On the notion of genus for division algebras and algebraic groups

(joint work with V. Chernousov and I. Rapinchuk)

Andrei S. Rapinchuk
University of Virginia

Amitsur Symposium – June 24, 2020

We start with the following question for division algebras:

We start with the following question for division algebras:

(∗) *Let* D_1 *and* D_2 *be finite-dimensional central division algebras over a field* K.

We start with the following question for division algebras:

(∗) *Let D_1 and D_2 be finite-dimensional central division algebras over a field K. How are D_1 and D_2 related if*

We start with the following question for division algebras:

($*$) *Let D_1 and D_2 be finite-dimensional central division algebras over a field K. How are D_1 and D_2 related if they have same maximal subfields?*

We start with the following question for division algebras:

(∗) *Let D_1 and D_2 be finite-dimensional central division algebras over a field K. How are D_1 and D_2 related if they have same maximal subfields?*

- D_1 and D_2 have same maximal subfields

1. $\deg D_1 = \deg D_2 =: n$;
2. for P/K of degree n,

We start with the following question for division algebras:

(∗) *Let D_1 and D_2 be finite-dimensional central division algebras over a field K. How are D_1 and D_2 related if they have same maximal subfields?*

- D_1 and D_2 have same maximal subfields **if**

1. $\deg D_1 = \deg D_2 =: n$;
2. for P/K of degree n,

We start with the following question for division algebras:

(∗) *Let D_1 and D_2 be finite-dimensional central division algebras over a field K. How are D_1 and D_2 related if they have same maximal subfields?*

• D_1 and D_2 have same maximal subfields **if**

1. $\deg D_1 = \deg D_2 =: n$;
2. for P/K of degree n,

We start with the following question for division algebras:

(∗) *Let D_1 and D_2 be finite-dimensional central division algebras over a field K. How are D_1 and D_2 related if they have same maximal subfields?*

- D_1 and D_2 have same maximal subfields **if**

1. $\deg D_1 = \deg D_2 =: n$;

2. for P/K of degree n,

We start with the following question for division algebras:

$(*)$ *Let D_1 and D_2 be finite-dimensional central division algebras over a field K. How are D_1 and D_2 related if they have same maximal subfields?*

- D_1 and D_2 have same maximal subfields **if**

1. $\deg D_1 = \deg D_2 =: n$;
2. for P/K of degree n, $P \hookrightarrow D_1$

We start with the following question for division algebras:

($*$) *Let D_1 and D_2 be finite-dimensional central division algebras over a field K. How are D_1 and D_2 related if they have <u>same</u> maximal subfields?*

- D_1 and D_2 have same maximal subfields **if**

1. $\deg D_1 = \deg D_2 =: n$;
2. for P/K of degree n, $P \hookrightarrow D_1 \quad \Leftrightarrow$

We start with the following question for division algebras:

($\ast$) *Let D_1 and D_2 be finite-dimensional central division algebras over a field K. How are D_1 and D_2 related if they have same maximal subfields?*

- D_1 and D_2 have same maximal subfields **if**

1. $\deg D_1 = \deg D_2 =: n$;
2. for P/K of degree n, $P \hookrightarrow D_1 \Leftrightarrow P \hookrightarrow D_2$.

Amitsur's Theorem

Amitsur's Theorem

Let D_1 and D_2 be central division algebras over K.

Amitsur's Theorem

Let D_1 and D_2 be central division algebras over K.

If D_1 and D_2 have same splitting fields,

Amitsur's Theorem

Let D_1 and D_2 be central division algebras over K.

If D_1 and D_2 have same splitting fields, i.e. for F/K we have

$$D_1 \otimes_K F \simeq M_{n_1}(F) \quad \Leftrightarrow \quad D_2 \otimes_K F \simeq M_{n_2}(F),$$

Amitsur's Theorem

Let D_1 and D_2 be central division algebras over K.
If D_1 and D_2 have same splitting fields, i.e. for F/K we have

$$D_1 \otimes_K F \simeq M_{n_1}(F) \quad \Leftrightarrow \quad D_2 \otimes_K F \simeq M_{n_2}(F),$$

then $n_1 = n_2$ and $\langle [D_1] \rangle = \langle [D_2] \rangle$ in $\mathrm{Br}(K)$.

Amitsur's Theorem

Let D_1 and D_2 be central division algebras over K.
If D_1 and D_2 have same splitting fields, i.e. for F/K we have

$$D_1 \otimes_K F \simeq M_{n_1}(F) \quad \Leftrightarrow \quad D_2 \otimes_K F \simeq M_{n_2}(F),$$

then $n_1 = n_2$ and $\langle [D_1] \rangle = \langle [D_2] \rangle$ in $\mathrm{Br}(K)$.

- Proof of Amitsur's Theorem uses *generic splitting fields* (function fields of Severi-Brauer varieties),

Amitsur's Theorem

Let D_1 and D_2 be central division algebras over K.

If D_1 and D_2 have same splitting fields, i.e. for F/K we have

$$D_1 \otimes_K F \simeq M_{n_1}(F) \quad \Leftrightarrow \quad D_2 \otimes_K F \simeq M_{n_2}(F),$$

then $n_1 = n_2$ and $\langle [D_1] \rangle = \langle [D_2] \rangle$ in $\mathrm{Br}(K)$.

- Proof of Amitsur's Theorem uses *generic splitting fields* (function fields of Severi-Brauer varieties), which are

infinite extensions of K.

Amitsur's Theorem

Let D_1 and D_2 be central division algebras over K.
If D_1 and D_2 have same splitting fields, i.e. for F/K we have

$$D_1 \otimes_K F \simeq M_{n_1}(F) \quad \Leftrightarrow \quad D_2 \otimes_K F \simeq M_{n_2}(F),$$

then $n_1 = n_2$ and $\langle [D_1] \rangle = \langle [D_2] \rangle$ in $\mathrm{Br}(K)$.

- Proof of Amitsur's Theorem uses *generic splitting fields* (function fields of Severi-Brauer varieties), which are

infinite extensions of K.

Can one prove Amitsur's Theorem using only

Amitsur's Theorem

Let D_1 and D_2 be central division algebras over K.

If D_1 and D_2 have same splitting fields, i.e. for F/K we have

$$D_1 \otimes_K F \simeq M_{n_1}(F) \quad \Leftrightarrow \quad D_2 \otimes_K F \simeq M_{n_2}(F),$$

then $n_1 = n_2$ and $\langle [D_1] \rangle = \langle [D_2] \rangle$ in $\mathrm{Br}(K)$.

- Proof of Amitsur's Theorem uses *generic splitting fields* (function fields of Severi-Brauer varieties), which are

infinite extensions of K.

Can one prove Amitsur's Theorem using only splitting fields of finite degree,

Amitsur's Theorem

Let D_1 and D_2 be central division algebras over K. **If** D_1 and D_2 have same splitting fields, i.e. for F/K we have

$$D_1 \otimes_K F \simeq M_{n_1}(F) \quad \Leftrightarrow \quad D_2 \otimes_K F \simeq M_{n_2}(F),$$

then $n_1 = n_2$ and $\langle [D_1] \rangle = \langle [D_2] \rangle$ in $\mathrm{Br}(K)$.

- Proof of Amitsur's Theorem uses *generic splitting fields* (function fields of Severi-Brauer varieties), which are

infinite extensions of K.

Can one prove Amitsur's Theorem using only splitting fields of finite degree, or just maximal subfields?

The answer is (strongly) negative.

The answer is (strongly) negative.

Recall that division algebras over $\mathbb{Q}$ can be described in terms of *local invariants:*

The answer is (strongly) negative.

Recall that division algebras over $\mathbb{Q}$ can be described in terms of *local invariants:* there is a *natural embedding*

$$\boxed{0 \to \mathrm{Br}(\mathbb{Q}) \xrightarrow{\theta} \bigoplus_p \mathrm{Br}(\mathbb{Q}_p)} \quad \text{(including } p = \infty)$$

The answer is (strongly) negative.

Recall that division algebras over $\mathbb{Q}$ can be described in terms of *local invariants:* there is a *natural embedding*

$$\boxed{0 \to \mathrm{Br}(\mathbb{Q}) \xrightarrow{\theta} \bigoplus_p \mathrm{Br}(\mathbb{Q}_p)} \quad \text{(including } p = \infty\text{)}$$

(Artin - Hasse - Brauer - Noether Theorem (ABHN)).

The answer is (strongly) negative.

Recall that division algebras over $\mathbb{Q}$ can be described in terms of *local invariants:* there is a *natural embedding*

$$\boxed{0 \to \mathrm{Br}(\mathbb{Q}) \xrightarrow{\theta} \bigoplus_p \mathrm{Br}(\mathbb{Q}_p)} \quad \text{(including } p = \infty\text{)}$$

(Artin - Hasse - Brauer - Noether Theorem (ABHN)).

For $p \neq \infty$, there is an isomorphism

$$\boxed{\mathrm{inv}_p \colon \mathrm{Br}(\mathbb{Q}_p) \longrightarrow \mathbb{Q}/\mathbb{Z}.}$$

The answer is (strongly) negative.

Recall that division algebras over $\mathbb{Q}$ can be described in terms of *local invariants:* there is a *natural embedding*

$$\boxed{0 \to \mathrm{Br}(\mathbb{Q}) \xrightarrow{\theta} \bigoplus_p \mathrm{Br}(\mathbb{Q}_p)} \quad \text{(including } p = \infty)$$

(Artin - Hasse - Brauer - Noether Theorem (ABHN)).

For $p \neq \infty$, there is an isomorphism

$$\boxed{\mathrm{inv}_p \colon \mathrm{Br}(\mathbb{Q}_p) \longrightarrow \mathbb{Q}/\mathbb{Z}.}$$

For $p = \infty$, there is an isomorphism

$$\boxed{\mathrm{inv}_\infty \colon \mathrm{Br}(\mathbb{R}) \longrightarrow \tfrac{1}{2}\mathbb{Z}/\mathbb{Z}.}$$

$$\operatorname{Im}\theta \;=\; \{(i_p) \in \bigoplus_p \mathbb{Q}/\mathbb{Z} \;\mid\; i_\infty \in \frac{1}{2}\mathbb{Z}/\mathbb{Z} \;\text{ and }\; \sum_p i_p = 0 \;\text{ in }\; \mathbb{Q}/\mathbb{Z}\}.$$

$$\operatorname{Im}\theta \ = \ \{(i_p) \in \bigoplus_p \mathbb{Q}/\mathbb{Z} \ \mid \ i_\infty \in \frac{1}{2}\mathbb{Z}/\mathbb{Z} \ \text{ and } \ \sum_p i_p = 0 \ \text{ in } \ \mathbb{Q}/\mathbb{Z}\}.$$

Note: if A is a central simple $\mathbb{Q}$-algebra of degree n then

$$\mathrm{inv}_p([A]) \ := \ \mathrm{inv}_p([A \otimes_{\mathbb{Q}} \mathbb{Q}_p]) \ = \ \frac{a_p}{n}, \ \ a_p \in \mathbb{Z}$$

$$\boxed{\operatorname{Im}\theta \;=\; \{(i_p) \in \bigoplus_p \mathbb{Q}/\mathbb{Z} \mid i_\infty \in \frac{1}{2}\mathbb{Z}/\mathbb{Z} \;\text{ and }\; \sum_p i_p = 0 \;\text{ in }\; \mathbb{Q}/\mathbb{Z}\}.}$$

Note: if A is a central simple $\mathbb{Q}$-algebra of degree n then

$$\mathrm{inv}_p([A]) \;:=\; \mathrm{inv}_p([A \otimes_{\mathbb{Q}} \mathbb{Q}_p]) \;=\; \frac{a_p}{n}, \;\; a_p \in \mathbb{Z}$$

Example (Cubic algebras with same maximal subfields)

$$\operatorname{Im}\theta \;=\; \{(i_p) \in \bigoplus_p \mathbb{Q}/\mathbb{Z} \;\mid\; i_\infty \in \frac{1}{2}\mathbb{Z}/\mathbb{Z} \;\text{ and }\; \sum_p i_p = 0 \;\text{ in }\; \mathbb{Q}/\mathbb{Z}\}.$$

Note: if A is a central simple $\mathbb{Q}$-algebra of degree n then

$$\mathrm{inv}_p([A]) \;:=\; \mathrm{inv}_p([A \otimes_{\mathbb{Q}} \mathbb{Q}_p]) \;=\; \frac{a_p}{n}, \;\; a_p \in \mathbb{Z}$$

Example (Cubic algebras with same maximal subfields)

Fix $r \geqslant 3$ and pick r distinct primes $p_1, \ldots, p_r$.

$$\operatorname{Im}\theta \;=\; \{(i_p) \in \bigoplus_p \mathbb{Q}/\mathbb{Z} \mid i_\infty \in \frac{1}{2}\mathbb{Z}/\mathbb{Z} \;\text{ and }\; \sum_p i_p = 0 \;\text{ in }\; \mathbb{Q}/\mathbb{Z}\}.$$

Note: if A is a central simple $\mathbb{Q}$-algebra of degree n then

$$\mathrm{inv}_p([A]) \;:=\; \mathrm{inv}_p([A \otimes_{\mathbb{Q}} \mathbb{Q}_p]) \;=\; \frac{a_p}{n}, \;\; a_p \in \mathbb{Z}$$

Example (Cubic algebras with same maximal subfields)

Fix $r \geqslant 3$ and pick r distinct primes $p_1, \ldots, p_r$.

Let $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_r)$ with $\varepsilon_i = \pm 1$ and $\sum_{i=1}^{r} \varepsilon_i \equiv 0 (\mathrm{mod}\, 3)$.

It follows from (AHBN) that there exists a <u>cubic</u> division algebra $D(\varepsilon)$ with local invariants

It follows from (AHBN) that there exists a <u>cubic</u> division algebra $D(\varepsilon)$ with local invariants

$$\mathrm{inv}_p\, D(\varepsilon) = \begin{cases} \frac{\varepsilon_i}{3}, & p = p_i \ \text{for} \ i = 1, \ldots, r; \\ 0, & p \notin \{p_1, \ldots, p_r\} \ \text{(including} \ p = \infty). \end{cases}$$

It follows from (AHBN) that there exists a cubic division algebra $D(\varepsilon)$ with local invariants

$$\operatorname{inv}_p D(\varepsilon) = \begin{cases} \frac{\varepsilon_i}{3}, & p = p_i \text{ for } i = 1, \ldots, r; \\ 0, & p \notin \{p_1, \ldots, p_r\} \text{ (including } p = \infty). \end{cases}$$

One shows that

1. For any $\varepsilon', \varepsilon''$ as above, $D(\varepsilon')$ and $D(\varepsilon'')$ have same finite-dimensional splitting fields,
2. For $\varepsilon' \neq \varepsilon''$, algebras $D(\varepsilon')$ and $D(\varepsilon'')$ are nonisomorphic.

It follows from (AHBN) that there exists a cubic division algebra $D(\varepsilon)$ with local invariants

$$\mathrm{inv}_p\, D(\varepsilon) = \begin{cases} \frac{\varepsilon_i}{3}, & p = p_i \text{ for } i = 1, \ldots, r; \\ 0, & p \notin \{p_1, \ldots, p_r\} \text{ (including } p = \infty). \end{cases}$$

One shows that

1. For any $\varepsilon', \varepsilon''$ as above, $D(\varepsilon')$ and $D(\varepsilon'')$ have same finite-dimensional splitting fields,
2. For $\varepsilon' \neq \varepsilon''$, algebras $D(\varepsilon')$ and $D(\varepsilon'')$ are nonisomorphic.

It follows from (AHBN) that there exists a cubic division algebra $D(\varepsilon)$ with local invariants

$$\mathrm{inv}_p\, D(\varepsilon) = \begin{cases} \frac{\varepsilon_i}{3}, & p = p_i \text{ for } i = 1, \ldots, r; \\ 0, & p \notin \{p_1, \ldots, p_r\} \text{ (including } p = \infty). \end{cases}$$

One shows that

1. For any $\varepsilon', \varepsilon''$ as above, $D(\varepsilon')$ and $D(\varepsilon'')$ have same finite-dimensional splitting fields, hence same maximal subfields;
2. For $\varepsilon' \neq \varepsilon''$, algebras $D(\varepsilon')$ and $D(\varepsilon'')$ are nonisomorphic.

It follows from (AHBN) that there exists a cubic division algebra $D(\varepsilon)$ with local invariants

$$\mathrm{inv}_p\, D(\varepsilon) = \begin{cases} \frac{\varepsilon_i}{3}, & p = p_i \text{ for } i = 1, \dots, r; \\ 0, & p \notin \{p_1, \dots, p_r\} \text{ (including } p = \infty). \end{cases}$$

One shows that

1. For any $\varepsilon', \varepsilon''$ as above, $D(\varepsilon')$ and $D(\varepsilon'')$ have same finite-dimensional splitting fields, hence same maximal subfields;
2. For $\varepsilon' \neq \varepsilon''$, algebras $D(\varepsilon')$ and $D(\varepsilon'')$ are nonisomorphic.

Taking r sufficiently big, we can construct *arbitrarily large* (but finite) families of

Taking r sufficiently big, we can construct *arbitrarily large* (but finite) families of pairwise nonisomorphic cubic division algebras over $\mathbb{Q}$

Taking r sufficiently big, we can construct *arbitrarily large* (but finite) families of pairwise nonisomorphic cubic division algebras over $\mathbb{Q}$ having same finite-dimensional splitting fields,

Taking r sufficiently big, we can construct *arbitrarily large* (but finite) families of pairwise nonisomorphic cubic division algebras over $\mathbb{Q}$ having same finite-dimensional splitting fields, hence same maximal subfields.

Taking r sufficiently big, we can construct *arbitrarily large* (but finite) families of pairwise nonisomorphic cubic division algebras over $\mathbb{Q}$ having same finite-dimensional splitting fields, hence same maximal subfields.

But for any such D, subgroup $\langle [D] \rangle \subset \mathrm{Br}(\mathbb{Q})$ has only *two* generators.

Taking r sufficiently big, we can construct *arbitrarily large* (but finite) families of pairwise nonisomorphic cubic division algebras over $\mathbb{Q}$ having same finite-dimensional splitting fields, hence same maximal subfields.

But for any such D, subgroup $\langle [D] \rangle \subset \mathrm{Br}(\mathbb{Q})$ has only *two* generators.

So, there are (many) such D' and D'' with $\langle [D'] \rangle \neq \langle [D''] \rangle$ that have same finite-dimensional splitting fields /maximal subfields.

Taking r sufficiently big, we can construct *arbitrarily large* (but finite) families of pairwise nonisomorphic cubic division algebras over $\mathbb{Q}$ having same finite-dimensional splitting fields, hence same maximal subfields.

But for any such D, subgroup $\langle [D] \rangle \subset \mathrm{Br}(\mathbb{Q})$ has only *two* generators.

So, there are (many) such D' and D'' with $\langle [D'] \rangle \neq \langle [D''] \rangle$ that have same finite-dimensional splitting fields /maximal subfields.

Thus, assertion of Amitsur's Theorem cannot be proven for two division algebra sharing only finite-dimensional splitting fields.

Taking r sufficiently big, we can construct *arbitrarily large* (but finite) families of pairwise nonisomorphic cubic division algebras over $\mathbb{Q}$ having same finite-dimensional splitting fields, hence same maximal subfields.

But for any such D, subgroup $\langle [D] \rangle \subset \mathrm{Br}(\mathbb{Q})$ has only *two* generators.

So, there are (many) such D' and D'' with $\langle [D'] \rangle \neq \langle [D''] \rangle$ that have same finite-dimensional splitting fields /maximal subfields.

Thus, assertion of Amitsur's Theorem cannot be proven for two division algebra sharing only finite-dimensional splitting fields. **So**, Amitsur found the right way of proving his theorem by using infinite-dimensional extensions.

What can one prove for two such division algebras?

What can one prove for two such division algebras?

But this is a reformulation of our original question:

What can one prove for two such division algebras?

But this is a reformulation of our original question:

(∗) *Let D_1 and D_2 be finite-dimensional central division algebras over a field K. How are D_1 and D_2 related if they have same maximal subfields?*

Geometry - Riemann surfaces

Geometry - Riemann surfaces

- Let $\mathbb{H} = \{ x + iy \mid y > 0 \}$.

Geometry - Riemann surfaces

- Let $\mathbb{H} = \{ x + iy \mid y > 0 \}$.

"Most" Riemann surfaces are of the form:

$$M = \mathbb{H}/\Gamma$$

Geometry - Riemann surfaces

- Let $\mathbb{H} = \{ x + iy \mid y > 0 \}$.

"Most" Riemann surfaces are of the form:

$$M = \mathbb{H}/\Gamma$$

where $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ is a *discrete torsion free subgroup*.

Geometry - Riemann surfaces

- Let $\mathbb{H} = \{ x + iy \mid y > 0 \}$.

"Most" Riemann surfaces are of the form:

$$M = \mathbb{H}/\Gamma$$

where $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ is a *discrete torsion free subgroup*.

- Some properties of M can be understood in terms of the

associated quaternion algebra A_Γ.

Geometry - Riemann surfaces

- Let $\mathbb{H} = \{ x + iy \mid y > 0 \}$.

"Most" Riemann surfaces are of the form:

$$M = \mathbb{H}/\Gamma$$

where $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ is a *discrete torsion free subgroup*.

- Some properties of M can be understood in terms of the *associated quaternion algebra* A_Γ.

Let

- $\pi : \mathrm{SL}_2(\mathbb{R}) \to \mathrm{PSL}_2(\mathbb{R})$;

Geometry - Riemann surfaces

- Let $\mathbb{H} = \{ x + iy \mid y > 0 \}$.

"Most" Riemann surfaces are of the form:

$$M = \mathbb{H}/\Gamma$$

where $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ is a *discrete torsion free subgroup*.

- Some properties of M can be understood in terms of the *associated quaternion algebra* A_Γ.

Let

- $\pi : \mathrm{SL}_2(\mathbb{R}) \to \mathrm{PSL}_2(\mathbb{R})$;
- $\tilde{\Gamma} = \pi^{-1}(\Gamma) \subset \mathrm{M}_2(\mathbb{R})$.

Set

$$A_\Gamma = \mathbb{Q}[\widetilde{\Gamma}^{(2)}], \quad \widetilde{\Gamma}^{(2)} \subset \widetilde{\Gamma} \text{ subgroups generated by squares.}$$

Set

$$A_\Gamma = \mathbb{Q}[\widetilde{\Gamma}^{(2)}], \quad \widetilde{\Gamma}^{(2)} \subset \widetilde{\Gamma} \ \text{subgroups generated by squares.}$$

One shows: A_Γ is a <u>quaternion</u> algebra

Set
$A_\Gamma = \mathbb{Q}[\widetilde{\Gamma}^{(2)}], \quad \widetilde{\Gamma}^{(2)} \subset \widetilde{\Gamma}$ subgroups generated by squares.

One shows: A_Γ is a quaternion algebra with center

$$K_\Gamma = \mathbb{Q}(\operatorname{tr}\gamma \mid \gamma \in \widetilde{\Gamma}^{(2)})$$

(trace field).

Set
$$A_\Gamma = \mathbb{Q}[\widetilde{\Gamma}^{(2)}], \quad \widetilde{\Gamma}^{(2)} \subset \widetilde{\Gamma} \text{ subgroups generated by squares.}$$

One shows: A_Γ is a quaternion algebra with center
$$K_\Gamma = \mathbb{Q}(\operatorname{tr}\gamma \mid \gamma \in \widetilde{\Gamma}^{(2)})$$
(trace field).

- If Γ is an *arithmetic* Fuchsian group, **then**

,

Set

$$A_\Gamma = \mathbb{Q}[\widetilde{\Gamma}^{(2)}], \quad \widetilde{\Gamma}^{(2)} \subset \widetilde{\Gamma} \text{ subgroups generated by squares.}$$

One shows: A_Γ is a quaternion algebra with center

$$K_\Gamma = \mathbb{Q}(\operatorname{tr}\gamma \mid \gamma \in \widetilde{\Gamma}^{(2)})$$

(trace field).

- If Γ is an *arithmetic* Fuchsian group, **then**

  (1) K_Γ is a number field,

Set
$A_\Gamma = \mathbb{Q}[\widetilde{\Gamma}^{(2)}], \quad \widetilde{\Gamma}^{(2)} \subset \widetilde{\Gamma}$ subgroups generated by squares.

One shows: A_Γ is a quaternion algebra with center

$$K_\Gamma = \mathbb{Q}(\operatorname{tr}\gamma \mid \gamma \in \widetilde{\Gamma}^{(2)})$$

(trace field).

- If Γ is an *arithmetic* Fuchsian group, **then**

  (1) K_Γ is a number field, and

Set

$A_\Gamma = \mathbb{Q}[\widetilde{\Gamma}^{(2)}]$, $\widetilde{\Gamma}^{(2)} \subset \widetilde{\Gamma}$ subgroups generated by squares.

One shows: A_Γ is a quaternion algebra with center

$$K_\Gamma = \mathbb{Q}(\mathrm{tr}\,\gamma \mid \gamma \in \widetilde{\Gamma}^{(2)})$$

(trace field).

- If Γ is an *arithmetic* Fuchsian group, **then**

  (1) K_Γ is a number field, and

  (2) A_Γ is the quaternion algebra involved in the description of Γ.

Set
$A_\Gamma = \mathbb{Q}[\widetilde{\Gamma}^{(2)}], \quad \widetilde{\Gamma}^{(2)} \subset \widetilde{\Gamma}$ subgroups generated by squares.

One shows: A_Γ is a quaternion algebra with center

$$K_\Gamma = \mathbb{Q}(\mathrm{tr}\,\gamma \mid \gamma \in \widetilde{\Gamma}^{(2)})$$

(trace field).

- If Γ is an *arithmetic* Fuchsian group, **then**
  (1) K_Γ is a number field, and
  (2) A_Γ is the quaternion algebra involved in the description of Γ.

Recall: A_Γ determines commensurability class of Γ.

Set

$A_\Gamma = \mathbb{Q}[\widetilde{\Gamma}^{(2)}], \quad \widetilde{\Gamma}^{(2)} \subset \widetilde{\Gamma}$ subgroups generated by squares.

One shows: A_Γ is a quaternion algebra with center

$$K_\Gamma = \mathbb{Q}(\mathrm{tr}\,\gamma \mid \gamma \in \widetilde{\Gamma}^{(2)})$$

(trace field).

- If Γ is an *arithmetic* Fuchsian group, **then**
  (1) K_Γ is a number field, and
  (2) A_Γ is the quaternion algebra involved in the description of Γ.

Recall: A_Γ determines commensurability class of Γ.

That is why we want to know A_Γ in this case!

- For a *non-arithmetic* Fuchsian group Γ:

- For a *non-arithmetic* Fuchsian group Γ:

  (1) K_Γ is **not** necessarily a number field;

- For a *non-arithmetic* Fuchsian group Γ:

  (1) K_Γ is **not** necessarily a number field;

  (In principle, can be any finitely generated subfield of $\mathbb{R}$)

- For a *non-arithmetic* Fuchsian group Γ:

  (1) K_Γ is **not** necessarily a number field;

  (In principle, can be any finitely generated subfield of $\mathbb{R}$)

  (2) A_Γ does **not** determine Γ up to commensurability.

- For a *non-arithmetic* Fuchsian group Γ:

  (1) K_Γ is **not** necessarily a number field;

  (In principle, can be any finitely generated subfield of $\mathbb{R}$)

  (2) A_Γ does **not** determine Γ up to commensurability.

Nevertheless, A_Γ remains an important *invariant* of the commensurability class of Γ.

- For a *non-arithmetic* Fuchsian group Γ:

  (1) K_Γ is **not** necessarily a number field;

  (In principle, can be any finitely generated subfield of $\mathbb{R}$)

  (2) A_Γ does **not** determine Γ up to commensurability.

Nevertheless, A_Γ remains an important *invariant* of the commensurability class of Γ.

So, when one cannot say much about Γ

- For a *non-arithmetic* Fuchsian group Γ:

  (1) K_Γ is **not** necessarily a number field;

  (In principle, can be any finitely generated subfield of $\mathbb{R}$)

  (2) A_Γ does **not** determine Γ up to commensurability.

Nevertheless, A_Γ remains an important *invariant* of the commensurability class of Γ.

So, when one cannot say much about Γ (which is typically the case for non-arithmetic Γ),

- For a *non-arithmetic* Fuchsian group Γ:

  (1) K_Γ is **not** necessarily a number field;

  (In principle, can be any finitely generated subfield of $\mathbb{R}$)

  (2) A_Γ does **not** determine Γ up to commensurability.

Nevertheless, A_Γ remains an important *invariant* of the commensurability class of Γ.

So, when one cannot say much about Γ (which is typically the case for non-arithmetic Γ), one would like to know at least A_Γ.

- For a *non-arithmetic* Fuchsian group Γ:
  (1) K_Γ is **not** necessarily a number field;
  (In principle, can be any finitely generated subfield of $\mathbb{R}$)
  (2) A_Γ does **not** determine Γ up to commensurability.

Nevertheless, A_Γ remains an important *invariant* of the commensurability class of Γ.

So, when one cannot say much about Γ (which is typically the case for non-arithmetic Γ), one would like to know at least A_Γ.

Bottom line: one would like to know A_Γ in all cases

- For a *non-arithmetic* Fuchsian group Γ:

  (1) K_Γ is **not** necessarily a number field;

  (In principle, can be any finitely generated subfield of $\mathbb{R}$)

  (2) A_Γ does **not** determine Γ up to commensurability.

Nevertheless, A_Γ remains an important *invariant* of the commensurability class of Γ.

So, when one cannot say much about Γ (which is typically the case for non-arithmetic Γ), one would like to know at least A_Γ.

Bottom line: one would like to know A_Γ in all cases (although maybe for different reasons)

Here is how A_Γ is linked with geometry of $M = \mathbb{H}/\Gamma$:

Here is how A_Γ is linked with geometry of $M = \mathbb{H}/\Gamma$:

To a (nontrivial) semi-simple $\gamma \in \tilde{\Gamma}^{(2)}$ there corresponds

Here is how A_Γ is linked with geometry of $M = \mathbb{H}/\Gamma$:

To a (nontrivial) semi-simple $\gamma \in \tilde{\Gamma}^{(2)}$ there corresponds

- *geometrically* – a closed geodesic $c_\gamma \subset M$,
  if $\gamma \sim \pm \begin{pmatrix} t_\gamma & 0 \\ 0 & t_\gamma^{-1} \end{pmatrix}$ $(t_\gamma > 1)$ then *length* $\ell(c_\gamma) = 2 \log t_\gamma$;

Here is how A_Γ is linked with geometry of $M = \mathbb{H}/\Gamma$:

To a (nontrivial) semi-simple $\gamma \in \tilde{\Gamma}^{(2)}$ there corresponds

- *geometrically* – a closed geodesic $c_\gamma \subset M$,
  if $\gamma \sim \pm \begin{pmatrix} t_\gamma & 0 \\ 0 & t_\gamma^{-1} \end{pmatrix}$ $(t_\gamma > 1)$ then *length* $\ell(c_\gamma) = 2 \log t_\gamma$;

- *algebraically*: a maximal etale subalgebra $K_\Gamma[\gamma] \subset A_\Gamma$.

Here is how A_Γ is linked with geometry of $M = \mathbb{H}/\Gamma$:

To a (nontrivial) semi-simple $\gamma \in \tilde{\Gamma}^{(2)}$ there corresponds

- *geometrically* – a closed geodesic $c_\gamma \subset M$, if $\gamma \sim \pm \begin{pmatrix} t_\gamma & 0 \\ 0 & t_\gamma^{-1} \end{pmatrix}$ $(t_\gamma > 1)$ then *length* $\ell(c_\gamma) = 2 \log t_\gamma$;

- *algebraically*: a maximal etale subalgebra $K_\Gamma[\gamma] \subset A_\Gamma$.

For a Riemannian manifold M:

Here is how A_Γ is linked with geometry of $M = \mathbb{H}/\Gamma$:

To a (nontrivial) semi-simple $\gamma \in \tilde{\Gamma}^{(2)}$ there corresponds

- *geometrically* – a closed geodesic $c_\gamma \subset M$, if $\gamma \sim \pm \begin{pmatrix} t_\gamma & 0 \\ 0 & t_\gamma^{-1} \end{pmatrix}$ $(t_\gamma > 1)$ then *length* $\ell(c_\gamma) = 2 \log t_\gamma$;

- *algebraically*: a maximal etale subalgebra $K_\Gamma[\gamma] \subset A_\Gamma$.

For a Riemannian manifold M:

$L(M)$ = set of lengths of closed geodesics in M

Here is how A_Γ is linked with geometry of $M = \mathbb{H}/\Gamma$:

To a (nontrivial) semi-simple $\gamma \in \tilde{\Gamma}^{(2)}$ there corresponds

- *geometrically* – a closed geodesic $c_\gamma \subset M$,
  if $\gamma \sim \pm \begin{pmatrix} t_\gamma & 0 \\ 0 & t_\gamma^{-1} \end{pmatrix}$ $(t_\gamma > 1)$ then *length* $\ell(c_\gamma) = 2 \log t_\gamma$;

- *algebraically*: a maximal etale subalgebra $K_\Gamma[\gamma] \subset A_\Gamma$.

For a Riemannian manifold M:

$L(M)$ = set of lengths of closed geodesics in M
((weak) length spectrum of M)

Definition.

Definition.

Riemannian manifolds M_1 and M_2 are

Definition.

Riemannian manifolds M_1 and M_2 are

- *iso-length spectral* if $L(M_1) = L(M_2)$;

Definition.

Riemannian manifolds M_1 and M_2 are

- *iso-length spectral* if $L(M_1) = L(M_2)$;
- *length-commensurable* if $\mathbb{Q} \cdot L(M_1) = \mathbb{Q} \cdot L(M_2)$.

Definition.

Riemannian manifolds M_1 and M_2 are

- *iso-length spectral* if $L(M_1) = L(M_2)$;
- *length-commensurable* if $\mathbb{Q} \cdot L(M_1) = \mathbb{Q} \cdot L(M_2)$.

Let $M_i = \mathbb{H}/\Gamma_i$ $(i = 1, 2)$ be Riemann surfaces.

Definition.

Riemannian manifolds M_1 and M_2 are

- *iso-length spectral* if $L(M_1) = L(M_2)$;
- *length-commensurable* if $\mathbb{Q} \cdot L(M_1) = \mathbb{Q} \cdot L(M_2)$.

Let $M_i = \mathbb{H}/\Gamma_i$ $(i = 1, 2)$ be Riemann surfaces.

If M_1 and M_2 are length-commensurable **then:**

Definition.

Riemannian manifolds M_1 and M_2 are

- *iso-length spectral* if $L(M_1) = L(M_2)$;
- *length-commensurable* if $\mathbb{Q} \cdot L(M_1) = \mathbb{Q} \cdot L(M_2)$.

Let $M_i = \mathbb{H}/\Gamma_i$ $(i = 1, 2)$ be Riemann surfaces.

If M_1 and M_2 are length-commensurable **then:**

- $K_{\Gamma_1} = K_{\Gamma_2} =: K$;

Definition.

Riemannian manifolds M_1 and M_2 are

- *iso-length spectral* if $L(M_1) = L(M_2)$;
- *length-commensurable* if $\mathbb{Q} \cdot L(M_1) = \mathbb{Q} \cdot L(M_2)$.

Let $M_i = \mathbb{H}/\Gamma_i$ $(i = 1, 2)$ be Riemann surfaces.

If M_1 and M_2 are length-commensurable **then:**

- $K_{\Gamma_1} = K_{\Gamma_2} =: K$;
- Given closed geodesics $c_{\gamma_i} \subset M_i$ for $i = 1, 2$ such that

$$\ell(c_{\gamma_2})/\ell(c_{\gamma_1}) = m/n \quad (m, n \in \mathbb{Z}),$$

Definition.

Riemannian manifolds M_1 and M_2 are

- *iso-length spectral* if $L(M_1) = L(M_2)$;
- *length-commensurable* if $\mathbb{Q} \cdot L(M_1) = \mathbb{Q} \cdot L(M_2)$.

Let $M_i = \mathbb{H}/\Gamma_i$ $(i = 1, 2)$ be Riemann surfaces.

If M_1 and M_2 are length-commensurable **then:**

- $K_{\Gamma_1} = K_{\Gamma_2} =: K$;
- Given closed geodesics $c_{\gamma_i} \subset M_i$ for $i = 1, 2$ such that

$$\ell(c_{\gamma_2})/\ell(c_{\gamma_1}) = m/n \quad (m, n \in \mathbb{Z}),$$

elements γ_1^m and γ_2^n are conjugate

Definition.

Riemannian manifolds M_1 and M_2 are

- *iso-length spectral* if $L(M_1) = L(M_2)$;
- *length-commensurable* if $\mathbb{Q} \cdot L(M_1) = \mathbb{Q} \cdot L(M_2)$.

Let $M_i = \mathbb{H}/\Gamma_i$ $(i = 1, 2)$ be Riemann surfaces.

If M_1 and M_2 are length-commensurable **then:**

- $K_{\Gamma_1} = K_{\Gamma_2} =: K$;
- Given closed geodesics $c_{\gamma_i} \subset M_i$ for $i = 1, 2$ such that
$$\ell(c_{\gamma_2})/\ell(c_{\gamma_1}) = m/n \quad (m, n \in \mathbb{Z}),$$
elements γ_1^m and γ_2^n are conjugate $\Rightarrow$

Definition.

Riemannian manifolds M_1 and M_2 are

- *iso-length spectral* if $L(M_1) = L(M_2)$;
- *length-commensurable* if $\mathbb{Q} \cdot L(M_1) = \mathbb{Q} \cdot L(M_2)$.

Let $M_i = \mathbb{H}/\Gamma_i$ $(i = 1, 2)$ be Riemann surfaces.

If M_1 and M_2 are length-commensurable **then:**

- $K_{\Gamma_1} = K_{\Gamma_2} =: K$;
- Given closed geodesics $c_{\gamma_i} \subset M_i$ for $i = 1, 2$ such that

$$\ell(c_{\gamma_2})/\ell(c_{\gamma_1}) = m/n \quad (m, n \in \mathbb{Z}),$$

  elements γ_1^m and γ_2^n are conjugate $\Rightarrow$

  $K[\gamma_1] \subset A_{\Gamma_1}$ and $K[\gamma_2] \subset A_{\Gamma_2}$ are isomorphic.

So, if M_1 and M_2 are length-commensurable then

So, if M_1 and M_2 are length-commensurable then

$$A_{\Gamma_1} \text{ and } A_{\Gamma_2}$$

have same etale subalgebras that intersect $\widetilde{\Gamma}_1^{(2)}$ and $\widetilde{\Gamma}_2^{(2)}$.

So, if M_1 and M_2 are length-commensurable then

$$A_{\Gamma_1} \text{ and } A_{\Gamma_2}$$

have same etale subalgebras that intersect $\widetilde{\Gamma}_1^{(2)}$ and $\widetilde{\Gamma}_2^{(2)}$.

(maybe not all subalgebras but let us ignore this for now ...)

So, if M_1 and M_2 are length-commensurable then

$$A_{\Gamma_1} \quad \text{and} \quad A_{\Gamma_2}$$

have same etale subalgebras that intersect $\widetilde{\Gamma}_1^{(2)}$ and $\widetilde{\Gamma}_2^{(2)}$.

(maybe not all subalgebras but let us ignore this for now ...)

Here we can see

So, if M_1 and M_2 are length-commensurable then

$$A_{\Gamma_1} \text{ and } A_{\Gamma_2}$$

have same etale subalgebras that intersect $\widetilde{\Gamma}_1^{(2)}$ and $\widetilde{\Gamma}_2^{(2)}$.

(maybe not all subalgebras but let us ignore this for now ...)

Here we can see

- how (quaternion) algebras sharing "lots" of etale subalgebras arise in "practice";

So, if M_1 and M_2 are length-commensurable then

$$A_{\Gamma_1} \quad \text{and} \quad A_{\Gamma_2}$$

have same etale subalgebras that intersect $\widetilde{\Gamma}_1^{(2)}$ and $\widetilde{\Gamma}_2^{(2)}$.

(maybe not all subalgebras but let us ignore this for now ...)

Here we can see

- how (quaternion) algebras sharing "lots" of etale subalgebras arise in "practice";
- how results on Question $(*)$ can be applied to Riemann surfaces.

Using (AHBN) one proves

Using (AHBN) one proves

Fact.

Using (AHBN) one proves

Fact.

Let D_1 and D_2 be central quaternion algebras over a number field K.

Using (AHBN) one proves

Fact.

Let D_1 and D_2 be central quaternion algebras over a number field K. If D_1 and D_2 have same maximal subfields

Using (AHBN) one proves

Fact.

Let D_1 and D_2 be central quaternion algebras over a number field K. If D_1 and D_2 have same maximal subfields then $D_1 \simeq D_2$.

Using (AHBN) one proves

Fact.

Let D_1 and D_2 be central quaternion algebras over a number field K. If D_1 and D_2 have same maximal subfields then $D_1 \simeq D_2$.

(Compare with above construction of families of pairwise nonisomorphic *cubic* algebras with same maximal subfields!)

Using (AHBN) one proves

Fact.

Let D_1 and D_2 be central quaternion algebras over a number field K. If D_1 and D_2 have same maximal subfields then $D_1 \simeq D_2$.

(Compare with above construction of families of pairwise nonisomorphic *cubic* algebras with same maximal subfields!)

Let

$$M_1 = \mathbb{H}/\Gamma_1 \quad \text{and} \quad M_2 = \mathbb{H}/\Gamma_2$$

be arithmetically defined Riemann surfaces,

Using (AHBN) one proves

Fact.

Let D_1 and D_2 be central quaternion algebras over a number field K. If D_1 and D_2 have same maximal subfields then $D_1 \simeq D_2$.

(Compare with above construction of families of pairwise nonisomorphic *cubic* algebras with same maximal subfields!)

Let

$$M_1 = \mathbb{H}/\Gamma_1 \quad \text{and} \quad M_2 = \mathbb{H}/\Gamma_2$$

be arithmetically defined Riemann surfaces, i. e. Γ_1 and Γ_2 are arithmetic Fuchsian groups.

Using (AHBN) one proves

Fact.

Let D_1 and D_2 be central quaternion algebras over a number field K. If D_1 and D_2 have same maximal subfields then $D_1 \simeq D_2$.

(Compare with above construction of families of pairwise nonisomorphic *cubic* algebras with same maximal subfields!)

Let

$$M_1 = \mathbb{H}/\Gamma_1 \quad \text{and} \quad M_2 = \mathbb{H}/\Gamma_2$$

be arithmetically defined Riemann surfaces, i. e. Γ_1 and Γ_2 are arithmetic Fuchsian groups.

Assume that M_1 and M_2 are length-commensurable.

Then A_{Γ_1} and A_{Γ_2} are quaternion algebras over <u>number field</u>

$$K := K_{\Gamma_1} = K_{\Gamma_2}$$

Then A_{Γ_1} and A_{Γ_2} are quaternion algebras over number field

$$K := K_{\Gamma_1} = K_{\Gamma_2}$$

sharing subfields that intersect certain arithmetic subgroups.

Then A_{Γ_1} and A_{Γ_2} are quaternion algebras over number field

$$K := K_{\Gamma_1} = K_{\Gamma_2}$$

sharing subfields that intersect certain arithmetic subgroups.

Using a variation of above fact, one concludes that $A_{\Gamma_1} \simeq A_{\Gamma_2}$.

Then A_{Γ_1} and A_{Γ_2} are quaternion algebras over number field

$$K := K_{\Gamma_1} = K_{\Gamma_2}$$

sharing subfields that intersect certain arithmetic subgroups.

Using a variation of above fact, one concludes that $A_{\Gamma_1} \simeq A_{\Gamma_2}$.

It follows that Γ_1 and Γ_2 are *commensurable*

Then A_{Γ_1} and A_{Γ_2} are quaternion algebras over number field

$$K := K_{\Gamma_1} = K_{\Gamma_2}$$

sharing subfields that intersect certain arithmetic subgroups.

Using a variation of above fact, one concludes that $A_{\Gamma_1} \simeq A_{\Gamma_2}$.

It follows that Γ_1 and Γ_2 are *commensurable* $\Rightarrow$ M_1 and M_2 are commensurable,

Then A_{Γ_1} and A_{Γ_2} are quaternion algebras over number field

$$K := K_{\Gamma_1} = K_{\Gamma_2}$$

sharing subfields that intersect certain arithmetic subgroups.

Using a variation of above fact, one concludes that $A_{\Gamma_1} \simeq A_{\Gamma_2}$.

It follows that Γ_1 and Γ_2 are *commensurable* $\Rightarrow$ M_1 and M_2 are commensurable, i. e. have a common finite-sheeted cover.

Then A_{Γ_1} and A_{Γ_2} are quaternion algebras over number field

$$K := K_{\Gamma_1} = K_{\Gamma_2}$$

sharing subfields that intersect certain arithmetic subgroups.

Using a variation of above fact, one concludes that $A_{\Gamma_1} \simeq A_{\Gamma_2}$.

It follows that Γ_1 and Γ_2 are *commensurable* $\Rightarrow$ M_1 and M_2 are commensurable, i. e. have a common finite-sheeted cover.

Theorem (A. Reid, 1992)

If two arithmetically defined Riemann surfaces are length-commensurable then they are commensurable.

In this talk we will see results on $(*)$ over arbitrary finitely generated field.

In this talk we will see results on $(*)$ over arbitrary finitely generated field. They have applications to Riemann surfaces without any assumptions on arithmeticity.

In this talk we will see results on $(*)$ over arbitrary finitely generated field. They have applications to Riemann surfaces without any assumptions on arithmeticity.

Theorem

Let $M_i = \mathbb{H}/\Gamma_i$ $(i \in I)$ be a family of length-commensurable *Riemann surfaces where $\Gamma_i \subset \mathrm{PSL}_2(\mathbb{R})$ is Zariski-dense. Then quaternion algebras A_{Γ_i} $(i \in I)$ split into finitely many isomorphism classes (over common center).*

Definition.

Definition.

Let D be a finite-dimensional central division algebra over K.

Definition.

Let D be a finite-dimensional central division algebra over K. The *genus* of D is

$$\mathbf{gen}(D) = \{ [D'] \in \mathrm{Br}(K) \mid D' \text{ has same maximal subfields as } D \}$$

Definition.

Let D be a finite-dimensional central division algebra over K. The *genus* of D is

$$\mathbf{gen}(D) = \{\, [D'] \in \mathrm{Br}(K) \mid D' \text{ has same maximal subfields as } D \,\}$$

Question 1. *When does* $\mathbf{gen}(D)$ *reduce to a single element?*

Definition.

Let D be a finite-dimensional central division algebra over K. The *genus* of D is

$$\mathbf{gen}(D) = \{ [D'] \in \mathrm{Br}(K) \mid D' \text{ has same maximal subfields as } D \}$$

Question 1. *When does* $\mathbf{gen}(D)$ *reduce to a single element?*

(This means that D is uniquely determined by maximal subfields.)

Definition.

Let D be a finite-dimensional central division algebra over K. The *genus* of D is

$$\mathbf{gen}(D) = \{ [D'] \in \mathrm{Br}(K) \mid D' \text{ has same maximal subfields as } D \}$$

Question 1. *When does* $\mathbf{gen}(D)$ *reduce to a single element?*

(This means that D is uniquely determined by maximal subfields.)

Question 2. *When is* $\mathbf{gen}(D)$ *finite?*

Definition.

Let D be a finite-dimensional central division algebra over K. The *genus* of D is

$$\mathbf{gen}(D) = \{ [D'] \in \mathrm{Br}(K) \mid D' \text{ has same maximal subfields as } D \}$$

Question 1. *When does* $\mathbf{gen}(D)$ *reduce to a single element?*

(This means that D is uniquely determined by maximal subfields.)

Question 2. *When is* $\mathbf{gen}(D)$ *finite?*

Over number fields:

Definition.

Let D be a finite-dimensional central division algebra over K. The *genus* of D is

$$\mathbf{gen}(D) = \{ [D'] \in \mathrm{Br}(K) \mid D' \text{ has same maximal subfields as } D \}$$

Question 1. *When does* $\mathbf{gen}(D)$ *reduce to a single element?*

(This means that D is uniquely determined by maximal subfields.)

Question 2. *When is* $\mathbf{gen}(D)$ *finite?*

Over number fields:

- genus of any quaternion algebra reduces to one element;
- genus of any division algebra is finite.

Definition.

Let D be a finite-dimensional central division algebra over K. The *genus* of D is

$$\mathbf{gen}(D) = \{ [D'] \in \mathrm{Br}(K) \mid D' \text{ has same maximal subfields as } D \}$$

Question 1. *When does* $\mathbf{gen}(D)$ *reduce to a single element?*

(This means that D is uniquely determined by maximal subfields.)

Question 2. *When is* $\mathbf{gen}(D)$ *finite?*

Over number fields:

- genus of any quaternion algebra reduces to one element;
- genus of any division algebra is finite.

Definition.

Let D be a finite-dimensional central division algebra over K. The *genus* of D is

$$\mathbf{gen}(D) = \{ [D'] \in \mathrm{Br}(K) \mid D' \text{ has same maximal subfields as } D \}$$

Question 1. *When does* $\mathbf{gen}(D)$ *reduce to a single element?*

(This means that D is uniquely determined by maximal subfields.)

Question 2. *When is* $\mathbf{gen}(D)$ *finite?*

Over number fields:

- genus of any quaternion algebra reduces to one element;
- genus of any division algebra is finite.

Both facts follow from (AHBN).

Let D be a quaternion algebra over $\mathbb{Q}$.

Let D be a quaternion algebra over $\mathbb{Q}$.

Then $\mathrm{inv}_p([D])$ can only be 0 or 1/2.

Let D be a quaternion algebra over $\mathbb{Q}$.

Then $\mathrm{inv}_p([D])$ can only be 0 or $1/2$.

So, by (AHBN), D is uniquely determined by finite set

$$\mathbf{Ram}(D) := \{\, p \mid \mathrm{inv}_p([D]) \neq 0 \,\}$$

Let D be a quaternion algebra over $\mathbb{Q}$.

Then $\mathrm{inv}_p([D])$ can only be 0 or 1/2.

So, by (AHBN), D is uniquely determined by finite set

$$\mathbf{Ram}(D) := \{\, p \mid \mathrm{inv}_p([D]) \neq 0 \,\} \text{ (ramified primes).}$$

Let D be a quaternion algebra over $\mathbb{Q}$.

Then $\mathrm{inv}_p([D])$ can only be 0 or 1/2.

So, by (AHBN), D is uniquely determined by finite set

$$\mathbf{Ram}(D) := \{ p \mid \mathrm{inv}_p([D]) \neq 0 \}$$ (ramified primes).

But maximal subfields of D determine $\mathbf{Ram}(D)$!

Let D be a quaternion algebra over $\mathbb{Q}$.

Then $\mathrm{inv}_p([D])$ can only be 0 or 1/2.

So, by (AHBN), D is uniquely determined by finite set

$$\mathbf{Ram}(D) := \{ \, p \mid \mathrm{inv}_p([D]) \neq 0 \, \} \ \text{(ramified primes).}$$

But maximal subfields of D determine $\mathbf{Ram}(D)$!

Namely, if quaternion algebras D_1 and D_2 are such that

$$\mathbf{Ram}(D_1) \neq \mathbf{Ram}(D_2),$$

Let D be a quaternion algebra over $\mathbb{Q}$.

Then $\mathrm{inv}_p([D])$ can only be 0 or 1/2.

So, by (AHBN), D is uniquely determined by finite set

$$\mathbf{Ram}(D) := \{ p \mid \mathrm{inv}_p([D]) \neq 0 \} \text{ (ramified primes).}$$

But maximal subfields of D determine $\mathbf{Ram}(D)$!

Namely, if quaternion algebras D_1 and D_2 are such that

$$\mathbf{Ram}(D_1) \neq \mathbf{Ram}(D_2),$$

then using weak approximation one finds a quadratic extension $L/\mathbb{Q}$ which embeds into one algebra but **not** into the other.

Example.

Example.

Let $D_1 = \left(\dfrac{-1\,,\,3}{\mathbb{Q}}\right)$ and $D_2 = \left(\dfrac{-1\,,\,7}{\mathbb{Q}}\right)$.

Example.

Let $D_1 = \left(\dfrac{-1\,,\,3}{\mathbb{Q}}\right)$ and $D_2 = \left(\dfrac{-1\,,\,7}{\mathbb{Q}}\right)$. We have

$$\mathbf{Ram}(D_1) = \{2\,,\,3\} \text{ and } \mathbf{Ram}(D_2) = \{2\,,\,7\}$$

Example.

Let $D_1 = \left(\dfrac{-1\,,\,3}{\mathbb{Q}}\right)$ and $D_2 = \left(\dfrac{-1\,,\,7}{\mathbb{Q}}\right)$. We have

$$\mathbf{Ram}(D_1) \;=\; \{2\,,\,3\} \;\text{ and }\; \mathbf{Ram}(D_2) \;=\; \{2\,,\,7\}$$

Extension $P = \mathbb{Q}(\sqrt{11})$ embeds into D_1 but **not** into D_2.

Example.

Let $D_1 = \left(\frac{-1\,,\,3}{\mathbb{Q}}\right)$ and $D_2 = \left(\frac{-1\,,\,7}{\mathbb{Q}}\right)$. We have

$$\mathbf{Ram}(D_1) = \{2\,,\,3\} \text{ and } \mathbf{Ram}(D_2) = \{2\,,\,7\}$$

Extension $P = \mathbb{Q}(\sqrt{11})$ embeds into D_1 but **not** into D_2.

This proves that $\mathbf{gen}(D) = \{[D]\}$ for D a *quaternion* algebra.

Example.

Let $D_1 = \left(\dfrac{-1\,,\,3}{\mathbb{Q}}\right)$ and $D_2 = \left(\dfrac{-1\,,\,7}{\mathbb{Q}}\right)$. We have

$$\mathbf{Ram}(D_1) \ = \ \{2\,,\,3\} \ \text{ and } \ \mathbf{Ram}(D_2) \ = \ \{2\,,\,7\}$$

Extension $P = \mathbb{Q}(\sqrt{11})$ embeds into D_1 but **not** into D_2.

This proves that $\mathbf{gen}(D) = \{[D]\}$ for D a *quaternion* algebra.

In fact, given a division algebra D of *any* degree n over a *number field* K,

Example.

Let $D_1 = \left(\frac{-1\,,\,3}{\mathbb{Q}}\right)$ and $D_2 = \left(\frac{-1\,,\,7}{\mathbb{Q}}\right)$. We have

$$\mathbf{Ram}(D_1) = \{2\,,\,3\} \text{ and } \mathbf{Ram}(D_2) = \{2\,,\,7\}$$

Extension $P = \mathbb{Q}(\sqrt{11})$ embeds into D_1 but **not** into D_2.

This proves that $\mathbf{gen}(D) = \{[D]\}$ for D a *quaternion* algebra.

In fact, given a division algebra D of *any* degree n over a *number field* K, for $D' \in \mathbf{gen}(D)$ we have

$$\mathbf{Ram}(D') = \mathbf{Ram}(D).$$

Example.

Let $D_1 = \left(\frac{-1\,,\,3}{\mathbb{Q}}\right)$ and $D_2 = \left(\frac{-1\,,\,7}{\mathbb{Q}}\right)$. We have

$$\mathbf{Ram}(D_1) = \{2\,,\,3\} \text{ and } \mathbf{Ram}(D_2) = \{2\,,\,7\}$$

Extension $P = \mathbb{Q}(\sqrt{11})$ embeds into D_1 but **not** into D_2.

This proves that $\mathbf{gen}(D) = \{[D]\}$ for D a *quaternion* algebra.

In fact, given a division algebra D of *any* degree n over a *number field* K, for $D' \in \mathbf{gen}(D)$ we have

$$\mathbf{Ram}(D') = \mathbf{Ram}(D).$$

But for each $p \in \mathbf{Ram}(D)$, there are only finitely many possibilities for $\mathrm{inv}_p([D'])$.

Example.

Let $D_1 = \left(\frac{-1\,,\,3}{\mathbb{Q}}\right)$ and $D_2 = \left(\frac{-1\,,\,7}{\mathbb{Q}}\right)$. We have

$$\mathbf{Ram}(D_1) \ = \ \{2\,,\,3\} \ \text{ and } \ \mathbf{Ram}(D_2) \ = \ \{2\,,\,7\}$$

Extension $P = \mathbb{Q}(\sqrt{11})$ embeds into D_1 but **not** into D_2.

This proves that $\mathbf{gen}(D) = \{[D]\}$ for D a *quaternion* algebra.

In fact, given a division algebra D of *any* degree n over a *number field* K, for $D' \in \mathbf{gen}(D)$ we have

$$\mathbf{Ram}(D') \ = \ \mathbf{Ram}(D).$$

But for each $p \in \mathbf{Ram}(D)$, there are only finitely many possibilities for $\mathrm{inv}_p([D'])$. **So,** $\theta(\mathbf{gen}(D))$ is finite,

Example.

Let $D_1 = \left(\frac{-1\,,\,3}{\mathbb{Q}}\right)$ and $D_2 = \left(\frac{-1\,,\,7}{\mathbb{Q}}\right)$. We have

$$\mathbf{Ram}(D_1) \ = \ \{2\,,\,3\} \ \text{ and } \ \mathbf{Ram}(D_2) \ = \ \{2\,,\,7\}$$

Extension $P = \mathbb{Q}(\sqrt{11})$ embeds into D_1 but **not** into D_2.

This proves that $\mathbf{gen}(D) = \{[D]\}$ for D a *quaternion* algebra.

In fact, given a division algebra D of *any* degree n over a *number field* K, for $D' \in \mathbf{gen}(D)$ we have

$$\mathbf{Ram}(D') \ = \ \mathbf{Ram}(D).$$

But for each $p \in \mathbf{Ram}(D)$, there are only finitely many possibilities for $\mathrm{inv}_p([D'])$. **So,** $\theta(\mathbf{gen}(D))$ is finite, hence $\mathbf{gen}(D)$ is finite since θ is injective by (AHBN).

Theorem 1 (Stability Theorem)

Let $\operatorname{char} k \neq 2$. *If* $|\mathbf{gen}(D)| = 1$ *for any quaternion algebra* D *over* k, *then* $|\mathbf{gen}(D')| = 1$ *for any quaternion algebra* D' *over* $k(x)$.

Theorem 1 (Stability Theorem)

Let $\operatorname{char} k \neq 2$*. If* $|\mathbf{gen}(D)| = 1$ *for any quaternion algebra* D *over* k*, then* $|\mathbf{gen}(D')| = 1$ *for any quaternion algebra* D' *over* $k(x)$*.*

- Same statement is true for division algebras of *exponent 2*.

Theorem 1 (Stability Theorem)

Let $\mathrm{char}\, k \neq 2$*. If* $|\mathbf{gen}(D)| = 1$ *for any quaternion algebra* D *over* k*, then* $|\mathbf{gen}(D')| = 1$ *for any quaternion algebra* D' *over* $k(x)$*.*

- Same statement is true for division algebras of *exponent 2*.
- $|\mathbf{gen}(D)| > 1$ if D is not of exponent 2.

Theorem 1 (Stability Theorem)

Let $\mathrm{char}\, k \neq 2$*. If* $|\mathbf{gen}(D)| = 1$ *for any quaternion algebra* D *over* k*, then* $|\mathbf{gen}(D')| = 1$ *for any quaternion algebra* D' *over* $k(x)$*.*

- Same statement is true for division algebras of *exponent 2*.
- $|\mathbf{gen}(D)| > 1$ if D is <u>not</u> of exponent 2.

- $\mathbf{gen}(D)$ can be infinite.

Theorem 1 (Stability Theorem)

Let char $k \neq 2$. *If* $|\mathbf{gen}(D)| = 1$ *for any quaternion algebra* D *over* k, *then* $|\mathbf{gen}(D')| = 1$ *for any quaternion algebra* D' *over* $k(x)$.

- Same statement is true for division algebras of *exponent 2*.
- $|\mathbf{gen}(D)| > 1$ if D is not of exponent 2.

- $\mathbf{gen}(D)$ can be infinite.

Generalizing construction used by Schacher, Garibaldi, Saltman, ...

Theorem 1 (Stability Theorem)

Let $\operatorname{char} k \neq 2$. *If* $|\mathbf{gen}(D)| = 1$ *for any quaternion algebra* D *over* k, *then* $|\mathbf{gen}(D')| = 1$ *for any quaternion algebra* D' *over* $k(x)$.

- Same statement is true for division algebras of *exponent 2*.
- $|\mathbf{gen}(D)| > 1$ if D is not of exponent 2.

- $\mathbf{gen}(D)$ can be infinite.

Generalizing construction used by Schacher, Garibaldi, Saltman, ...
Meyer constructed quaternion algebras over "large" fields with infinite genus,

Theorem 1 (Stability Theorem)

Let $\operatorname{char} k \neq 2$. *If* $|\mathbf{gen}(D)| = 1$ *for any quaternion algebra* D *over* k, *then* $|\mathbf{gen}(D')| = 1$ *for any quaternion algebra* D' *over* $k(x)$.

- Same statement is true for division algebras of *exponent 2*.
- $|\mathbf{gen}(D)| > 1$ if D is not of exponent 2.

- $\mathbf{gen}(D)$ can be infinite.

Generalizing construction used by Schacher, Garibaldi, Saltman, ...
Meyer constructed quaternion algebras over "large" fields with infinite genus,

Tikhonov extended construction to algebras of prime degree.

Construction

Construction

- Start with nonisomorphic quaternion algebras D_1 and D_2 over K (char $K \neq 2$) having a common maximal subfield.

Construction

- Start with nonisomorphic quaternion algebras D_1 and D_2 over K (char $K \neq 2$) having a common maximal subfield.

(E.g., take $D_1 = \left(\dfrac{-1,3}{\mathbb{Q}}\right)$ and $D_2 = \left(\dfrac{-1,7}{\mathbb{Q}}\right)$ over $K = \mathbb{Q}$)

Construction

- Start with nonisomorphic quaternion algebras D_1 and D_2 over K (char $K \neq 2$) having a common maximal subfield.

(E.g., take $D_1 = \left(\frac{-1,3}{\mathbb{Q}}\right)$ and $D_2 = \left(\frac{-1,7}{\mathbb{Q}}\right)$ over $K = \mathbb{Q}$)

- Want: find $\mathcal{K}/K$ so that $D_1 \otimes_K \mathcal{K} \not\simeq D_2 \otimes_K \mathcal{K}$,

Construction

- Start with nonisomorphic quaternion algebras D_1 and D_2 over K (char $K \neq 2$) having a common maximal subfield.

(E.g., take $D_1 = \left(\frac{-1,3}{\mathbb{Q}}\right)$ and $D_2 = \left(\frac{-1,7}{\mathbb{Q}}\right)$ over $K = \mathbb{Q}$)

- Want: find $\mathcal{K}/K$ so that $D_1 \otimes_K \mathcal{K} \not\simeq D_2 \otimes_K \mathcal{K}$, and have *same* maximal subfields.

Construction

- Start with nonisomorphic quaternion algebras D_1 and D_2 over K (char $K \neq 2$) having a common maximal subfield.

(E.g., take $D_1 = \left(\frac{-1,3}{\mathbb{Q}}\right)$ and $D_2 = \left(\frac{-1,7}{\mathbb{Q}}\right)$ over $K = \mathbb{Q}$)

- Want: find $\mathcal{K}/K$ so that $D_1 \otimes_K \mathcal{K} \not\simeq D_2 \otimes_K \mathcal{K}$, and have *same* maximal subfields.

- If D_1 and D_2 already have same maximal subfields, we are done ($\mathcal{K} = K$).

Construction

- Start with nonisomorphic quaternion algebras D_1 and D_2 over K ($\mathrm{char}\, K \neq 2$) having a common maximal subfield.

(E.g., take $D_1 = \left(\frac{-1,3}{\mathbb{Q}}\right)$ and $D_2 = \left(\frac{-1,7}{\mathbb{Q}}\right)$ over $K = \mathbb{Q}$)

- Want: find $\mathcal{K}/K$ so that $D_1 \otimes_K \mathcal{K} \not\simeq D_2 \otimes_K \mathcal{K}$, and have *same* maximal subfields.

- If D_1 and D_2 already have same maximal subfields, we are done ($\mathcal{K} = K$).

  Otherwise, pick $K(\sqrt{d_1}) \hookrightarrow D_1$ such that $K(\sqrt{d_1}) \not\hookrightarrow D_2$.

Construction

- Start with nonisomorphic quaternion algebras D_1 and D_2 over K (char $K \neq 2$) having a common maximal subfield.

(E.g., take $D_1 = \left(\frac{-1,3}{\mathbb{Q}}\right)$ and $D_2 = \left(\frac{-1,7}{\mathbb{Q}}\right)$ over $K = \mathbb{Q}$)

- Want: find $\mathcal{K}/K$ so that $D_1 \otimes_K \mathcal{K} \not\simeq D_2 \otimes_K \mathcal{K}$, and have *same* maximal subfields.

- If D_1 and D_2 already have same maximal subfields, we are done $(\mathcal{K} = K)$.

Otherwise, pick $K(\sqrt{d_1}) \hookrightarrow D_1$ such that $K(\sqrt{d_1}) \not\hookrightarrow D_2$.

(E.g., $\mathbb{Q}(\sqrt{11}) \hookrightarrow D_1$ but $\mathbb{Q}(\sqrt{11}) \not\hookrightarrow D_2$.)

- Find K_1/K such that

1. $D_1 \otimes_K K_1 \not\simeq D_2 \otimes_K K_1$;
2. $K_1(\sqrt{d_1}) \hookrightarrow D_2 \otimes_K K_1$.

- Find K_1/K such that

1. $D_1 \otimes_K K_1 \not\simeq D_2 \otimes_K K_1$;
2. $K_1(\sqrt{d_1}) \hookrightarrow D_2 \otimes_K K_1$.

- Find K_1/K such that

  1. $D_1 \otimes_K K_1 \not\simeq D_2 \otimes_K K_1$;
  2. $K_1(\sqrt{d_1}) \hookrightarrow D_2 \otimes_K K_1$.

- Find K_1/K such that

1. $D_1 \otimes_K K_1 \not\simeq D_2 \otimes_K K_1$;
2. $K_1(\sqrt{d_1}) \hookrightarrow D_2 \otimes_K K_1$.

For K_1 one can take the function field of a quadric.

- Find K_1/K such that

  1. $D_1 \otimes_K K_1 \not\simeq D_2 \otimes_K K_1$;
  2. $K_1(\sqrt{d_1}) \hookrightarrow D_2 \otimes_K K_1$.

For K_1 one can take the function field of a quadric.

In our example, K_1 is function field of

$$-x_1^2 + 7x_2^2 + 7x_3^2 = 11x_4^2$$

- Find K_1/K such that
  1. $D_1 \otimes_K K_1 \not\simeq D_2 \otimes_K K_1$;
  2. $K_1(\sqrt{d_1}) \hookrightarrow D_2 \otimes_K K_1$.

For K_1 one can take the function field of a quadric.

In our example, K_1 is function field of

$$-x_1^2 + 7x_2^2 + 7x_3^2 = 11x_4^2$$

Then (2) is obvious, and (1) follows from the fact that

$$x_0^2 + x_1^2 - 21x_2^2 - 21x_3^2$$

remains anisotropic over K_1.

- If there exists $K_1(\sqrt{d_2}) \hookrightarrow D_1 \otimes_K K_1$ and $K_1(\sqrt{d_2}) \not\hookrightarrow D_2 \otimes_K K_1$ we construct K_2/K_1 similarly.

- If there exists $K_1(\sqrt{d_2}) \hookrightarrow D_1 \otimes_K K_1$ and $K_1(\sqrt{d_2}) \not\hookrightarrow D_2 \otimes_K K_1$ we construct K_2/K_1 similarly.

This generates a tower $K \subset K_1 \subset K_2 \subset \cdots$

- If there exists $K_1(\sqrt{d_2}) \hookrightarrow D_1 \otimes_K K_1$ and $K_1(\sqrt{d_2}) \not\hookrightarrow D_2 \otimes_K K_1$ we construct K_2/K_1 similarly.

This generates a tower $K \subset K_1 \subset K_2 \subset \cdots$

Set $\mathcal{K} = \bigcup_{i=1}^{\infty} K_i$.

- If there exists $K_1(\sqrt{d_2}) \hookrightarrow D_1 \otimes_K K_1$ and $K_1(\sqrt{d_2}) \not\hookrightarrow D_2 \otimes_K K_1$ we construct K_2/K_1 similarly.

This generates a tower $K \subset K_1 \subset K_2 \subset \cdots$

Set $\mathcal{K} = \bigcup_{i=1}^{\infty} K_i$.

- **Then** $\mathcal{K}$ is as required.

- If there exists $K_1(\sqrt{d_2}) \hookrightarrow D_1 \otimes_K K_1$ and $K_1(\sqrt{d_2}) \not\hookrightarrow D_2 \otimes_K K_1$ we construct K_2/K_1 similarly.

This generates a tower $K \subset K_1 \subset K_2 \subset \cdots$

Set $\mathcal{K} = \bigcup_{i=1}^{\infty} K_i.$

- **Then** $\mathcal{K}$ is as required.

For *infinite* genus, one starts with $D_p = \left(\frac{-1, p}{\mathbb{Q}}\right)$, $p \equiv 3 (\mathrm{mod}\, 4)$.

- If there exists $K_1(\sqrt{d_2}) \hookrightarrow D_1 \otimes_K K_1$ and $K_1(\sqrt{d_2}) \not\hookrightarrow D_2 \otimes_K K_1$ we construct K_2/K_1 similarly.

This generates a tower $K \subset K_1 \subset K_2 \subset \cdots$

Set $\mathcal{K} = \bigcup_{i=1}^{\infty} K_i$.

- **Then** $\mathcal{K}$ is as required.

For *infinite* genus, one starts with $D_p = \left(\frac{-1, p}{\mathbb{Q}}\right)$, $p \equiv 3 (\mathrm{mod}\, 4)$.

Note that $\mathcal{K}$ is infinitely generated.

Theorem 2.

Let K be a finitely generated *field. Then for any central division K-algebra D the genus* $\mathbf{gen}(D)$ *is* *finite.*

Theorem 2.

Let K be a finitely generated *field. Then for any central division K-algebra D the genus* $\mathbf{gen}(D)$ *is finite.*

- Proofs of both theorems use *analysis of ramification* and info about *unramified Brauer group*.

Theorem 2.

Let K be a finitely generated *field. Then for any central division K-algebra D the genus* $\mathbf{gen}(D)$ *is* *finite.*

- Proofs of both theorems use *analysis of ramification* and info about *unramified Brauer group*.

In general case, *definition of unramified algebra* needs to be changed.

Theorem 2.

Let K be a finitely generated field. Then for any central division K-algebra D the genus $\mathbf{gen}(D)$ is finite.

- Proofs of both theorems use *analysis of ramification* and info about *unramified Brauer group*.

In general case, *definition of unramified algebra* needs to be changed.

- A central simple algebra D over $\mathbb{Q}$ is unramified at p if

$$\mathrm{inv}_p([D \otimes_{\mathbb{Q}} \mathbb{Q}_p]) = 0,$$

Theorem 2.

Let K be a finitely generated field. Then for any central division K-algebra D the genus $\mathbf{gen}(D)$ is finite.

- Proofs of both theorems use *analysis of ramification* and info about *unramified Brauer group*.

In general case, *definition of unramified algebra* needs to be changed.

- A central simple algebra D over $\mathbb{Q}$ is unramified at p if

$$\mathrm{inv}_p([D \otimes_{\mathbb{Q}} \mathbb{Q}_p]) = 0,$$

i. e., $D \otimes_{\mathbb{Q}} \mathbb{Q}_p \simeq \mathrm{M}_n(\mathbb{Q}_p)$.

- For D a central simple algebra over an arbitrary field K with a discrete valuation v,

- For D a central simple algebra over an arbitrary field K with a discrete valuation v, *being unramified* can be defined in several (essentially, equivalent) ways:

- For D a central simple algebra over an arbitrary field K with a discrete valuation v, *being unramified* can be defined in several (essentially, equivalent) ways:

1. using valuations;
2. using residue maps;
3. using Azumaya algebras.

- For D a central simple algebra over an arbitrary field K with a discrete valuation v, *being unramified* can be defined in several (essentially, equivalent) ways:

1. using valuations;
2. using residue maps;
3. using Azumaya algebras.

- For D a central simple algebra over an arbitrary field K with a discrete valuation v, *being unramified* can be defined in several (essentially, equivalent) ways:

1. using valuations;
2. using residue maps;
3. using Azumaya algebras.

- For D a central simple algebra over an arbitrary field K with a discrete valuation v, *being unramified* can be defined in several (essentially, equivalent) ways:

1. using valuations;
2. using residue maps;
3. using Azumaya algebras.

Definition.

- For D a central simple algebra over an arbitrary field K with a discrete valuation v, *being unramified* can be defined in several (essentially, equivalent) ways:

1. using valuations;
2. using residue maps;
3. using Azumaya algebras.

Definition.

A central simple algebra K-algebra A is unramified at a discrete valuation v of K

- For D a central simple algebra over an arbitrary field K with a discrete valuation v, *being unramified* can be defined in several (essentially, equivalent) ways:
  1. using valuations;
  2. using residue maps;
  3. using Azumaya algebras.

Definition.

A central simple algebra K-algebra A is unramified at a discrete valuation v of K if there exists an *Azumaya algebra* $\mathcal{A}$ over valuation ring $\mathcal{O}_v \subset K_v$ such that

$$\mathcal{A} \otimes_{\mathcal{O}_v} K_v \simeq A \otimes_K K_v.$$

Fact

If D is a division K-algebra which is unramified at a discrete valuation v of K

Fact

If D is a division K-algebra which is unramified at a discrete valuation v of K then every $D' \in \mathbf{gen}(D)$ *is also unramified at v.*

Fact

If D is a division K-algebra which is unramified at a discrete valuation v of K then every $D' \in \mathbf{gen}(D)$ *is also unramified at v.*

If degree n of D is prime to $\mathrm{char}\, K^{(v)}$ (residue field), we can relate *ramification data* of D and D' even when D is ramified.

Fact

If D is a division K-algebra which is unramified at a discrete valuation v of K then every $D' \in \mathbf{gen}(D)$ *is also unramified at v.*

If degree n of D is prime to $\mathrm{char}\, K^{(v)}$ (residue field), we can relate *ramification data* of D and D' even when D is ramified.

Suppose K is equipped with a set V of discrete valuations.

Fact

If D is a division K-algebra which is unramified at a discrete valuation v of K then every $D' \in \mathbf{gen}(D)$ *is also unramified at v.*

If degree n of D is prime to $\mathrm{char}\, K^{(v)}$ (residue field), we can relate *ramification data* of D and D' even when D is ramified.

Suppose K is equipped with a set V of discrete valuations.

Instead of analyzing kernel of global-to-local map (as in (ABHN)),

Fact

If D is a division K-algebra which is unramified at a discrete valuation v of K then every $D' \in \mathbf{gen}(D)$ *is also unramified at v.*

If degree n of D is prime to $\mathrm{char}\, K^{(v)}$ (residue field), we can relate *ramification data* of D and D' even when D is ramified.

Suppose K is equipped with a set V of discrete valuations.

Instead of analyzing kernel of global-to-local map (as in (ABHN)), we consider *unramified Brauer group* w. r. t. V:

Fact

If D is a division K-algebra which is unramified at a discrete valuation v of K then every $D' \in \mathbf{gen}(D)$ *is also unramified at v.*

If degree n of D is prime to $\mathrm{char}\, K^{(v)}$ (residue field), we can relate *ramification data* of D and D' even when D is ramified.

Suppose K is equipped with a set V of discrete valuations.

Instead of analyzing kernel of global-to-local map (as in (ABHN)), we consider *unramified Brauer group* w. r. t. V:

$$\mathrm{Br}(K)_V = \{ x \in \mathrm{Br}(K) \mid x \text{ is unramified at all } v \in V\}.$$

Note that if D of degree n is unramified at all $v \in V$ then

Note that if D of degree n is unramified at all $v \in V$ then

$$\mathbf{gen}(D) \subset {}_n\mathrm{Br}(K)_V \quad (n\text{-torsion subgroup})$$

Note that if D of degree n is unramified at all $v \in V$ then

$$\mathbf{gen}(D) \subset {}_n\mathrm{Br}(K)_V \quad (n\text{-torsion subgroup})$$

Theorem 3

Let K be a finitely generated *field, and $n \geqslant 2$ be an integer prime to* char K.

Note that if D of degree n is unramified at all $v \in V$ then

$$\mathbf{gen}(D) \subset {}_n\mathrm{Br}(K)_V \quad (n\text{-torsion subgroup})$$

Theorem 3

Let K be a finitely generated *field, and $n \geqslant 2$ be an integer prime to* $\operatorname{char} K$. *Then for any divisorial set of places V of K, the groups ${}_n\mathrm{Br}(K)_V$ is* finite.

Note that if D of degree n is unramified at all $v \in V$ then

$$\mathbf{gen}(D) \subset {}_n\mathrm{Br}(K)_V \quad (n\text{-torsion subgroup})$$

Theorem 3

Let K be a finitely generated *field, and $n \geqslant 2$ be an integer prime to* $\mathrm{char}\, K$. *Then for any divisorial set of places V of K, the groups ${}_n\mathrm{Br}(K)_V$ is* finite.

This implies finiteness of $\mathbf{gen}(D)$ over a f. g. field K of characteristic prime to degree n of D.

Note that if D of degree n is unramified at all $v \in V$ then

$$\mathbf{gen}(D) \subset {}_n\mathrm{Br}(K)_V \quad (n\text{-torsion subgroup})$$

Theorem 3

Let K be a finitely generated *field, and $n \geqslant 2$ be an integer prime to* char K. *Then for any divisorial set of places V of K, the groups ${}_n\mathrm{Br}(K)_V$ is* finite.

This implies finiteness of $\mathbf{gen}(D)$ over a f. g. field K of characteristic prime to degree n of D.

In fact, in this case one can give an estimate on size of $\mathbf{gen}(D)$ that depends on seize of ${}_n\mathrm{Br}(K)_V$ for a fixed V:

$$| \, \mathbf{gen}(D) \, | \ \leqslant \ \varphi(n)^r \cdot | \, {}_n\mathrm{Br}(K)_V \, |$$

$$|\,\mathbf{gen}(D)\,| \leqslant \varphi(n)^r \cdot |\,{}_n\mathrm{Br}(K)_V\,|$$

where φ is Euler's function, r is # of $v \in V$ where D ramifies.

$$|\,\mathbf{gen}(D)\,| \ \leqslant \ \varphi(n)^r \cdot |\,{}_n\mathrm{Br}(K)_V\,|$$

where φ is Euler's function, r is # of $v \in V$ where D ramifies.

In particular, we have $|\mathbf{gen}(D)| \leqslant {}_2\mathrm{Br}(K)_V$ for <u>all</u> quaternion K-algebras D

$$|\,\mathbf{gen}(D)\,| \ \leqslant\ \varphi(n)^r \cdot |\,{}_n\mathrm{Br}(K)_V\,|$$

where φ is Euler's function, r is # of $v \in V$ where D ramifies.

In particular, we have $|\mathbf{gen}(D)| \leqslant {}_2\mathrm{Br}(K)_V$ for <u>all</u> quaternion K-algebras D (in fact, for all algebras of *exponent* 2).

$$|\, \mathbf{gen}(D) \,| \ \leqslant \ \varphi(n)^r \cdot |\, {}_n\mathrm{Br}(K)_V \,|$$

where φ is Euler's function, r is # of $v \in V$ where D ramifies.

In particular, we have $|\mathbf{gen}(D)| \leqslant {}_2\mathrm{Br}(K)_V$ for all quaternion K-algebras D (in fact, for all algebras of *exponent* 2).

Note that Theorem 2 has no restrictions on characteristic!

$$|\,\mathbf{gen}(D)\,| \leqslant \varphi(n)^r \cdot |\,{}_n\mathrm{Br}(K)_V\,|$$

where φ is Euler's function, r is # of $v \in V$ where D ramifies.

In particular, we have $|\mathbf{gen}(D)| \leqslant {}_2\mathrm{Br}(K)_V$ for all quaternion K-algebras D (in fact, for all algebras of *exponent* 2).

Note that Theorem 2 has no restrictions on characteristic!

Recently we gave a variation of the above argument (Israel J. Math. **236** (2020)) that works in all characteristics

$$|\,\mathbf{gen}(D)\,| \ \leqslant \ \varphi(n)^r \cdot |\,{}_n\mathrm{Br}(K)_V\,|$$

where φ is Euler's function, r is # of $v \in V$ where D ramifies.

In particular, we have $|\mathbf{gen}(D)| \leqslant {}_2\mathrm{Br}(K)_V$ for <u>all</u> quaternion K-algebras D (in fact, for all algebras of *exponent* 2).

Note that Theorem 2 has no restrictions on characteristic!

Recently we gave a variation of the above argument (Israel J. Math. **236** (2020)) that works in all characteristics but does not give an estimate on size of $\mathbf{gen}(D)$.

$$|\,\mathbf{gen}(D)\,| \ \leqslant\ \varphi(n)^r \cdot |\, {}_n\mathrm{Br}(K)_V\,|$$

where φ is Euler's function, r is # of $v \in V$ where D ramifies.

In particular, we have $|\mathbf{gen}(D)| \leqslant {}_2\mathrm{Br}(K)_V$ for <u>all</u> quaternion K-algebras D (in fact, for all algebras of *exponent* 2).

Note that Theorem 2 has no restrictions on characteristic!

Recently we gave a variation of the above argument (Israel J. Math. **236** (2020)) that works in all characteristics but does not give an estimate on size of $\mathbf{gen}(D)$.

It is based on finiteness of certain subgroups of ${}_2\mathrm{Br}(K)_V$.

To prove Theorem 1 (Stability Theorem) over $K = k(x)$, one analyzes ramification w. r. t. set V of *geometric* places of K

To prove Theorem 1 (Stability Theorem) over $K = k(x)$, one analyzes ramification w. r. t. set V of *geometric* places of K and uses Faddeev's Theorem which yields that

$$ {}_2\mathrm{Br}(K)_V = {}_2\mathrm{Br}(k) $$

if $\operatorname{char} k \neq 2$.

To prove Theorem 1 (Stability Theorem) over $K = k(x)$, one analyzes ramification w. r. t. set V of *geometric* places of K and uses Faddeev's Theorem which yields that

$$ {}_2\mathrm{Br}(K)_V = {}_2\mathrm{Br}(k) $$

if $\mathrm{char}\, k \neq 2$.

Open question. *Does there exist a quaternion division algebra D over $K = k(C)$, where C is a smooth geometrically integral curve over a number field k, such that*

$$ |\mathbf{gen}(D)| > 1? $$

To prove Theorem 1 (Stability Theorem) over $K = k(x)$, one analyzes ramification w. r. t. set V of *geometric* places of K and uses Faddeev's Theorem which yields that

$$_2\mathrm{Br}(K)_V = {}_2\mathrm{Br}(k)$$

if $\mathrm{char}\, k \neq 2$.

Open question. *Does there exist a quaternion division algebra D over $K = k(C)$, where C is a smooth geometrically integral curve over a number field k, such that*

$$|\mathbf{gen}(D)| > 1?$$

- The answer is **not** known for *any* finitely generated K.

To prove Theorem 1 (Stability Theorem) over $K = k(x)$, one analyzes ramification w. r. t. set V of *geometric* places of K and uses Faddeev's Theorem which yields that

$$ {}_2\mathrm{Br}(K)_V = {}_2\mathrm{Br}(k) $$

if $\mathrm{char}\, k \neq 2$.

Open question. *Does there exist a quaternion division algebra D over $K = k(C)$, where C is a smooth geometrically integral curve over a number field k, such that*

$$ |\mathbf{gen}(D)| > 1? $$

- The answer is **not** known for *any* finitely generated K.
- One can construct examples where ${}_2\mathrm{Br}(K)_V$ is "large."

- To define the genus of an algebraic group, we replace maximal subfields with *maximal tori* in the definition of genus of division algebra.

- To define the genus of an algebraic group, we replace maximal subfields with *maximal tori* in the definition of genus of division algebra.

- Let G_1 and G_2 be semi-simple groups over a field K.

- To define the genus of an algebraic group, we replace maximal subfields with *maximal tori* in the definition of genus of division algebra.

- Let G_1 and G_2 be semi-simple groups over a field K.

  G_1 & G_2 have *same isomorphism classes of maximal K-tori* **if** every maximal K-torus T_1 of G_1 is K-isomorphic to a maximal K-torus T_2 of G_2, and vice versa.

- To define the genus of an algebraic group, we replace maximal subfields with *maximal tori* in the definition of genus of division algebra.

- Let G_1 and G_2 be semi-simple groups over a field K.

  G_1 & G_2 have *same isomorphism classes of maximal K-tori* **if** every maximal K-torus T_1 of G_1 is K-isomorphic to a maximal K-torus T_2 of G_2, and vice versa.

- Let G be an absolutely almost simple K-group.

- To define the genus of an algebraic group, we replace maximal subfields with *maximal tori* in the definition of genus of division algebra.

- Let G_1 and G_2 be semi-simple groups over a field K.

  G_1 & G_2 have *same isomorphism classes of maximal K-tori* **if** every maximal K-torus T_1 of G_1 is K-isomorphic to a maximal K-torus T_2 of G_2, and vice versa.

- Let G be an absolutely almost simple K-group.

  $\mathbf{gen}_K(G) =$ set of isomorphism classes of K-forms G' of G having same K-isomorphism classes of maximal K-tori.

Question 1′. *When does* $\mathbf{gen}_K(G)$ *reduce to a single element?*

Question 1′. *When does* $\mathbf{gen}_K(G)$ *reduce to a single element?*

Question 2′. *When is* $\mathbf{gen}_K(G)$ *finite?*

Question 1′. *When does* $\mathbf{gen}_K(G)$ *reduce to a single element?*

Question 2′. *When is* $\mathbf{gen}_K(G)$ *finite?*

Theorem 4 (Prasad-A.R.)

Let G *be an absolutely almost simple simply connected algebraic group over a number field* K.

Question 1′. *When does* $\mathbf{gen}_K(G)$ *reduce to a single element?*

Question 2′. *When is* $\mathbf{gen}_K(G)$ *finite?*

Theorem 4 (Prasad-A.R.)

Let G be an absolutely almost simple simply connected algebraic group over a number field K.

(1) $\mathbf{gen}_K(G)$ *is finite;*

Question 1′. *When does* $\mathbf{gen}_K(G)$ *reduce to a single element?*

Question 2′. *When is* $\mathbf{gen}_K(G)$ *finite?*

Theorem 4 (Prasad-A.R.)

Let G be an absolutely almost simple simply connected algebraic group over a number field K.

(1) $\mathbf{gen}_K(G)$ *is finite;*

(2) *If G is not of type* A_n, D_{2n+1} *or* E_6, *then* $|\mathbf{gen}_K(G)| = 1$.

Question 1′. *When does* $\mathbf{gen}_K(G)$ *reduce to a single element?*

Question 2′. *When is* $\mathbf{gen}_K(G)$ *finite?*

Theorem 4 (Prasad-A.R.)

Let G *be an absolutely almost simple simply connected algebraic group over a number field* K.

(1) $\mathbf{gen}_K(G)$ *is finite;*

(2) *If* G *is not of type* A_n, D_{2n+1} *or* E_6, *then* $|\mathbf{gen}_K(G)| = 1$.

Conjecture. (1) *For* $K = k(x)$, k *a number field, and* G *an absolutely almost simple simply connected* K*-group with* $|Z(G)| \leqslant 2$, *we have* $|\mathbf{gen}_K(G)| = 1$;

Question 1′. *When does* $\mathbf{gen}_K(G)$ *reduce to a single element?*

Question 2′. *When is* $\mathbf{gen}_K(G)$ *finite?*

Theorem 4 (Prasad-A.R.)

Let G be an absolutely almost simple simply connected algebraic group over a number field K.

(1) $\mathbf{gen}_K(G)$ *is finite;*

(2) *If G is not of type* A_n, D_{2n+1} *or* E_6, *then* $|\mathbf{gen}_K(G)| = 1$.

Conjecture. (1) *For* $K = k(x)$, *k a number field, and G an absolutely almost simple simply connected K-group with* $|Z(G)| \leqslant 2$, *we have* $|\mathbf{gen}_K(G)| = 1$;

(2) *If G is an absolutely almost simple group over a finitely generated field K of "good" characteristic then* $\mathbf{gen}_K(G)$ *is finite.*

- Results for division algebras do **not** automatically imply results for $G = \mathrm{SL}_{m,D}$.

- Results for division algebras do **not** automatically imply results for $G = \mathrm{SL}_{m,D}$.

Theorem 5.

- Results for division algebras do **not** automatically imply results for $G = \mathrm{SL}_{m,D}$.

Theorem 5.

(1) *Let D be a central division algebra of exponent 2 over $K = k(x_1, \ldots, x_r)$ where k is a number field or a finite field of characteristic $\neq 2$. Then for $G = \mathrm{SL}_{m,D}$ $(m \geqslant 1)$ we have $|\mathbf{gen}_K(G)| = 1$.*

- Results for division algebras do **not** automatically imply results for $G = \mathrm{SL}_{m,D}$.

Theorem 5.

(1) *Let* D *be a central division algebra of exponent* 2 *over* $K = k(x_1, \ldots, x_r)$ *where* k *is a number field or a finite field of characteristic* $\neq 2$. *Then for* $G = \mathrm{SL}_{m,D}$ $(m \geqslant 1)$ *we have* $|\mathbf{gen}_K(G)| = 1$.

(2) *Let* $G = \mathrm{SL}_{m,D}$, *where* D *is a central division algebra over a finitely generated field* K. *Then* $\mathbf{gen}_K(G)$ *is finite.*

Theorem 6.

Theorem 6.

Let $K = k(C)$ *where* C *is a geometrically integral smooth curve over a number field* k*,*

Theorem 6.

Let $K = k(C)$ where C is a geometrically integral smooth curve over a number field k, and let G be either

- $\mathrm{Spin}_n(q)$, *q a quadratic form over K and n is odd,*

Theorem 6.

Let $K = k(C)$ *where* C *is a geometrically integral smooth curve over a number field* k*, and let* G *be either*

- $\mathrm{Spin}_n(q)$, q *a quadratic form over* K *and* n *is* odd*, or*
- $\mathrm{SU}_n(h)$, h *hermitian form over a quadratic extension* L/K.

Theorem 6.

Let $K = k(C)$ *where* C *is a geometrically integral smooth curve over a number field* k*, and let* G *be either*

- $\mathrm{Spin}_n(q)$, q *a quadratic form over* K *and* n *is* *odd*, *or*
- $\mathrm{SU}_n(h)$, h *hermitian form over a quadratic extension* L/K.

Then $\mathbf{gen}_K(G)$ *is finite.*

Theorem 6.

Let $K = k(C)$ where C is a geometrically integral smooth curve over a number field k, and let G be either

- $\mathrm{Spin}_n(q)$, *q a quadratic form over K and n is <u>odd</u>, or*
- $\mathrm{SU}_n(h)$, *h hermitian form over a quadratic extension L/K.*

Then $\mathbf{gen}_K(G)$ is finite.

Theorem 7.

Let G be a simple algebraic group of type G_2.

Theorem 6.

Let $K = k(C)$ where C is a geometrically integral smooth curve over a number field k, and let G be either

- $\mathrm{Spin}_n(q)$, *q a quadratic form over K and n is odd, or*
- $\mathrm{SU}_n(h)$, *h hermitian form over a quadratic extension L/K.*

Then $\mathbf{gen}_K(G)$ is finite.

Theorem 7.

Let G be a simple algebraic group of type G_2.

(1) *If $K = k(x)$, where k is a number field, then $|\mathbf{gen}_K(G)| = 1$;*

Theorem 6.

Let $K = k(C)$ where C is a geometrically integral smooth curve over a number field k, and let G be either

- $\mathrm{Spin}_n(q)$, *q a quadratic form over K and n is odd, or*
- $\mathrm{SU}_n(h)$, *h hermitian form over a quadratic extension L/K.*

Then $\mathbf{gen}_K(G)$ is finite.

Theorem 7.

Let G be a simple algebraic group of type G_2.

(1) *If $K = k(x)$, where k is a number field, then $|\mathbf{gen}_K(G)| = 1$;*

(2) *If $K = k(x_1, \ldots, x_r)$ or $k(C)$, where k is a number field, then $\mathbf{gen}_K(G)$ is finite.*

- Adequate substitute for unramified algebras in case of algebraic groups is algebraic groups with good reduction.

- Adequate substitute for unramified algebras in case of algebraic groups is algebraic groups with good reduction.

Definition.

- Adequate substitute for unramified algebras in case of algebraic groups is algebraic groups with good reduction.

Definition.

Let G be an absolutely almost simple algebraic group over K, v be a discrete valuation of K.

- Adequate substitute for unramified algebras in case of algebraic groups is algebraic groups with good reduction.

Definition.

Let G be an absolutely almost simple algebraic group over K, v be a discrete valuation of K.

G has good reduction at v if there exists a reductive *group scheme $\mathcal{G}$ over valuation ring $\mathcal{O}_v \subset K_v$ with generic fibre*

$$G \times_K K_v.$$

- Adequate substitute for unramified algebras in case of algebraic groups is algebraic groups with good reduction.

Definition.

Let G be an absolutely almost simple algebraic group over K, v be a discrete valuation of K.

G has good reduction at v if there exists a reductive *group scheme $\mathcal{G}$ over valuation ring $\mathcal{O}_v \subset K_v$ with generic fibre*

$$G \times_K K_v.$$

Then special fiber (reduction)

$$\underline{G}^{(v)} = \mathcal{G} \otimes_{\mathcal{O}_v} K^{(v)}$$

is a connected simple group of same type as G.

Examples.

Examples.

0. Let G be a simply connected *split* K-group.

.

Examples.

0. Let G be a simply connected *split* K-group. Then G has good reduction at *any* discrete valuation v of K.

Examples.

0. Let G be a simply connected *split* K-group. Then G has good reduction at *any* discrete valuation v of K.
   (The $\mathcal{O}_v$-scheme $\mathcal{G}$ is provided by Chevalley construction.)

Examples.

0. Let G be a simply connected *split* K-group. Then G has good reduction at *any* discrete valuation v of K.
   (The $\mathcal{O}_v$-scheme $\mathcal{G}$ is provided by Chevalley construction.)

1. Let A be a central simple K-algebra.

Examples.

0. Let G be a simply connected *split* K-group. Then G has good reduction at *any* discrete valuation v of K.
   (The $\mathcal{O}_v$-scheme $\mathcal{G}$ is provided by Chevalley construction.)

1. Let A be a central simple K-algebra. Then $G = \mathrm{SL}_{1,A}$ has good reduction at v

Examples.

0. Let G be a simply connected *split* K-group. Then G has good reduction at *any* discrete valuation v of K.
   (The $\mathcal{O}_v$-scheme $\mathcal{G}$ is provided by Chevalley construction.)

1. Let A be a central simple K-algebra. Then $G = \mathrm{SL}_{1,A}$ has good reduction at v $\Leftrightarrow$

Examples.

0. Let G be a simply connected *split* K-group. Then G has good reduction at *any* discrete valuation v of K.
   (The $\mathcal{O}_v$-scheme $\mathcal{G}$ is provided by Chevalley construction.)

1. Let A be a central simple K-algebra. Then $G = \mathrm{SL}_{1,A}$ has good reduction at v $\Leftrightarrow$ there exists an Azumaya $\mathcal{O}_v$-algebra $\mathcal{A}$ such that

Examples.

0. Let G be a simply connected *split* K-group. Then G has good reduction at *any* discrete valuation v of K.
   (The $\mathcal{O}_v$-scheme $\mathcal{G}$ is provided by Chevalley construction.)

1. Let A be a central simple K-algebra. Then $G = \mathrm{SL}_{1,A}$ has good reduction at v $\Leftrightarrow$ there exists an Azumaya $\mathcal{O}_v$-algebra $\mathcal{A}$ such that

$$A \otimes_K K_v \simeq \mathcal{A} \otimes_{\mathcal{O}_v} K_v$$

Examples.

0. Let G be a simply connected *split* K-group. Then G has good reduction at *any* discrete valuation v of K.
   (The $\mathcal{O}_v$-scheme $\mathcal{G}$ is provided by Chevalley construction.)

1. Let A be a central simple K-algebra. Then $G = \mathrm{SL}_{1,A}$ has good reduction at v $\Leftrightarrow$ there exists an Azumaya $\mathcal{O}_v$-algebra $\mathcal{A}$ such that

$$A \otimes_K K_v \simeq \mathcal{A} \otimes_{\mathcal{O}_v} K_v$$

   (i. e., A is *unramified* at v)

Examples.

0. Let G be a simply connected *split* K-group. Then G has good reduction at *any* discrete valuation v of K.
(The $\mathcal{O}_v$-scheme $\mathcal{G}$ is provided by Chevalley construction.)

1. Let A be a central simple K-algebra. Then $G = \mathrm{SL}_{1,A}$ has good reduction at v $\Leftrightarrow$ there exists an Azumaya $\mathcal{O}_v$-algebra $\mathcal{A}$ such that
$$A \otimes_K K_v \simeq \mathcal{A} \otimes_{\mathcal{O}_v} K_v$$
(i. e., A is *unramified* at v)

2. Assume char $K^{(v)} \neq 2$.

Examples.

0. Let G be a simply connected *split* K-group. Then G has good reduction at *any* discrete valuation v of K.
   (The $\mathcal{O}_v$-scheme $\mathcal{G}$ is provided by Chevalley construction.)

1. Let A be a central simple K-algebra. Then $G = \mathrm{SL}_{1,A}$ has good reduction at v $\Leftrightarrow$ there exists an Azumaya $\mathcal{O}_v$-algebra $\mathcal{A}$ such that

$$A \otimes_K K_v \simeq \mathcal{A} \otimes_{\mathcal{O}_v} K_v$$

   (i. e., A is *unramified* at v)

2. Assume char $K^{(v)} \neq 2$. Then $G = \mathrm{Spin}_n(q)$ has good reduction at v

Examples.

0. Let G be a simply connected *split* K-group. Then G has good reduction at *any* discrete valuation v of K.
   (The $\mathcal{O}_v$-scheme $\mathcal{G}$ is provided by Chevalley construction.)

1. Let A be a central simple K-algebra. Then $G = \mathrm{SL}_{1,A}$ has good reduction at v $\Leftrightarrow$ there exists an Azumaya $\mathcal{O}_v$-algebra $\mathcal{A}$ such that
$$A \otimes_K K_v \simeq \mathcal{A} \otimes_{\mathcal{O}_v} K_v$$
   (i. e., A is *unramified* at v)

2. Assume char $K^{(v)} \neq 2$. Then $G = \mathrm{Spin}_n(q)$ has good reduction at v $\Leftrightarrow$

Examples.

0. Let G be a simply connected *split* K-group. Then G has good reduction at *any* discrete valuation v of K.
   (The $\mathcal{O}_v$-scheme $\mathcal{G}$ is provided by Chevalley construction.)

1. Let A be a central simple K-algebra. Then $G = \mathrm{SL}_{1,A}$ has good reduction at v $\Leftrightarrow$ there exists an Azumaya $\mathcal{O}_v$-algebra $\mathcal{A}$ such that
   $$A \otimes_K K_v \simeq \mathcal{A} \otimes_{\mathcal{O}_v} K_v$$
   (i. e., A is *unramified* at v)

2. Assume char $K^{(v)} \neq 2$. Then $G = \mathrm{Spin}_n(q)$ has good reduction at v $\Leftrightarrow$
   $$q \sim_{K_v} \lambda(u_1x_1^2 + \cdots + u_nx_n^2), \quad u_i \in \mathcal{O}_v^\times, \quad \lambda \in K_v.$$

Examples.

0. Let G be a simply connected *split* K-group. Then G has good reduction at *any* discrete valuation v of K.
   (The $\mathcal{O}_v$-scheme $\mathcal{G}$ is provided by Chevalley construction.)

1. Let A be a central simple K-algebra. Then $G = \mathrm{SL}_{1,A}$ has good reduction at v $\Leftrightarrow$ there exists an Azumaya $\mathcal{O}_v$-algebra $\mathcal{A}$ such that
$$A \otimes_K K_v \simeq \mathcal{A} \otimes_{\mathcal{O}_v} K_v$$
   (i. e., A is *unramified* at v)

2. Assume $\mathrm{char}\, K^{(v)} \neq 2$. Then $G = \mathrm{Spin}_n(q)$ has good reduction at v $\Leftrightarrow$
$$q \sim_{K_v} \lambda(u_1x_1^2 + \cdots + u_nx_n^2), \quad u_i \in \mathcal{O}_v^\times, \quad \lambda \in K_v.$$

A. R., I. Rapinchuk, *Linear algebraic groups with good reduction*, arXiv: 2005.05484

Theorem 8.

Let G be an absolutely almost simple simply connected group over K, and v be a discrete valuation of K.

Theorem 8.

Let G be an absolutely almost simple simply connected group over K, and v be a discrete valuation of K.

Assume that $K^{(v)}$ is finitely generated*, and G has good reduction at v.*

Theorem 8.

Let G be an absolutely almost simple simply connected group over K, and v be a discrete valuation of K.

Assume that $K^{(v)}$ is finitely generated*, and G has good reduction at v.*

Then *every* $G' \in \mathbf{gen}_K(G)$ *has good reduction at v.*

Theorem 8.

Let G be an absolutely almost simple simply connected group over K, and v be a discrete valuation of K.

Assume that $K^{(v)}$ is finitely generated, *and G has good reduction at v.*

Then *every* $G' \in \mathbf{gen}_K(G)$ *has good reduction at v.*

For applications to genus problem assume K is equipped with a set V of discrete valuations satisfying:

Theorem 8.

Let G be an absolutely almost simple simply connected group over K, and v be a discrete valuation of K.

Assume that $K^{(v)}$ is finitely generated, *and G has good reduction at v.*

Then *every* $G' \in \mathbf{gen}_K(G)$ *has good reduction at v.*

For applications to genus problem assume K is equipped with a set V of discrete valuations satisfying:

(I) for any $a \in K^{\times}$, set $V(a) := \{v \in V \mid v(a) \neq 0\}$ is finite;

Theorem 8.

Let G be an absolutely almost simple simply connected group over K, and v be a discrete valuation of K.

Assume that $K^{(v)}$ is finitely generated, *and G has good reduction at v.*

Then *every* $G' \in \mathbf{gen}_K(G)$ *has good reduction at v.*

For applications to genus problem assume K is equipped with a set V of discrete valuations satisfying:

(I) for any $a \in K^{\times}$, set $V(a) := \{v \in V \mid v(a) \neq 0\}$ is finite;

(II) for every $v \in V$, residue field $K^{(v)}$ is finitely generated.

Corollary

Let G be an absolutely almost simple simply connected K-group.

Corollary

Let G be an absolutely almost simple simply connected K-group. There exists a finite *subset $S \subset V$ (depending on G) such that <u>every</u> $G' \in \mathbf{gen}_K(G)$ has good reduction at <u>all</u> $v \in V \setminus S$.*

Corollary

Let G be an absolutely almost simple simply connected K-group. There exists a finite *subset $S \subset V$ (depending on G) such that every $G' \in \mathbf{gen}_K(G)$ has good reduction at all $v \in V \setminus S$.*

So, question about finiteness of $\mathbf{gen}_K(G)$ reduces to

Corollary

Let G be an absolutely almost simple simply connected K-group. There exists a finite *subset $S \subset V$ (depending on G) such that every $G' \in \mathbf{gen}_K(G)$ has good reduction at all $v \in V \setminus S$.*

So, question about finiteness of $\mathbf{gen}_K(G)$ reduces to

Problem

Corollary

Let G be an absolutely almost simple simply connected K-group. There exists a finite *subset $S \subset V$ (depending on G) such that every $G' \in \mathbf{gen}_K(G)$ has good reduction at all $v \in V \setminus S$.*

So, question about finiteness of $\mathbf{gen}_K(G)$ reduces to

Problem

Let K be a finitely generated field.

Corollary

Let G be an absolutely almost simple simply connected K-group. There exists a finite *subset $S \subset V$ (depending on G) such that* <u>*every*</u> *$G' \in \mathbf{gen}_K(G)$ has good reduction at* <u>*all*</u> *$v \in V \setminus S$.*

So, question about finiteness of $\mathbf{gen}_K(G)$ reduces to

Problem

Let K be a finitely generated field. Can one equip K with a set V of discrete valuations that satisfies (I) and (II) *and also*

Corollary

Let G be an absolutely almost simple simply connected K-group. There exists a finite *subset $S \subset V$ (depending on G) such that every $G' \in \mathbf{gen}_K(G)$ has good reduction at all $v \in V \setminus S$.*

So, question about finiteness of $\mathbf{gen}_K(G)$ reduces to

Problem

Let K be a finitely generated field. Can one equip K with a set V of discrete valuations that satisfies (I) and (II) *and also*

(Φ) *For any finite subset $S \subset V$, set of K-isomorphism classes of K-forms G' of G having good reduction at all $v \in V \setminus S$ is finite?*

Corollary

Let G be an absolutely almost simple simply connected K-group. There exists a finite *subset $S \subset V$ (depending on G) such that every $G' \in \mathbf{gen}_K(G)$ has good reduction at all $v \in V \setminus S$.*

So, question about finiteness of $\mathbf{gen}_K(G)$ reduces to

Problem

Let K be a finitely generated field. Can one equip K with a set V of discrete valuations that satisfies (I) and (II) *and also*

(Φ) *For any finite subset $S \subset V$, set of K-isomorphism classes of K-forms G' of G having good reduction at all $v \in V \setminus S$ is finite?*

Variations: one can consider only groups of a specific type, only inner forms, impose restrictions on characteristic etc.

Every finitely generated field K can be equipped with an *almost canonical* set V of discrete valuations called divisorial.

Every finitely generated field K can be equipped with an *almost canonical* set V of discrete valuations called divisorial.

Valuation $v \in V$ correspond to prime divisors on a *model* $\mathfrak{X}$ for K

Every finitely generated field K can be equipped with an *almost canonical* set V of discrete valuations called divisorial.

Valuation $v \in V$ correspond to prime divisors on a *model* $\mathfrak{X}$ for K (normal separated scheme of finite type over $\mathbb{Z}$)

Every finitely generated field K can be equipped with an *almost canonical* set V of discrete valuations called divisorial.

Valuation $v \in V$ correspond to prime divisors on a *model* $\mathfrak{X}$ for K (normal separated scheme of finite type over $\mathbb{Z}$)

Example

Every finitely generated field K can be equipped with an *almost canonical* set V of discrete valuations called divisorial.

Valuation $v \in V$ correspond to prime divisors on a *model* $\mathfrak{X}$ for K (normal separated scheme of finite type over $\mathbb{Z}$)

Example

For $K = \mathbb{Q}(x)$ one can take $\mathfrak{X} = \operatorname{Spec} \mathbb{Z}[x]$.

Every finitely generated field K can be equipped with an *almost canonical* set V of discrete valuations called divisorial.

Valuation $v \in V$ correspond to prime divisors on a *model* $\mathfrak{X}$ for K (normal separated scheme of finite type over $\mathbb{Z}$)

Example

For $K = \mathbb{Q}(x)$ one can take $\mathfrak{X} = \operatorname{Spec} \mathbb{Z}[x]$. Then

$$V = V_0 \cup V_1$$

Every finitely generated field K can be equipped with an *almost canonical* set V of discrete valuations called divisorial.

Valuation $v \in V$ correspond to prime divisors on a *model* $\mathfrak{X}$ for K (normal separated scheme of finite type over $\mathbb{Z}$)

Example

For $K = \mathbb{Q}(x)$ one can take $\mathfrak{X} = \operatorname{Spec} \mathbb{Z}[x]$. Then

$$V = V_0 \cup V_1$$

where valuations in V_0 correspond to rational primes,

Every finitely generated field K can be equipped with an *almost canonical* set V of discrete valuations called divisorial.

Valuation $v \in V$ correspond to prime divisors on a *model* $\mathfrak{X}$ for K (normal separated scheme of finite type over $\mathbb{Z}$)

Example

For $K = \mathbb{Q}(x)$ one can take $\mathfrak{X} = \operatorname{Spec} \mathbb{Z}[x]$. Then

$$V = V_0 \cup V_1$$

where valuations in V_0 correspond to rational primes, and valuations in V_1 to irreducible polynomials $p(x) \in \mathbb{Z}[x]$ with content 1.

Every finitely generated field K can be equipped with an *almost canonical* set V of discrete valuations called divisorial.

Valuation $v \in V$ correspond to prime divisors on a *model* $\mathfrak{X}$ for K (normal separated scheme of finite type over $\mathbb{Z}$)

Example

For $K = \mathbb{Q}(x)$ one can take $\mathfrak{X} = \operatorname{Spec} \mathbb{Z}[x]$. Then

$$V = V_0 \cup V_1$$

where valuations in V_0 correspond to rational primes, and valuations in V_1 to irreducible polynomials $p(x) \in \mathbb{Z}[x]$ with content 1.

One expects that divisorial sets are as required.

Finiteness Conjecture

Finiteness Conjecture

Let G be a reductive algebraic group over a finitely generated *field K, and let V be a* divisorial set *of discrete valuations of K.*

Finiteness Conjecture

Let G be a reductive algebraic group over a finitely generated *field K, and let V be a* divisorial set *of discrete valuations of K.*

Then *set of K-isomorphism classes of (inner) K-forms of G that have good reduction at all $v \in V$ is finite*

Finiteness Conjecture

Let G be a reductive algebraic group over a finitely generated *field K, and let V be a* divisorial set *of discrete valuations of K.*

Then *set of K-isomorphism classes of (inner) K-forms of G that have good reduction at all $v \in V$ is finite (at least when* $\mathrm{char}\, K$ *is "good.")*

Finiteness Conjecture

Let G be a reductive algebraic group over a finitely generated *field K, and let V be a* divisorial set *of discrete valuations of K.*

Then *set of K-isomorphism classes of (inner) K-forms of G that have good reduction at all $v \in V$ is finite (at least when* $\mathrm{char}\, K$ *is "good.")*

For G absolutely almost simple, $p = \mathrm{char}\, K$ is "good" if either $p = 0$ or p does not divide order of Weyl group;

Finiteness Conjecture

Let G be a reductive algebraic group over a finitely generated *field K, and let V be a* divisorial set *of discrete valuations of K.*

Then *set of K-isomorphism classes of (inner) K-forms of G that have good reduction at all $v \in V$ is finite (at least when* $\mathrm{char}\, K$ *is "good.")*

For G absolutely almost simple, $p = \mathrm{char}\, K$ is "good" if either $p = 0$ or p does not divide order of Weyl group; for G non-semi-simple, only $p = 0$ is "good."

Finiteness Conjecture

Let G be a reductive algebraic group over a finitely generated *field K, and let V be a* divisorial set *of discrete valuations of K.*

Then *set of K-isomorphism classes of (inner) K-forms of G that have good reduction at all $v \in V$ is finite (at least when* $\mathrm{char}\, K$ *is "good.")*

For G absolutely almost simple, $p = \mathrm{char}\, K$ is "good" if either $p = 0$ or p does not divide order of Weyl group; for G non-semi-simple, only $p = 0$ is "good."

This conjecture can be viewed as an analog of Shafarevich's Conjecture for abelian varieties proved by Faltings,

Finiteness Conjecture

Let G be a reductive algebraic group over a finitely generated *field K, and let V be a* divisorial set *of discrete valuations of K.*

Then *set of K-isomorphism classes of (inner) K-forms of G that have good reduction at all $v \in V$ is finite (at least when* $\mathrm{char}\, K$ *is "good.")*

For G absolutely almost simple, $p = \mathrm{char}\, K$ is "good" if either $p = 0$ or p does not divide order of Weyl group; for G non-semi-simple, only $p = 0$ is "good."

This conjecture can be viewed as an analog of Shafarevich's Conjecture for abelian varieties proved by Faltings, but presents a different set of challenges.

- Finiteness Conjecture $\Rightarrow$ finiteness of $\mathbf{gen}_K(G)$ in "good" characteristic

- Finiteness Conjecture $\Rightarrow$ finiteness of $\mathbf{gen}_K(G)$ in "good" characteristic
- Consider

$$\iota_{G,V}\colon H^1(K,G) \longrightarrow \prod_{v\in V} H^1(K_v,G).$$

- Finiteness Conjecture $\Rightarrow$ finiteness of $\mathbf{gen}_K(G)$ in "good" characteristic

- Consider

$$\iota_{G,V} \colon H^1(K,G) \longrightarrow \prod_{v \in V} H^1(K_v, G).$$

Finiteness Conjecture for a semi-simple G $\Rightarrow$ $\iota_{\overline{G},V}$ is proper

- Finiteness Conjecture $\Rightarrow$ finiteness of $\mathbf{gen}_K(G)$ in "good" characteristic

- Consider

$$\iota_{G,V} \colon H^1(K,G) \longrightarrow \prod_{v \in V} H^1(K_v, G).$$

Finiteness Conjecture for a semi-simple $G \Rightarrow \iota_{\overline{G},V}$ is proper (preimage of a finite set is finite) for adjoint $\overline{G}$.

- Finiteness Conjecture $\Rightarrow$ finiteness of $\mathbf{gen}_K(G)$ in "good" characteristic

- Consider

$$\iota_{G,V} \colon H^1(K, G) \longrightarrow \prod_{v \in V} H^1(K_v, G).$$

Finiteness Conjecture for a semi-simple $G \Rightarrow \iota_{\overline{G},V}$ is proper (preimage of a finite set is finite) for adjoint $\overline{G}$. In particular,

$$Ш(\overline{G}, V) \ := \ \mathrm{Ker}\, \iota_{\overline{G},V} \ \text{ is } \ \text{finite.}$$

- Finiteness Conjecture $\Rightarrow$ finiteness of $\mathbf{gen}_K(G)$ in "good" characteristic
- Consider

$$\iota_{G,V} \colon H^1(K,G) \longrightarrow \prod_{v \in V} H^1(K_v, G).$$

Finiteness Conjecture for a semi-simple G $\Rightarrow$ $\iota_{\overline{G},V}$ is proper (preimage of a finite set is finite) for adjoint $\overline{G}$. In particular,

$$Ш(\overline{G}, V) \ := \ \mathrm{Ker}\, \iota_{\overline{G},V} \ \ \text{is} \ \ \text{finite.}$$

One expects $\iota_{G,V}$ to be proper for **all** reductive G.

- Finiteness Conjecture $\Rightarrow$ finiteness of $\mathbf{gen}_K(G)$ in "good" characteristic
- Consider

$$\iota_{G,V} \colon H^1(K, G) \longrightarrow \prod_{v \in V} H^1(K_v, G).$$

Finiteness Conjecture for a semi-simple G $\Rightarrow$ $\iota_{\overline{G},V}$ is proper (preimage of a finite set is finite) for adjoint $\overline{G}$. In particular,

$$\text{Ш}(\overline{G}, V) \ := \ \mathrm{Ker}\, \iota_{\overline{G},V} \ \text{ is finite.}$$

One expects $\iota_{G,V}$ to be proper for **all** reductive G.

These and other connections shift focus of current work to Finiteness Conjecture.

- Finiteness Conjecture and properness of $\iota_{G,V}$ is known for **all** tori

- Finiteness Conjecture and properness of $\iota_{G,V}$ is known for **all** tori

- Finiteness Conjecture is known for inner forms of type A_{n-1} over all finitely generated K provided that $(\mathrm{char}\, K\,,\ n) = 1$

- Finiteness Conjecture and properness of $\iota_{G,V}$ is known for **all** tori

- Finiteness Conjecture is known for inner forms of type A_{n-1} over all finitely generated K provided that $(\mathrm{char}\, K\,,\, n) = 1 \;\Rightarrow\; \iota_{\mathrm{PSL}_n, V}$ is proper

- Finiteness Conjecture and properness of $\iota_{G,V}$ is known for **all** tori

- Finiteness Conjecture is known for inner forms of type A_{n-1} over all finitely generated K provided that $(\mathrm{char}\, K\,,\ n) = 1 \ \Rightarrow\ \iota_{\mathrm{PSL}_n,V}$ is proper

- Over $K = k(C)$, function field of an irreducible curve C over a number field k, Finiteness Conjecture is known for spinor groups, some unitary groups and groups of type G_2

- Finiteness Conjecture and properness of $\iota_{G,V}$ is known for **all** tori

- Finiteness Conjecture is known for inner forms of type A_{n-1} over all finitely generated K provided that $(\mathrm{char}\, K\,,\ n) = 1 \ \Rightarrow\ \iota_{\mathrm{PSL}_n, V}$ is proper

- Over $K = k(C)$, function field of an irreducible curve C over a number field k, Finiteness Conjecture is known for spinor groups, some unitary groups and groups of type G_2

General case is work in progress ...

- Finiteness Conjecture and properness of $\iota_{G,V}$ is known for **all** tori

- Finiteness Conjecture is known for inner forms of type A_{n-1} over all finitely generated K provided that $(\mathrm{char}\, K\,,\, n) = 1 \;\Rightarrow\; \iota_{\mathrm{PSL}_n, V}$ is proper

- Over $K = k(C)$, function field of an irreducible curve C over a number field k, Finiteness Conjecture is known for spinor groups, some unitary groups and groups of type G_2

General case is work in progress ...

Finiteness of $\mathbf{gen}_K(G)$ is derived from results on Finiteness Conjecture

Challenges:

Challenges:

Results for $G = \mathrm{SL}_{1,A}$ are based on (known) finiteness of

$$ {}_n\mathrm{Br}(K)_V \; = \; H^2(K, \mu_n)_V \quad \text{(unramified cohomology)} $$

Challenges:

Results for $G = \mathrm{SL}_{1,A}$ are based on (known) finiteness of

$$ {}_n\mathrm{Br}(K)_V = H^2(K, \mu_n)_V \quad \text{(unramified cohomology)} $$

where μ_n is group of nth roots of unity (assuming that n is prime to char K)

Challenges:

Results for $G = \mathrm{SL}_{1,A}$ are based on (known) finiteness of

$${}_n\mathrm{Br}(K)_V = H^2(K, \mu_n)_V \quad \text{(unramified cohomology)}$$

where μ_n is group of nth roots of unity (assuming that n is prime to char K)

Consideration of type G_2 requires finiteness of $H^3(K, \mu_2)_V$,

Challenges:

Results for $G = \mathrm{SL}_{1,A}$ are based on (known) finiteness of

$$ {}_n\mathrm{Br}(K)_V = H^2(K, \mu_n)_V \quad \text{(unramified cohomology)} $$

where μ_n is group of nth roots of unity (assuming that n is prime to char K)

Consideration of type G_2 requires finiteness of $H^3(K, \mu_2)_V$, and of spinor groups — finiteness of

$$ H^i(K, \mu_2)_V \quad \text{for} \quad i = 1, \ldots, [\log_2 n] + 1. $$

Challenges:

Results for $G = \mathrm{SL}_{1,A}$ are based on (known) finiteness of

$$ {}_n\mathrm{Br}(K)_V = H^2(K, \mu_n)_V \ \ \text{(unramified cohomology)} $$

where μ_n is group of nth roots of unity (assuming that n is prime to char K)

Consideration of type G_2 requires finiteness of $H^3(K, \mu_2)_V$, and of spinor groups — finiteness of

$$ H^i(K, \mu_2)_V \ \ \text{for} \ \ i = 1, \ldots, [\log_2 n] + 1. $$

However, proving finiteness of unramified cohomology groups $H^i(K, \mu_n^{\otimes j})_V$ is a very difficult problem for $i \geqslant 3$ resolved only is special cases!

At the same time, a precise description of K-forms in terms of *commutative Galois cohomology* is available only for certain types.

At the same time, a precise description of K-forms in terms of *commutative Galois cohomology* is available only for certain types.

So, to handle general case in Finiteness Conjecture one will need to develop an *intrinsic approach* to analysis of forms with good reduction.

Question. *What happens to genus under base change?*

Question. *What happens to genus under base change?*

Using (ABHN), one can construct a *cubic* division algebra D over $\mathbb{Q}$ and finite extensions F_1 and F_2 of $\mathbb{Q}$ such that

$$|\,\mathbf{gen}(D \otimes_K F_1)\,| \ < |\,\mathbf{gen}(D)\,| \ < \ |\,\mathbf{gen}(D \otimes_K F_2)\,|.$$

Question. *What happens to genus under base change?*

Using (ABHN), one can construct a *cubic* division algebra D over $\mathbb{Q}$ and finite extensions F_1 and F_2 of $\mathbb{Q}$ such that

$$|\,\mathbf{gen}(D \otimes_K F_1)\,| \;<\; |\,\mathbf{gen}(D)\,| \;<\; |\,\mathbf{gen}(D \otimes_K F_2)\,|.$$

Similar examples can be constructed for algebraic groups.

Question. *What happens to genus under base change?*

Using (ABHN), one can construct a *cubic* division algebra D over $\mathbb{Q}$ and finite extensions F_1 and F_2 of $\mathbb{Q}$ such that

$$|\,\mathbf{gen}(D \otimes_K F_1)\,| \ < |\,\mathbf{gen}(D)\,| \ < \ |\,\mathbf{gen}(D \otimes_K F_2)\,|.$$

Similar examples can be constructed for algebraic groups.

So, there appears to be **no** regularity for *finite* extensions of base field.

Question. *What happens to genus under base change?*

Using (ABHN), one can construct a *cubic* division algebra D over $\mathbb{Q}$ and finite extensions F_1 and F_2 of $\mathbb{Q}$ such that

$$|\,\mathbf{gen}(D \otimes_K F_1)\,| \; < |\,\mathbf{gen}(D)\,| \; < \; |\,\mathbf{gen}(D \otimes_K F_2)\,|.$$

Similar examples can be constructed for algebraic groups.

So, there appears to be **no** regularity for *finite* extensions of base field.

However, for *purely transcendental* extensions we have:

Theorem 9

Theorem 9

Let D be a central division algebra over a field K.

Theorem 9

Let D be a central division algebra over a field K. Then for any $\Delta \in \mathbf{gen}(D \otimes_K K(x))$ there exists $\Delta_0 \in \mathbf{gen}(D)$ such that

$$\Delta = \Delta_0 \otimes_K K(x).$$

Theorem 9

Let D be a central division algebra over a field K. Then for any $\Delta \in \mathbf{gen}(D \otimes_K K(x))$ there exists $\Delta_0 \in \mathbf{gen}(D)$ such that

$$\Delta = \Delta_0 \otimes_K K(x).$$

Theorem 9′

Theorem 9

Let D be a central division algebra over a field K. Then for any $\Delta \in \mathbf{gen}(D \otimes_K K(x))$ *there exists* $\Delta_0 \in \mathbf{gen}(D)$ *such that*

$$\Delta = \Delta_0 \otimes_K K(x).$$

Theorem 9′

Let G be an absolutely almost simple simply connected algebraic group over a finitely generated *field K.*

Theorem 9

Let D be a central division algebra over a field K. Then for any $\Delta \in \mathbf{gen}(D \otimes_K K(x))$ there exists $\Delta_0 \in \mathbf{gen}(D)$ such that

$$\Delta = \Delta_0 \otimes_K K(x).$$

Theorem 9′

Let G be an absolutely almost simple simply connected algebraic group over a finitely generated *field K. Then any*

$$H \in \mathbf{gen}_{K(x)}(G \times_K K(x))$$

is of the form $H = H_0 \times_K K(x)$ for some $H_0 \in \mathbf{gen}_K(G)$.

Theorem 9

Let D be a central division algebra over a field K. Then for any $\Delta \in \mathbf{gen}(D \otimes_K K(x))$ there exists $\Delta_0 \in \mathbf{gen}(D)$ such that

$$\Delta = \Delta_0 \otimes_K K(x).$$

Theorem 9′

Let G be an absolutely almost simple simply connected algebraic group over a finitely generated *field K. Then any*

$$H \in \mathbf{gen}_{K(x)}(G \times_K K(x))$$

is of the form $H = H_0 \times_K K(x)$ for some $H_0 \in \mathbf{gen}_K(G)$.

Corollary

Theorem 9

Let D be a central division algebra over a field K. Then for any $\Delta \in \mathbf{gen}(D \otimes_K K(x))$ there exists $\Delta_0 \in \mathbf{gen}(D)$ such that

$$\Delta = \Delta_0 \otimes_K K(x).$$

Theorem 9′

Let G be an absolutely almost simple simply connected algebraic group over a finitely generated *field K. Then any*

$$H \in \mathbf{gen}_{K(x)}(G \times_K K(x))$$

is of the form $H = H_0 \times_K K(x)$ for some $H_0 \in \mathbf{gen}_K(G)$.

Corollary

In above notations, if $\mathbf{gen}_K(G)$ is finite, then so is

$$\mathbf{gen}_{K(x)}(G \times_K K(x)).$$

Theorem 9

Let D be a central division algebra over a field K. Then for any $\Delta \in \mathbf{gen}(D \otimes_K K(x))$ there exists $\Delta_0 \in \mathbf{gen}(D)$ such that

$$\Delta = \Delta_0 \otimes_K K(x).$$

Theorem 9′

Let G be an absolutely almost simple simply connected algebraic group over a finitely generated *field K. Then any*

$$H \in \mathbf{gen}_{K(x)}(G \times_K K(x))$$

is of the form $H = H_0 \times_K K(x)$ for some $H_0 \in \mathbf{gen}_K(G)$.

Corollary

In above notations, if $\mathbf{gen}_K(G)$ is finite, then so is

$$\mathbf{gen}_{K(x)}(G \times_K K(x)).$$

In particular, the latter is finite if K is a number field.

Both Theorems and Corollary remain valid over $K(x_1, \ldots, x_r)$.

Both Theorems and Corollary remain valid over $K(x_1, \ldots, x_r)$.

Does genus remain same or shrink?

Both Theorems and Corollary remain valid over $K(x_1, \ldots, x_r)$.

Does genus remain same or shrink?

We are not saying that

Both Theorems and Corollary remain valid over $K(x_1, \ldots, x_r)$.

Does genus remain same or shrink?

We are not saying that

if $\Delta_0 \in \mathbf{gen}(D)$ then $\Delta_0 \otimes_K K(x_1, \ldots, x_r) \in \mathbf{gen}(\Delta_0 \otimes_K K(x_1, \ldots, x_r))$.

Both Theorems and Corollary remain valid over $K(x_1, \ldots, x_r)$.

Does genus remain same or shrink?

We are not saying that

if $\Delta_0 \in \mathbf{gen}(D)$ then $\Delta_0 \otimes_K K(x_1, \ldots, x_r) \in \mathbf{gen}(\Delta_0 \otimes_K K(x_1, \ldots, x_r))$.

In fact, the *opposite* is true:

Both Theorems and Corollary remain valid over $K(x_1, \ldots, x_r)$.

Does genus remain same or shrink?

We are not saying that

if $\Delta_0 \in \mathbf{gen}(D)$ then $\Delta_0 \otimes_K K(x_1, \ldots, x_r) \in \mathbf{gen}(\Delta_0 \otimes_K K(x_1, \ldots, x_r))$.

In fact, the *opposite* is true:

Theorem 10

Both Theorems and Corollary remain valid over $K(x_1, \ldots, x_r)$.

Does genus remain same or shrink?

We are not saying that

if $\Delta_0 \in \mathbf{gen}(D)$ then $\Delta_0 \otimes_K K(x_1, \ldots, x_r) \in \mathbf{gen}(\Delta_0 \otimes_K K(x_1, \ldots, x_r))$.

In fact, the *opposite* is true:

Theorem 10

Let D be a central division algebra of degree n over a field K.

Both Theorems and Corollary remain valid over $K(x_1, \ldots, x_r)$.

Does genus remain same or shrink?

We are not saying that

if $\Delta_0 \in \mathbf{gen}(D)$ then $\Delta_0 \otimes_K K(x_1, \ldots, x_r) \in \mathbf{gen}(\Delta_0 \otimes_K K(x_1, \ldots, x_r))$.

In fact, the *opposite* is true:

Theorem 10

Let D be a central division algebra of degree n over a field K. Then

$$\mathbf{gen}(D \otimes_K K(x_1, \ldots, x_{n-1}))$$

consists of $D' \otimes_K K(x_1, \ldots, x_{n-1})$ where $\langle [D'] \rangle = \langle [D] \rangle$ in $\mathrm{Br}(K)$.

So,

So,

- After base change $K(x_1, \ldots, x_{n-1})/K$, genus becomes *finite* and *"minimal possible"*;

So,

- After base change $K(x_1, \ldots, x_{n-1})/K$, genus becomes *finite* and *"minimal possible"*;
- If D is a quaternion algebra, then $|\,\mathbf{gen}(D \otimes_K K(x))\,| = 1$.

So,

- After base change $K(x_1, \ldots, x_{n-1})/K$, genus becomes *finite* and *"minimal possible"*;
- If D is a quaternion algebra, then $|\,\mathbf{gen}(D \otimes_K K(x))\,| = 1$.

Informally: genus of D can be "killed" by a purely transcendental extension.

So,

- After base change $K(x_1, \ldots, x_{n-1})/K$, genus becomes *finite* and *"minimal possible"*;
- If D is a quaternion algebra, then $|\,\mathbf{gen}(D \otimes_K K(x))\,| = 1$.

Informally: genus of D can be "killed" by a purely transcendental extension.

Proofs use Amitsur's Theorem and results of Saltman and Matzri on function fields of Severi-Brauer varieties.

So,

- After base change $K(x_1, \ldots, x_{n-1})/K$, genus becomes *finite* and *"minimal possible"*;
- If D is a quaternion algebra, then $|\,\mathbf{gen}(D \otimes_K K(x))\,| = 1$.

Informally: genus of D can be "killed" by a purely transcendental extension.

Proofs use Amitsur's Theorem and results of Saltman and Matzri on function fields of Severi-Brauer varieties.

Can one extend this phenomenon to all simple algebraic groups?

We have analog of Theorem 10 for groups $G = \mathrm{SL}_{1,A}$.

We have analog of Theorem 10 for groups $G = \mathrm{SL}_{1,A}$.

We also have the following:

We have analog of Theorem 10 for groups $G = \mathrm{SL}_{1,A}$.

We also have the following:

Theorem 11

We have analog of Theorem 10 for groups $G = \mathrm{SL}_{1,A}$.

We also have the following:

Theorem 11

Let G be a group of type G_2 over a finitely generated field K of characteristic $\neq 2, 3$.

We have analog of Theorem 10 for groups $G = \mathrm{SL}_{1,A}$.

We also have the following:

Theorem 11

Let G be a group of type G_2 over a finitely generated field K of characteristic $\neq 2, 3$. Then

$$|\,\mathbf{gen}_{K(x_1,\ldots,x_6)}(G \times_K K(x_1,\ldots,x_6))\,| = 1.$$

We have analog of Theorem 10 for groups $G = \mathrm{SL}_{1,A}$.

We also have the following:

Theorem 11

Let G be a group of type G_2 over a finitely generated field K of characteristic $\neq 2, 3$. Then

$$|\, \mathbf{gen}_{K(x_1,\ldots,x_6)}(G \times_K K(x_1, \ldots, x_6))\, | = 1.$$

Proof uses properties of Pfister forms.

We have analog of Theorem 10 for groups $G = \mathrm{SL}_{1,A}$.

We also have the following:

Theorem 11

Let G be a group of type G_2 over a finitely generated field K of characteristic $\neq 2, 3$. Then

$$|\,\mathbf{gen}_{K(x_1,\ldots,x_6)}(G \times_K K(x_1,\ldots,x_6))\,| = 1.$$

Proof uses properties of Pfister forms.

General case remains open.