**The Open University of Israel**
**Department of Mathematics and Computer Science**

# Approximating Survivable Networks with Minimum Number of Steiner Points

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By

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## Abstract

Given a graph $H = (U, E)$ and connectivity requirements $r = \{r(u, v) : u, v \in R \subseteq U\}$, we say that $H$ satisfies $r$ if it contains $r(u, v)$ internally disjoint $uv$-paths for all $u, v \in R$. We consider the Survivable Network with Minimum Number of Steiner Points (SN-MSP) problem: given a finite set $V$ of points in a normed space $(M, \|\cdot\|)$, and connectivity requirements, find a minimum size set $S \subseteq M - V$ of additional points, such that the unit disc graph induced by $V \cup S$ satisfies the requirements. In the Survivable Network Design Problem (SNDP) we are given a graph $G = (V, E)$ with edge costs and connectivity requirements, and seek a min-cost subgraph $H$ of $G$ that satisfies the requirements. Let $k = \max_{u,v \in V} r(u, v)$ denote the maximum connectivity requirement. Given a normed space, $(M, \|\cdot\|)$, let $\Delta$ be the minimum number so that for every set $V \subseteq M$ contained in a unit ball, the unit disc graph induced by $V$ has a dominating set of size at most $\Delta$ ($\Delta = 5$ in the Euclidean plane $\mathbb{R}^2$, $\Delta$ is at most the Hadwiger Number in $\mathbb{R}^\ell$, and $\Delta$ is a constant for any normed space of finite dimension). We will show a natural transformation of an SN-MSP instance $V, r$ into an SNDP instance $G = (V, E), c, r$ so that an $\alpha$-approximation for the SNDP instance implies an $\alpha \cdot \rho(k)$-approximation algorithm for the SN-MSP instance, where $\rho(k) = O(\Delta k^2) = O(k^2)$. Specifically, using known approximation algorithms for SNDP, we obtain the following approximation ratios for SN-MSP:

- $O(k^2 \ln k) \cdot \rho(k) = O(k^4 \ln k)$ for *subset uniform requirements*, when there is $R \subseteq V$ such that $r(u, v) = k$ for all $u, v \in R$, and $r(u, v) = 0$ otherwise. In the case of *uniform requirements*, when $R = V$, our ratio is $O\left(\ln \frac{|V|}{|V|-k} \cdot \ln k\right) \cdot \rho(k) = O(k^2 \ln k)$. In particular, this solves an open problem from [3] for the Euclidean plane.
- $O(k^2) \cdot \rho(k) = O(k^4)$ for *rooted (single source/sink) requirements*, when there is $s \in V$ so that $r(u, v) > 0$ implies $u = s$ or $v = s$. In the case $r(s, v) = k$ for all $v \in R \setminus \{s\} \subseteq V$ the ratio is $O(k \ln k) \cdot \rho(k) = O(k^3 \ln k)$, and in the case $R = V$ the ratio is just $2\rho(k)$.

We also obtain the ratio $O(k^3 \ln |R|) \cdot \rho(k) = O(k^5 \ln |R|)$ for arbitrary requirements.

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# 1 Introduction and motivation

In the past several decades, the area of wireless communication has developed rapidly, and the uses of wireless networks have grown significantly. They range from civilian cellular networks through battlefield military applications, and disaster relief communication (see [20], [5]).

However, wireless networks are affected by various constraints which do not affect wired networks. Signal quality and intensity decays with inverse ratio of the square of the distance (in free space) [17], and in other media may decay with inverse ratio to the distance to the power of 4 [7]. In addition, wireless transmission is subject to interference by environmental factors such as electromagnetic interruption and physical obstacles, both natural and man-made (widely discussed in [17]).

We consider our networks to be *ad hoc*, i.e. networks consisting of mobile units communicating via radio transceivers. Each radio transceiver is assigned transmission and reception power, along with transmission and reception orientation. For two transceivers, $u$ and $v$, $u$ can transmit to $v$ if $v$ is in the transmission radius of $u$, $u$ is in the reception radius of $v$, and their transmission and reception orientations are correlated.

In most networks, if $v$ cannot directly receive the transmission from $u$, the transceivers are able to cooperate in order to transmit the message from $u$ to $v$. In this case we say the network is *multi hop*.

For simplicity of the model, we also assume that the transceivers cannot relocate once placed, and that transmission and reception are equally powerful in all directions, i.e. the network is *static*, *symmetric* and *omnidirectional*.

The model of an ad hoc, static, symmetric, multi hop wireless network with omnidirectional transmitters and receivers was considered in [2], [13], [1] and [3]. One known way to increase connectivity is power assignment. Given the set of terminals, $V$, we wish to assign each terminal with transmission and reception power to satisfy the connectivity requirements. This problem was considered and studied in [18], [21] and [10]. Recall, however, that the power needed to transmit through a given distance, $d$, has at least the growth order of $d^2$, and might be proportional with $d^4$.

Since energy budget and battery time is a primary constraint in designing wireless networks, one might prefer adding sensors rather than increasing power [7]. Thus the problem of adding sensors to increase connectivity arises. We therefore assume transceivers are assigned constant power for both transmission and reception. In our model, we translate fixed power to a fixed transmission radius.

Given the previous characterization, the model is well described by the *unit disc graph* in a metric space, in which two distinct nodes are adjacent if and only if their distance is no grater than 1. Given a metric space, $(M, d)$, and finite set $V \subseteq M$ with connectivity requirements $\{r(u, v) \mid u, v \in V\}$, we think of elements of $V$ as terminals of a wireless network and wish to adjust the network to satisfy these requirements.

# 2 Preliminaries

## 2.1 Connectivity Problems in Wired Networkss

Let $H = (U, E)$ be an undirected graph, possibly with parallel edges. For $u, v \in U$ let $\kappa_H(u, v)$ denote the maximum number of pairwise internally vertex disjoint $uv$-paths in $H$. Given a set of non-negative integer connectivity requirements $r = \{r(u, v) : u, v \in R \subseteq U\}$, we say that $H$ *satisfies* $r$ if $\kappa_H(u, v) \geq r(u, v)$ for all $u, v \in R$. Let $k = \max_{u,v \in R} r(u, v)$ denote the maximum connectivity requirement.

In wired networks, connecting two terminals, if possible, incurs a cost. Therefore an inherent optimization problem for wired networks is the following.

Survivable Network Design Problem (SNDP):
*Instance:* A multi-graph $G = (V, E)$ with edge costs $\{c_e : e \in E\}$ and pairwise connectivity requirements $r = \{r(u, v) : u, v \in R \subseteq V\}$.
*Objective:* Find a minimum cost subgraph $H$ of $G$ that satisfies $r$.

Several important types of requirements are considered in the literature.

- *Subset uniform requirements:* there is $R \subseteq V$, such that $r(u, v) = k$ for all $u, v \in R$, and $r(u, v) = 0$ otherwise;
  Requiremenrs are called *uniform* if $R = V$.
- *Rooted requirements:* there is $s \in V$ so that $r(u, v) > 0$ implies $u = s$ or $v = s$;
  Requirements are called *rooted subset uniform* if there is $R \subseteq V$ such that $r(s, v) = k$ for all $v \in R \setminus \{s\}$, and $r(u, v) = 0$ otherwise; If $R = V$ requirements are called *rooted-uniform*.

The problem is widely studied, and several approximation results have been achieved. Let $k = \max_{u,v \in V} r(u, v)$. For *simple graphs*, the currently best known approximation ratios for SNDP are $O(k^3 \ln |R|)$ for general requirements [4] and:

- $O(k^2 \ln k)$ for subset uniform requirements [16], and $O(\ln \frac{|V|}{|V|-k} \cdot \ln k)$ (which is $O(\ln k)$ unless $k = |V| - o(|V|)$) for uniform requirements [15].
- $O(k^2)$ for rooted requirements, $O(k \ln k)$ for rooted subset-uniform requirements [16], and 2 for rooted-uniform requirements [9].

However, as will be explained in Section 2.3, we are interested in approximation ratios on multigraphs. The following statement shows that most of known approximation algorithms for simple graphs extend to multigraphs.

**Lemma 2.1** *For* SNDP*, an* $\alpha$*-approximation algorithm on simple graphs implies an* $\alpha$*-approximation algorithm on multigraphs; this is so also for subset uniform, uniform, rooted, and rooted subset uniform requirements. In the case of rooted uniform requirements,* SNDP *on multigraphs admits a* 2*-approximation algorithm.*

Lemma 2.1 is proved later in Section 3.

## 2.2 Problems Considered - Connectivity in Wireless Networks

**Definition 2.1** *Given a finite set of points* $V \subset M$ *in a metric space* $(M, d)$*, the* unit disc graph induced by $V$*, denoted by* $G[V]$*, has node set* $V$ *and edge set*
$\{uv : 0 < d(u, v) \leq 1, u, v \in V\}$.

As stated earlier, in wireless networks, adding relay stations incurs a cost. Therefore a natural optimization problem in wireless networks is the following.

Survivable Network with Minimum Number of Steiner Points (SN-MSP)
*Instance:* A finite set $V$ of points in a metric space $(M, d)$ and pairwise connectivity requirements $r = \{r(u, v) : u, v \in R \subseteq V\}$.
*Objective:* Find a minimum size set of points $S \subset M - V$ such that $G[V \cup S]$ satisfies $r$.

We consider the problem with every special type of requirement mentioned for SNDP. As in practical networks $k$ is rather small, we focus on obtaining approximation ratios that depend on $k$ only. For $k = 1$, SN-MSP with uniform requirements is the Steiner Tree with Minimum Number of Steiner Points problem (ST-MSP). In the Euclidean plane, this problem admits a 2.5-approximation algorithm [6]. On graphs with unit edge lengths ST-MSP includes the Set-Cover problem [14], and thus has an $\Omega(\ln |V|)$-approximation threshold. Hence for SN-MSP one cannot expect in arbitrary metric spaces a ratio that depends on $k$ only. We will consider instances of SN-MSP defined on a normed space $(M, \|\cdot\|)$, when the metric $d$ is induced by the norm $\|\cdot\|$. Obtaining a non-trivial (with ratio depending on $k$ only) approximation algorithm for SN-MSP with uniform requirements and $k \geq 2$ in the Euclidean plane was posed as an open problem in [3].

## 2.3 Our Results

We will prove a much more general result than the open problem presented in [3], that resolves this open problem as a particular case.

### 2.3.1 The Covering Parameter $\Delta$

Our approximation ratios are expressed in terms of $k$ and the following parameter that depends on the normed space. Let $\Delta = \Delta(M, \|\cdot\|)$ be the minimum number such that for every $V \subseteq M$ contained in a unit ball, $G[V]$ has a dominating set of size at most $\Delta$. For $\mathbb{R}^2$ with the Euclidean norm, it is well known that $\Delta = 5$. For $\mathbb{R}^3$ with the Euclidean norm, it is well known that $\Delta = 11$.

For $\ell, p \in \mathbb{N}$, let $\mathbb{R}^\ell_p$ be the normed space induced on $\mathbb{R}^\ell$ by the norm $\|(x_1, x_2, \ldots, x_\ell)\|_p = \left(\sum_{i=1}^{\ell} |x_i|^p\right)^{1/p}$. A collection of open convex subsets of $\mathbb{R}^\ell$ is called a *packing* if the sets are mutually pairwise disjoint. Two sets in a packing which share a boundry point are called *neighbours*. Let $B$ be an open convex subset of $\mathbb{R}^\ell$. The *Hadwiger Number of* $B$ is the maximum number of neighbours of $B$ over all packings composed of translations of $B$. In [19], Robins and Salowe proved that for $\mathbb{R}^\ell_p$, $\Delta$ is upper bounded by the Hadwiger Number of the unit ball in $\mathbb{R}^\ell_p$; in particular, for $\mathbb{R}^\ell_2$ (i.e. $\mathbb{R}^\ell$ with the Euclidean Norm), $\Delta \leq 2^{0.401\ell(1+o(1))}$, by [12].

### 2.3.2 The Main Theorem

Let $\rho(k) = (\Delta + 3)k^2 + 7k + 2$. Our main result is:

**Theorem 2.2** *An $\alpha$-approximation algorithm for* SNDP *on multigraphs implies an $\alpha \cdot \rho(k)$-approximation algorithm for* SN-MSP*, and this is so also for subset uniform, uniform, rooted, rooted subset uniform, and rooted uniform requirements.*

Using Lemma 2.1, and substituting the currently best known values of $\alpha$ in Theorem 2.2, we obtain:

**Corollary 2.3** SN-MSP *admits the following approximation ratios:*

- $O(k^2 \ln k) \cdot \rho(k) = O(k^4 \ln k)$ *for* subset uniform requirements *and* $O\left(\ln \frac{|V|}{|V|-k} \cdot \ln k\right) \cdot \rho(k) = O(k^2 \ln k)$ *for uniform requirements.*
- $O(k^2) \cdot \rho(k) = O(k^4)$ *for rooted requirements,* $O(k \ln k) \cdot \rho(k) = O(k^3 \ln k)$ *for rooted*

*subset uniform requirements, and* $2\rho(k) \leq 2\Delta k^2 = O(k^2)$ *for rooted uniform requirements.*

Theorem 2.2 together with the $O(k^3 \ln|R|)$-approximation algorithm for SNDP of [4] implies the ratio $O(k^3 \ln|R|) \cdot \rho(k) = O(k^5 \ln|R|)$ for SN-MSP with arbitrary requirements. But in this case we conjecture that a ratio $O(k^3) \cdot \rho(k)$ can be achieved.

# 3 Proof of the Main Result

## 3.1 The Main Lemma

We will prove the following statement that implies Theorem 2.2.

**Lemma 3.1** *There exists a polynomial time algorithm that given an instance $V, r$ of* SN-MSP *constructs an instance $G = (V, E), c, r$ of* SNDP *so that: any solution of cost $C$ to* SNDP *can be converted in polynomial time to a solution of size $\leq C$ to* SN-MSP*, and for every solution $S$ to* SN-MSP *there exists a solution of cost $\leq |S| \cdot \rho(k)$ to* SNDP*. Furthermore, the construction preserves the requirement type (subset uniform, uniform, rooted, rooted subset uniform, and rooted uniform).*

**Claim 3.2** *Lemma 3.1 implies Theorem 2.2.*

**Proof:** Assuming validity of Lemma 3.1, consider the following approximation algorithm for SN-MSP:

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**Algorithm 1** *Approximation algorithm for* SN-MSP

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**Approximate-**SN-MSP$(V \subset M, r = \{r(u, v) : u, v \in R \subseteq V\})$

1. Let $G, c$ be the SNDP instance constructed from $V, r$.
2. Let $J \subseteq G$ be a subgraph satisfying $r$ with cost not greater than $\alpha$ times the optimal value of the SNDP problem.
3. Let $S \subseteq G$ be the feasible solution to SN-MSP constructed from $J$.

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By Lemma 3.1, the algorithm runs in polynomial time. In addition, the Lemma ensures that the constructed set $S$ is a feasible solution to SN-MSP on $V, r$. It remains to show the approximation ratio. To see this, let $J^*$ be a minimum cost subgraph of $G$ satisfying $r$, and let $S^*$ be a minimum size set of points so that $G[V \cup S^*]$ satisfies $r$. By the Lemma, there is a feasible solution $J_0 \subseteq G$ such that $c(J_0) \leq \rho(k) \cdot |S^*|$. Since $J^*$ is a solution of minimal cost, we get

$$c(J^*) \leq \rho(k) \cdot |S^*|$$

In the notaion of the algorithm, $J$ is a feasible solution constructed using an $\alpha$-approximation algorithm. Therefore

$$c(J) \leq \alpha \cdot c(J^*) \ .$$

Since $|S| \leq c(J)$, we get

$$|S^*| \leq |S|\alpha \cdot |S^*| \cdot \rho(k) \ .$$

□

In the rest of this section we prove Lemma 3.1. We divide the proof into three parts:

1. First we show how to build the SNDP instance in polynomial time.
2. The second part show how to cunstruct an SN-MSP feasible solution from an SNDP solution of no lesser cost.
3. The third part proves the lower bound on SN-MSP guaranteed by the lemma.

### 3.1.1 Instance Construction

**Definition 3.1** *Given a finite set of points $V \subset M$ and an integer $k \geq 1$, the graph $K_V$ is obtained by connecting every $u, v \in V$ by $k$ parallel edges, one of cost $\lceil d(u,v) \rceil - 1$ the others of cost $\lceil d(u,v) \rceil$.*

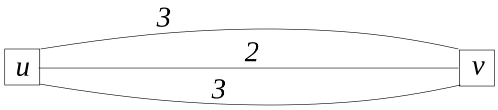
Clearly, given an SN-MSP instance, $V, r$, the graph $K_V$ with the corresponding costs $c$ can be constructed in polynomial time. The triple $K_V, c, r$ will serve as the SNDP instance guaranteed in Lemma 3.1. The construction preserves each of the requirement types listed in Lemma 3.1.

### 3.1.2 Solution Construction

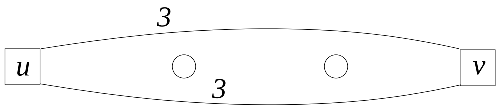
Let $J$ be a subgraph of $K_V$. Let $u, v \in V$ be adjacent in $J$ by $j + 1 \leq k$ edges. Place $\lceil d(u,v) \rceil - 1$ new points uniformly on the line segment between $u$ and $v$, dividing the segment to $\lceil d(u,v) \rceil$ subsegments, each of length $\frac{d(u,v)}{\lceil d(u,v) \rceil} \leq 1$. On each subsegment, place uniformly $j$ new points. Denote the set of added nodes $S(u,v)$. Denote by $S(J)$ the union over all adjacent pairs $u, v \in V$ of $S(u,v)$. An example for the process with $j = 2$ and $2 < d(u,v) \leq 3$ is shown in Figure 1. The following statement is straightforward.

**Claim 3.3** *$|S(J)| \leq c(J)$ holds for any subgraph $J$ of $K_V$. Furthermore, if $H = G[V \cup S(J)]$ is the unit disc graph induced by $V \cup S(J)$ then $\kappa_H(u,v) \geq \kappa_J(u,v)$ for all $u, v \in V$.*

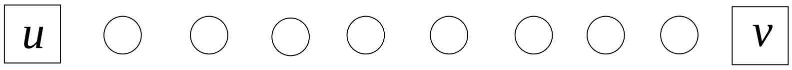
Clearly, $S(J)$ can be computed from $J$ in polynomial time.



(a) Edges of $J$ between two nodes $u$ and $v$. Here $2 < d(u, v) \leq 3$.



(b) Replacing the short edge with additional points.



(c) Replacing the rest of the edges.

Figure 1: Construction of $S(u, v)$

### 3.1.3 Lower Bounding SN-MSP

For a subset $C$ of nodes of a graph $G$ let $\Gamma_G(C)$ denote the set of neighbors of $C$ in $G$. We need the following lemma on connectivity of graphs.

**Lemma 3.4** *Let $V$ be a subset of nodes of a graph $G$, let $k \geq 1$ be an integer, and let $C$ be a connected component of $G - V$. Let $J_C$ be a set of new edges on $\Gamma_G(C)$ such that the following holds:*

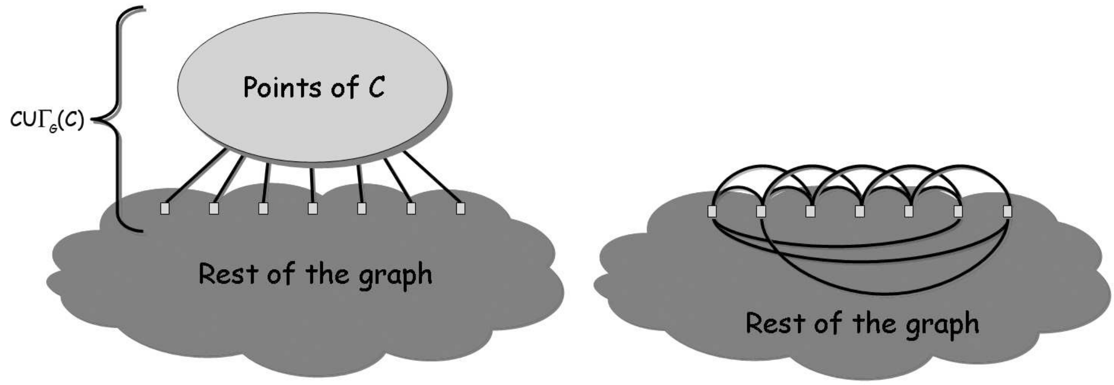
(i) *If $|\Gamma_G(C)| \leq k$ then $J_C$ has $\min\{\ell_{uv}, k - |I_{uv}|\}$ $uv$-edges for any $u, v \in \Gamma_G(C)$, where $I_{uv}$ is the set of $uv$-edges in $G$ and $\ell_{uv}$ is the maximum number of internally disjoint $uv$-paths in the subgraph of $G$ induced by $\{u, v\} \cup C$.*

(ii) *If $|\Gamma_G(C)| \geq k + 1$ then the graph induced by $\Gamma_G(C)$ in $G + J_C$ is $k$-connected.*

*Let $J = G - C + J_C$. Then $\kappa_J(u, v) \geq \min\{\kappa_G(u, v), k\}$ for all $u, v \in V$.*

**Proof:** The case $|\Gamma_G(C)| \leq k$ easily follows from the following construction. Let $u, v \in G - C$. Given a set $\Pi$ of at most $k$ internally disjoint $uv$-paths in $G$, for every $P \in \Pi$ do the following. For every maximal $u'v'$-subpath of $P$ that visits $C$ and has all its internal nodes in $C$, replace this subpath by a $u'v'$-edge $e$ not used by any other path in $\Pi$. Such $e$ is chosen to be an edge of $G$ if $\{u', v'\} \neq \{u, v\}$ and $I_{u'v'} \neq \emptyset$ or if $\{u', v'\} = \{u, v\}$ and $P$ is composed of only $u$ and $v$. Otherwise, $e$ is a new edge added to $G$. This gives a set of $|\Pi|$ internally disjoint $uv$-paths that do not visit $C$. Since the paths in $\Pi$ are internally disjoint, the set of edges added to $G$ may have parallel edges only between $u$ and $v$, and by the construction, the number of $uv$-edges added, if any, can be at most $\min\{\ell_{uv}, |\Pi| - |I_{uv}|\} \leq \min\{\ell_{uv}, k - |I_{uv}|\}$.

Now suppose that $|\Gamma_G(C)| \geq k+1$, so $\Gamma_G(C)$ induces in $G + J_C$ a $k$-connected graph. Let $u, v \in G - C$. Let $I_{uv}$ be a set of $uv$-edges in $J$. Let $A$ be a minimum size subset of nodes of




(a) Original Graph (b) Replacing Points with Edges

Figure 2: Replacing Steiner points with edges while maintaining relevant connectivity

$J$ so that $J-(A+I_{uv})$ has no $uv$-path. By Menger's Theorem $\kappa_J(u,v)=|A|+|I_{uv}|$. Thus if $|A|+|I_{uv}|\geq k$ then $\kappa_J(u,v)\geq k\geq \min\{\kappa_G(u,v),k\}$. We claim that if $|A|+|I_{uv}|\leq k-1$ then $G-(A+I_{uv})$ has no $uv$-path, hence by Menger's Theorem $\kappa_J(u,v)=|A|+|I_{uv}|\geq \kappa_G(u,v)$. Suppose to the contrary that $G-(A+I_{uv})$ has a $uv$-path $P$. Going along $P$ from $u$ to $v$, let $u'$ be the first and $v'$ the last node in $\Gamma_G(C)$; such $u',v'$ exist since $P$ must contain at least one node from $C$, as $P$ is not a $uv$-path in $J-(A+I_{uv})$. As $J$ has $k$ internally disjoint $u'v'$-paths and $|A|+|I_{uv}|\leq k-1$, the graph $J-(A+I_{uv})$ has at least one $u'v'$-path $P'$. Replacing the $u'v'$-subpath of $P$ by $P'$ gives a $uv$-path in $J-(A+I_{uv})$, contradicting the definition of $A$. □

Let $S$ be a feasible solution to an SN-MSP instance, so $G=G[V\cup S]$ satisfies $r$. The key step in constructing a solution $J$ to SNDP of cost $c(J)\leq |S|\cdot\rho(k)$ is replacing every connected component $C$ of $G-V$ by an edge set $J_C$ as in Lemma 3.4. Obviously, $\Gamma_G(C)\subseteq V$, and thus $J_C\subseteq K_V$. A general example of this process is shown in Figure 2. The following lemma shows that there exists such $J_C$ of low cost.

**Lemma 3.5** *For every connected component $C$ of $G-V$ there exists a subset $J_C$ of edges of $K_V$ as in Lemma 3.4 of cost $c(J_C)\leq\rho(k)\cdot|C|$.*

The proof of Lemma 3.5 is somewhat long, so we prove it after the following corollary, which easily implies the last part of Lemma 3.1.

**Corollary 3.6** *Let $\mathcal{C}$ be the set of connected components of $G-V$. For $C\in\mathcal{C}$ let $J_C$ be as in Lemma 3.5. Then $J=G-S+(\bigcup_{C\in\mathcal{C}} J_C)$ is a subgraph of $K_V$ of cost $c(J)\leq\rho(k)\cdot|S|$ that satisfies $r$.*

**Proof:** It is easy to see that for any $u, v \in V$ the number of $uv$-edges in $J$ is at most $k$. Hence $J$ is a subgraph of $K_V$. As $\mathcal{C}$ is a partition of $S$, we have by Lemma 3.5 $c(J) = c(\bigcup_{C \in \mathcal{C}} J_C) \leq \sum_{C \in \mathcal{C}} c(J_C) \leq \sum_{C \in \mathcal{C}} \rho(k) \cdot |C| = \rho(k) \cdot |S|$. To prove that $J$ satisfies $r$, let $\mathcal{C} = \{C_1, C_2, \ldots, C_m\}$. For $1 \leq j \leq m$ let $G_j = G - (\bigcup_{i=1}^{j} C_i) + (\bigcup_{i=1}^{j} J_{C_i})$. Using Lemma 3.4, a simple induction shows that for all $1 \leq j \leq m$, $G_j$ satisfies $r$. In particular, this is so for $J = G_m$. □

Now we prove Lemma 3.5. Let $C \in \mathcal{C}$. We start with the easier case $|\Gamma_G(C)| \leq k$. Then $J_C$ consists of $\min\{\ell_{uv}, k\}$ edges for every $u, v \in \Gamma_G(C)$. Let $u, v \in \Gamma_G(C)$. Since there are $\ell_{uv}$ internally disjoint $uv$-paths in the subgraph of $G$ induced by $\{u, v\} \cup C$, there is one such path containing no more than $\lfloor |C|/\ell_{uv} \rfloor$ points in $C$. Therefore $d(u, v) \leq |C|/\ell_{uv} + 1$. Consequently, the total cost of $uv$-edges in $J_C$ is bounded by $\ell_{uv} \cdot (\frac{|C|}{\ell_{uv}} + 1) = |C| + \ell_{uv} \leq 2|C|$. Thus as $|\Gamma_G(C)| \leq k$ we have

$$c(J_C) \leq \binom{k}{2} \cdot 2|C| \leq \frac{1}{2} k(k-1) \cdot 2|C| \leq k^2 \cdot |C| \leq \rho(k) \cdot |C| \; .$$

This finishes the proof of Lemma 3.5 for the case $|\Gamma_G(C)| \leq k$.

We now turn to prove Lemma 3.5 for the case $|\Gamma_G(C)| \geq k+1$. We will use the following two easy claims (Claim 3.7 is well known and therefore its proof is omitted):

**Claim 3.7** *Let $H$ be a $k$-connected graph on at least $k+1$ nodes. Then the graph obtained from $H$ by adding a new node and joining it to some $k$ nodes of $H$ is also $k$-connected.*

**Claim 3.8** *Let $U', U''$ be two subsets of the node set $V$ of a graph $H$ so that their union is $V$, and so that each of $U', U''$ has at least $k+1$ nodes and induces in $H$ a $k$-connected graph. If $H$ contains a matching $M$ between $U' - U''$ and $U'' - U'$ of size at least $k - |U' \cap U''|$ then $H$ is $k$-connected.*

**Proof:** By a theorem of Whitney (c.f. [8]), a graph $H$ on at least $k+1$ nodes is $k$-connected if, and only if, $\kappa_H(u, v) \geq k$ for every $u, v \in V$ so that $uv \notin H$. Therefore, it is sufficient to prove that if $u, v \in V$ and $uv \notin H$ then $H - A$ contains a $uv$-path for any $A \subseteq V - \{u, v\}$ so that $|A| \leq k - 1$. If $u, v \in U'$ or if $u, v \in U''$ then the statement is obvious, since both $U'$ and $U''$ induce in $H$ a $k$-connected graph. Therefore assume $u \in U' - U''$ and $v \in U'' - U'$. If there is $w \in (U' \cap U'') - A$, then, by the previouse argument, there exist a $uw$-path and a $wv$-path in $H - A$. Therefore there is a $uv$-path in $H - A$. Otherwise, $U' \cap U'' \subseteq A$, and in particular $|U' \cap U''| \leq k - 1$. Thus $H - A$ has an edge $e = w'w'' \in M$. We then obtain a $uv$-path in $H - A$ by the same argument as before. □

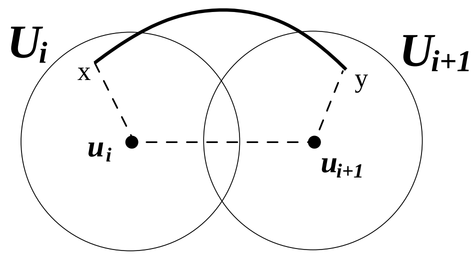



Figure 3: Bounding Cost on Matching Edges

$d(x, u_i) \leq 1$, $d(u_i, u_{i+1}) \leq |P_i|$, $d(u_{i+1}, y) \leq 1$

**Lemma 3.9** *Let $C_U = \{u \in C : |\Gamma_G(u) \cap V| \geq k+1\}$ and $C_W = \{w \in C : |\Gamma_G(w) \cap V| \leq k\} = C - C_U$. Let $U = \bigcup_{u \in C_U} \Gamma_G(u) \cap V$ and $W = \bigcup_{w \in C_W} \Gamma_G(w) \cap V$.*

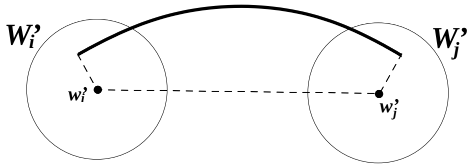
(i) *If $U$ is non-empty then $|U| \geq k+1$ and there is a subset $E_U$ of edges of $K_V$ on $U$ such that the graph $(U, E_U)$ is $k$-connected and $c(E_U) \leq |C_U|\,(\Delta k^2 + 2k + 2) + 2k|C|$.*

(ii) *If $|W| \geq k+1$ then there is a subset $E_W$ of edges of $K_V$ on $W$ such that the graph $(W, E_W)$ is $k$-connected and $c(E_W) \leq |C_W|(2k^2 + 2k) + |C|(3k^2 + 2k) - k$.*

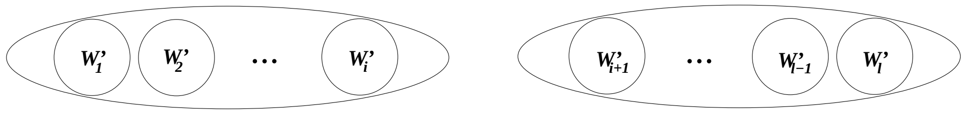
**Proof:** Let $T$ be a spanning tree in the subgraph induced in $G$ by $C$. Order the nodes in $C_U$ and in $C_W$ in the order of some Eulerian tour of $T$, say $C_U = \{u_1, \ldots, u_p\}$ and $C_W = \{w_1, \ldots, w_q\}$. Let $U_i = \Gamma_G(u_i) \cap V$ and $W_i = \Gamma_G(w_i) \cap V$. Let $P_i$ be the part of the Eulerian Tour from $u_i$ to $u_{i+1}$, and let $Q_i$ be the part of the Eulerian Tour from $w_i$ to $w_{i+1}$, Clearly, $\sum_{i=1}^{p-1} |P_i| = 2|C| - 2 \leq 2|C|$ and $\sum_{i=1}^{q-1} |Q_i| = 2|C| - 2 \leq 2|C|$.

To prove part (i) of the lemma, assume $U$ is non-empty. By the definition of $C_U$, $|U| \geq k+1$. For every $1 \leq i \leq p$, $|U_i| \geq k+1$, and we will construct a $k$-connected graph on $U_i$ of cost $\leq \Delta k^2 + 2$. Then we will add a matching $M_i$ between $U_i - U_{i+1}$ and $U_{i+1} - U_i$ so that $|M_i| \geq k - |U_i \cap U_{i+1}|$. By the triangle inequality, every matching edge is of cost no greater than $|P_i| + 2$ as described in figure 3, thus $c(M_i) \leq k(|P_i| + 2)$. The union of the constructed graphs will be a $k$-connected graph, by Claim 3.8. The total cost of the matchings is $\leq 2k|C| + 2kp$. Consequently, we get a $k$-connected graph on $U$ of cost $\leq p(\Delta k^2 + 2) + 2k|C| + 2kp \leq |C_U|\,(\Delta k^2 + 2k + 2) + 2k|C|$ as required.

Fix $1 \leq i \leq t$. We now construct a $k$-connected graph on $U_i$. By the definition of $\Delta$, since $U_i \subseteq \Gamma_G(u_i)$, there is a dominating set $U_i^1$ of size at most $\Delta$ in $G[U_i]$. By the same arguments, if $U_i - U_i^1$ is non-empty, there is a dominating set $U_i^2$ of size at most $\Delta$ in $G[U_i - U_i^1]$. Repeating the process $k$ times, and accumulating the dominating sets, we obtain a set $U_i'$, so that $|U_i'| \leq \Delta k$ and for every $u \in U_i - U_i'$, $u$ has at least $k$ neighbors from $U_i'$




(a) Evaluating the cost of a single edge of the clique.




(b) $Q_i$ is shortened by all edges from lefthand to righthand side.

Figure 4: Bounding cost of clique edges of one block.

in $G[U_i]$. By a theorem of Harary (c.f. [11]), there is a $k$-connected graph on $U'_i$ containing $\lceil \Delta k^2/2 \rceil$ edges. Since $U_i \subseteq \Gamma_G(u_i)$, and by the triangle inequality, every such edge has cost $\leq 2$. We get a $k$-connected graph on $U'_i$ of cost $\leq 2\lceil \Delta k^2/2 \rceil \leq \Delta k^2 + 2$. Every node in $U_i - U'_i$ is connected to at least $k$ nodes in the constructed graph, therefore by Claim 3.7, the constructed graph is a $k$-connected graph on $U_i$.

Assume $|W| \geq k + 1$. We construct a $k$-connected graph on $W$. The construction is as follows. Let $W'_i = W_i - \bigcup_{j=1}^{i} W_j$. Then the nonempty sets from $W'_1, \ldots, W'_q$ partition $W$. Traversing the sequence $W'_1, W'_2, \ldots, W'_q$ from left to right, we can partition it into blocks, each consisting of consecutive sets from the sequence, such that: the number of nodes in the union of the sets in each block is between $k+1$ and $2k$, except maybe that the last block has less than $k+1$ nodes. We will construct a clique on the nodes of each block. We then add a matching $M_t$ as in Claim 3.8 between each block $t$ and block $t+1$, except that if the last block has less than $k+1$ nodes, then we connect each of its nodes to the preceding block as described in Claim 3.7.

Consider the first block, say $W'_1, \ldots, W'_\ell$, and let $B_1$ be the union of the sets in this block. Note that $k+1 \leq |B_1| \leq 2k$. We bound the cost of a clique on $B_1$ as follows. In $G$, each $W'_i$ is connected by a star with center $w_i$, and $w_i$ is joined to $w_{i+1}$ by the path $Q_i$. An edge connecting a node in $W'_i$ to a node in $W'_{i+j}$ shortcuts at most one edge from the star of each of $W'_i, W'_{i+j}$, and each of the paths $Q_i, \ldots, Q_{i+j-1}$, as in figure 4(a). Thus by the triangle inequality, each such edge adds at most $2 + |Q_i| + \cdots + |Q_{i+j-1}|$ to the cost. Clearly, over all edges, we shortcut every $Q_i$ at most

$$\left(\sum_{j=1}^{i} |W'_j|\right)\left(\sum_{j=i+1}^{\ell} |W'_j|\right) \leq \frac{1}{4}\left(\sum_{j=1}^{\ell} |W'_j|\right)^2 = \frac{1}{4}|B_1|^2$$

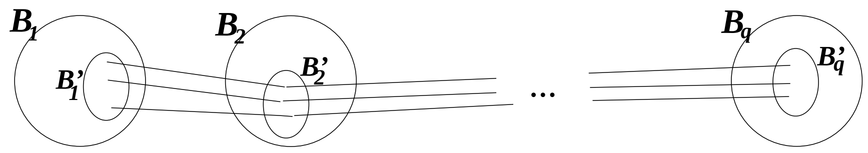



Figure 5: Bounding cost of matching edges between blocks (here $k = 3$).

times, as described in figure 4(b). In addition, every edge adds at most 2 to the cost, which sums to at most $2\binom{|B_1|}{2} \leq |B_1|^2$ for all edges. Denoting $L_1 = \sum_{i=1}^{\ell} |Q_i|$ and recalling that $|B_1| \leq 2k$ we obtain that the cost of a clique on $B_1$ is bounded by

$$\frac{|B_1|^2}{4} \cdot \sum_{i=1}^{\ell-1} |Q_i| + |B_1|^2 \leq k^2 L_1 + 2k|B_1| \ .$$

A similar argument applies on every block $t$. Since $\sum_t L_t = 2|C| - 2$ and $\sum_t |B_t| \leq |W| \leq k|C_W|$, the overall cost of the cliques on the blocks is bounded by

$$k^2 \sum_t L_t + 2k \sum_t |B_t| \leq 2k^2|C| - 2k^2 + 2k^2|C_W|.$$

For every $t$, we choose a set $B'_t$ of $k$ nodes from $B_t$ arbitrarily. Next we construct consecutive matchings between $B'_t$ and $B'_{t+1}$ for all $t$. For all $i$, $Q_i$ is shortened at most $k$ times as shown in figure 5. In addition, by previous arguments, each edge may shortcut at most one edge from the stars around some $w'_i$ and $w'_j$. Thus the cost of all matchings is bounded by

$$k \sum_{i=1}^{q} |Q_i| + 2k|C_W| \leq 2k|C| - 2k + 2k|C_W| \ .$$

Finally, if the last block has at most $k$ nodes, we connect every its node to $k$ nodes from the preceding block, thus constructing a $k$ connected graph on $W$ by Claim 3.7. By the triangle inequality, for every $u, v \in \Gamma_G(C)$, $d(u, v) \leq |C| + 1$, thus every edge is of cost no greater than $|C| + 1$, and the added edges add a cost of at most

$$k^2(|C| + 1) \leq k^2|C| + k^2 \ .$$

The total cost of the edges added is bounded by

$$|C_W|(2k^2 + 2k) + |C|(3k^2 + 2k) - k^2 - 2k \leq |C_W|(2k^2 + 2k) + |C|(3k^2 + 2k) - k$$

This completes the proof of Lemma 3.9. $\square$

Now we finish the proof of Lemma 3.5. If $|W| \geq k+1$ and $U$ is non-empty, then there is a matching $E_{UW}$ of size $k$ between $U$ and $W$. By the triangle inequality $c(E_{UW}) \leq k(|C|+1)$. By Lemma 3.9 and Claim 3.8, the edge set $J_C = E_U \cup E_W \cup E_{UW}$ forms a $k$-connected graph on $\Gamma_G(C)$, of cost at most (assuming $\Delta \geq 2$):

$$\begin{aligned} c(E_U) + c(E_W) + c(E_{UW}) &\leq |C_U|(\Delta k^2 + 2k + 2) + |C_W|(2k^2 + 2k) + |C|(3k^2 + 5k) \leq \\ &\leq (|C_U| + |C_W|)(\Delta k^2 + 2k + 2) + |C|(3k^2 + 5k) \leq \\ &\leq |C| \left( (\Delta + 3)k^2 + 7k + 2 \right) = \rho(k)|C| \ . \end{aligned}$$

If $|W| \leq k$, then $U$ is non-empty, since $U \cup W = \Gamma_G(C)$ and $|\Gamma_G(C)| \geq k+1$. Then in addition to $E_U$, we connect every node in $W$ to $k$ arbitrary nodes in $U$. This gives a $k$-connected graph on $\Gamma_G(C)$, by Claim 3.7. By the triangle inequality, the cost of added edges is $\leq k^2(|C|+1) \leq 2k^2|C|$. Thus the total cost is $\leq \rho(k)|C|$.

This finishes the proof of Lemma 3.5, and thus also the proof of Lemma 3.1 is complete.

**A tight example:** The following example shows that our bound $\rho(k) = O(k^2)$ is tight (up to constants). Given $k$ points in a ball of radius 1/2 with uniform requirements as an instance for SN-MSP, an optimal solution size is 1 – add one Steiner point in the ball. An optimal solution for the SNDP instance has cost $\binom{k}{2}$, as it is a union of two cliques on $V$: in one clique every edge $uv$ has cost $\lceil d(u,v) \rceil - 1 = 0$, while in the other every edge $uv$ has cost $\lceil d(u,v) \rceil = 1$.

## 3.2 Approximating SNDP on Multigraphs

Finally, we will prove Lemma 2.1, which is restated here for the convenience of the reader.

**Lemma 1.2** *For* SNDP*, an $\alpha$-approximation algorithm on simple graphs implies an $\alpha$-approximation algorithm on multigraphs; this is so also for subset uniform, uniform, rooted, and rooted subset uniform requirements. In the case of rooted uniform requirements,* SNDP *on multigraphs admits a 2-approximation algorithm.*

**Proof:** Given an SNDP instance (with parallel edges), insert a new node into every edge, and divide (arbitrarily) the cost of the edge between the corresponding two new edges. Clearly, the obtained graph is simple. It is easy to see that an $\alpha$-approximation for the modified instance implies an $\alpha$-approximation for the original instance, and that this transformation is requirement type preserving for subset uniform, rooted, and rooted subset uniform requirements. It remains therefore to consider uniform and rooted uniform requirements.

We now consider the case of uniform requirements, where feasible solutions of the problem are $k$-connected spanning subgraphs of $G$. Let $H = (V, E)$ be a minimally $k$-connected multi-graph (so $H - e$ is not $k$-connected for every $e \in E$). We claim that if $|V| \geq k + 1$ then $H$ is simple (thus we can keep for every maximal set of pairwise parallel edges of $G$ only the cheapest one), and if $|V| \leq k$ then $H$ has exactly $k + 2 - |V|$ edges between every of it nodes (thus an optimal solution is found by taking the $k + 2 - |V|$ cheapest edges in $G$ between every pair of nodes). Assume that $|V| \geq k + 1$. Then the simple underlying graph $H'$ of $H$ is $k$-connected by a theorem of Whitney (c.f. [8]): If $\kappa_{H'}(u, v) \geq k$ for every $u, v \in V$ so that $uv \notin E'$, then $H'$ is $k$-connected. This holds in our case, since $k$ pairwise internally disjoint $uv$-paths in $H$ have no parallel edges. If $|V| \leq k$, then note that if $H$ has exactly $k + 2 - |V|$ edges between every pair of its nodes then $H$ is $k$-connected. Hence it is sufficient to prove that there are at least $k + 2 - |V|$ edges between every two nodes of $H$. To see this, consider a set of $k$ internally disjoint $uv$-paths in $H$. At most $|V| - 2$ of these paths may not be edges between $u, v$, thus at least $k - (|V| - 2)$ of these paths are edges between $u, v$.

Finally, for rooted uniform requirements, we note that the existing 2-approximation algorithms do not have the restriction that $G$ is simple, and hence work also for multi-graphs. □

# 4 Conclusions and Open Problems

We have presented a polynomial time approximation algorithm for SN-MSP in normed spaces with variouse connectivity requirements types, using known approximation algorithms for SNDP as a subroutine.

The approximation ratio achieved are a multiplication of two factors. One is the approximation ratio of the corresponding SNDP algorithm, which we denoted by $\alpha$, and the other is a "reduction fee", denoted as $\rho(k) = O(k^2)$. Finding better approximation ratios for SNDP problems will improve the "$\alpha$" factor of our approximation ratios.

As shown earlier, using a feasible solution of SNDP on $K_V$ incurs an $\Omega(k^2)$ reduction fee. Along with the analysis of our approximation algorithm, we get that the reduction fee is $\Theta(k^2)$. Constructing a less natural instance of SNDP from the SN-MSP instance may improve this threshold.

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