ON THE HOMOGENIZATION OF OSCILLATORY SOLUTIONS TO NONLINEAR CONVECTION-DIFFUSION EQUATIONS[1]

Eitan Tadmor and Tamir Tassa

Abstract

We study the behavior of oscillatory solutions to convection-diffusion problems, subject to initial and forcing data with modulated oscillations. We *quantify* the weak convergence in $W^{-1,\infty}$ to the 'expected' averages and obtain a sharp $W^{-1,\infty}$-convergence rate of order $\mathcal{O}(\varepsilon)$ – the small scale of the modulated oscillations. Moreover, in case the solution operator of the equation is compact, this weak convergence is translated into a strong one. Examples include nonlinear conservation laws, equations with nonlinear degenerate diffusion, etc. In this context, we show how the regularizing effect built-in such compact cases smoothes out initial oscillations and, in particular, outpaces the persisting generation of oscillations due to the source term. This yields a precise description of the weakly convergent initial layer which filters out the initial oscillations and enables the strong convergence in later times.

In memory of Haim Nessyahu, a dearest friend and research colleague.

1 Introduction

In this paper we study the behavior of oscillatory solutions for equations of the form

$$u_t = K(u, u_x)_x + h(x,t), \qquad (x,t) \in \mathbb{R} \times \mathbb{R}^+ \ , \tag{1.1}$$

where $K = K(u,p)$, is a nondecreasing function in $p := u_x$,

$$K_p \geq 0 \quad \forall (u,p) \ . \tag{1.2}$$

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This large family includes equations which mix both types – hyperbolic equations dominated by purely convective terms ($K_p \equiv 0$), or, parabolic equations dominated by possibly degenerate diffusive terms ($K_p \geq 0$). Due to the possible degeneracy, weak entropy solutions are sought; i.e., $u = \lim_{\delta \downarrow 0} u^\delta$, where u^δ is the classical solution which corresponds to $K^\delta = K + \delta p$.

We are concerned with the initial value problem for (1.1) where the initial data, $u_0^\varepsilon(x)$, and the forcing data, $h^\varepsilon(x,t)$, are subject to modulated oscillations. Specifically, we are interested in the behavior of u^ε, the entropy solution of

$$u_t^\varepsilon = K(u^\varepsilon, u_x^\varepsilon)_x + h^\varepsilon(x,t), \qquad u^\varepsilon(x,0) = u_0^\varepsilon(x), \tag{1.3}$$

where the modulation of the initial and forcing data takes the form

$$u_0^\varepsilon(x) = u_0(x, \frac{x}{\varepsilon}), \qquad h^\varepsilon(x,t) = \frac{1}{\varepsilon^\lambda} h(x, \frac{x}{\varepsilon}, t) \ , \qquad \text{fixed } \lambda \in [0,1), \quad \varepsilon \downarrow 0. \tag{1.4}$$

<u>Assumptions</u>.
{i} **smoothness**. The data, u_0 and h, are assumed to have a minimal *necessary* amount of smoothness. Thus, throughout the paper we assume $u_0(x,y) \in BV_x(\Omega \times [0,1])$ and $h(x,y,t) \in BV_x(\Omega(t) \times [0,1])$, where $\Omega, \Omega(t)$ denote bounded intervals in $\mathbb{R}_x$, and $BV_x(\Omega \times [0,1])$ denotes the space of all bounded functions which are 1-periodic in y, have a bounded variation in x and are constant for $x \notin \Omega$ (the last assumption covers the case of compactly supported data).
{ii} **compatibility**. There holds

$$\lambda \cdot \bar{h}(x,t) \equiv 0, \qquad \bar{h}(x,t) = \int_0^1 h(x,y,t)dy.$$

Thus, in the case of 'amplified' modulation ($\lambda > 0$), the average $\bar{h}(x,t)$ is assumed to vanish – a necessary compatibility requirement for the convergence statements stated below.

As $\varepsilon \downarrow 0$, $u_0^\varepsilon(x)$ and $h^\varepsilon(x,t)$ approach the corresponding averages,

$$u_0^\varepsilon(x) \rightharpoonup \bar{u}_0(x) := \int_0^1 u_0(x,y)dy, \qquad h^\varepsilon(x,t) \rightharpoonup \bar{h}(x,t) := \int_0^1 h(x,y,t)dy \ .$$

Note that this convergence statement (and similarly, the ones that follow), makes sense for $\lambda > 0$ only when $\bar{h}(x,t) \equiv 0$. Then, the entropy solution, $u^\varepsilon(x,t)$, is shown to approach the corresponding entropy solution of the homogenized problem

$$u_t = K(u, u_x)_x + \bar{h}(x,t), \qquad u(x,0) = \bar{u}_0(x) \ . \tag{1.5}$$

We *quantify* the convergence rate of u^ε towards u in the weak $W^{-1,\infty}$-topology[2]. Furthermore, in case the solution operator is compact, we are able to translate this weak convergence into a strong one, with L^p-convergence rate estimates for every $t > 0$. We also provide a precise

[2] $\|g\|_{W^{-1,r}(a,b)} := \| \int_a^x g \|_{L^r(a,b)}$, $r \in [1,\infty]$. In case we do not specify the interval we refer to the whole real line.

description of the initial layer in which the weakly convergent oscillations are filtered out to enable the strong convergence which follows.

The paper is organized as follows. In §2 we show the $W^{-1,\infty}$-convergence of u^ε to u, proving a sharp convergence rate estimate of order $\mathcal{O}(\varepsilon^{1-\lambda})$ (Theorem 2.1). The proof is based upon two ingredients: a precise $W^{-1,\infty}$-error estimate for modulated limits (Lemma 2.1), and a familiar $W^{-1,\infty}$-stability of (1.1) with respect to both the initial and forcing data (Proposition 2.1).

This weak $W^{-1,\infty}$-convergence need not imply strong convergence unless the solution operators associated with (1.3) and (1.5) are compact. Specifically, we seek solution operators which are $W^{s,r}$-regular, in the sense that they map initial data in L^∞-bounded sets into bounded sets in $W^{s,r}_{loc}$, $s>0, r\in[1,\infty]$ [3]. Such a regularizing effect is clearly linked to the *nonlinear* nature of the equations and is responsible for the immediate cancellation of initial oscillations, as well as the forcing oscillations.

In §3 we note that if we are granted such regularizing property (mapping $L^\infty \to W^{s,r}, s > 0$), then we may interpolate our weak $W^{-1,\infty}$-error estimate and the $W^{s,r}_{loc}$-bound to obtain strong L^p-convergence, $u^\varepsilon(\cdot,t)\to u(\cdot,t)$, $t>0$, as well as convergence rate estimates. We are therefore led to study the regularizing effect of convective-diffusive equations. There are numerous works in this direction and we refer to [12] for a recent contribution and for a partial list of relevant references.

In the next sections we demonstrate our results for a variety of convection-diffusion equations (1.1) which are equipped with a certain $W^{s,r}$-regularity. We begin, in §4, with convex hyperbolic conservation laws which render BV-regular solutions. In §4.1 we deal with the homogeneous case (no forcing term, $h\equiv 0$). Here, we obtain L^p-convergence rate estimates of $u^\varepsilon(\cdot,t)$ to $u(\cdot,t)$ for a fixed $t>0$, as well as a precise description of the initial layer $t\sim 0$. In §4.2 we study the inhomogeneous case. We show how the nonlinear regularizing effect outpaces the persisting generation of modulated oscillations due to the oscillatory forcing term, $\varepsilon^{-\lambda}h(x,x/\varepsilon,t)$, and still yields strong convergence, though of a slower rate than in the homogeneous case.

In §5 we consider various types of nonlinear, mixed convection-diffusion equations with possibly degenerate diffusion, and we link their nonlinearity to an appropriate $W^{s,r}$-regularity. Our first examples, in §5.1, consist of degenerate parabolic equations augmenting a convex hyperbolic flux. These equations are BV-regular and therefore admit convergence rate estimates similar to the ones obtained in §4 for the purely convective conservation laws. In §5.2 we extend these results to a rather general class of *nonlinear* convective fluxes, where convexity is relaxed by requiring only a non-vanishing high-order(≥ 2) derivative. Next, we focus on the regularizing effect due to the nonlinearity of the degenerate diffusivity. In §5.3 we deal with the prototype porous media equation, $u_t=(u^m)_{xx}$, $m>1$, $u\geq 0$. In the context of its regularizing effect, we identify $m=2$ as a critical exponent: when $m>2$ the equation is known to posses $W^{s,\infty}$-regularity with $s=\frac{1}{m-1}<1$, consult [1]; when $m\leq 2$, however, we have an improved $W^{2,1}$-regularity which results in better convergence rate estimates. We close this section, in §5.4, with a revisit of the general mixed convection-diffusion equations, this time quantifying their regularizing effect (and hence convergence estimates)

[3]Throughout this paper we identify $W^{s,r}$ with the homogeneous space $\dot{W}^{s,r}$, e.g. (for $s<1$), the space equipped with the seminorm $\|g\|_{W^{s,r}} := (\int\int |g(x)-g(y)|^r/|x-y|^{1+sr}dxdy)^{1/r}$.

due to the nonlinearity of the degenerate diffusion. The $W^{s,r}$- regularity of the general mixed convective-diffusive case is analyzed in terms of the velocity averaging lemma along the lines of [12].

Finally, in §6, we provide illustrated examples for our convergence analysis.

2 $W^{-1,\infty}$-Stability and Convergence

In this section we prove that u^ε, the solution of the oscillatory equation (1.3)–(1.4), converges in $W^{-1,\infty}$ to u, the solution of the homogenized equation (1.5). To this end we start by proving the following fundamental lemma which is interesting for its own sake:

Lemma 2.1 *Assume that $g(x,y) \in BV_x(\Omega \times [0,1])$, Ω being a possibly unbounded interval in $\mathbb{R}_x$, and let $g_\varepsilon(x) := g(x, \frac{x}{\varepsilon})$ and $\bar{g}(x) := \int_0^1 g(x,y)dy$. Then*

$$\|g_\varepsilon(x) - \bar{g}(x)\|_{W^{-1,\infty}} \leq C\varepsilon, \qquad C = \|g\|_{L^1([0,1];BV(\mathbb{R}_x))}. \tag{2.1}$$

Proof. For each fixed $x_0 \in \Omega$ we let $a = a(x_0, \varepsilon)$ denote the largest value in the left complement of Ω for which $n := \frac{x_0 - a}{\varepsilon}$ is integral ($a = -\infty$ if Ω is left unbounded). This enables us to break the primitive of $g_\varepsilon(x) - \bar{g}(x)$ over consecutive intervals of size ε as follows:

$$\int_{-\infty}^{x_0} (g_\varepsilon(x) - \bar{g}(x))dx = \sum_{j=-\infty}^{n} \int_{I_j} (g_\varepsilon(x) - \bar{g}(x))dx, \quad I_j = [a_{j-1}, a_j], \qquad a_j := a + j\varepsilon.$$

Change of variable and the 1-periodicity of $g(x, \cdot)$ yield that

$$\int_{I_j} g_\varepsilon(x)dx = \varepsilon \int_{j-1+a/\varepsilon}^{j+a/\varepsilon} g(\varepsilon y, y)dy = \varepsilon \int_0^1 g(y_j, y^\varepsilon)dy, \qquad y_j := a_{j-1} + \varepsilon y \in I_j, \quad y^\varepsilon := \frac{a}{\varepsilon} + y.$$

The 1-periodicity of $g(x, \cdot)$ enables us to express $\bar{g}(x)$ as $\bar{g}(x) = \int_0^1 g(x, y^\varepsilon)dy$; using Fubini's Theorem we get that

$$\int_{I_j} \bar{g}(x)dx = \int_{I_j} \int_0^1 g(x, y^\varepsilon)dydx = \int_0^1 \int_{I_j} g(x, y^\varepsilon)dxdy = \int_0^1 \varepsilon \tilde{g}_j(y^\varepsilon)dy \ ,$$

where $\tilde{g}_j(y^\varepsilon)$ is some intermediate value in $[\operatorname{ess\,inf}_{I_j} g(\cdot, y^\varepsilon), \operatorname{ess\,sup}_{I_j} g(\cdot, y^\varepsilon)]$. Finally, using the last three equalities, we conclude that

$$|\int_{-\infty}^{x_0} (g_\varepsilon(x) - \bar{g}(x))dx| \leq \varepsilon \int_0^1 \sum_{j=-\infty}^{n} |g(y_j, y^\varepsilon) - \tilde{g}_j(y^\varepsilon)|dy \leq$$

$$\varepsilon \int_0^1 \sum_{j=-\infty}^{n} \|g(\cdot, y^\varepsilon)\|_{BV(I_j)} \leq \|g\|_{L^1([0,1];BV(\mathbb{R}_x))} \cdot \varepsilon \ .$$

$\square$

Remarks.

1. Let $f(x) \in BV$ and $g(x,y) \in BV_x(\Omega \times [0,1])$ have a zero average, $\int_0^1 g(x,y)dy \equiv 0$. Applying Lemma 2.1 to $G(x,y) = f(x)g(x,y)$, we conclude that for every a and b there exists a constant C such that

$$\left| \int_a^b f(x) g(x, \frac{x}{\varepsilon}) dx \right| \leq C\varepsilon \ .$$

   This result plays a key role in previous works on homogenization by B. Engquist and T.Y. Hou (e.g., [6, Lemma 2.1], [9, Lemma 2.1]). Here we improve in both generality and simplicity: the corresponding result in [6, 9] was restricted to $f(x), g(x,y) \in C^1$.

2. The sharpness of estimate (2.1) is illustrated by the following example. Assume that $\alpha(x) \in BV$ and $\beta(y)$ is a bounded 2π-periodic function. Let $\bar{\alpha}$, $\bar{\beta}$ denote, respectively, the averages of α and β in $[0, 2\pi]$. Then, by taking $g(x,y) = \alpha(x)\beta(y)$ and $\varepsilon = 1/n$, it follows from Lemma 2.1 that

$$\lim_{n\to\infty} \frac{1}{2\pi} \int_0^{2\pi} \alpha(x)\beta(nx)dx = \bar{\alpha} \cdot \bar{\beta} \ ,$$

   and furthermore, thanks to the bounded variation of α,

$$\left| \frac{1}{2\pi} \int_0^{2\pi} \alpha(x)\beta(nx)dx - \bar{\alpha} \cdot \bar{\beta} \right| \leq \frac{Const}{n} \ .$$

   This result generalizes and illuminates the Riemann-Lebesgue Lemma, where $\beta(y) = e^{iy}$ (see also [21, Theorem (4.15)]).

3. In the simpler case with no x-dependence, i.e, for $g_\varepsilon(x) = g(\frac{x}{\varepsilon})$, a shorter alternative proof of $\mathcal{O}(\varepsilon)$ error estimate is provided in Theorem 8.1 in Appendix B below.

We proceed with a brief proof of the $W^{-1,\infty}$-stability of the solution operator associated with (1.1) with respect to both the initial and forcing data. This $W^{-1,\infty}$-stability agrees with the L^∞-stability for viscosity solutions of Hamilton-Jacobi equations, consult M.G. Crandall, H. Ishii and P.L. Lions [2]. We also refer the reader to [10] for (a qualitative statement of) $W^{-1,\infty}$-stability in the context of of hyperbolic conservation laws.

Proposition 2.1 *($W^{-1,\infty}$-Stability). Let u and v be entropy solutions of the following equations:*

$$u_t = K(u, u_x)_x + g(x,t) \ ; \tag{2.2}$$

$$v_t = K(v, v_x)_x + h(x,t) \ . \tag{2.3}$$

Then, for $t > 0$,

$$\|u(\cdot,t) - v(\cdot,t)\|_{W^{-1,\infty}} \leq \|u(\cdot,0) - v(\cdot,0)\|_{W^{-1,\infty}} + \int_0^t \|g(\cdot,\tau) - h(\cdot,\tau)\|_{W^{-1,\infty}} d\tau \ . \tag{2.4}$$

Proof. Let u^δ and v^δ, $\delta > 0$, be the corresponding regularized solutions, associated with $K^\delta = K + \delta p$. The primitive of the error, $E^\delta := \int_{-\infty}^{x}(u^\delta - v^\delta)$, satisfies the convection-diffusion equation

$$E_t^\delta = q_1 \cdot E_x^\delta + (q_2 + \delta) \cdot E_{xx}^\delta + D \ . \tag{2.5}$$

Here, $q_1 = K_u(w_1, u_x^\delta)$, $q_2 = K_p(v^\delta, w_2)$, with appropriate mid-values w_j, $j = 1, 2$, and $D = \int_{-\infty}^{x}(g(\xi,t) - h(\xi,t))d\xi$. Since, in view of (1.2), $q_2 \geq 0$, we conclude that

$$\frac{d}{dt}\|E^\delta(\cdot,t)\|_{L^\infty} \leq \|D(\cdot,t)\|_{L^\infty} \ ,$$

which, by letting δ go to zero, implies (2.4). □

Finally, combining Proposition 2.1 and Lemma 2.1, we conclude the following:

Theorem 2.1 *($W^{-1,\infty}$-Convergence). Let u^ε be the entropy solution of*

$$u_t^\varepsilon = K(u^\varepsilon, u_x^\varepsilon)_x + h^\varepsilon(x,t), \qquad u^\varepsilon(x,0) = u_0^\varepsilon(x), \tag{2.6}$$

with modulated initial and forcing data, $u_0^\varepsilon(x)$ and $h^\varepsilon(x,t)$, outlined in (1.4). Let u be the entropy solution of the corresponding homogenized equation

$$u_t = K(u, u_x)_x + \bar{h}(x,t), \qquad u(x,0) = \bar{u}_0(x), \tag{2.7}$$

associated with the respective averages,

$$\bar{u}_0(x) = \int_0^1 u_0(x,y)dy, \qquad \bar{h}(x,t) = \int_0^1 h(x,y,t)dy \ .$$

Then, for every $t > 0$ there exists a constant $C(t) > 0$ such that

$$\|u^\varepsilon(\cdot,t) - u(\cdot,t)\|_{W^{-1,\infty}} \leq C(t)\varepsilon^{1-\lambda} \ . \tag{2.8}$$

Moreover, in the homogeneous case (where $h \equiv 0$ and $\lambda = 0$) the constant $C(t)$ does not depend on t and we have

$$\|u^\varepsilon(\cdot,t) - u(\cdot,t)\|_{W^{-1,\infty}} \leq C\varepsilon \ . \tag{2.9}$$

Proof. Lemma 2.1 with $g(x,y) = u_0(x,y)$ and $g(x,y) = h(x,y,t)$ with fixed $t > 0$, tells us that

$$\|u_0^\varepsilon(x) - \bar{u}_0(x)\|_{W^{-1,\infty}} \leq C\varepsilon \quad ; \quad \|\frac{1}{\varepsilon^\lambda}h(x,\frac{x}{\varepsilon},t) - \frac{1}{\varepsilon^\lambda}\bar{h}(x,t)\|_{W^{-1,\infty}} \leq \frac{1}{\varepsilon^\lambda}\cdot c(t)\varepsilon \ .$$

By our assumption, since either λ or $\bar{h}$ vanish, we have $\varepsilon^{-\lambda}\bar{h} = \bar{h}$; hence

$$\|h^\varepsilon(x,t) - \bar{h}\|_{W^{-1,\infty}} \leq c(t)\varepsilon^{1-\lambda} \ .$$

Finally, (2.8) and (2.9) follow in view of Proposition 2.1 with $C(t) = C + \int_0^t c(\tau)d\tau$. □

Remark. We may extend Theorem 2.1 by allowing amplified initial data; i.e., $u_0^\varepsilon = \varepsilon^{-\mu}u_0(x, \frac{x}{\varepsilon})$ with fixed $\mu \in [0,1)$ such that $\mu \cdot \bar{u}_0 \equiv 0$. In that case, the $W^{-1,\infty}$-error in (2.8) would be of order $\mathcal{O}(\varepsilon^{1-\max(\mu,\lambda)})$.

3 Strong Convergence to the Homogenized Solution

Our aim in this section is to translate the weak $W^{-1,\infty}$-convergence rate estimate, (2.8), into strong L^p-convergence rate estimates. To this end we focus our attention on nonlinear equations for which the solution operator is compact. Specifically, we concentrate on solution operators, $S(t) : u(\cdot, 0) \mapsto u(\cdot, t)$, which map bounded sets in L^∞ into bounded sets in the regularity spaces, $W^{s,r}_{loc}$, $s > 0$, $1 \leq r \leq \infty$. This compactness is clearly of a nonlinear nature and it implies that the solution operator immediately cancels out oscillations which may have been present at $t = 0$. For future reference, we refer to such equations as $W^{s,r}$-*regular*. We remark that *nonlinearity* is essential for such $W^{s,r}$-regularity in the scalar case. For the interaction of a *linearly degenerate* field with oscillatory nonlinear fields in hyperbolic systems, we refer to [3],[15] and the error estimate in [8].

The following theorem translates, for $W^{s,r}$-regular equations, the weak $W^{-1,\infty}$-convergence into *strong* L^p-convergence rate estimates.

Theorem 3.1 *Let u^ε be the solution of equation (2.6) subject to modulated data, (1.4), and assume that the equation possesses a $W^{s,r}$-regularizing effect. Then, u^ε converges to u – the solution of the homogenized equation (2.7), and the following error estimates hold*

$$\|u^\varepsilon(\cdot, t) - u(\cdot, t)\|_{L^p(\Omega)} \leq C \cdot B^{s,r}_\varepsilon(t)^{1-\theta} \cdot \varepsilon^{\theta(1-\lambda)} \qquad \forall p \in [1, (\frac{1}{r} - s)^{-1}_+] \ . \tag{3.1}$$

Here, θ, p_* and $B^{s,r}_\varepsilon$ are given by

$$\theta = \frac{\frac{1}{p_*} - \frac{1}{r} + s}{1 - \frac{1}{r} + s} \in [0, 1], \qquad p_* := \max\{p, r(s + 1)\} \ , \tag{3.2}$$

$$B^{s,r}_\varepsilon(t) = \|u^\varepsilon(\cdot, t) - u(\cdot, t)\|_{W^{s,r}} \ , \tag{3.3}$$

and C is some constant which depends on p, $|\Omega|^{\frac{1}{p} - \frac{1}{p_*}}$ and t.

Proof. By Gagliardo-Nirenberg inequality, e.g., [7, Theorem 9.3], interpolation between the $W^{-1,\infty}$ and $W^{s,r}$-bounds yields for the intermediate L^p-norms,

$$\|v\|_{L^p} \leq c_p \cdot \|v\|^\theta_{W^{-1,\infty}} \|v\|^{1-\theta}_{W^{s,r}}, \qquad \theta = \frac{\frac{1}{p} - \frac{1}{r} + s}{1 - \frac{1}{r} + s}; \tag{3.4}$$

this inequality holds for all $p \in [r(s + 1), (\frac{1}{r} - s)^{-1}_+]$. Since by our assumption the solution operator associated with (2.6) is $W^{s,r}$-regular, so does the solution operator associated with (2.7), and hence their difference is bounded, (3.3). We may now use (3.4) with $v = u^\varepsilon(\cdot, t) - u(\cdot, t)$, together with the $W^{-1,\infty}$-error estimate, (2.8), to conclude the L^p-error estimate (3.1) for all $p \geq r(s + 1)$ in the relevant range. For the remaining values of $p < r(s + 1)$, the L^p-errors are dominated by the one obtained already for the $L^{r(s+1)}$-norm, $\| \cdot \|_{L^p(\Omega)} \leq |\Omega|^{\frac{1}{p} - \frac{1}{r(s+1)}} \| \cdot \|_{L^{r(s+1)}(\Omega)}$. □Moreover, on p,

The particular homogeneous case, $h \equiv 0$, where the oscillations are introduced only at $t = 0$ via the initial data, is of special interest. In this case, the solution operator of (2.6) does not depend on ε and coincides with that of (2.7). Since the initial data for those equations, $u_0(x, \frac{x}{\varepsilon})$ and $\bar{u}_0(x)$, are uniformly bounded in L^∞, we conclude that $B^{s,r}_\varepsilon(t)$, given in (3.3), is uniformly bounded with respect to ε. Hence, we arrive at the following simplified version of Theorem 3.1 for homogeneous problems:

Corollary 3.1 *(Initial Oscillations). Under the assumptions of Theorem 2.1, if equations (2.6) and (2.7) are homogeneous and $W^{s,r}$-regular, then for every $t > 0$ and $p \in [1, (\frac{1}{r} - s)^{-1}_+]$ there exists a constant C such that*

$$\|u^\varepsilon(\cdot, t) - u(\cdot, t)\|_{L^p(\Omega)} \leq C \cdot \varepsilon^\theta \ , \tag{3.5}$$

where θ is given in (3.2).

In the inhomogeneous case, the solution operator of (2.6) depends on ε. Hence, due to the persisting generation of oscillations by the oscillatory source term, $\varepsilon^{-\lambda}h(x, x/\varepsilon, t)$, the $W^{s,r}$-bound, $B^{s,r}_\varepsilon(t)$, may grow when $\varepsilon \downarrow 0$. Therefore, in order to have strong convergence in this case, we need a moderate growth of $B^{s,r}_\varepsilon(t)$ so that $B^{s,r}_\varepsilon(t)^{1-\theta}\varepsilon^{\theta(1-\lambda)} \underset{\varepsilon\to 0}{\longrightarrow} 0$.

In the following sections we give examples of equations, both hyperbolic and parabolic, homogeneous and inhomogeneous, which are $W^{s,r}$- regular and derive strong convergence estimates for them.

4 Applications to Hyperbolic Conservation Laws

In this section we demonstrate our results in the context of hyperbolic conservation laws with convex flux f,

$$u_t + f(u)_x = h, \qquad f'' \geq \alpha > 0 \ .$$

The convexity of the flux f implies that these equations are BV- regular – consult Proposition 4.1 below. Granted this BV-regularity which we identify with the $W^{1,1}$-regularity, we may invoke the L^p-error estimates (3.1)–(3.3) which now read,

$$\|u^\varepsilon(\cdot, t) - u(\cdot, t)\|_{L^p(\Omega)} \leq C \cdot B_\varepsilon(t)^{1-\frac{1}{p_*}} \cdot \varepsilon^{\frac{1-\lambda}{p_*}} \qquad \forall p \in [1, \infty) \ \ ; \ \ p_* := \max\{p, 2\}. \tag{4.1}$$

Here, $B_\varepsilon(t)$ abbreviates the BV-size of the difference,

$$B_\varepsilon(t) = B^{1,1}_\varepsilon(t) = \|u^\varepsilon(\cdot, t) - u(\cdot, t)\|_{BV} \ , \tag{4.2}$$

and the constant C depends on p, $|\Omega|^{\frac{1}{p} - \frac{1}{p_*}}$, and (in the inhomogeneous case) also on t.

In the remaining of this section we take a closer look at the convergence rate estimate (4.1). In §4.1 we study the homogeneous case ($h \equiv 0$); §4.2 is devoted for the more intricate case with inhomogeneous oscillatory data.

4.1 The Homogeneous Case

Let u^ε and u be the entropy solutions of the corresponding initial value problems,

$$u_t^\varepsilon + f(u^\varepsilon)_x = 0, \qquad u^\varepsilon(x,0) = u_0(x, \frac{x}{\varepsilon}) \ , \tag{4.3}$$

$$u_t + f(u)_x = 0, \qquad u(x,0) = \bar{u}_0(x) = \int_0^1 u_0(x,y)dy \ , \tag{4.4}$$

where, as usual, $u_0 \in BV_x(\Omega \times [0,1])$. Since $u^\varepsilon(\cdot,0) - u(\cdot,0)$ vanish outside Ω, $u^\varepsilon(\cdot,t) - u(\cdot,t)$ is compactly supported, say on $\Omega(t), \forall t > 0$ (thanks to the finite speed of propagation), and therefore,

$$B_\varepsilon(t) = \|u^\varepsilon(\cdot,t) - u(\cdot,t)\|_{BV} \leq \|u^\varepsilon(\cdot,t)\|_{BV(\Omega(t))} + \|u(\cdot,t)\|_{BV(\Omega(t))} \ . \tag{4.5}$$

If we let D denote the difference between the far right and far left values of $u(\cdot,t)$ and $u^\varepsilon(\cdot,t)$, then the BV-norms of $u^\varepsilon(\cdot,t)$ and of $u(\cdot,t)$ can be upper-bounded in terms of their Lip^+-(semi)-norms,[4]

$$\|u^\varepsilon(\cdot,t)\|_{BV(\Omega(t))} \leq D + 2|\Omega(t)| \cdot \|u^\varepsilon(\cdot,t)\|_{Lip^+} \ , \quad \|u(\cdot,t)\|_{BV(\Omega(t))} \leq D + 2|\Omega(t)| \cdot \|u(\cdot,t)\|_{Lip^+}, \tag{4.6}$$

and since $f'' \geq \alpha > 0$, both u and u^ε are Lip^+-stable – consult e.g. [16],

$$\|u(\cdot,t)\|_{Lip^+} \leq (\|u(\cdot,0)\|_{Lip^+}^{-1} + \alpha t)^{-1} \quad , \quad \|u^\varepsilon(\cdot,t)\|_{Lip^+} \leq (\|u^\varepsilon(\cdot,0)\|_{Lip^+}^{-1} + \alpha t)^{-1} \ . \tag{4.7}$$

Finally, since $\|u(\cdot,0)\|_{Lip^+} \leq \mathcal{O}(1)$ and $\|u^\varepsilon(\cdot,0)\|_{Lip^+} \leq \mathcal{O}(\varepsilon^{-1})$, we conclude by (4.5)–(4.7), that the term $B_\varepsilon(t)$ in (4.1) does not exceed

$$B_\varepsilon(t) \leq 2D + Const \cdot |\Omega(t)| \cdot (\alpha t + \mathcal{O}(\varepsilon))^{-1} \ . \tag{4.8}$$

We now distinguish between three different regimes:

(1) <u>Small $t > 0$ – the initial layer.</u>
For small values of t we get by (4.1) and (4.8) that

$$\|u^\varepsilon(\cdot,t) - u(\cdot,t)\|_{L^p} \sim (t+\varepsilon)^{\frac{1}{p_*}-1} \cdot \varepsilon^{\frac{1}{p_*}} \qquad \forall p \in [1,\infty).$$

Hence, for a fixed value of $\varepsilon > 0$, the initial layer is of width $\mathcal{O}(\varepsilon)$. More precisely, the width of the initial layer in which there is no strong L^p-convergence is $\mathcal{O}(\varepsilon^{\frac{1}{p_*-1}})$.

(2) <u>Fixed $t > 0$ – cancellation of oscillations.</u>
B. Engquist and W. E proved the strong convergence, $u^\varepsilon(\cdot,t) \to u(\cdot,t)$ in $L^1_{loc}(\mathbb{R}), \forall t > 0$, [5, Theorem 4.1]. Here, we are able to quantify the convergence rate in L^p, $1 \leq p \leq \infty$, whenever the flux f is convex: the convergence rate implied by (4.1) and (4.8) is bounded by

$$\|u^\varepsilon(\cdot,t) - u(\cdot,t)\|_{L^p} \leq Const \cdot \varepsilon^{\frac{1}{p_*}} \qquad p_* = \max\{p,2\} \qquad \forall p \in [1,\infty) \ . \tag{4.9}$$

---

[4] Lip^+ abbreviates the semi-norm, $\|w\|_{Lip^+} := \sup_{x \neq y} \left(\frac{w(x)-w(y)}{x-y}\right)_+$.

Remarks.

1. The convergence result in [5, §4] assumes the nonlinearity of f to be weaker than convexity. An extension of the $W^{s,r}$-regularity (which in turn implies strong L^p-convergence estimates) to a larger family of nonlinear fluxes in the spirit of [5] is outlined in §5.2 below.

2. A further improvement of (4.9) is available whereever the homogenized solution is smooth. To this end we employ a localized version of a *one-sided* interpolation inequality due to [18], stating that

$$\|v\|_{L^\infty_{loc}} \leq Const \cdot \|v\|^{\frac{1}{2}}_{W^{-1,\infty}_{loc}} \|v\|^{\frac{1}{2}}_{Lip^+_{loc}}. \tag{4.10}$$

We remark that (4.10) is the analogue of Gagliardo-Nirenberg inequality (3.4) with $p = r = \infty, s = 1$. However, here only *one-sided* bound (on the first derivative) is assumed. Such local error estimates in the presence of one-sided bounds were first used in [16, §4].

Equipped with (4.10), we conclude that in any interval of C^1- smoothness of $u(\cdot, t)$, the one-sided Lip^+-bound of the *difference* $\|u^\varepsilon(\cdot, t) - u(\cdot, t)\|_{Lip^+(\Omega)}$ is bounded *independently* of ε. This, together with (2.9) imply that

$$|u^\varepsilon(x, t) - u(x, t)| \leq Const \cdot |u(\cdot, t)|_{C^1_{loc(x)}} \cdot \varepsilon^{\frac{1}{2}},$$

which improves estimate (4.9).

The one-sided inequality (4.10) may be used similarly to localize the strong error estimates discussed below. We omit the details.

(3) Large $t > 0$ – asymptotic behavior.

We fix $\varepsilon > 0$ and consider large values of $t > 0$. For simplicity, let us concentrate on the case where the initial data admits the *same* constant value outside (the left and right of –) Ω, say $u_0|_{\Omega^c} \equiv A$. In this case, the constant D in (4.8) vanishes, and the time decay of $\|u^\varepsilon(\cdot, t)\|_{BV}$ implies that the solution tends to its constant initial average, $u^\varepsilon(\cdot, t \uparrow \infty) \to A$. The error estimates (4.1) and (4.8) then imply that

$$\|u^\varepsilon(\cdot, t) - u(\cdot, t)\|_{L^p} \sim |\Omega(t)|^{1+\frac{1}{p}-\frac{2}{p*}} t^{\frac{1}{p*}-1} \qquad \forall p \in [1, \infty]. \tag{4.11}$$

Since $|\Omega(t)| = |\Omega(0)| + \mathcal{O}(t^{\frac{1}{2}})$, e.g. [11], we conclude that

$$\|u^\varepsilon(\cdot, t) - u(\cdot, t)\|_{L^p} \leq \mathcal{O}(t^{\frac{1}{2}(\frac{1}{p}-1)}) \qquad \forall p \in [1, \infty]. \tag{4.12}$$

In particular, (4.12) with $p = \infty$ yields a uniform error estimate of order $\mathcal{O}(t^{-\frac{1}{2}})$. In fact, this reflects the uniform large time decay of $\|u(\cdot, t) - A\|_{L^\infty}$ and $\|u^\varepsilon(\cdot, t) - A\|_{L^\infty}$ – each of which decays like $\mathcal{O}(t^{-\frac{1}{2}})$.

4.2 The Inhomogeneous Case

Let u^ε and u be the entropy solutions of the following initial value problems,

$$u_t^\varepsilon + f(u^\varepsilon)_x = \frac{1}{\varepsilon^\lambda} h(x, \frac{x}{\varepsilon}, t), \qquad u^\varepsilon(x,0) = u_0(x, \frac{x}{\varepsilon}) \; ; \tag{4.13}$$

$$u_t + f(u)_x = \bar{h}(x,t) := \int_0^1 h(x,y,t)dy, \qquad u(x,0) = \bar{u}_0(x) := \int_0^1 u_0(x,y)dy \; , \tag{4.14}$$

with $u_0(x,y)$, $h(x,y,t)$ as in (1.4) and $f'' \geq \alpha > 0$. Recall our assumption that either λ or $\bar{h}$ vanish, and in any case, $\lambda < 1$. The case $\lambda = 1$ is different, consult [4]: in this context, E and Serre provided a rigorous justification of the asymptotic expansion (under extra compatibility requirements), $u^\varepsilon(x,t) \sim U(x, x/\varepsilon, t)$.

We begin by studying the Lip^+-behavior in the presence of an oscillatory force. To this end we state the following Lip^+-stability estimate for inhomogeneous conservation laws, which is a special case of Proposition 7.1 in §7.

Proposition 4.1 *Let v be the entropy solution of*

$$v_t + f(v)_x = g(x,t), \qquad f''(v) \geq \alpha \; , \tag{4.15}$$

subject to the initial condition $v(x,0) = v_0(x)$. Then

$$\|v(\cdot,t)\|_{Lip^+} \leq c \cdot \frac{\|v_0\|_{Lip^+} + c + (\|v_0\|_{Lip^+} - c)e^{-2\alpha ct}}{\|v_0\|_{Lip^+} + c - (\|v_0\|_{Lip^+} - c)e^{-2\alpha ct}} \leq c \cdot \frac{1 + e^{-2\alpha ct}}{1 - e^{-2\alpha ct}} \; , \tag{4.16}$$

where

$$c = c(t) := \max_{0 \leq \tau \leq t} \sqrt{\frac{\|g(\cdot,\tau)\|_{Lip^+}}{\alpha}} \; . \tag{4.17}$$

Remarks.

1. In the particular case of homogeneous data, $g \equiv c = 0$, Proposition 4.1 recovers the usual homogeneous Lip^+-decay (4.7).

2. Key features of Proposition 4.1 to be used later are
{i} that the dependence of the Lip^+-bound on the inhomogeneous term, $\|g\|_{Lip^+}$, is proportional to the *square root* of the latter, $c \sim \sqrt{\|g\|_{Lip^+}}$, rather than the expected $\|g\|_{Lip^+}$.
{ii} that the second upper bound for $\|v(\cdot,t)\|_{Lip^+}$ on the right of (4.16) is independent of the initial data (and hence, even if the initial data was Lip^+-unbounded, the solution $v(\cdot,t)$ will be Lip^+-bounded for all $t > 0$.)

Corollary 4.1 *(Lip^+-estimate). Let u^ε be the entropy solution of (4.13). Then for any fixed $t > 0$ it holds that*

$$\|u^\varepsilon(\cdot,t)\|_{Lip^+} \le \mathcal{O}(\varepsilon^{-\frac{1+\lambda}{2}}) \ . \tag{4.18}$$

Proof. Since $\|u^\varepsilon(\cdot,0)\|_{Lip^+} \le \mathcal{O}(\varepsilon^{-1})$ and, for any fixed $t > 0$, $\|\varepsilon^{-\lambda}h(\cdot,\cdot/\varepsilon,t)\|_{Lip^+} \le \mathcal{O}(\varepsilon^{-(1+\lambda)})$, (4.18) follows from (4.16). □

Remark. We recall that in the absence of a forcing term, the convexity of f implies according to (4.7), that $\|u^\varepsilon(\cdot,t)\|_{Lip^+} \le \mathcal{O}(1)$. If, on the other hand, f does not render any regularizing effect (such as linear f's), then the presence of such an oscillatory forcing term implies $\|u^\varepsilon(\cdot,t)\|_{Lip^+} \sim \mathcal{O}(\varepsilon^{-(1+\lambda})$. With this in mind, Corollary 4.1 states that the $\mathcal{O}(\varepsilon^{-(1+\lambda)})$-modulated oscillations due to the forcing term are *relaxed*, thanks to the convexity of the equation, resulting in Lip^+ bound of order $\mathcal{O}(\varepsilon^{-\frac{1+\lambda}{2}})$.

Since $\|u(\cdot,t)\|_{Lip^+}$ is independent of ε we conclude, in view of (4.5), (4.6) and Corollary 4.1, the BV-upper bound

$$B_\varepsilon(t) = \|u^\varepsilon - u\|_{BV} \le \mathcal{O}(\varepsilon^{-\frac{1+\lambda}{2}}) \ . \tag{4.19}$$

Though estimate (4.19) does not provide a $B_\varepsilon(t)$-bound which remains bounded as $\varepsilon \downarrow 0$, it suffices in order to obtain strong L^p-convergence. Indeed, combining it with the L^p-error estimates (4.1) we conclude the following.

Proposition 4.2 *Let u^ε and u be the entropy solutions of (4.13) and (4.14), respectively. Then the following L^p-error estimates hold for every fixed $t > 0$:*

$$\|u^\varepsilon(\cdot,t) - u(\cdot,t)\|_{L^p} \le Const \cdot \varepsilon^{\frac{3-\lambda}{2p_*} - \frac{1+\lambda}{2}} \qquad p_* = \max\{p, 2\}, \tag{4.20}$$

We conclude with the following remarks.

{i} In case $\lambda = 0$ we obtain an error bound of order $\mathcal{O}(\varepsilon^{\frac{3}{2p_*} - \frac{1}{2}})$. Comparing this to the analogous estimate in the homogeneous case, (4.9), we see that the oscillatory source term, $h(x, \frac{x}{\varepsilon}, t)$, decelerates the rate of convergence; moreover, the error bound in (4.20) (with $\lambda = 0$) is limited to strong L^p-convergence as long as $p < 3$.

{ii} In case the forcing oscillations are amplified ($\lambda > 0$) we obtain an L^2-estimate of order $\mathcal{O}(\varepsilon^{\frac{1-3\lambda}{4}})$. In this case (4.20) is limited to strong L^2-convergence as long as $0 < \lambda < \frac{1}{3}$. In general, (4.20) is limited to strong L^p-convergence as long as $p_* < \frac{3-\lambda}{1+\lambda}$.

{iii} A final note on the initial layer: using (4.16)– (4.17), we may study the behavior of $\|u^\varepsilon(\cdot,t)\|_{Lip^+}$ and, therefore, also of $B_\varepsilon(t)$ as $t \downarrow 0$ and find that $B_\varepsilon(t \sim \varepsilon^\eta) \sim \varepsilon^{-\max(\eta,(1+\lambda)/2)}$. With that and (4.1) it is possible to determine the width of the initial layer near $t = 0$, in which there is no strong L^p-convergence. A simple though tedious computation which we omit shows that the width of the initial layer is $\mathcal{O}(\varepsilon^{\frac{1-\lambda}{p_*-1}})$ (where $p_* < \frac{3-\lambda}{1+\lambda}$). Note that when $\lambda = 0$, it is of the same order as in the homogeneous case, namely, $\mathcal{O}(\varepsilon^{\frac{1}{p_*-1}})$.

5 Applications to Convection-Diffusion Equations

Here we demonstrate our results in the context of convection-diffusion equations of the form,

$$u_t^\varepsilon + f(u^\varepsilon)_x = Q(u^\varepsilon, p^\varepsilon)_x \ , \quad Q_p \geq 0 \ ; \qquad u^\varepsilon(x,0) = u_0(x, \frac{x}{\varepsilon}). \tag{5.1}$$

Thus, here we rewrite (1.1) with $K(u,p) = Q(u,p) - f(u)$ where we distinguish between the convective flux, $f(u)$, and the diffusive part, $Q(u,p)$. We concentrate on the homogeneous case and obtain strong convergence rate estimates of the entropy solution which corresponds to the oscillatory initial data,$u^\varepsilon(\cdot,t)$, to the entropy solution which corresponds to the averaged data, $u(\cdot,t)$. equations A similar program can be carried out for convection-diffusion equations in the presence of oscillatory forcing terms.

Note that in case of uniform parabolicity, $Q_p \geq Const > 0$, the solution becomes C^∞-smooth at $t > 0$ and therefore equation (5.1) is $W^{s,\infty}$-regular for all $s > 0$. This optimal regularity implies, in view of Theorem 3.1, the full recovery of strong convergence of first-order,

$$\|u^\varepsilon(\cdot,t) - u(\cdot,t)\|_{L^\infty_{loc}} \leq Const \cdot \varepsilon \ .$$

Consequently, our main concern below is with *degenerate* diffusivity, where we separate our discussion to two types of equations: those dominated by a nonlinear convective flux (in §5.1 and §5.2), and those whose regularizing effect is due to a *degenerate* diffusive term (in §5.3 and §5.4).

5.1 Convection-diffusion equations with convex flux

We begin with examples of convective-diffusive equations which are dominated by a convex flux, $f'' \geq \alpha > 0$. The convexity of the convective flux enables us to prove, in §7 below, the *Lip*$^+$-stability of those equations. As in §4, this *Lip*$^+$-stability implies BV-regularity which in turn yields error estimate (4.9),

$$\|u^\varepsilon(\cdot,t) - u(\cdot,t)\|_{L^p} \leq Const \cdot \varepsilon^{\frac{1}{p_*}} \qquad p_* = \max\{p, 2\} \qquad \forall p \in [1,\infty) \ . \tag{5.2}$$

Let us quote two examples. First, convex conservation laws augmented with possibly degenerate viscosity,

$$u_t + f(u)_x = Q(u)_{xx}, \qquad f'' \geq \alpha > 0 \ , \quad Q' \geq 0 \geq Q'''. \tag{5.3}$$

For instance, the convective porous media equation which consists of a convex flux augmented with subquadratic diffusion, $Q(u) = cu^m$, $1 \leq m \leq 2$ ($u \geq 0$), falls into this category.
As a second example we mention conservation laws with degenerate pseudo-viscosity, [14],

$$u_t + f(u)_x = Q(u_x)_x, \qquad f'' \geq \alpha > 0 \ , \quad Q' \geq 0 \ . \tag{5.4}$$

The *Lip*$^+$-stability of (5.3) and (5.4) is a consequence of Proposition 7.1, with $K(u,p) = Q'(u)p - f(u)$ in the first case and $K(u,p) = Q(p) - f(u)$ in the second case; in both cases $K_{uu} \leq -\alpha < 0$ for all $p \geq 0$ so that the requirement (7.3) for *Lip*$^+$-stability holds.

5.2 Convection-diffusion equations with general nonlinear flux

We consider the viscous conservation law (5.3),

$$u_t + f(u)_x = Q(u)_{xx}, \quad Q' \geq 0. \tag{5.5}$$

This time, convexity is relaxed by assuming the following:
Assumption (nonlinear hyperbolic flux). The flux f is *nonlinear* in the sense it has some high-order nonvanishing derivative; i.e., there exists $k \geq 2$ such that

$$f^{(k)}(v) \neq 0 \qquad \forall v. \tag{5.6}$$

According to [12, Theorem 4], the convection-diffusion equation (5.1) is $W^{s,1}$-regular with $s = \frac{1}{2k-1}$, and Corollary 3.1 yields the error estimate

$$\|u^\varepsilon(\cdot,t) - u(\cdot,t)\|_{L^p} \leq Const. \begin{cases} \varepsilon^{\frac{s}{s+1}} & \forall p \in [1, s+1) \ , \\ \\ \varepsilon^{\frac{1-p(1-s)}{sp}} & \forall p \in [s+1, \frac{1}{1-s}) \ . \end{cases} \tag{5.7}$$

Remark. The regularity result stated above is not sharp: as noted in [12] one expects $W^{s,1}$-regularity of order $s = \frac{1}{k-1}$. In this case one obtains an L^1-error estimate of order $\mathcal{O}(\varepsilon^{\frac{1}{k}})$. Also, for convex fluxes ($k = 2$, $s = 1$), one recovers the L^p-error estimate of order $\mathcal{O}(\varepsilon^{\frac{1}{p*}})$ stated in (5.2).

5.3 The Porous Media Equation

Here, we consider the porous media equation,

$$u_t = (u^m)_{xx} \ , \quad u \geq 0 \ , \quad m > 1 \ , \tag{5.8}$$

as a prototype model example for parabolic, 'convection-free' equations with degenerate diffusion.

D.G. Aronson, [1], proved that for every $t > 0$, $u(\cdot,t)$ is uniformly Hölder continuous with Hölder exponent $s = \min\{1, (m-1)^{-1}\}$ (a generalization for convective porous media type equations can be found in [19]).

In case $m \geq 2$, it implies that (5.8) is $W^{s,\infty}$- regular, $s = (m-1)^{-1} < 1$. With this Hölder $W^{s,\infty}$-regularity, the L^p-error estimates (3.1)–(3.3) take the form:

$$\|u^\varepsilon(\cdot,t) - u(\cdot,t)\|_{L^p(\Omega)} \leq C \cdot B_\varepsilon(t)^{\frac{1}{s+1}} \cdot \varepsilon^{\frac{s}{s+1}} \qquad \forall p \in [1, \infty], \tag{5.9}$$

where $B_\varepsilon(t) = \|u^\varepsilon(\cdot,t) - u(\cdot,t)\|_{W^{s,\infty}}$ and the constant C depends on p and $|\Omega|^{\frac{1}{p}}$. Since the last upper-bound is independent of p, we summarize the case of $m \geq 2$ with a *uniform* error estimate

$$\|u^\varepsilon(\cdot,t) - u(\cdot,t)\|_{L^\infty(\Omega)} \leq Const \cdot \varepsilon^{\frac{1}{m}} \qquad m \geq 2. \tag{5.10}$$

The case $m \le 2$ is different (note that $m = 2$ is already hinted as a critical exponent in example (5.3) where $Q(u) = cu^m$ satisfies the condition $Q''' \le 0$ only if $m \le 2$). In this case, Aronson's result tells us that the porous media equation with subquadratic diffusion is $W^{1,\infty}$-regular. We claim that in fact more is true, namely, that the solution operator of (5.8) with $m \le 2$ is even $W^{2,1}$-regular:

Proposition 5.1 *Let $u \ge 0$ be the entropy solution of*

$$u_t = (u^m)_{xx} , \quad m \le 2 , \quad u(\cdot, 0) = u_0 \in L^\infty(\Omega) , \tag{5.11}$$

where, as usual, $u_0\Big|_{\Omega^c} \equiv Const$. Then, for every $t > 0$, $\|u_{xx}(\cdot, t)\|_{L^1} < \infty$.

Proof. We recall that the pressure, $v := \frac{m}{m-1}u^{m-1}$, satisfies the one sided estimate [20, Proposition 5]

$$v_{xx} \ge -\frac{1}{(m+1)t} . \tag{5.12}$$

Next, we invoke the identity,

$$v_{xx} = mu^{m-2}u_{xx} + m(m-2)u^{m-3}u_x^2 . \tag{5.13}$$

Since $m \le 2$, the second term on the right of (5.13) is nonpositive. Hence, we conclude in view of (5.12) and (5.13) that

$$u^{m-2}u_{xx} \ge -\frac{1}{m(m+1)t} . \tag{5.14}$$

Using the maximum principle and, once more, that $m \le 2$, we conclude by (5.14) that

$$u_{xx} \ge -\frac{u^{2-m}}{m(m+1)t} \ge -\frac{\|u_0\|_{L^\infty}^{2-m}}{m(m+1)t}. \tag{5.15}$$

The fact that equation (5.11) is conservative – which we express as $\int_{\mathbb{R}}(u_{xx})_+dx = \int_{\mathbb{R}}(u_{xx})_-dx$, implies

$$\|u_{xx}(\cdot, t)\|_{L^1} = 2\int_{\mathbb{R}} |(u_{xx})_-|dx . \tag{5.16}$$

Due to the finite speed of propagation, $u(\cdot, t)$ is constant outside some bounded interval $\Omega(t)$ and therefore $u_{xx}(\cdot, t)$ is compactly supported on $\Omega(t)$. Hence, (5.15) and (5.16) imply

$$\|u_{xx}(\cdot, t)\|_{L^1} = 2\int_{\Omega(t)} |(u_{xx})_-|dx \le 2|\Omega(t)|\frac{\|u_0\|_{L^\infty}^{2-m}}{m(m+1)t} \tag{5.17}$$

and we are done. □

Equipped with the $W^{2,1}$-regularity derived in Proposition 5.1, the L^p-error estimates (3.1)–(3.3) take the form

$$\|u^\varepsilon(\cdot, t) - u(\cdot, t)\|_{L^p(\Omega)} \le C \cdot B_\varepsilon(t)^{\frac{1-\frac{1}{p_*}}{2}} \cdot \varepsilon^{\frac{1+\frac{1}{p_*}}{2}}, \quad p_* := \max\{p, 3\} \qquad \forall p \in [1, \infty], \tag{5.18}$$

where $B_\varepsilon(t) = \|u^\varepsilon(\cdot,t) - u(\cdot,t)\|_{W^{2,1}}$, and the constant C depends on p and $|\Omega|^{\frac{1}{p}-\frac{1}{p_*}}$. Hence, for any fixed $t > 0$, it holds that

$$\|u^\varepsilon(\cdot,t) - u(\cdot,t)\|_{L^p(\Omega)} \leq Const \cdot \varepsilon^{\frac{p_*+1}{2p_*}}, \quad p_* = \max\{p, 3\} \qquad \forall p \in [1,\infty] \ . \tag{5.19}$$

Finally, we combine the two error estimates, (5.10) for $m \geq 2$ and (5.19) for $m \leq 2$, as follows:

Theorem 5.1 *Let u^ε and u be an oscillatory and the corresponding homogenized solutions of the porous media equation (5.8). Then for any fixed $t > 0$ it holds that*

$$\|u^\varepsilon(\cdot,t) - u(\cdot,t)\|_{L^\infty(\Omega)} \leq Const \cdot \varepsilon^{\min\{\frac{1}{m},\frac{1}{2}\}}. \tag{5.20}$$

5.4 Convection-diffusion equations with nonlinear diffusion

We revisit the viscous conservation law,

$$u_t + f(u)_x = Q(u)_{xx}, \quad Q' \geq 0. \tag{5.21}$$

This time the C^1 flux f could be arbitrary and the nonlinearity of the equation is related to the possibly degenerate diffusion – nonlinearity quantified by:
Assumption (Nonlinear diffusion). The diffusion term, $Q(u)$, is *nonlinear* in the sense that

$$\exists \alpha \in (0,1)\ ,\ \delta_0 > 0 \quad : \quad meas\{u :\ 0 \leq Q'(u) \leq \delta\} \leq Const \cdot \delta^\alpha, \qquad \forall \delta \leq \delta_0 \ . \tag{5.22}$$

If (5.22) holds then equation (5.21) is at least $W^{s,1}$-regular with $s = \frac{2\alpha}{\alpha+4}$, by arguing along the lines of [12, §4-5]. Hence, we end up with L^p error estimate

$$\|u^\varepsilon(\cdot,t) - u(\cdot,t)\|_{L^p} \leq Const \cdot \begin{cases} \varepsilon^{\frac{s}{s+1}} & \forall p \in [1, s+1)\ , \\ \varepsilon^{\frac{1-p(1-s)}{sp}} & \forall p \in [s+1, \frac{1}{1-s})\ . \end{cases} \tag{5.23}$$

Remark. As before, we do not claim this regularity to be sharp: by borrowing a bootstrap argument from [12], one obtains $W^{s,1}$-regularity of order $s = \min\{1, \frac{8\alpha}{3\alpha+4}\}$. An even sharper regularity result of order $W^{2\alpha,1}$ is expected in this case, [17]. For example, for the porous media equation (where $Q(u) \sim u^m$, with $m > 2$ and consequently $\alpha = \frac{1}{m-1} < 1$), a regularity of order $W^{s,1}$ with $s = \frac{2}{m-1}$ yields L^1-error estimate of order $\mathcal{O}(\varepsilon^{\frac{2}{m+1}})$. Note that when $m \to 2^+$, this L_1-error estimate coincides with the one in (5.18).

6 Examples

In the first two examples, we consider the inhomogeneous Burgers' equation,

$$u_t^\varepsilon + f(u^\varepsilon)_x = \frac{1}{2\varepsilon^\lambda}\sin(2\pi\frac{x}{\varepsilon}) \qquad , \qquad f(u) = \frac{u^2}{2} \ , \tag{6.1}$$

with oscillatory initial data,

$$u^\varepsilon(x,0) = x + \cos(2\pi\frac{x}{\varepsilon}) \quad x \in [0,1] \qquad , \qquad u^\varepsilon(x+1,0) = u^\varepsilon(x,0) \ , \tag{6.2}$$

(the value of ε in all examples is $\varepsilon = 0.0408$). The corresponding homogenized problem is

$$u_t + f(u)_x = 0 \tag{6.3}$$

$$u(x,0) = x \quad x \in [0,1] \qquad , \qquad u(x+1,0) = u(x,0) \ . \tag{6.4}$$

First, we consider the case where the forcing data are not amplified, i.e., $\lambda = 0$. In Figure 1 we plot the oscillatory solution, $u^\varepsilon(\cdot,t)$, and the homogenized one, $u(\cdot,t)$ (in solid and dashed lines, respectively) for four values of t. The cancellation of the oscillations is reflected in the figures and we note that at $t = 0.04 \approx \varepsilon$, the two solutions are close in the strong L^∞-norm.

In Figure 2 we depict the two solutions when the oscillatory solution is subject to amplified forcing data, $\lambda = \frac{1}{2}$. The effect of that amplification is notable at $t = 0.1$.

Finally, we consider the porous media equation,

$$u_t = (|u|^{m-1}u)_{xx} \qquad m = 2 \ . \tag{6.5}$$

Here, u^ε is the solution of (6.5) subject to the oscillatory initial data,

$$u^\varepsilon(x,0) = \left\{\frac{x}{\varepsilon}\right\} \cdot \cos(2\pi x) \ , \tag{6.6}$$

where $\{y\}$ is the fractional part of y. Since $\int_0^1 \{y\}dy = \frac{1}{2}$, u^ε approaches u, the solution of (6.5) with the averaged initial data,

$$u(x,0) = \frac{1}{2}\cos(2\pi x) \ . \tag{6.7}$$

Both solutions are depicted in Figure 3.

The numerical results were obtained by the non-oscillatory high order central difference scheme in [13].

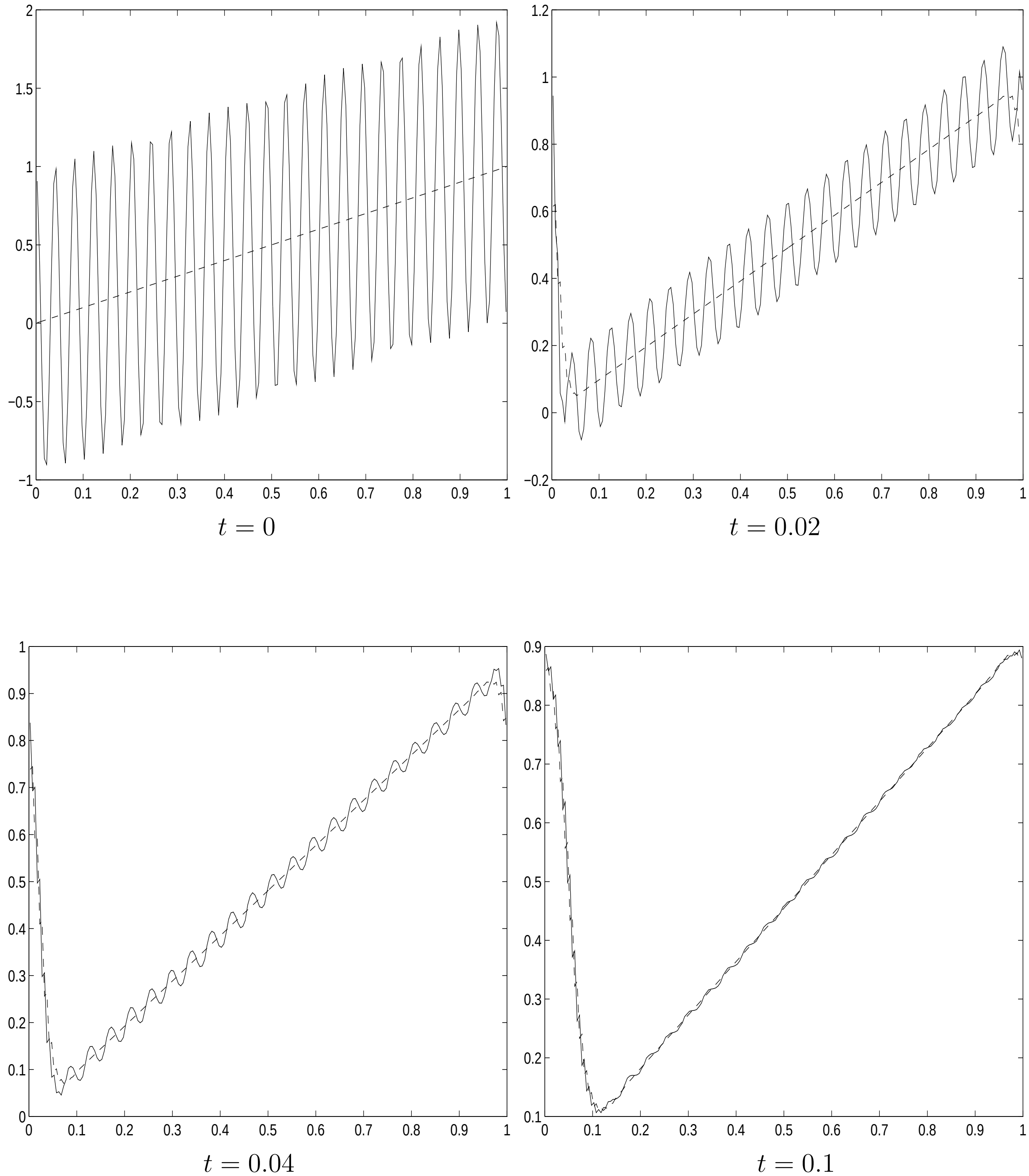

Figure 1

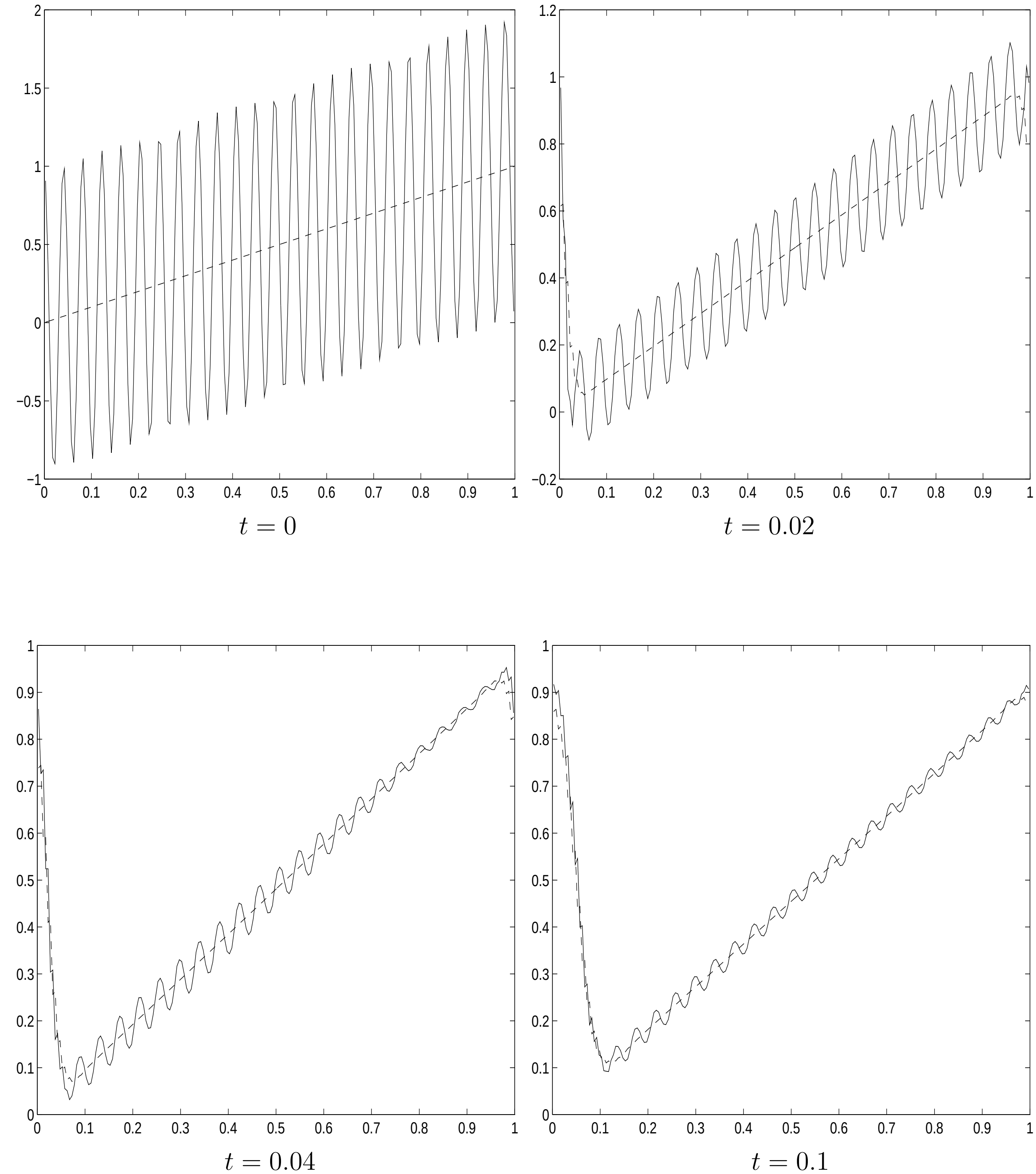

Figure 2

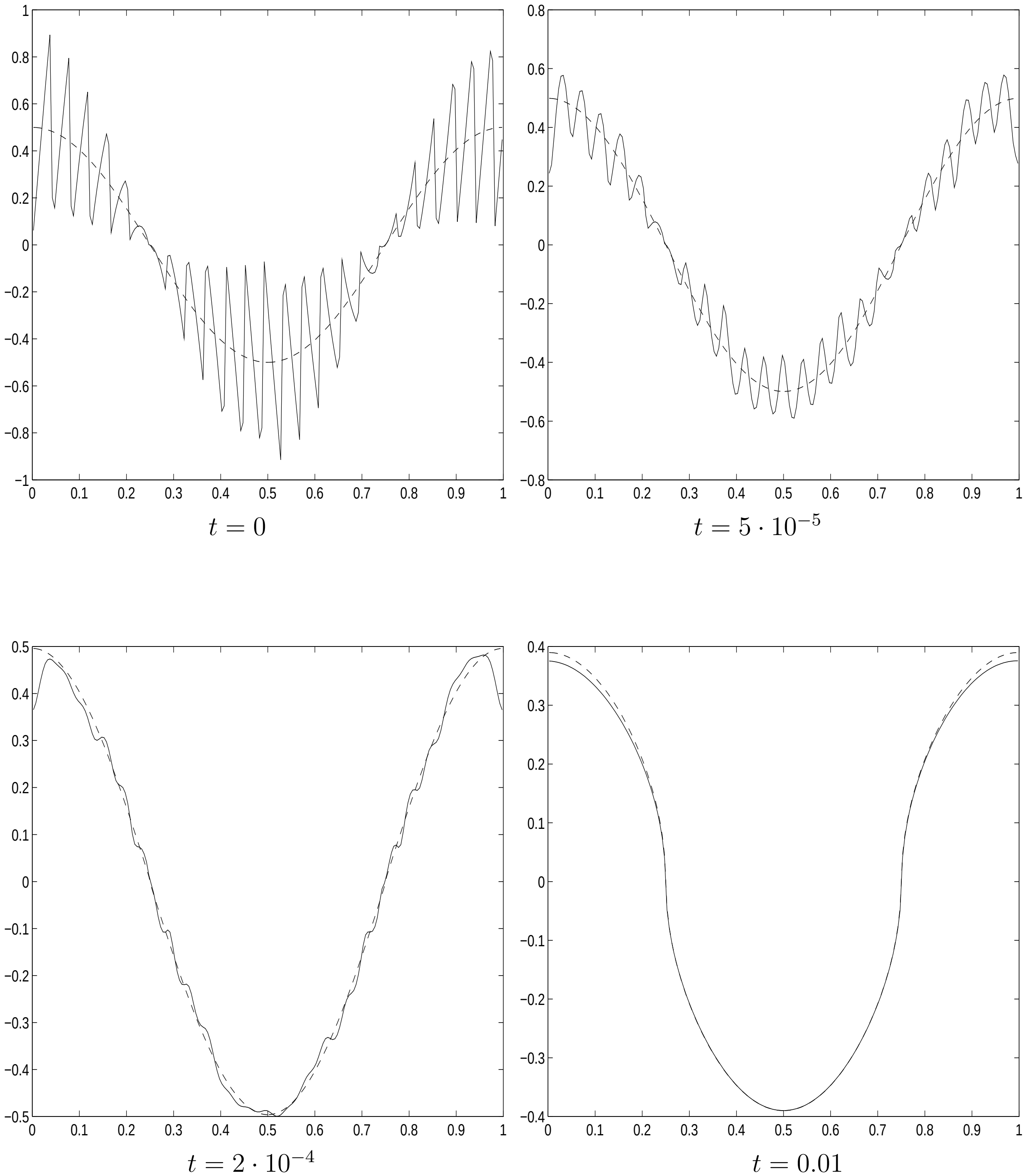

Figure 3

7 Appendix A: Lip^+-Stability

In this section we prove the Lip^+-stability of some (possibly degenerate) parabolic equations which were discussed in §5.

Proposition 7.1 *Consider the convective-diffusive equation (1.1),*

$$u_t = K(u,p)_x + h(x,t), \quad K_p \geq 0, \quad (x,t) \in \mathbb{R} \times \mathbb{R}^+, \tag{7.1}$$

with a Lip^+-bounded source term,

$$h_x(x,t) \leq c(t) < \infty \qquad \forall (x,t) \in \mathbb{R} \times \mathbb{R}^+, \tag{7.2}$$

and assume that $K(u, p \geq 0)$ is concave in u,

$$-K_{uu}(u,p) \geq \alpha > 0 \qquad \forall (u,p) \in \mathbb{R} \times \mathbb{R}^+. \tag{7.3}$$

Then the equation is Lip^+-stable and, for all $T > 0$, $\|u(\cdot,T)\|_{Lip^+}$ is bounded independently of the initial data as follows:

$$\|u(\cdot,T)\|_{Lip^+} \leq c \cdot \frac{\|u(\cdot,0)\|_{Lip^+} + c + (\|u(\cdot,0)\|_{Lip^+} - c)e^{-2\alpha cT}}{\|u(\cdot,0)\|_{Lip^+} + c - (\|u(\cdot,0)\|_{Lip^+} - c)e^{-2\alpha cT}} \leq c \cdot \frac{1 + e^{-2\alpha cT}}{1 - e^{-2\alpha cT}}\ , \tag{7.4}$$

where $c = c_T := \max_{0 \leq t \leq T} \sqrt{\dfrac{c(t)_+}{\alpha}}$.

Proof. We assume that $K_p > 0$; the degenerate case, $K_p \geq 0$, is treated by the standard procedure of replacing K by $K^\delta = K + \delta p$, $\delta \downarrow 0$.

Differentiating (7.1) with respect to x we find that $p = u_x$ is governed by

$$p_t = K_u \cdot p_x + (K_{uu} \cdot p + K_{up} \cdot p_x) \cdot p + K_p \cdot p_{xx} + \frac{dK_p}{dx} \cdot p_x + h_x\ .$$

Since $K_p > 0$, it follows that nonnegative maximal values of p satisfy

$$\frac{dp}{dt} \leq K_{uu} \cdot p^2 + h_x\ .$$

Hence, by (7.2) and (7.3), we get that in *positive* local maximal points,

$$\frac{dp}{dt} \leq -\alpha p^2 + c(t)\ .$$

Finally, estimate (7.4) follows from the last inequality in view of Lemma 7.1 below. $\square$

For the sake of completeness, we now prove an upper-bound estimate for a general Riccati ODE of the type encountered above.

Lemma 7.1 *Assume that $p = p(t)$ satisfies the Riccati-type inequality*

$$\frac{dp}{dt} \leq -a(t)p^2 + b(t)p + c(t) \ , \tag{7.5}$$

where $a(t)$ is uniformly positive,

$$a(t) \geq \alpha > 0 \qquad \forall t \geq 0, \tag{7.6}$$

and $b(t), c(t)$ are locally upper bounded functions. Then $p(t)_+$, $t > 0$, is upper-bounded <u>*independently*</u> *of the initial value $p(0)_+$, and the following estimate holds for all $T > 0$:*

$$p(T)_+ \leq b + c \cdot \frac{p(0)_+ - b + c + (p(0)_+ - b - c)e^{-2\alpha cT}}{p(0)_+ - b + c - (p(0)_+ - b - c)e^{-2\alpha cT}} \leq b + c \cdot \frac{1 + e^{-2\alpha cT}}{1 - e^{-2\alpha cT}}, \tag{7.7}$$

where

$$b = b_T := \frac{1}{2\alpha} \max_{0 \leq t \leq T} b(t), \qquad c = c_T := \max_{0 \leq t \leq T} \sqrt{b_T^2 + \frac{c(t)_+}{\alpha}}. \tag{7.8}$$

Proof. We fix $T > 0$ and denote by β_T and γ_T the upper bounds of $b(t)$ and $c(t)_+$, respectively, in $[0, T]$:

$$\beta_T := \max_{0 \leq t \leq T} b(t) \ , \quad \gamma_T := \max_{0 \leq t \leq T} c(t)_+ \ . \tag{7.9}$$

Using (7.6) and (7.9) in (7.5) we conclude that

$$\frac{dp}{dt} \leq -\alpha p^2 + b(t)p + \gamma_T \qquad \forall t \in [0, T] \ . \tag{7.10}$$

By standard arguments (which we omit), the positive part of $p(t)$ is majorized by $P(t)$, $p(t)_+ \leq P(t)$, where

$$\frac{dP}{dt} = -\alpha P^2 + \beta_T P + \gamma_T \qquad t \in [0, T] \ , \tag{7.11}$$

subject to the same initial value, $P(0) = p(0)_+$. Equation (7.11) may be now rewritten in the equivalent form

$$\frac{dP}{dt} = -\alpha(P - b_T)^2 + \alpha c_T^2 \qquad t \in [0, T] \ , \tag{7.12}$$

where the constants, $b = b_T$ and $c = c_T$, are specified in (7.8). The solution of this equation gives

$$P(t) = b + c \cdot \frac{P(0) - b + c + (P(0) - b - c)e^{-2\alpha ct}}{P(0) - b + c - (P(0) - b - c)e^{-2\alpha ct}} \qquad t \in [0, T] \ . \tag{7.13}$$

We conclude that $p(T)_+$, being dominated by $P(T)$, is bounded by

$$p(T)_+ \leq b + c \cdot \frac{p(0)_+ - b + c + (p(0)_+ - b - c)e^{-2\alpha cT}}{p(0)_+ - b + c - (p(0)_+ - b - c)e^{-2\alpha cT}}. \tag{7.14}$$

Finally, we observe that the right hand side of (7.14) may be upper-bounded independently of $p(0)_+$ and, consequently,

$$p(T)_+ \leq b + c \cdot \frac{1 + e^{-2\alpha cT}}{1 - e^{-2\alpha cT}}, \tag{7.15}$$

which completes the proof. $\square$

8 Appendix B

Here, we would like to concentrate on the special case where there is no explicit dependence on x in (2.6),

$$u^\varepsilon_t = K(u^\varepsilon, u^\varepsilon_x)_x + h(\frac{x}{\varepsilon}, t), \qquad u^\varepsilon(x,0) = u_0(\frac{x}{\varepsilon}) \ ,$$

and propose an alternative simpler proof of Theorem 2.1 (for the sake of simplicity we concentrate on the case $\lambda = 0$; the case of amplified modulations, $0 < \lambda < 1$, may be easily treated in the same manner as before). In this case, the solution $u^\varepsilon(\cdot, t)$ is ε-periodic for all $t \geq 0$ (since $u^\varepsilon(\cdot, 0)$ is and the equation remains invariant under translations $x \mapsto x + \varepsilon$). The homogenized problem takes the form (compare to (2.7))

$$u_t = K(u, u_x)_x + \bar{h}(t), \qquad u(x,0) = \bar{u}_0 \ ,$$

where

$$\bar{h}(t) = \int_0^1 h(y,t)dy \quad \text{and} \quad \bar{u}_0 = \int_0^1 u_0(y)dy \ .$$

The solution of that problem does not depend on x and is given by

$$u(x,t) = u(t) = \bar{u}_0 + \int_0^t \bar{h}(\tau)d\tau \ .$$

This value of the homogenized solution at time t equals, as can be easily seen, to the averaged value of the oscillatory solution at the same time, i.e.,

$$u(\cdot, t) = \frac{1}{\varepsilon}\int_x^{x+\varepsilon} u^\varepsilon(y,t)dy \ .$$

Therefore, the $W^{-1,\infty}$-error estimate, (2.8), is a direct consequence in this case of the following simple proposition:

Proposition 8.1 *Let $g(y)$ be a bounded 1-periodic function; let $\bar{g}$ denote its average, $\bar{g} := \int_0^1 g(y)dy$, and $g_\varepsilon(x) := g(\frac{x}{\varepsilon})$. Then there exists a constant $C > 0$, independent of ε, such that for all $1 \leq p \leq \infty$:*

$$\|g_\varepsilon(x) - \bar{g}\|_{W^{-1,p}[0,1]} \leq C \cdot \varepsilon \ . \tag{8.1}$$

Before proving this proposition, we state and prove a useful lemma which is interesting for its own:

Lemma 8.1 *Let $w(x)$ be a function in $L^p(I)$ where $I = (a,b)$ is a (possibly unbounded) interval in $\mathbb{R}$ and $1 \leq p \leq \infty$. Let $W(x) := \int_a^x w(\xi)d\xi$ be the primitive of w. Consider the division of I into subintervals, I_j, induced by the zeroes of W, i.e.,*

$$I = \uplus_{j \in J} I_j \qquad I_j = [x_j, x_{j+1})$$

where, for all $j \in J$,

$$W(x_j) = 0 \quad \text{and} \quad W(x) \neq 0 \ \ \forall x \in (x_j, x_{j+1}) \ .$$

Then

$$\|w\|_{W^{-1,p}(I)} \leq \max_{j \in J} |I_j| \cdot \|w\|_{L^p(I)} \ . \tag{8.2}$$

Proof. For any $p < \infty$ (– the conclusion for $p \uparrow \infty$ is thus straightforward) we have

$$\|w\|^p_{W^{-1,p}(I)} = \sum_{j\in J}\int_{I_j} |W(x)|^p dx = \sum_{j\in J}\int_{I_j}\left|\int_{x_j}^x w(y)dy\right|^p dx \le \sum_{j\in J}\int_{I_j}\left(\int_{x_j}^x |w(y)|dy\right)^p dx.$$

If we let K denote the size of the maximal subinterval, $K = \max_{j\in J}|I_j|$, we get by Hölder inequality that for $x \in I_j$,

$$\int_{x_j}^x |w(y)|dy \le \int_{I_j} |w(y)|dy \le K^{\frac{1}{p'}}\|w\|_{L^p(I_j)}, \qquad \frac{1}{p}+\frac{1}{p'} = 1.$$

Combining the two last inequalities, we obtain the desired result (8.2):

$$\|w\|^p_{W^{-1,p}(I)} \le \sum_{j\in J}\int_{I_j} K^{\frac{p}{p'}}\|w\|^p_{L^p(I_j)}dx \le \sum_{j\in J} K^{\frac{p}{p'}+1}\|w\|^p_{L^p(I_j)} = K^p\|w\|^p_{L^p(I)}\ .$$

Proof of Proposition 8.1. Denote $w_\varepsilon(x) := g_\varepsilon(x) - \bar{g}$. It can be easily seen that for all $1 \le p \le \infty$,

$$\|w_\varepsilon\|_{L^p[0,1]} \le 2\|g\|_{L^p[0,1]} + |\bar{g}| \le C, \qquad C := 3\|g\|_{L^\infty[0,1]}.$$

The key point is that due to the 1-periodicity of $g(x)$, the primitive $W_\varepsilon(x) := \int_0^x w_\varepsilon$ vanishes at the points $j\varepsilon$ for any integer j. Hence, (8.1) follows from the simplest version of (8.2) with equidistant zeroes at a distance of $|I_j| = \varepsilon$. □

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Eitan Tadmor
School of Mathematical Sciences
Tel-Aviv University

Tel-Aviv 69978
Israel

Tamir Tassa
Department of Mathematics
University of California
Los Angeles, CA 90095
U.S.A