecting every node in $R - r$ to $r$, picking a maximum matching on $S$, and if $k-1$ is odd adding one more link from the unmatched node in $S$ to $r$.
**4.** Consider the tree $G$ obtained by connecting a node $r$ to the root of a complete binary tree with $\nu$ leaves, where $k - 1 = 1$. Then $\nu(G) = \nu$, $b(G) - 1 = \beta(G) = 2$, but $opt(G) = \lceil (\nu+1)/2 \rceil$.

## 3.2 Main results

A tight set is a *core* if it contains a unique minimal tight set. By Lemma 3.2(i), the union and the intersection of any two intersecting cores are also cores. Thus for every minimal core $C$ (that is, a minimal tight set) there exists a unique maximal core $S$ containing it. Let $C_1, \ldots, C_\nu$ be the minimal cores of $G$. By Fact 1.2 $G + F$ is $k$-connected to $r$ if, and only if, $G + F$ has no cores; thus the graph $G + \{v_i r : v_i \in C_i, i = 1, \ldots, \nu\}$ is $k$-connected to $r$, and $opt(G) \leq \nu(G)$. We say that a link $e$ is $(\nu, 2)$-*reducing* for $G$ if $\nu(G+e) \leq \nu(G) - 2$. To prove Theorem 3.1 we use Proposition 3.6, Lemma 3.4, and the following two theorems:

**Theorem 3.7** *Let $G = (V + r, E)$ be $(k-1)$ connected to $r$ and let $\mathcal{F}$ be a subfamily of the family of maximal cores of $G$. Then exactly one of the following holds:*

*(i) there is a $(\nu, 2)$-reducing link for $G$ connecting two distinct cores in $\mathcal{F}$, or*

*(ii) $\mathcal{F}$ is strongly independent.*

*In particular, if (ii) holds and if $|\mathcal{F}| \geq k$ then the sets in $\mathcal{F}$ are sides of the same shredder.*

**Theorem 3.8** *Let $G = (V + r, E)$ be an undirected graph which is $(k-1)$ connected to $r$. If $b(G) \geq k$, then there exists a polynomial time algorithm that finds a link set $F$ of size at most* $\max\{\lceil (\nu(G)+1)/2 \rceil, b(G) - 1\}$ *such that $G + F$ is $k$-connected to $r$.*

**Proof of Theorem 3.1:** The algorithm is as follows:

*If* $b(G) \geq k$, find an augmenting link set as in Theorem 3.8.
*Else*, perform the following two steps:
1. Find and add a $(\nu, 2)$-reducing link, as long as one exists.
2. In the resulting graph, add one link from every minimal core to $r$.

If $b(G) \geq k$ the algorithm finds an augmenting link set as required, by Theorem 3.8. Suppose that $b(G) \leq k-1$, and let $F_1, F_2$ be the link sets added at steps $1, 2$, respectively. Then the resulting graph $G + F_1 + F_2$ is $k$-connected to $r$, by Fact 1.2. By Theorem 3.7, Proposition 3.6, and Lemma 3.4 $|F_2| = \nu(G + F_1) \leq \min\{opt, k-1\}$. Thus

$$|F_1| + |F_2| = (\nu - |F_2|)/2 + |F_2| = \lfloor(\nu + |F_2|)/2\rfloor = \lceil \nu/2 \rceil + \min\{opt/2, \lceil k/2 \rceil - 1\}.$$

□

An edge $e$ of a graph $H$ is *critical* w.r.t. a certain property if $H$ satisfies this property but $H - e$ does not. *Splitting off* two edges $su, sv$ means replacing them by a new edge $uv$. Using Theorem 3.7 we prove the following "splitting off" theorem:

**Theorem 3.9** *Let $H = (V + r, E)$ be $k$-connected from $V - s$ to $r$, where $s$ is a neighbor of $r$, such that every edge $sv$ of $H$, $v \neq r$ is critical with respect to $k$-connectivity from $V - s$ to $r$. Then either*

*(i) there exists a pair of edges $su, sv$ with $u, v \in \Gamma_H(s)$ that can be split-off while preserving the $k$-connectivity from $V - s$ to $r$, or*

*(ii) the family of maximal cores of $G = H - s$ is independent.*

**Proof:** Let $H$ and $G$ be as in the theorem. Clearly, $G$ is $(k-1)$-connected from $V - s$ to $r$.

*Claim:* $\nu(G) = |\Gamma_H(s)|$ and every minimal core of $G$ contains exactly one node from $\Gamma_H(s)$.

*Proof:* It is clear that every tight set $X$ of $G$ contains at least one node from $\Gamma_H(s)$, as otherwise $X$ is a tight set of $H$ intersecting $V - s$, contradicting that $H$ is $k$-connected from $V - s$ to $r$. For an edge $sv$ of $H$, let $X_v \subseteq V - s$ be a minimal tight set of $H - sv$ that intersects $V - s$; such exists, by the criticality of the edges incident to $s$. Then exactly one of the following holds: (i) $s \notin X_v$ and $X_v \cap \Gamma_H(s) = \{v\}$, or (ii) $s \in X_v$ (and $X_v \cap \Gamma_H(v) = \{s\}$).

If case (i) holds for every $v \in \Gamma_H(s)$, then every minimal core of $G$ contains at most, and thus exactly, one node from $\Gamma_H(s)$, as required. We will show that case (ii) cannot hold. Suppose to the contrary that $s \in X_v$ for some node $v \in \Gamma_H(s)$. Since $X_v$ intersects $V - s$, and since $s$ is a neighbor of $r$ in $H - sv$, then $X_v - s$ is also a tight set of $H - sv$ that intersects $V - s$, contradicting the minimality of $X_v$. $\square$

The above claim implies that there exists a pair of edges $su, sv$ of $H$ that can be split-off while preserving $k$-connectivity from $V - s$ to $r$ if and only if $G + uv$ has exactly $|\Gamma_H(s)| - 2 = \nu(G) - 2$ minimal cores, that is if and only if the edge $uv$ is $(\nu, 2)$-reducing for $G$. The proof of the theorem follows via Theorem 3.7. $\square$

We note that Theorem 3.9 is related to (but is also independent of) similar theorems in [1], [9, 10], and [2]. Provided that (A) $\deg(s) \geq k + 2$ and (B) $|V| \geq 2k$, these theorems give a characterization when there exists a pair of edges incident to $s$ that can be split-off while preserving: "global" $k$-connectivity in [1] and [9, 10], and $k$-connectivity from $V$ to $s = r$ in [2]. Our Theorem 3.9 which considers a different but related setting, gives a necessary and sufficient condition without restrictions (A) and (B). However, if (A) holds, then our characterization takes a similar form to the one given in [2, Theorem 3].

Let us call a sequence $F^* = (e_1, \ldots, e_p)$ of links $(\nu, 2)$-reducing for $G$ if $e_i$ is $(\nu, 2)$-reducing for $G + \{e_1, \ldots, e_{i-1}\}$, $i = 1, \ldots, p$. Let $\zeta(G)$ denote the maximum length of a $(\nu, 2)$-reducing link sequence for $G$. Using Theorem 3.7 and Lemma 3.4 we prove the following theorem:

**Theorem 3.10** *Let $G$ be $(k-1)$-connected to $r$. Then $opt(G) = \nu(G) - \zeta(G)$.*

Note that computing a maximum length $(\nu, 2)$-reducing sequence for $G$ is not equivalent to finding a maximum matching in the graph induced on the minimal cores by the $(\nu, 2)$-reducing links (formally, the nodes of this graph are the minimal cores of $G$, and we connect two cores by an edge if and only if there exists a $(\nu, 2)$-reducing link connecting them); see Example 4 in Section 3.1.

We also derive the following upper bound on the number of shredders.

**Theorem 3.11** *Let $G = (V + r, E)$ be $(k-1)$-connected to $r$. Then the number of shredders in $G$ is at most $(2|V| - 2|\Gamma(r)| - 1)/3 \leq (2|V| - 2k + 1)/3$.*

Theorems 3.7 and 3.10 are proved in Section 3.3. Theorems 3.8 and 3.11 are proved in Section 3.4.

## 3.3 Proof of Theorems 3.7 and 3.10

Let $C_1, \ldots, C_\nu$ be the minimal cores of $G$. For $I \subseteq \{1, \ldots, \nu\}$, let $\mathcal{S}_I$ denote the collection of tight sets containing $\cup_{i \in I} C_i$ and not containing any other minimal core; note that $\mathcal{S}_\emptyset = \emptyset$. Let $S_I$ be the union of the sets in $\mathcal{S}_I$; we set $S_I = \emptyset$ if $\mathcal{S}_I = \emptyset$. By Lemma 3.2(i), if $\mathcal{S}_I \neq \emptyset$

then $S_I$ is tight, and thus it is the inclusion maximal set in $\mathcal{S}_I$. Also, for any $I' \subset I$, $S_{I'} \subset S_I$ holds, unless $S_I = \emptyset$. For simplicity, $S_{ij}$ means $S_{\{i,j\}}$ and $S_i = S_{\{i\}} = S_{ii}$. Note that $S_I \cap S_J = S_{I \cap J}$ for any $I, J \subseteq \{1, \dots, \nu\}$ with $\mathcal{S}_I, \mathcal{S}_J \neq \emptyset$. Thus we have:

**Proposition 3.12** *(i) The sets $S_i$ are pairwise disjoint.*

*(ii) If $\mathcal{S}_{ip}, \mathcal{S}_{pj} \neq \emptyset$, then $S_{ip} \cap S_{pj} = S_p$.*

Clearly, if there is a $(\nu, 2)$ reducing link, then its endnodes belong to distinct minimal cores $C_i, C_j$.

**Proposition 3.13** *Let $C_i, C_j$ be distinct minimal cores. Then the following are equivalent:*

*(i) There exists a $(\nu, 2)$-reducing link connecting $C_i$ and $C_j$.*

*(ii) (A) $S_i - \Gamma(S_j)$ and $S_j - \Gamma(S_i)$ are both nonempty, and (B) $\mathcal{S}_{ij} = \emptyset$.*

*(iii) Any link connecting $C_i$ and $C_j$ is $(\nu, 2)$-reducing.*

**Proof:** It is easy to see that (i)⇒(ii), and clearly (iii)⇒(i). We will show that (ii)⇒(iii). Assume to the contrary that (ii) holds, but there are $s_i \in C_i$ and $s_j \in C_j$ such that the link $e = s_i s_j$ is not $(\nu, 2)$-reducing. Note that any minimal core $C$ of $G$ that is distinct from $C_i, C_j$ is also a minimal core of $G + e$. Thus our assumption implies that there is a tight set $X$ in $G$, that contains at least one of $C_i, C_j$ and disjoint to all the other minimal cores, such that $e$ has its both endpoints in $X \cup \Gamma(X)$. By condition (iiB), $\mathcal{S}_{ij} = \emptyset$, hence either $X \in \mathcal{S}_i$ or $X \in \mathcal{S}_j$, say $X \in \mathcal{S}_i$. In particular, $X \subseteq S_i$. By Lemma 3.2(ii) condition (iiA) implies that $S_j \cap S_i^*$ is tight; thus $C_j \subseteq S_j \cap S_i^*$. Consequently, $X \subseteq S_i$ and $C_j \subseteq S_i^*$, implying that $C_j \subseteq X^*$. This contradicts that $e$ has its both endpoints in $X \cup \Gamma(X)$. □

**Lemma 3.14** *Let $\mathcal{F}$ be a subfamily of the family of maximal cores of $G$. If no $(\nu, 2)$-reducing link that connects two cores in $\mathcal{F}$ exists, then the relation $\mathcal{R}(\mathcal{F})$ is an equivalence.*

**Proof:** Symmetry and reflexivity are obvious, so we need to prove transitivity. Suppose therefore that $\{S_i, S_p\}, \{S_p, S_j\} \in \mathcal{R}(\mathcal{F})$ for distinct $S_i, S_p, S_j \in \mathcal{F}$. Then $\mathcal{S}_{ip}, \mathcal{S}_{pj} \neq \emptyset$ by Proposition 3.13. By Proposition 3.12 $S_p = S_{ip} \cap S_{pj}$, and by Proposition 1.1 equality holds in (1) for $X = S_{ip}$ and $Y = S_{pj}$. Thus $C_i \subseteq S_{ip} \cap S_{pj}^* \subseteq S_{pj}^*$, and $S_j \subseteq S_{pj}$, by Lemma 3.2(ii). Thus we must have $C_i \cap \Gamma(S_j) = \emptyset$. For a similar reason, $C_j \cap \Gamma(S_i) = \emptyset$. This proves transitivity. □

The following two lemmas are used to establish that the equivalence classes of size at least three of $\mathcal{R}(\mathcal{F})$ correspond to sides of a shredder.

**Lemma 3.15** *Let $A, B$ be disjoint nonadjacent tight sets. If $A \cup B$ is tight, then $\Gamma(A) = \Gamma(B)$ and $d_r(A) = d_r(B) = 0$.*

**Proof:** Since $A, B$ are nonadjacent $|\Gamma(A \cup B)| = |\Gamma(A)| + |\Gamma(B)| - |\Gamma(A) \cap \Gamma(B)|$, implying

$$\begin{aligned} k-1 &= g(A \cup B) = d_r(A \cup B) + |\Gamma(A \cup B)| = \\ &= d_r(A) + d_r(B) + |\Gamma(A)| + |\Gamma(B)| - |\Gamma(A) \cap \Gamma(B)| = \\ &= g(A) + g(B) - |\Gamma(A) \cap \Gamma(B)| = 2(k-1) - |\Gamma(A) \cap \Gamma(B)|. \end{aligned}$$

Thus $|\Gamma(A) \cap \Gamma(B)| = k - 1$. Since $g(A) = g(B) = k - 1$, the claim follows. $\square$

**Lemma 3.16** *Let $A, B, C$ be pairwise disjoint tight sets such that none of them is contained in the set of neighbors of the other, and such that the union of any two of them is tight. Then $d_r(A) = d_r(B) = d_r(C) = 0$ and $\Gamma(A) = \Gamma(B) = \Gamma(C)$.*

**Proof:** We prove that $d_r(A) = 0$ (the proof for $B, C$ is similar). Consider the tight sets $X = A \cup B, Y = A \cup C$. Then $X \cap Y = A$, and equality holds in each one of (1),(2), by Lemma 3.2(i). Since $B - \Gamma(A), C - \Gamma(A)$ are both nonempty, we must have that $X \cap Y^*, Y \cap X^*$ are both nonempty. Thus $d_r(A) = 0$ and equality holds in each one of (3),(4), by Lemma 3.2(ii).

By Lemma 3.15, to finish the proof it is enough to show that $A, B, C$ are pairwise nonadjacent. We prove that $A, B$ are nonadjacent, and the proof for other pairs is similar. Since each one of (2),(3),(4) holds with equality, then $\Gamma(X \cup Y) \subseteq \Gamma(B \cup C)$. Since $X \cup Y, B \cup C$ are tight and since $d_r(X \cup Y) = 0$, we must have $\Gamma(X \cup Y) = \Gamma(B \cup C)$. Therefore, $B \cup C$ and $A$ are nonadjacent, implying that $A$ and $B$ are nonadjacent. $\square$

**Corollary 3.17** *Let $\mathcal{F}$ be a subfamily of the family of maximal cores of $G$ such that no $(\nu, 2)$-reducing link that connects two cores in $\mathcal{F}$ exists, and let $\mathcal{P}$ be an equivalence class of $\mathcal{R}(\mathcal{F})$. Then:*

*(i) If $|\mathcal{P}| \geq 3$ then $\mathcal{P}$ consists of some sides of the same shredder.*

*(ii) If $\mathcal{P} = \{S_i, S_j\}$ then $\mathcal{S}_{ij} \neq \emptyset$.*

**Proof:** Part (ii) follows from Proposition 3.13 and we will prove Part (i). We will show that if distinct $S_i, S_j, S_p$ belong to the same class of $\mathcal{R}(\mathcal{F})$ then they satisfy the conditions of Lemma 3.16. Since $S_i, S_j, S_p$ are distinct, they are pairwise disjoint (by Proposition 3.12(i)), and by the definition of $\mathcal{R}(\mathcal{F})$, none of them is contained in the set of neighbors of the other. It remains therefore to show that the union of any two of them, say $S_i \cup S_j$, is tight. By Proposition 3.13 and the definition of $\mathcal{R}(\mathcal{F})$, each one of the sets $S_{ij}, S_{jp}, S_{pi}$ exists. Thus $(S_{ip} \cup S_{pj}) \cap S_{ij}$ is tight, by Lemma 3.2(i), and $(S_{ip} \cup S_{pj}) \cap S_{ij} = (S_{ip} \cap S_{ij}) \cup (S_{pj} \cap S_{ij}) = S_i \cup S_j$, where the last equation follows from Proposition 3.12(ii). Thus $S_i \cup S_j$ is tight, and the proof for the other pairs is similar. $\Box$

**Proof of Theorem 3.7:** Let $\mathcal{F}$ be as in Theorem 3.7. From Proposition 3.13 and the definition of an independent family it follows that if case (ii) of Theorem 3.7 holds then case (i) cannot hold. The rest of the proof shows that if case (i) does not hold, then case (ii) must hold. Suppose therefore that no $(\nu, 2)$-reducing link connecting two distinct cores in $\mathcal{F}$ exists. By Lemma 3.14, $\mathcal{R}(\mathcal{F})$ is an equivalence, and let $\Pi$ be its partition into the corresponding equivalence classes. For an equivalence class $\mathcal{P}$, let $S_{\mathcal{P}}$ be the union of the sets in $\mathcal{P}$ if $|\mathcal{P}| \neq 2$, and $S_{\mathcal{P}} = S_{ij}$ if $\mathcal{P} = \{S_i, S_j\}$. Combining this setting with Corollary 3.17, we conclude that condition (i) in the definition of an independent family is satisfied for $\mathcal{F}$, $\Pi$, and $\mathcal{S}$, and, moreover, if $\mathcal{F}$ is independent, then it is strongly independent. We show that condition (ii) is also satisfied. Let $X, Y \in \mathcal{F} \cup \mathcal{S}$ be disjoint with $X - \Gamma(Y), Y - \Gamma(X) \neq \emptyset$. Then $X \cap Y^*$ and $Y \cap X^*$ are both nonempty, thus by Lemma 3.2 (ii) they are both tight. Let $S_X$ be an arbitrary maximal core intersecting $X \cap Y^*$, and let $S_Y$ be an arbitrary maximal core intersecting $Y \cap X^*$. Note that $S_i \subseteq S$ and $S_i \in \mathcal{F}$ for any maximal core $S_i$ and $S \in \mathcal{F} \cup \mathcal{S}$ that intersect. Thus $S_X \subseteq X$ and $S_Y \subseteq Y$, and $S_X, S_Y \in \mathcal{F}$. However, $S_X$ intersects $X \cap Y^*$, $S_Y$ intersects $Y \cap X^*$, implying that $S_X - \Gamma(S_Y), S_Y - \Gamma(S_X)$ are both nonempty; therefore, $S_X, S_Y$ belong to the same class of $\mathcal{R}(\mathcal{F})$. Since $X, Y$ are disjoint, $X = S_X$ and $Y = S_Y$, which finishes the proof. $\Box$

We now prove Theorem 3.10. A link set is *basic* if every its link connects two minimal cores of $G$ or connects a minimal core of $G$ to $r$.

**Lemma 3.18** *There exists a basic augmenting link set of size opt$(G)$.*

**Proof:** Let $F$ be an optimal augmenting link set with maximal number of basic links. If $F$ is basic, we are done. Otherwise, let $e' \in F$ be a nonbasic link. Then $(G + F) - e'$ has either two minimal cores $S'_1, S'_2$ or one minimal core $S'$. In each case, there is a basic link $e$ such that $G + (F - e' + e)$ is $k$-connected to $r$. In the first case $e$ connects arbitrary minimal cores $S_1, S_2$ of $G$ with $S_1 \subseteq S'_1$, $S_2 \subseteq S'_2$, and in the second case $e$ connects an arbitrary minimal core $S \subseteq S'$ of $G$ to $r$. This contradicts our choice of $F$. $\Box$

**Proof of Theorem 3.10:** Among all basic augmenting link sets of size *opt*$(G)$, let $F$ be one with the maximal number of links incident to $r$. We claim that then an arbitrary ordering of the set $F^*$ of links in $F$ that are not incident to $r$ is a $(\nu, 2)$-reducing sequence for $G$. In particular, $|F^*| \leq \zeta(G)$. Clearly, $opt(G) \geq \nu(G) - \zeta(G)$. Thus we have

$$\nu(G) - \zeta(G) \geq opt(G) = |F| = \nu(G) - |F^*| \geq \nu(G) - \zeta(G).$$

Consequently, equality holds everywhere, implying $|F^*| = \zeta(G)$ and $opt(G) = \nu(G) - \zeta(G)$.

We now prove that an arbitrary ordering of $F^*$ is a $(\nu, 2)$-reducing sequence for $G$. We start by showing that any $f \in F^*$ is $(\nu, 2)$-reducing for $G$. Observe that $\nu(G + F - f) = 2$, that is $f$ is $(\nu, 2)$-reducing for $H = G+F-f$; otherwise, similarly to the proof of Lemma 3.18, one can replace $f$ by a basic link $f'$ incident to $r$ such that $G+(F-f+f')$ is $k$-connected to $r$, contradicting our choice of $F$. Let $X_i, X_j$ be the maximal and $Y_i, Y_j$ the minimal cores of $H$, and let $C_i, C_j$ be minimal cores of $G$ that $f$ connects, where $C_i \subseteq Y_i \subseteq X_i$ and $C_j \subseteq Y_j \subseteq X_j$. Let $S_i$ and $S_j$ be the maximal cores of $G$ with $C_i \subseteq S_i$ and $C_j \subseteq S_j$. Note that $S_i \subseteq X_i$ and $S_j \subseteq X_j$. By Proposition 3.13, $C_i \subseteq Y_i \subseteq X_i - \Gamma_H(X_j)$ and $C_j \subseteq Y_j \subseteq X_j - \Gamma_H(Y_i)$, hence $C_i \subseteq S_i - \Gamma_G(S_j)$ and $C_j \subseteq S_j - \Gamma_G(S_i)$. Thus by Proposition 3.13, if $e$ is not $(\nu, 2)$-reducing for $G$ then $\mathcal{S}_{ij} \neq \emptyset$ in $G$. By Lemma 3.2 (i) $X_{ij} = X_i \cup S_{ij} \cup X_j$ is tight in $G$. Since $f$ is $(\nu, 2)$-reducing for $H$, $X_{ij}$ is not tight in $H$. Thus there is $xv \in F - f$ with $x \in X_{ij}$ and $v \in (V + r) - (X_{ij} + \Gamma_H(X_{ij}))$. Note that every minimal core of $G$ contained in $X_{ij}$ must be contained in one of $X_i, X_j$. Since $F$ is basic we conclude that for $X = X_i$ or $X = X_j$ holds $x \in X$ and $x \in (V + r) - (X + \Gamma_H(X))$, contradicting that $X_i, X_j$ are both tight in $H$.

Let $f \in F^*$ be arbitrary. Then among all basic augmenting link sets for $G + f$ of size $opt(G + f)$, $F - f$ has the maximal number of links incident to $r$. By repeatedly applying this argument, we obtain that any ordering of $F^*$ is a $(\nu, 2)$-reducing sequence for $G$. $\square$

## 3.4 Proof of Theorems 3.8 and 3.11

**Lemma 3.19** *Let $T$ be a shredder and let $Y$ be a tight set.*

*(i) If $\Gamma(Y) = T$, then $Y$ is a union of some sides of $T$.*

*(ii) If $Y$ intersects two distinct sides $X_i, X_j$ of $T$, then $X_i, X_j \subseteq Y$.*

**Proof:** Part (i) is obvious, and we will prove part (ii). By Lemma 3.2(i), the sets $Y \cap X_i, Y \cap X_j$ are tight, and their union (which is the intersection of two intersecting tight sets $Y, X_i \cup X_j$) is also tight. Moreover, $Y \cap X_i, Y \cap X_j$ are nonadjacent, since $X_i, X_j$ are nonadjacent. Thus $\Gamma(Y \cap X_i) = \Gamma(Y \cap X_j) = T$, by Lemma 3.15. Part (ii) follows now from part (i). $\square$

Two intersecting sets $X, Y$ are *crossing*, (or $Y$ *crosses* $X$) if none of them contains the other.

**Lemma 3.20** *No tight set crosses a side of a shredder or the union of sides of a shredder.*

**Proof:** Let $Y$ be a tight set intersecting some side $X$ of a shredder $T$. By Lemma 3.19(ii), if $Y$ intersects all sides of $T$, then it contains all of them. Assume therefore that there is a side $X'$ of $T$ disjoint to $Y$. Let $Z = X \cup Y$. Then (i) $d_r(Z \cup X') = d_r(Z)$ (since $d_r(X') = 0$); (ii) $\Gamma(Z \cup X') \subseteq \Gamma(Z)$ and $\Gamma(Z \cup X') = \Gamma(Z)$ if, and only if, $Z$ and $X'$ are nonadjacent (since $\Gamma(X) = \Gamma(X') = T$ and $X \subseteq Z$). Thus $Z$ and $X'$ are tight and nonadjacent. Moreover, $Z \cup X'$ is tight (since $Z, X \cup X'$ are intersecting and tight, and since $X \subseteq Z$). Thus $\Gamma(Z) = \Gamma(X') = T$, by Lemma 3.15. Consequently, $Z$ must be a union of some sides of $T$, by Lemma 3.19(i). Now, if $Y$ intersects a side of $T$ distinct from $X$, then $X \subset Y$, by Lemma 3.19(i); otherwise, we have $Y \subseteq X$, and the proof is complete. $\square$

Given a nontrivial partition $\mathcal{W}$ of a groundset $W$, a link set $F$ on $W$ is a *$\mathcal{W}$-connecting cover* (*of* $W$) if the following three conditions hold: (a) $\deg_F(w) \geq 1$ for every $w \in W$; (b) every link in $F$ connects distinct parts of $\mathcal{W}$; (c) $F$ induces a connected graph on the parts of $\mathcal{W}$. Let $\max(\mathcal{W})$ denote the largest cardinality of a set in $\mathcal{W}$.

**Lemma 3.21** *Let $\mathcal{W}$ be a nontrivial partition of a groundset $W$. Then the minimum cardinality of a $\mathcal{W}$-connecting cover equals* $\max\{\lceil |W|/2 \rceil, \max(\mathcal{W}), |\mathcal{W}| - 1\}$*, and given $\mathcal{W}$ a minimum cardinality $\mathcal{W}$-connecting cover can be found in linear time.*

**Proof:** Let $F$ be a $\mathcal{W}$-connecting cover (satisfying conditions (a),(b),(c) above). Then: (a) implies $|F| \geq \lceil |W|/2 \rceil$, (a) and (b) imply $|F| \geq \max(\mathcal{W})$, and (c) implies $|F| \geq |\mathcal{W}| - 1$; hence $|F| \geq \max\{\lceil |W|/2 \rceil, \max(\mathcal{W}), |\mathcal{W}| - 1\}$. The following algorithm starts with $F = \emptyset$ and computes a $\mathcal{W}$-connecting cover for which equality holds.

*While* $|\mathcal{W}| \geq 2$ and $\max(\mathcal{W}) \geq 2$ do:
  add a link $zw$ to $F$ where $z$ belongs to the largest set $Z \in \mathcal{W}$, and $w$ belongs to:
  - the largest set in $\mathcal{W} - Z$ if $\max(\mathcal{W}) \geq |\mathcal{W}|$;
  - to the smallest set in $\mathcal{W}$ otherwise.
  $W \leftarrow W - \{z, w\}$, and replace $\mathcal{W}$ by its restriction to $W$ (discarding empty sets).
*End while*
*If* $|\mathcal{W}| = 1$ then for every $z \in W$ add to $F$ an arbitrary link $zw$ that satisfies condition (b);
*Else* (applies if $|W| \geq 2$ and $\max(\mathcal{W}) = 1$) add to $F$ an arbitrary tree on $W$.

It is easy to see that at every iteration of the loop the bound $\max\{\lceil |W|/2 \rceil, \max(\mathcal{W}), |\mathcal{W}| - 1\}$ decreases by 1. Thus at the end of the algorithm $F$ has size as claimed. Also, (a) and (b) hold for $F$ by the construction, while (c) can be easily proved by induction on the number of iterations in the loop. Thus at the end of the algorithm $F$ is as required. The algorithm can be implemented to run in linear time, by maintaining an array $A$ of size $|W|$, where

$A[i]$ has a pointer to a linked list of the sets in $\mathcal{W}$ of size $i$, pointers to the sizes in $A$ of the largest, the second largest, and the smallest sets in $\mathcal{W}$, and a variable indicating $|W|$. It is easy to see that this data structure enables to answer every query during the algorithm in $O(1)$ time, and can be maintained during the algorithm in $O(|W|)$ total time. $\square$

**Corollary 3.22** *Let $T$ be a shredder with $b(T) \geq k$ and suppose that every $T$-component contains at most $b(T) - 1$ minimal cores. Then given $T$, an augmenting link set for $G$ of size* $\max\{\lceil(\nu(G)+1)/2\rceil, b(T)-1\}$ *can be found in polynomial time.*

**Proof:** Let $R$ be the $T$-component that contains $r$, let $W'$ be the set of minimal cores of $G$, and let $W = W' + r$. By Lemma 3.20 and the assumption $b(T) \geq k$ the inclusion in the $T$-components induces a partition $\mathcal{W}$ of $W$, and let $F$ be a minimum cardinality $\mathcal{W}$-connecting cover. Note that $\mathcal{W}$, and thus also $F$, can be computed in polynomial time. Then $G + F$ is $k$-connected to $r$, since by Lemma 3.20, for any tight set $Y$ of $G$ either $Y$ is properly contained in a side of $T$ or is a union of some sides of $T$, and thus $F$ has an edge connecting $Y$ and $Y^*$. Note that $|W| = \nu(G) + 1$, $|\mathcal{W}| = b(T)$, and $\max(\mathcal{W}) \leq b(T) - 1 = |\mathcal{W}| - 1$. Hence, by Lemma 3.21, $|F| = \max\{\lceil|W|/2\rceil, |\mathcal{W}|-1\} = \max\{\lceil(\nu(G)+1)/2\rceil, b(T)-1\}$. $\square$

We note that a shredder $T$ with $b(T) = b(G)$ can be found in polynomial time. In fact, all the shredders can be found in polynomial time (their number is at most $(2|V|-2k+1)/3$, by Theorem 3.11). This can be done using the algorithm of [3] where a corresponding problem in a $(k-1)$-connected graph was considered. Specifically, let $r, s$ be two nonadjacent nodes in an undirected graph $G$, and let $T$ be an $(r,s)$-cut; that is, $r, s$ belong to distinct $T$-components $R, S$, respectively. Note that any path between $r$ and $s$ that contains a node from a $T$-component distinct from $R$ and from $S$ must contain at least two nodes from $T$. Thus if $T$ is a minimum $(r,s)$-cut, and $P$ is the union of the nodes of $|T|$ internally disjoint paths between $r$ and $s$, then $T \subseteq P - \{r,s\}$ and $X \cap P = \emptyset$ for every $T$-component $X$ distinct from $R, S$. Consequently, every $T$-component distinct from $R, S$ is a connected component of $G - (P - \{r,s\})$. Therefore the following algorithm computes in polynomial time all the shredders of a graph $G$ which is $(k-1)$-connected to $r$. For every node $s$ not adjacent to $r$ do the following. First, compute a set of $k-1$ internally disjoint paths between $r$ and $s$, and set $P$ to be the the union of the nodes of these paths. Second, for every connected component $X$ of $G - (P - \{r,s\})$ check whether $\Gamma(X)$ is a shredder.

**Proof of Theorem 3.8:** Consider the following algorithm applied on a shredder $T$ with $b(T) = b(G) \geq k$.

**Phase 1:** *While* there exists a $T$-component $X$ that contains $b(T)$ minimal cores do:

add to $G$ a $(\nu, 2)$-reducing link connecting two cores contained in $X$.

*End While*

**Phase 2:** Add to $G$ a link set as in Corollary 3.22.

The condition in the loop of Phase 1 ensures that a $(\nu, 2)$-reducing link connecting two cores in $X$ exists; otherwise by Theorem 3.7 the maximal cores contained in $X$ are sides of the same shredder with at least $b(T)$ sides, while $T$ has $b(T) - 1$ sides; this contradicts the maximality of $b(T)$. Consequently, the algorithm correctly finds an augmenting link set of size at most $\max\{\lceil(\nu(G) + 1)/2\rceil, b(T) - 1\}$, by Corollary 3.22. $\square$

**Proof of Theorem 3.11:** Consider the family $\mathcal{L}$ obtained by picking for every shredder its sides and the union of its sides; we color the former blue and the latter red. Let $U$ be the union of the sets in $\mathcal{L}$, and note that $|U| \leq |V| - |\Gamma(r)| \leq |V| - k + 1$. Note that $\mathcal{L}$ is laminar (that is, its members are pairwise noncrossing), by Lemma 3.20. It is well known that a laminar family on $U$ has at most $2|U| - 1$ members, thus $|\mathcal{L}| \leq 2(|V| - |\Gamma(r)|) - 1$. We can represent $\mathcal{L}$ as a forest of rooted trees if we order the sets in $\mathcal{L}$ by inclusion: $X$ is a child of $Y$ if $X$ is the largest set in $\mathcal{L}$ properly contained in $Y$. Then this forest has the following properties: (i) every set is either blue or red, but not both; (ii) the children of every red set are all blue, and there are at least two of them. Properties (i) and (ii) imply that the number of red sets, which is exactly as the number of shredders in the graph, is at most half the number of blue sets, which implies the statement. $\square$

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