The Open University of Israel
Department of Mathematics and Computer Science

Approximating Minimum Power Edge-Multi-Covers

Thesis submitted as partial fulfillment of the requirements
towards an M.Sc. degree in Computer Science
The Open University of Israel
Computer Science Division

By

Nachshon Cohen

Prepared under the supervision of Prof. Zeev Nutov

October 2011

Abstract

Given a graph with edge costs, the *power* of a node is the maximum cost of an edge incident to it, and the power of a graph is the sum of the powers of its nodes. Motivated by applications in wireless networks, we consider the following fundamental problem in wireless network design. Given a graph $G = (V, E)$ with edge costs and degree bounds $\{r(v) : v \in V\}$, the Minimum-Power Edge-Multi-Cover (MPEMC) problem is to find a minimum-power subgraph J of G such that the degree of every node v in J is at least $r(v)$. Let $k = \max_{v \in V} r(v)$. For $k = \Omega(\log n)$, the previous best approximation ratio for MPEMC was $O(\log n)$, even for uniform costs [3]. Our main result improves this ratio to $O(\log k)$ for general costs, and to $O(1)$ for uniform costs. This also implies ratios $O(\log k)$ for the Minimum-Power k-Outconnected Subgraph and $O\left(\log k \log \frac{n}{n-k}\right)$ for the Minimum-Power k-Connected Subgraph problems; the latter is the currently best known ratio for the min-cost version of the problem. In addition, for small values of k, we improve the previously best ratio $k+1$ to $k+1/2$.

Acknowledgments

I wish to express my warm and sincere thanks to my thesis advisor, Prof. Zeev Nutov. This thesis would never have taken shape without his kind and limitless help on every possible level.

Contents

1 Introduction **4**

1.1 Motivation and problems considered 4

1.2 Our Results 5

1.3 Overview of the techniques 6

2 Proof of Theorem 1.1 **7**

2.1 Reduction to bipartite graphs 7

2.2 An $O(\log k)$-approximation algorithm for general costs 7

2.3 A constant ratio approximation algorithm for unit costs 10

3 Proof of Theorem 1.2 **13**

4 Conclusions and open problems **18**

List of Figures

1 Illustration to the definition of the U-cover I'. 16

1 Introduction

1.1 Motivation and problems considered

Wireless networks are studied extensively due to their wide applications. The power consumption of a station determines its transmission range, and thus also the stations it can send messages to; the power typically increases at least quadratically in the transmission range. Assigning power levels to the stations (nodes) determines the resulting communication network. Conversely, given a communication network, the power required at v only depends on the farthest node reached directly by v. This is in contrast with wired networks, in which every pair of stations that communicate directly incurs a cost. An important network property is fault-tolerance, which is often measured by minimum degree or node-connectivity of the network. Node-connectivity is much more central here than edge-connectivity, as it models stations failures. Such power minimization problems were vastly studied; see for example [1, 2, 5, 8, 9] and the references therein for a small sample of papers in this area. The first problem we consider is finding a low power network with specified lower degree bounds. The second problem is the Min-Power k-Connected Subgraph problem. We give approximation algorithms for these problems, improving the previously best known ratios.

Definition 1.1 *Let (V,J) be a graph with edge-costs $\{c(e) : e \in J\}$. For a node $v \in V$ let $\delta_J(v)$ denote the set of edges incident to v in J. The* power *$p_J(v)$ of v is the maximum cost of an edge in J incident to v, or 0 if v is an isolated node of J; i.e., $p_J(v) = \max_{e \in \delta_J(v)} c(e)$ if $\delta_J(v) \neq \emptyset$, and $p_J(v) = 0$ otherwise. For $V' \subseteq V$ the power of V' w.r.t. J is the sum $p_J(V') = \sum_{v \in V'} p_J(v)$ of the powers of the nodes in V'.*

Unless stated otherwise, all graphs are assumed to be undirected and simple. Let $n = |V|$. Given a graph $G = (V,E)$ with edge-costs $\{c(e) : e \in E\}$, we seek to find a low power subgraph (V,J) of G that satisfies some prescribed property. One of the most fundamental problems in Combinatorial Optimization is finding a minimum-cost subgraph that obeys specified degree constraints (sometimes called also "matching problems") c.f. [10]. Another fundamental property is fault-tolerance (connectivity). In fact, these problems are related, and we use our algorithm for the former as a tool for approximating the latter.

Definition 1.2 *Given degree bounds $r = \{r(v) : v \in V\}$, we say that an edge-set J on V is an* r-edge cover *if $d_J(v) \geq r(v)$ for every $v \in V$, where $d_J(v) = |\delta_J(v)|$ is the degree of v in the graph (V,J).*

Minimum-Power Edge-Multi-Cover (MPEMC):
Instance: A graph $G = (V,E)$ with edge-costs $\{c(e) : e \in E\}$, degree bounds $r = \{r(v) : v \in V\}$.
Objective: Find a minimum power r-edge cover $J \subseteq E$.

Given an instance of MPEMC, let $k = \max_{v \in V} r(v)$ denote the maximum requirement.

We now define our connectivity problems. A graph is *k-outconnected from s* if it contains k internally-disjoint sv-paths for all $v \in V \setminus \{s\}$. A graph is *k-connected* if it is k-outconnected from every node, namely, if it contains k internally-disjoint uv-paths for all $u, v \in V$.

Minimum-Power k-Outonnected Subgraph (MPkOS):
Instance: A graph $G = (V, E)$ with edge-costs $\{c(e) : e \in E\}$, a root $s \in V$, and an integer k.
Objective: Find a minimum-power k-outconnected from s spanning subgraph J of G.

Minimum-Power k-Connected Subgraph (MPkCS):
Instance: A graph $G = (V, E)$ with edge-costs $\{c(e) : e \in E\}$ and an integer k.
Objective: Find a minimum-power k-connected spanning subgraph J of G.

1.2 Our Results

For large values of $k = \Omega(\log n)$, the previous best approximation ratio for MPEMC was $O(\log n)$, even for uniform costs [3]. Our main result improves this ratio to $O(\log k)$ for general costs, and to $O(1)$ for uniform costs.

Theorem 1.1 MPEMC *admits an $O(\log k)$-approximation algorithm. For uniform costs,* MPEMC *admits a randomized approximation algorithm with expected approximation ratio $\rho < 2.16851$, where ρ is the real root of the qubic equation $e(\rho - 1)^3 = 2\rho$.*

For small values of k, the problem admits also the ratios $k+1$ for arbitrary k [2], while for $k = 1$ the best known ratio is $k + 1/2 = 3/2$ [4]. Our second result extends the latter ratio to arbitrary k.

Theorem 1.2 MPEMC *admits a $(k + 1/2)$-approximation algorithm.*

For small values of k, say $k \leq 6$, the ratio $(k + 1/2)$ is better than $O(\log k)$ because of the constant hidden in the $O(\cdot)$ term. And overall, our paper gives the currently best known ratios for all values $k \geq 2$.

In [5] it is proved that an α-approximation for MPEMC implies an $(\alpha + 4)$-approximation for MPkOS. The previous best ratio for MPkOS was $O(\log n) + 4 = O(\log n)$ [5] for large values of $k = \Omega(\log n)$, and $k + 1$ for small values of k [9]. From Theorem 1.1 we obtain the following.

Theorem 1.3 MPkOS *admits an $O(\log k)$-approximation algorithm.*

In [2] it is proved that an α-approximation for MPEMC and a β-approximation for Min-Cost k-Connected Subgraph implies a $(\alpha + 2\beta)$-approximation for MPkCS. Thus the previous best ratio for MPkCS was $2\beta + O(\log n)$ [3], where β is the best ratio for MCkCS (for small values of k better ratios for MPkCS are given in [9]). The currently best known value of β is $O\left(\log k \log \frac{n}{n-k}\right)$ [7], which is $O(\log k)$, unless $k = n - o(n)$. From Theorem 1.1 we obtain the following.

Theorem 1.4 MPkCS *admits an* $O(\beta + \log k)$*-approximation algorithm, where* β *is the best ratio for* MCkCS*. In particular,* MPkCS *admits an* $O\left(\log k \log \frac{n}{n-k}\right)$*-approximation algorithm.*

1.3 Overview of the techniques

Let the *trivial solution* for MPEMC be obtained by picking for every node $v \in V$ the cheapest $r(v)$ edges incident to v. It is known and easy to see that this produces an edge set of power at most $(k+1) \cdot \mathsf{opt}$, see [2].

Our $O(\log k)$-approximation algorithm uses the following idea. Extending and generalizing an idea from [3], we show how to find an edge set $I \subseteq E$ of power $O(\mathsf{opt})$ such that for the residual instance, the trivial solution value is reduced by a constant fraction. We repeatedly find and add such an edge set I to the constructed solution, while updating the degree bounds accordingly to $r(v) \leftarrow \max\{r(v) - d_I(v), 0\}$. After $O(\log k)$ steps, the trivial solution value is reduced to opt, and the total power of the edges we picked is $O(\log k) \cdot \mathsf{opt}$. At this point we add to the constructed solution the trivial solution of the residual problem, which at this point has value opt, obtaining an $O(\log k)$-approximate solution.

Our algorithm for uniform costs has two phases. In the first phase we compute an optimal solution x to a certain LP-relaxation for the problem and round it to 1 with probability $\min\{\rho \cdot x, 1\}$. In the second phase we add to the obtained partial solution the trivial solution to the residual problem.

Our $(k + 1/2)$-approximation algorithm uses a two-stage reduction. The first reduction reduces MPEMC to a constrained version of MPEMC with $k = 1$, where we also have lower bounds ℓ_v on the power of each node $v \in V$; these lower bounds are determined by the trivial solution to the problem. We will show that a ρ-approximation algorithm to this constrained version implies a $(k - 1 + \rho)$-approximation algorithm for MPEMC. The second reduction reduces the constrained version to the Minimum-Cost Edge Cover problem with a loss of $3/2$ in the approximation ratio. As Minimum-Cost Edge Cover admits a polynomial time algorithm, we get a ratio $\rho = 3/2$ for the constrained problem, which in turn gives the ratio $k - 1 + \rho = k + 1/2$ for MPEMC.

2 Proof of Theorem 1.1

2.1 Reduction to bipartite graphs

Let Bipartite MPEMC be the restriction of MPEMC to instances for which the input graph $G = (V, E)$ is a bipartite graph with sides A, B, and with $r(a) = 0$ for every $a \in A$ (so, only the nodes in B may have positive degree bound).

As in [3], we can reduce MPEMC to Bipartite MPEMC, by taking two copies $A = \{a_v : v \in V\}$ and $B = \{b_v : v \in V\}$ of V, for every edge $e = uv \in E$ adding the two edges $a_u b_v$ and $a_v b_u$ of cost $c(e)$ each, and for every $v \in V$ setting $r(b_v) = r(v)$ and $r(a_v) = 0$. It is proved in [3] that this reduction invokes a factor of 2 in the approximation ratio, namely, that a ρ-approximation for bipartite MPEMC implies a 2ρ-approximation for general MPEMC.

In the case of uniform costs, we can save a factor of 2 using a different reduction.

Proposition 2.1 *Ratio ρ for* Bipartite MPEMC *with unit costs implies ratio ρ for* MPEMC *with uniform costs.*

Proof: Clearly, the case of uniform costs is equivalent to the case of unit costs. Now we show that for unit costs, MPEMC can be reduced to Bipartite MPEMC. Let $G = (V, E), r$ be an instance of MPEMC with unit costs. If there is an edge $e = uv \in E$ with $r(u), r(v) \geq 1$ or with $r(u) = r(v) = 0$, then we can obtain an equivalent instance by removing e from G, and in the case $r(u), r(v) \geq 1$ also decreasing each of $r(u), r(v)$ by 1. Hence we may assume that every $e \in E$ has one end in $A = \{a \in V : r(a) = 0\}$ and the other end in $B = \{b \in V : r(b) \geq 1\}$. The statement follows. $\square$

2.2 An $O(\log k)$-approximation algorithm for general costs

Let opt denote the optimal solution value of a problem instance at hand. For $v \in V$, let w_v be the cost of the $r(v)$-th least cost edge incident to v in E if $r(v) \geq 1$, and $w_v = 0$ otherwise. Given a partial solution J to Bipartite MPEMC let $r_J(v) = \max\{r(v) - d_J(v), 0\}$ be the *residual bound* of v w.r.t. J. Let

$$R_J = \sum_{b \in B} w_b r_J(b) \ .$$

The main step in our algorithm is given in the following lemma, which will be proved later.

Lemma 2.2 *There exists a polynomial time algorithm that given an edge set $J \subseteq E$, an integer τ, and a parameter $\gamma > 1$, either correctly establishes that $\tau <$ opt, or returns an edge set $I \subseteq E \setminus J$ such that $p_I(V) \leq (1+\gamma)\tau$ and $R_{J \cup I} \leq \theta R_J$, where $\theta = 1 - \left(1 - \frac{1}{\gamma}\right)\left(1 - \frac{1}{e}\right)$.*

Lemma 2.3 *Let $J \subseteq E$ and let $F \subseteq E \setminus J$ be an edge set obtained by picking $r_J(b)$ least cost edges in $\delta_{E \setminus J}(b)$ for every $b \in B$. Then $J \cup F$ is an r-edge-cover and: $p_F(B) \leq$ opt, $p_F(A) \leq R_J \leq k \cdot$opt.*

Proof: Since F is an r_J-edge-cover, $J \cup F$ is an r-edge-cover. By the definition of F, for any r-edge-cover I, $p_F(b) \leq w_b \leq p_I(b)$ for all $b \in B$. In particular, if I is an optimal r-edge-cover, then

$$p_F(B) \leq \sum_{b \in B} w_b \leq \sum_{b \in B} p_I(b) = p_I(B) \leq \mathsf{opt} \ .$$

Also,

$$R_J = \sum_{b \in B} w_b r_J(b) \leq k \cdot \sum_{b \in B} w_b \leq k \cdot \mathsf{opt} \ .$$

Finally, $p_F(A) \leq R_J$ since

$$p_F(A) = \sum_{a \in A} p_F(a) \leq \sum_{a \in A} \sum_{e \in \delta_F(a)} c(e) = \sum_{e \in F} c(e) \leq \sum_{b \in B} w_b r_J(b) = R_J \ .$$

This concludes the proof of the lemma. □

Theorem 1.1 is deduced from Lemmas 2.2 and 2.3 as follows. We set γ to be constant strictly greater than 1, say $\gamma = 2$. Then $\theta = 1 - \frac{1}{2}\left(1 - \frac{1}{e}\right)$. Using binary search, we find the least integer τ such that the following procedure computes an edge set J satisfying $R_J \leq \tau$.

Initialization: $J \leftarrow \emptyset$.
Loop: Repeat $\lceil \log_{1/\theta} k \rceil$ times:
Apply the algorithm from Lemma 2.3:
- If it establishes that $\tau < \mathsf{opt}$ then return "ERROR" and STOP.
- Else do $J \leftarrow J \cup I$.

After computing J as above, we compute an edge set $F \subseteq E \setminus J$ as in Lemma 2.3. The edge-set $J \cup F$ is a feasible solution, by Lemma 2.3. We claim that for any $\tau \geq \mathsf{opt}$ the above procedure returns an edge set J satisfying $R_J \leq \tau$; thus binary search indeed applies. To see this, note that $R_\emptyset \leq k \cdot \mathsf{opt}$ and thus

$$R_J \leq R_\emptyset \cdot \theta^{\lceil \log_{1/\theta} k \rceil} \leq k \cdot \mathsf{opt} \cdot 1/k = \mathsf{opt} \leq \tau \ .$$

Consequently, the least integer τ for which the above procedure does not return "ERROR" satisfies $\tau \leq \mathsf{opt}$. Thus $p_J(V) \leq \lceil \log_{1/\theta} k \rceil \cdot (1+\gamma) \cdot \tau = O(\log k) \cdot \mathsf{opt}$. Also, by Lemma 2.3, $p_F(V) \leq \mathsf{opt} + R_J \leq 2\mathsf{opt}$. Consequently,

$$p_{J \cup F}(V) \leq p_J(V) + p_F(V) = O(\log k) \cdot \mathsf{opt} + 2\mathsf{opt} = O(\log k) \cdot \mathsf{opt} \ .$$

In the rest of this section we prove Lemma 2.2. It is sufficient to prove the statement in the lemma for the residual instance $((V, E \setminus J), r_J)$ with edge-costs restricted to $E \setminus J$; namely, we may assume that $J = \emptyset$. Let $R = R_\emptyset = \sum_{b \in B} w_b r(b)$.

Definition 2.1 *An edge $e \in E$ incident to a node $b \in B$ is* τ-cheap *if $c(e) \leq \frac{\tau\gamma}{R} \cdot w_b r(b)$.*

Lemma 2.4 *Let F be an r-edge-cover, let $\tau \geq p_F(B)$, and let*

$$I = \bigcup_{b \in B} \{e \in \delta_E(b) : c(e) \leq \frac{\tau\gamma}{R} \cdot w_b r(b)\}$$

be the set of τ-cheap edges in E. Then $R_{I\cap F} \leq R/\gamma$ and $p_I(B) \leq \gamma\tau$.

Proof: Let $D = \{b \in B : \delta_{F\setminus I}(b) \neq \emptyset\}$. Since for every $b \in D$ there is an edge $e \in F \setminus I$ incident to b with $c(e) > \frac{\tau\gamma}{R} \cdot w_b r(b)$, we have $p_{F\setminus I}(b) \geq \frac{\tau\gamma}{R} \cdot w_b r(b)$ for every $b \in D$. Thus

$$\tau \geq p_F(B) \geq p_{F\setminus I}(B) = \sum_{b\in D} p_{F\setminus I}(b) \geq \tau \cdot \frac{\gamma}{R} \sum_{b \in D} w_b r(b) \ .$$

This implies $\sum_{b\in D} w_b r(b) \leq R/\gamma$. Note that for every $b \in B \setminus D$, $\delta_F(b) \subseteq \delta_I(b)$ and hence $r_{I\cap F}(b) = r_F(b) = 0$. Thus we obtain:

$$R_{I\cap F} = \sum_{b\in B} w_b r_{I\cap F}(b) = \sum_{b \in D} w_b r_{I \cap F}(b) \leq \sum_{b\in D} w_b r(b) \leq R/\gamma \ .$$

To see that $p_I(B) \leq \gamma\tau$ note that

$$p_I(B) = \sum_{b\in B} p_I(b) \leq \frac{\tau\gamma}{R} \sum_{b\in B} w_b r(b) = \frac{\tau\gamma}{R} \cdot R = \tau\gamma \ .$$

This concludes the proof of the lemma. $\square$

In [3] it is proved that the following problem, which is a particular case of submodular function minimization subject to matroid and knapsack constraint (see [6]) admits a $\left(1 - \frac{1}{e}\right)$-approximation algorithm.

Bipartite Power-Budgeted Maximum Edge-Multi-Coverage (BPBMEM):
Instance: A bipartite graph $G = (A \cup B, E)$ with edge-costs $\{c(e) : e \in E\}$ and node-weights $\{w_v : v \in B\}$, degree bounds $\{r(v) : v \in B\}$, and a budget τ.
Objective: Find $I \subseteq E$ with $p_I(A) \leq \tau$ that maximizes

$$\mathsf{val}(I) = \sum_{v \in B} w_v \cdot \min\{d_I(v), r(v)\} \ .$$

The following algorithm computes an edge set as in Lemma 2.2.

1. Among the τ-cheap edges, compute a $\left(1 - \frac{1}{e}\right)$-approximate solution I to BPBMEM.
2. If $R_I \leq \theta R$ then return I, where $\theta = 1 - \left(1 - \frac{1}{\gamma}\right)\left(1 - \frac{1}{e}\right)$;
   Else declare "$\tau < \mathsf{opt}$".

Clearly, $p_I(A) \leq \tau$. By Lemma 2.4, $p_I(B) \leq \gamma\tau$. Thus $p_I(V) \leq p_I(A) + p_I(B) \leq (1+\gamma)\tau$.

Now we show that if $\tau \geq \mathsf{opt}$ then $R_I \leq \theta R$. Let F be the set of cheap edges in some optimal solution. Then $p_F(A) \leq \mathsf{opt} \leq \tau$. By Lemma 2.4 $R_F \leq R/\gamma$, namely, F reduces R by at least $R\left(1 - \frac{1}{\gamma}\right)$. Hence our $\left(1 - \frac{1}{e}\right)$-approximate solution I to BPBMEM reduces R by at least $R\left(1 - \frac{1}{e}\right)\left(1 - \frac{1}{\gamma}\right)$. Consequently, we have $R_I \leq R - R\left(1 - \frac{1}{e}\right)\left(1 - \frac{1}{\gamma}\right) = \theta R$, as claimed.

The proof of Theorem 1.1 for the case of general costs is complete.

2.3 A constant ratio approximation algorithm for unit costs

Bipartite MPEMC with unit costs is closely related to the Set-Multicover problem, that can be casted as follows.

Set-Multicover
Instance: A bipartite graph $G = (A \cup B, E)$ and demands $\{r(b) : b \in B\}$.
Objective: Find a subgraph H of G with $\deg_H(b) \geq r(b)$ for every $b \in B$; minimize $|H \cap A|$.

In fact, it is easy to see that Bipartite MPEMC with unit costs is equivalent to the following modification of Set-Multicover, where instead of minimizing $|H \cap A|$ we seek to minimize $|H \cap A| + |B|$; namely, the problem we consider is as follows.

Set-Multicover+
Instance: A bipartite graph $G = (A \cup B, E)$ and demands $\{r(b) : b \in B\}$.
Objective: Find a subgraph H of G with $\deg_H(b) \geq r(b)$ for every $b \in B$; minimize $|H \cap A| + |B|$.

Clearly, ratio ρ for Set-Multicover implies ratio ρ for Set-Multicover+. As Set-Multicover admits ratio $H(|B|)$, so is Set-Multicover+. On the other hand, Set-Multicover+ is APX-hard even for instances with $\max_{a \in A} \deg_G(a) = 3$, by a reduction from 3-Set-Cover. If $|A| = O(|B|)$ then the problem is clearly approximable within a constant; but we may have $|A| >> |B|$, if $k = \max_{b \in B} r(b)$ is large. We prove the following theorem that implies the second part of Theorem 1.1, and is also of independent interest.

Theorem 2.5 Set-Multicover+ *admits a randomized approximation algorithm with expected approximation ratio ρ, where $\rho < 2.16851$ is the real root of the qubic equation $e(\rho - 1)^3 = 2\rho$.*

Let $\Gamma(a)$ denote the set of neighbors of a in G. Consider the following LP-relaxation for both Set-Multicover and Set-Multicover+

$$\min \left\{ \sum_{a \in A} x_a : \sum_{a \in \Gamma(b)} x_a \geq r(b) \ \forall b \in B, 0 \leq x_a \leq 1 \ \forall a \in A \right\}. \tag{1}$$

The value of a solution x to LP (1) is $x(A) = \sum_{a \in A} x_a$ in the Set-Multicover case, and $x(A) + |B|$

in the Set-Multicover+ case. Given a partial cover $S \subseteq A$, the residual demand of $b \in B$ is $r_S(b) = \max\{r(b) - |\Gamma(b) \cap S|, 0\}$. Let $\rho > 1$ be a parameter eventually set to be as in Theorem 2.5. Let $\gamma = \gamma(\rho) = \frac{(\rho-1)^2}{2\rho}$. Note that $\gamma = 1$ if, and only if, $\rho = 2+\sqrt{3}$, and that the value of ρ in Theorem 2.5 is less than $2+\sqrt{3}$. Let x be a feasible solution to LP (1), and let $S \subseteq A$ be obtained by choosing every $a \in A$ with probability $\min\{\rho \cdot x_a, 1\}$.

Lemma 2.6 *If $x_a < 1/\rho$ for all $a \in A$ then $\Pr[r_S(b) \geq 1] \leq e^{-\gamma \cdot r(b)}$ for every $b \in B$.*

Proof: Let $C(b) = \Gamma(b) \cap S$ be a random variable that counts the number of times b is "covered" by S. Clearly, $r_S(b) \geq 1$ if, and only if, $C(b) < r(b)$. The expectation of $C(b)$ is $\mu_b = \mathbb{E}[C(b)] = \sum_{a \in \Gamma(b)} \rho \cdot x_a \geq \rho \cdot r(b)$. Since the nodes in $\Gamma(b)$ are chosen independently, $C(b)$ is a sum of independent Bernoulli random variables. The statement now follows by applying the Chernoff bound:

$$
\begin{aligned}
\Pr[C(b) < r(b)] &= \Pr\left[C(b) < \left(1 - \frac{\rho-1}{\rho}\right) \cdot \rho \cdot r(b)\right] \leq \\
&\leq \Pr\left[C(b) < \left(1 - \frac{\rho-1}{\rho}\right) \cdot \mu_b\right] \leq \\
&\leq e^{-\frac{1}{2}\left(\frac{\rho-1}{\rho}\right)^2 \mu_b} \leq e^{-\gamma \cdot r(b)} .
\end{aligned}
$$

□

Corollary 2.7 *$\mathbb{E}[r_S(B)] \leq f(\rho)|B|$, where $f(\rho) = \frac{1}{e\gamma}$ if $1 < \rho \leq 2+\sqrt{3}$. and $f(\rho) = e^{-\gamma}$ if $\rho \geq 2+\sqrt{3}$.*

Proof: Let $S' = \{a : x_a \geq 1/\rho\}$, let $r'(b) = r_{S'}(b) = \max\{r(b) - |\Gamma(b) \cap S'|, 0\}$, and let x' be defined by $x'_a = 0$ if $a \in S'$ and $x'_a = x_a$ otherwise. Note that x' is a feasible solution to LP (1) with the residual requirements r', and that $x'_a < 1/\rho$ for all $a \in A$. Thus by Lemma 2.6 we have

$$
\mathbb{E}\left[r'(B)\right] = \sum_{b \in B} \mathbb{E}\left[r'(b)\right] \leq \sum_{b \in B} \Pr\left[r'(b) \geq 1\right] \cdot r'(b) \leq \sum_{b \in B} e^{-\gamma \cdot r'(b)} \cdot r'(b) .
$$

Let $z = r'(b) \geq 1$ and $f(z) = e^{-\gamma z} \cdot z$. Then $f'(z) = e^{-\gamma z}(1 - \gamma z)$. Hence in the range $z \geq 1$, the function $f(z)$ has maximum value:

- $\frac{1}{e\gamma}$ if $\gamma \leq 1$ (namely, if $1 < \rho \leq 2+\sqrt{3}$), attained at $z = 1/\gamma$.
- $e^{-\gamma}$ if $\gamma \geq 1$ (namely, if $\rho \geq 2+\sqrt{3}$), attained at $z = 1$.

The statement follows. □

Now we finish the proof of Theorem 2.5. The algorithm is as follows. We compute an optimal solution x to LP (1), and then an edge set S as in Corollary 2.7. For every $b \in B$ let A_b be a set of $r_S(b)$ neighbors in $\Gamma(b) \setminus S$, and let $S' = \bigcup_{b \in B} A_b$. The solution returned is $S \cup S'$. Note that $\mathbb{E}[|S'|] \leq \mathbb{E}[r_S(B)]$. Thus by Corollary 2.7, the expected size of our solution is bounded by

$$\mathbb{E}(|S|) + \mathbb{E}(r_S(B)) + |B| \leq \rho x(A) + f(\rho)|B| + |B| \leq \max\{\rho, f(\rho) + 1\}(x(A) + |B|) \ .$$

Consequently, as $x(A) + |B|$ is a lower bound on the optimal solution value, the approximation ratio is bounded by $\max\{\rho, f(\rho) + 1\}$. Solving the equation $\rho = f(\rho) + 1$ for $f(\rho) = \frac{1}{e\gamma} = \frac{2\rho}{e(\rho-1)^2}$ gives the result.

The proof of Theorem 2.5, and thus also of Theorem 1.1 for the case of uniform cost, is complete.

3 Proof of Theorem 1.2

We say that an edge set $F \subseteq E$ *covers* a node set $U \subseteq V$, or that F is a U*-cover*, if $\delta_F(v) \neq \emptyset$ for every $v \in U$. Consider the following auxiliary problem:

Restricted Minimum-Power Edge-Cover

Instance: A graph $G = (V, E)$ with edge-costs $\{c(e) : e \in E\}$, $U \subseteq V$, and degree bounds $\{\ell_v : v \in U\}$.

Objective: Find a power assignment $\{\pi(v) : v \in V\}$ that minimizes $\sum_{v \in V} \pi(v)$, such that $\pi(v) \geq \ell_v$ for all $v \in U$, and such that the edge set $F = \{e = uv \in E : \pi(u), \pi(v) \geq c(e)\}$ covers U.

Later, we will prove the following lemma.

Lemma 3.1 Restricted Minimum-Power Edge-Cover *admits a* $3/2$*-approximation algorithm.*

Theorem 1.2 is deduced from Lemma 3.1 and the following statement.

Lemma 3.2 *If* Restricted Minimum-Power Edge-Cover *admits a* ρ*-approximation algorithm, then* Minimum-Power Edge-Multi-Cover *admits a* $(k - 1 + \rho)$*-approximation algorithm.*

Proof: Consider the following algorithm.

1. Let $\pi(v)$ be the power assignment computed by the ρ-approximation algorithm for Restricted Minimum-Power Edge-Cover with $U = \{v \in V : r(v) \geq 1\}$ and bounds $\ell_v = w_v$ for all $v \in U$. Let $F = \{e = uv \in E : \pi(u), \pi(v) \geq c(e)\}$.

2. For every $v \in V$ let I_v be the edge-set obtained by picking the least cost $r_F(v)$ edges in $\delta_{E \setminus F}(v)$ and let $I = \cup_{v \in V} I_v$.

Clearly, $F \cup I$ is a feasible solution to Minimum-Power Edge-Multi-Cover. Let opt denote the optimal solution value for Minimum-Power Edge-Multi-Cover. In what follows note that $\pi(V) \leq \rho \cdot \mathsf{opt}$ and that $\sum_{v \in V} w_v \leq \mathsf{opt}$.

We claim that

$$p_{I \cup F}(V) \leq \pi(V) + (k-1) \cdot \mathsf{opt} \ .$$

As $\pi(V) \leq \rho \cdot \mathsf{opt}$, this implies $p_{I \cup F}(V) \leq (\rho + k - 1) \cdot \mathsf{opt}$.

For $v \in V$ let Γ_v be the set of neighbors of v in the graph (V, I_v). The contribution of each edge set I_v to the total power is at most $p_{I_v}(\Gamma_v) + p_{I_v}(v)$. Note that $\pi(v) \geq p_{I_v}(v)$ and $\pi(v) \geq p_F(v)$ for every $v \in V$, hence $p_{F \cup I_v}(v) \leq \pi(v)$. This implies

$$p_{F \cup I}(V) \leq \sum_{v \in V} (\pi(v) + p_{I_v}(\Gamma_v)) = \pi(V) + \sum_{v \in V} p_{I_v}(\Gamma_v) \ .$$

Now observe that $|\Gamma_v| = |I_v| = r_F(v) \leq k-1$ and that $p_{I_v}(u) \leq w_v$ for every $u \in \Gamma_v$. Thus

$$p_{I_v}(\Gamma_v) \leq (k-1) \cdot w_v \qquad \forall v \in V \ .$$

Finally, using the fact that $\sum_{v \in V} w_v \leq \mathsf{opt}$, we obtain

$$p_{F \cup I}(V) \leq \pi(V) + \sum_{v \in V} p_{I_v}(\Gamma_v) \leq \pi(V) + (k-1) \sum_{v \in V} w_v \leq \pi(V) + (k-1) \cdot \mathsf{opt} \ .$$

This finishes the proof of the lemma. □

In the rest of this section we prove Lemma 3.1.

We reduce Restricted Minimum-Power Edge-Cover to the following problem that admits an exact polynomial time algorithm, c.f. [10].

Minimum-Cost Edge-Cover:
Instance: A multi-graph (possibly with loops) $G = (U, E)$ with edge-costs $\{c(e) : e \in E\}$.
Objective: Find a minimum cost edge-set $F \subseteq E$ that covers U.

Our reduction is not approximation ratio preserving, but incurs a loss of 3/2 in the approximation ratio. That is, given an instance (G, c, U, ℓ) of Restricted Minimum-Power Edge-Cover, we construct in polynomial time an instance (G', c') of Minimum-Cost Edge-Cover such that:

(i) For any U-cover I' in G' corresponds a feasible solution π to (G, c, U, ℓ) with $\pi(V) \leq c'(I')$.

(ii) $\mathsf{opt}' \leq 3\mathsf{opt}/2$, where opt is an optimal solution value to Restricted Minimum-Power Edge-Cover and opt' is the minimum cost of a U-cover in G'.

Hence if I' is an optimal (min-cost) solution to (G', c'), then $\pi(V) \leq c'(I') \leq 3\mathsf{opt}/2$.

Clearly, we may set $\ell_v = 0$ for all $v \in V \setminus U$. For $I \subseteq E$ let

$$D(I) = \sum_{v \in V} \max\{p_I(v) - \ell_v, 0\} \ .$$

Here is the construction of the instance (G', c'), where $G' = (U, E')$ and E' consists of the following three types of edges, where for every edge $e' \in E'$ corresponds a set $I(e') \subseteq E$ of one edge or of two edges.

1. For every $v \in U$, E' has a loop-edge $e' = vv$ with $c'(vv) = \ell_v + D(\{vu\})$ where vu is is an arbitrary chosen minimum cost edge in $\delta_E(v)$.
   Here $I(e') = \{vu\}$.

2. For every $uv \in E$ such that $u, v \in U$, E' has an edge $e' = uv$ with $c'(uv) = \ell_u + \ell_v + D((\{uv\})$.
   Here $I(e') = \{uv\}$.

3. For every pair of edges $ux, xv \in E$ such that $c(ux) \geq c(xv)$, E' has an edge $e' = uv$ with $c'(uv) = \ell_v + \ell_u + D(\{ux, xv\})$.
   Here $I(e') = \{ux, xv\}$.

Lemma 3.3 *Let $I' \subseteq E'$ be a U-cover in G', let $I = \cup_{e \in I'} I(e)$, and let π be a power assignment defined on V by $\pi(v) = \max\{p_I(v), \ell_v\}$. Then $\pi(V) \leq c'(I')$, I is a U-cover in G, and π is a feasible solution to (G, c, U, ℓ).*

Proof: We have that I is a U-cover in G, by the definition of I and since $I(e')$ covers both endnodes of every $e' \in E'$. By the definition of π, we have that $I \subseteq \{e = uv \in E : \pi(u), \pi(v) \geq c(e)\}$. Hence π is a feasible solution to (G, c, U, ℓ), as claimed.

We prove that $\pi(V) \leq c'(I')$. For $e' = uv \in E'$ let $\ell(e') = \ell_v$ if e' is a type 1 edge, and $\ell(e') = \ell_u + \ell_v$ otherwise. Note that $\pi(v) = \max\{p_I(v), \ell(v)\} = \ell_v + \max\{p_I(v) - \ell(v), 0\}$, hence

$$\pi(V) = \sum_{v \in U} \ell_v + \sum_{v \in V} \max\{p_I(v) - \ell(v), 0\} = \sum_{v \in U} \ell_v + D(I) \ .$$

By the definition of $\ell(e')$ and since I' is a U-cover $\sum_{v \in U} \ell_v \leq \sum_{e' \in I'} \ell(e')$. Also, $D(I) = D\left(\cup_{e' \in I'} I(e')\right)$, by the definition of I. Thus we have

$$\sum_{v \in U} \ell_v + D(I) \leq \sum_{e' \in I'} \ell(e') + D\left(\cup_{e' \in I'} I(e')\right) \ .$$

It is easy to see that

$$D\left(\cup_{e' \in I'} I(e')\right) \leq \sum_{e' \in I'} D(I(e')) \ .$$

Finally, note that $\ell(e') + D(I(e')) = c'(e')$ for every $e' \in I'$ (if e' is a type 1 edge, this follows from our assumption that $\ell_v \geq \min\{c(e) : e \in \delta_E(v)\}$). Combining we get

$$\begin{aligned}
\pi(V) &= \sum_{v \in U} \ell_v + D(I) \leq \\
&\leq \sum_{e' \in I'} \ell(e') + D\left(\cup_{e' \in I'} I(e')\right) \leq \\
&\leq \sum_{e' \in I'} \ell(e') + \sum_{e' \in I'} D(I(e')) = \\
&= \sum_{e' \in I'} \left(\ell(e') + D(I(e'))\right) = \\
&= \sum_{e' \in I'} c'(e') = c'(I') \ .
\end{aligned}$$

□

Lemma 3.4 *Let $\{\pi(v) : v \in V\}$ be a feasible solution to an instance (G, c, U, ℓ) of* Restricted Minimum-Power Edge-Cover. *Then there exists a U-cover I' in G' such that $c'(I') \leq 3\pi(V)/2$.*

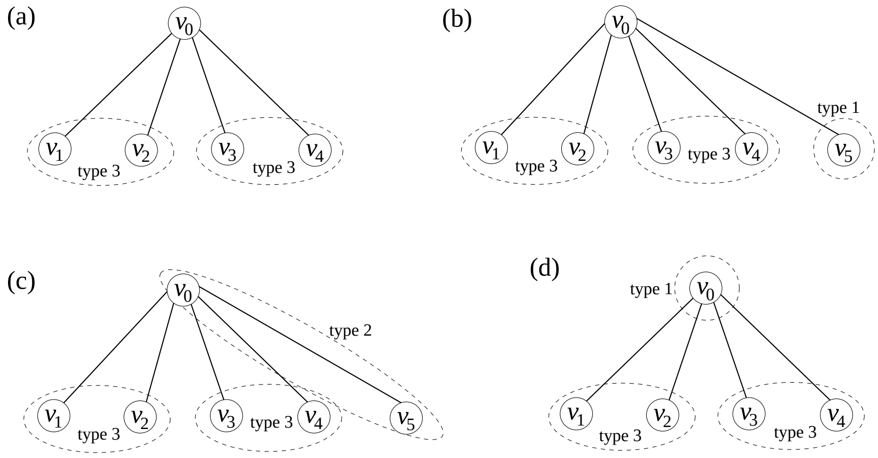

Figure 1: Illustration to the definition of the U-cover I'.

Proof: Let $I \subseteq \{e = uv \in E : \pi(u), \pi(v) \geq c(e)\}$ be an inclusion minimal U-cover. We may assume that $\pi(v) = \max\{p_I(v), \ell_v\}$ for every $v \in V$. Since any inclusion minimal U-cover is a collection of node disjoint stars, it is sufficient to prove the statement for the case when I is a star. Then I has at most one node not in U, and if there is such a node, then it is the center of the star, if $|I| \geq 2$; in the case I consists of a single edge e, then we define the center of I to be the endnode of e in $V \setminus U$ if such exists, or an arbitrary endnode of e otherwise.

We define a U-cover I' in G', and show that

$$c'(I') \leq \frac{3}{2} \sum_{v \in V} \max\{p_I(v), \ell_v\} = \frac{3}{2}\pi(V) \ . \tag{2}$$

Let v_0 be the center of I and let $\{v_i : 1 \leq i \leq d\}$ be the leaves of I ordered by descending order of costs $c(v_0v_i) \geq c(v_0v_{i+1})$. The U-cover $I' \subseteq E'$ is defined as follows. We cover each pair v_{2i-1}, v_{2i}, $i = 1, \ldots, \lfloor d/2 \rfloor$, by a type 3 edge. This covers all the nodes except v_0, and maybe v_d if d is odd. We add an additional edge f of type 1 or 2, if there are nodes in U (v_0 and/or v_d) that remain uncovered by the picked type 3 edges. Formally, we have the following 4 cases, see Figure 1.

1. d is even and $v_0 \notin U$, see Figure 1(a). Then U is covered by type 3 edges.

2. d is odd, and $v_0 \notin U$, see Figure 1(b). Then we add a type 1 edge f to cover v_d.

3. d is odd and $v_0 \in U$, see Figure 1(c). Then we add a type 2 edge f to cover v_0, v_d.

4. d is even and $v_0 \in U$, see Figure 1(d). Then we add a type 1 edge f to cover v_0.

Consider a type 3 edge $v_{2i-1}v_{2i} \in I'$. Let $q_i = \max\{c(v_{2i-1}v_0) - \ell_{v_0}, 0\}$. Note that $c'(v_{2i-1}v_{2i}) \leq \pi(v_{2i-1}) + \pi(v_{2i}) + q_i$. The key point is that

$$q_i \leq \frac{1}{2}(\pi(v_{2i-3}) + \pi(v_{2i-2})) \quad i = 2, \ldots, \lfloor d/2 \rfloor \ .$$

This is since $q_i \le c(v_0v_{2i-1}) \le \frac{1}{2}\left(c(v_0v_{2i-3}) + c(v_0v_{2i-2})\right)$ while $c(v_0v_j) \le \pi(v_j)$. Therefore,

$$\sum_{i=1}^{d/2} c'(v_{2i-1}v_{2i}) \le \sum_{i=1}^{d/2} [\pi(v_{2i-1}) + \pi(v_{2i}) + q_i] \le \sum_{i=1}^{2\lfloor d/2 \rfloor} \pi(v_i) + q_1 + \frac{1}{2}\sum_{i=1}^{d-2} \pi(v_i)$$

Now, we prove that (2) hold in each one of our four cases.

1. $v_0 \notin U$ and d is even. Note that $q_1 \le c(v_0v_1) \le \pi(v_0)$. Then:

$$c'(I') = \sum_{i=1}^{d/2} c'(e_i) \le \frac{3}{2}\sum_{i=1}^{d} \pi(v_i) + q_1 \le \frac{3}{2}\sum_{i=1}^{d} \pi(v_i) + \pi(v_0) \le \frac{3}{2}\sum_{i=0}^{d} \pi(v_i)$$

2. $v_0 \notin U$ and d is odd. In this case $f = v_dv_d$ is a loop type 1 edge, so $c'(f) \le \pi(v_d) + \max(c(v_0v_d) - \ell_{v_0}, 0)$. This implies

$$\begin{aligned} q_1 + c'(f) &\le c(v_0v_1) + c(v_0v_d) + \pi(v_d) \le \pi(v_0) + \frac{1}{2}[\pi(v_0) + \pi(v_d)] + \pi(v_d) \\ &= \frac{3}{2}\left(\pi(v_0) + \pi(v_d)\right) . \end{aligned}$$

   Thus

$$c'(I') = \sum_{i=1}^{d/2} c'(e_i) + c'(f) \le \frac{3}{2}\sum_{i=1}^{d-1} \pi(v_i) + c'(f) + q_1 \le \frac{3}{2}\sum_{i=0}^{d} \pi(v_i)$$

3. $v_0 \in U$ and d is odd. In this case $f = v_0v_d$, so $c'(f) \le \max(\ell_{v_0}, c(v_0v_d)) + \pi(v_d)$. This implies $q_1 + c'(f) \le c(v_0v_1) + c(v_0v_d) + \pi(v_d) \le \frac{3}{2}\left(\pi(v_0) + \pi(v_d)\right)$. Thus

$$c'(I') = \sum_{i=1}^{d/2} c'(e_i) + c'(f) \le \frac{3}{2}\sum_{i=1}^{d-1} \pi(v_i) + c'(f) + q_1 \le \frac{3}{2}\sum_{i=0}^{d} \pi(v_i) .$$

4. $v_0 \in U$ and d is even. In this case $f = v_0v_0$ is a loop type 1 edge, so $c'(f) \le \ell_{v_0} + c(v_0v_d) \le \ell_{v_0} + \frac{1}{2}\left(\pi(v_{d-1}) + \pi(v_d)\right)$. This implies $q_1 + c'(f) \le c(v_0v_1) + \frac{1}{2}\left(\pi(v_{d-1}) + \pi(v_d)\right)$. Thus

$$\begin{aligned} c'(I') &= \sum_{i=1}^{d/2} c'(e_i) + c'(f) \le \sum_{i=1}^{d} \pi(v_i) + \frac{1}{2}\sum_{i=1}^{d-2} \pi(v_i) + q_1 + c'(f) \\ &\le \frac{3}{2}\sum_{i=1}^{d} \pi(v_i) + \pi(v_0) \le \sum_{i=0}^{d} \pi(v_i) . \end{aligned}$$

This concludes the proof of the lemma. $\square$

As was mentioned, Lemmas 3.3 and 3.4 imply Lemma 3.1. Lemmas 3.1 and 3.2 imply Theorem 1.2, hence the proof of Theorem 1.2 is now complete.

4 Conclusions and open problems

The main results of this paper are two new approximation algorithm for MPEMC: one with ratio $O(\log k)$ for general costs, and the other with constant ratio for uniform costs. This improves the ratio $O(\log(nk)) = O(\log n)$ of [3]. We also gave a $(k + 1/2)$-approximation algorithm, which is better than our $O(\log k)$-approximation algorithm for small values of k (roughly $k \leq 6$).

The main open problem is whether for general costs, the ratio $O(\log k)$ shown in this paper is tight, or the problem admits a constant ratio approximation algorithm.

References

[1] E. Althaus, G. Calinescu, I. Mandoiu, S. Prasad, N. Tchervenski, and A. Zelikovsky. Power efficient range assignment for symmetric connectivity in static ad-hoc wireless networks. *Wireless Networks*, 12(3):287–299, 2006.

[2] M. Hajiaghayi, G. Kortsarz, V. Mirrokni, and Z. Nutov. Power optimization for connectivity problems. *Math. Programming*, 110(1):195–208, 2007.

[3] G. Kortsarz, V. Mirrokni, Z. Nutov, and E. Tsanko. Approximating minimum-power degree and connectivity problems. *Algorithmica*, 58, 2010.

[4] G. Kortsarz and Z. Nutov. Approximating minimum-power edge-covers and $2,3$-connectivity. *Discrete Applied Mathematics*, 157:1840–1847, 2009.

[5] Y. Lando and Z. Nutov. On minimum power connectivity problems. *J. of Discrete Algorithms*, 8(2):164–173, 2010.

[6] J. Lee, V. Mirrokni, V. Nagarajan, and M. Sviridenko. Maximizing nonmonotone submodular functions under matroid or knapsack constraints. *SIAM J. Discrete Mathematics*, 23(4):20532078, 2010.

[7] Z. Nutov. An almost $O(\log k)$-approximation for k-connected subgraphs. In *SODA*, pages 912–921, 2009.

[8] Z. Nutov. Approximating minimum power covers of intersecting families and directed edge-connectivity problems. *Theoretical Computer Science*, 411(26-28):2502–2512, 2010.

[9] Z. Nutov. Approximating minimum power k-connectivity. *Ad Hoc & Sensor Wireless Networks*, 9(1-2):129–137, 2010.

[10] A. Schrijver. *Combinatorial Optimization, Polyhedra and Efficiency*. Springer-Verlag Berlin, Heidelberg New York, 2004.