Finding All Maximally-Matchable Edges in a Bipartite Graph

Tamir Tassa*

Abstract

We consider the problem of finding all maximally-matchable edges in a bipartite graph $G = (V, E)$, i.e., all edges that are included in some maximum matching. We show that given any maximum matching in the graph, it is possible to perform this computation in linear time $O(n + m)$ (where $n = |V|$ and $m = |E|$). Hence, the time complexity of finding all maximally-matchable edges reduces to that of finding a single maximum matching, which is $O(n^{1/2}m)$ (Hopcroft and Karp [12]), or $O((n/\log n)^{1/2}m)$ for dense graphs with $m = \Theta(n^2)$ (Alt et al. [2]). This time complexity improves upon that of the best known algorithms for the problem, which is $O(nm)$ (Costa [5] for bipartite graphs, and Carvalho and Cheriyan [6] for general graphs). Other algorithms for solving that problem are randomized algorithms due to Rabin and Vazirani [15] and Cheriyan [3], the runtime of which is $\tilde{O}(n^{2.376})$. Our algorithm, apart from being deterministic, improves upon that time complexity for bipartite graphs when $m = O(n^r)$ and $r < 1.876$. In addition, our algorithm is elementary, conceptually simple, and easy to implement.

Keywords. Bipartite graphs, perfect matchings, maximum matchings, maximally-matchable edges

1   Introduction

Let V_1 be a set of men and V_2 be a set of women that are registered in a matchmaking agency. Each of the men (women) has an associated description and given preferences regarding the woman (man) with whom he (she) wishes to get acquainted. Those descriptions and preferences induce a bipartite graph G of compatibilities on the set of nodes $V = V_1 \cup V_2$. The agency then presents to the clients the links that are relevant to them, so that they can choose whom to meet. Some of those links may not be extended to a maximum matching in the bipartite graph. The agency has an incentive to identify those links upfront and not offer them to its clients, since if they would be offered and one of them would be successful, it would prevent the possibility of achieving a maximum matching. Therefore, given the full bipartite graph G, it is necessary to remove from it all edges that are not part of a maximum matching, or, alternatively speaking, it is needed to compute the subgraph of G that equals the union of all maximum matchings in G.

In order to formalize this problem, we recall some of the basic terminology regarding matchings in graphs.

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*Department of Mathematics and Computer Science, The Open University, 1 University Road, Ra'anana, Israel 43537. Telephone: +972-9-7781272. Fax: +972-9-7780706. Email: `tamirta@openu.ac.il`

Definition 1.1. *Let $G = (V, E)$ be a graph.*

- *A matching in G is a collection $M \subseteq E$ of non-adjacent edges.*

- *A matching M is called maximal if it is not a proper subset of any other matching.*

- *A maximal matching M is called a maximum (cardinality) matching if there does not exist a matching with a greater cardinality.*

- *A maximum matching M is called perfect if it covers all of the nodes in V.*

- *An edge $e \in E$ is a maximally-matchable edge if it is included in some maximum matching.*[1]

We study here one of the fundamental problems in matching theory, namely, the identification of all maximally-matchable edges in a given graph G. We focus here on bipartite graphs, i.e. $G = (V, E)$ where $V = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$, and $E \subseteq V_1 \times V_2$, and devise an efficient algorithm for that problem in such graphs.

Our main contributions herein are summarized as follows: In case we are given one of the maximum matchings in the bipartite graph $G = (V, E)$, Algorithm 3 in Section 2 finds all maximally-matchable edges in linear time $O(n + m)$, where $n = |V|$ and $m = |E|$. (In the application that motivated this study there was indeed one "natural" maximum matching that was known without any computation, see Section 5.) Without such prior knowledge on the graph, we may first find any maximum matching in the graph, and then proceed to apply Algorithm 3 in order to complete the computation in additional linear time. Algorithm 1 implements this approach. Its runtime is dominated by the runtime of the procedure for finding a maximum matching.

Algorithm 1 Finding all maximally-matchable edges in a bipartite graph $G = (V, E)$.

Input: A bipartite graph $G = (V, E)$.
Output: All maximally-matchable edges in E.
  1: Find a maximum matching in G, say M.
  2: Using M, invoke Algorithm 3 in order to find all maximally-matchable edges in G.

The Hopcroft-Karp algorithm [12] offers the best known worst-case performance for finding a maximum matching in a bipartite graph, with a runtime of $O(n^{1/2}m)$. For dense bipartite graphs, a slightly better alternative exists: An algorithm by Alt et al. [2] finds a maximum matching in a bipartite graph in time $O\left(n^{3/2}\left(\frac{m}{\log n}\right)^{1/2}\right)$. In cases where $m = \Theta(n^2)$, it becomes $O((n/\log n)^{1/2}m)$, whence it is a $(\log n)^{1/2}$-factor faster than the Hopcroft-Karp algorithm.

A faster algorithm for finding a maximum matching in bipartite graphs was recently proposed by Goel et al. [10] for the special case of d-regular graphs (graphs in which all nodes have degree d). However, in such graphs all edges are maximally-matchable (see more on that in Section B).

The paper is organized as follows. The algorithm for finding all maximally-matchable edges from any maximum matching in the bipartite graph is given in Section 2. In Section 3, we consider the problem in a dynamic setting and discuss an efficient implementation in such settings. In Section 4, we discuss related work and compare our algorithm to the leading algorithms. We conclude in Section 5 by describing problems from data privacy that motivated this study, and we suggest future

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[1]Such edges are sometimes called *allowed* edges, e.g. [6, 13].

research. In the Appendix, we comment on the size of a matching to which a given set of non-adjacent edges can be extended (Section A), and prove that in regular bipartite graphs all edges are maximally-matchable (Section B).

2 A linear time algorithm for finding all maximally-matchable edges from any maximum matching

We begin by considering in Section 2.1 the case of balanced graphs ($|V_1| = |V_2|$) with a perfect matching. In Section 2.2 we consider the general case.

2.1 Bipartite graphs with a perfect matching

Let us denote the nodes of the graph as follows: $V_1 = \{v_1, \ldots, v_{\tilde{n}}\}$ and $V_2 = \{v'_1, \ldots, v'_{\tilde{n}}\}$ (here, $\tilde{n} = \frac{n}{2}$). The graph G is assumed to have at least one known perfect matching; without loss of generality, we assume that it is $M := \{(v_1, v'_1), \ldots, (v_{\tilde{n}}, v'_{\tilde{n}})\}$.

Definition 2.1. *A set of $\ell \geq 1$ edges in the graph G is called an* alternating cycle *(with respect to M) if there exist ℓ distinct indices, $1 \leq i_1, \ldots, i_\ell \leq \tilde{n}$, such that the ℓ edges are (v_{i_1}, v'_{i_2}), (v_{i_2}, v'_{i_3}),..., $(v_{i_{\ell-1}}, v'_{i_\ell})$, (v_{i_ℓ}, v'_{i_1}).*

Theorem 2.2. *Let $G = (V, E)$ be a balanced bipartite graph as described above, and assume that $M := \{(v_1, v'_1), \ldots, (v_{\tilde{n}}, v'_{\tilde{n}})\} \subseteq E$. Then an edge $e \in E$ is maximally-matchable if and only if it is included in an alternating cycle.*

Proof. Assume that $e = (v_{i_1}, v'_{i_2})$ is part of an alternating cycle, say (v_{i_1}, v'_{i_2}), (v_{i_2}, v'_{i_3}),..., $(v_{i_{\ell-1}}, v'_{i_\ell})$, (v_{i_ℓ}, v'_{i_1}). If we augment this alternating cycle with the $\tilde{n} - \ell$ edges (v_i, v'_i) for all $i \notin \{i_1, \ldots, i_\ell\}$ we get a perfect matching. Hence, e is maximally-matchable.

Assume next that $e = (v_{i_1}, v'_{i_2})$ is maximally-matchable. Then it is included in some perfect matching M'. We proceed to define a sequence of edges $S_e \subseteq M'$ in the following manner: $e_1 := e = (v_{i_1}, v'_{i_2})$; then, for all $j \geq 1$, if $e_j = (v_{i_j}, v'_{i_{j+1}})$, we set $e_{j+1} := (v_{i_{j+1}}, v'_{i_{j+2}})$, where $v'_{i_{j+2}}$ is the node that M' matches with $v_{i_{j+1}}$. Since M' is finite, the sequence must repeat itself at some point. Namely, there must exist a minimal index $j_0 \geq 2$ such that e_{j_0} equals one of the previous edges in S_e. It is easy to see that e_{j_0} must coincide with e_1 since if e_{j_0} would coincide with, say e_2, then M' would have included two different edges that are adjacent to v_{i_2}. But then the sequence S_e up to that point, $S_e = \{e_1 = e, e_2, \ldots, e_{j_0-1}\}$ is an alternating cycle with respect to M. Hence, every maximally-matchable edge must be contained in an alternating cycle. $\square$

Next, we define the directed graph $H = (U, F)$ that is induced by the bipartite graph $G = (V, E)$. In the directed graph $H = (U, F)$ the set of nodes is $U = \{u_1, \ldots, u_{\tilde{n}}\}$ and (u_i, u_j) is a directed edge in that graph if and only if $i \neq j$ and $(v_i, v'_j) \in E$. See example of a bipartite graph G and the corresponding directed graph H in Figure 1.

It is easy to see that, in view of Theorem 2.2, an edge $(v_i, v'_j) \in E$ is maximally-matchable in G if and only if $i = j$ or the edge $(u_i, u_j) \in F$ is part of a directed cycle in H. For example, the edge (v_3, v'_4) is maximally-matchable in G that is given in Figure 1 since it is part of the perfect matching $\{(v_1, v'_1), (v_2, v'_3), (v_3, v'_4), (v_4, v'_2)\}$ (in which the last three edges are an alternating cycle), and indeed the corresponding edge (u_3, u_4) is part of a directed cycle of length 3 in H.

Therefore, the problem of finding all maximally-matchable edges in G reduces to the problem of finding all edges in the directed graph H that are part of a directed cycle. This may be achieved as
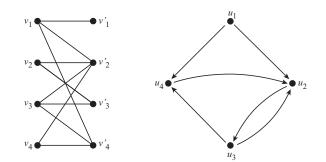
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Figure 1. A bipartite graph G and the corresponding directed graph H

follows: First, one has to find all strongly connected components of H; namely, all maximal strongly connected subgraphs of H (recall that a directed graph is strongly connected if there is a path from each node in the graph to every other node). If each strongly connected component is contracted to a single node, the resulting graph is a directed acyclic graph. Consequently, a given edge in H is part of a directed cycle if and only if it connects two nodes in the same strongly connected component. There are several efficient algorithms for finding the strongly connected components of a given directed graph. Tarjan's algorithm [17] and Cheriyan-Mehlhorn-Gabow algorithm [4] are both equally efficient with a linear runtime. Finally, we use those findings to mark all maximally-matchable edges in the original bipartite graph G. Algorithm 2 does all of the above. Its runtime is $O(n + m)$.

Algorithm 2 Finding all maximally-matchable edges in a bipartite graph, given a perfect matching.

Input: A bipartite graph $G = (V, E)$ where $V = V_1 \cup V_2$, $V_1 = \{v_1, \ldots, v_{\tilde{n}}\}$, $V_2 = \{v'_1, \ldots, v'_{\tilde{n}}\}$, $E \subseteq V_1 \times V_2$, and for all $1 \leq i \leq \tilde{n}$, $(v_i, v'_i) \in E$.

Output: All maximally-matchable edges in E.

 1: Mark all edges in E as not maximally-matchable.
 2: Construct the directed graph $H = (U, F)$ that corresponds to G.
 3: Find all strongly connected components of H.
 4: **for** all edges $(u_i, u_j) \in F$ **do**
 5:    **if** u_i and u_j belong to the same strongly connected component in H **then**
 6:       Mark the edge $(v_i, v'_j) \in E$ as maximally-matchable.
 7:    **end if**
 8: **end for**
 9: Mark all edges $(v_i, v'_i) \in E$ as maximally-matchable, $1 \leq i \leq \tilde{n}$.

2.2 General bipartite graphs

After dealing with balanced bipartite graphs that have a perfect matching, we turn our attention to the general case. Hereinafter we let $G = (V, E)$ be a bipartite graph where the two parts of the graph are $V_1 = \{v_1, \ldots, v_{n_1}\}$ and $V_2 = \{v'_1, \ldots, v'_{n_2}\}$, $n_1 \leq n_2$, and the maximum matchings are of size $t \leq n_1$. We may assume that $t < n_2$ since if $t = n_2$, G is a balanced bipartite graph with a perfect matching, and we already solved the problem for such graphs. Let M be a given maximum matching in G. We may assume, without loss of generality, that $M := \{(v_1, v'_1), \ldots, (v_t, v'_t)\}$.

Definition 2.3. *A node v_i or v_i' is called an M-upper node if $i \leq t$ and an M-lower node if $i > t$. An edge (v_i, v_j') in G is called an M-upper edge if it connects two M-upper nodes; all other edges are called M-lower edges.*

In other words, a node is called M-upper if it is covered by M, and called M-lower otherwise. (The terminology simply reflects the fact that the so-called M-upper nodes appear in the upper part of the graph.) Consider, for example, the graph in the left of Figure 2. In that graph $n_1 = n_2 = 4$, $t = 3$, and $M = \{(v_1, v_1'), (v_2, v_2'), (v_3, v_3')\}$; the graph has six M-upper nodes and two M-lower ones. The edges (v_3, v_4') and (v_4, v_1') are M-lower edges, while all other five edges are M-upper.
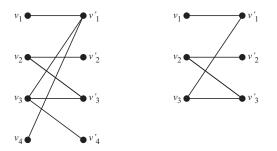

Figure 2. A bipartite graph G and the corresponding restriction G_u

For simplicity, we shall use hereinafter the terms upper and lower without specifying the prefix M. It should always be understood that a node or an edge are upper or lower with respect to the given M.

Our first observation towards a classification of all maximally-matchable edges in such a graph is as follows.

Proposition 2.4. *Let $G = (V, E)$ be a bipartite graph as described above. Then:*

1. *The graph has no lower edges that connect two lower nodes.*

2. *All lower edges are maximally-matchable.*

Proof. The first claim is obvious since if there was a lower edge (v_i, v_j') where $i, j > t$, then $M \cup \{(v_i, v_j')\}$ would have been a matching, thus contradicting the assumed maximality of M. As for the second claim, let (v_i, v_j') be a lower edge with $i > t$. In view of the first claim, v_j' must be an upper node. Therefore, $\left(M \cup \{(v_i, v_j')\} \right) \setminus \{(v_j, v_j')\}$ is also a maximum matching, whence (v_i, v_j') is a maximally-matchable edge. $\qquad\square$

As Proposition 2.4 determines that all lower edges are maximally-matchable, it remains to identify the maximally-matchable edges among the upper edges. Let G_u denote the restriction of G to the upper nodes, $\{v_1, v_1', \ldots, v_t, v_t'\}$. We distinguish between two types of maximally-matchable edges among the upper edges:

Definition 2.5. *An upper edge is a maximally-matchable edge of type I (with respect to M) if it is included in a maximum matching that consists only of upper edges. An upper edge is a maximally-matchable edge of type II (with respect to M) if it is maximally-matchable, but all maximum matchings that include it include also a lower edge.*

For example, consider the graph G on the left of Figure 2 and the corresponding G_u on the right. The edge (v_3, v_1') is not maximally-matchable, but all other six edges are maximally-matchable:

1. The edges (v_3, v_4') and (v_4, v_1') are maximally-matchable since they are lower edges. (For example, (v_3, v_4') may be extended to a maximum matching with (v_1, v_1') and (v_2, v_2').)

2. The edges (v_1, v_1'), (v_2, v_2'), and (v_3, v_3') are maximally-matchable edges of type I since they are maximally-matchable edges also in G_u.

3. The edge (v_2, v_3') is a maximally-matchable edge of type II since it is maximally-matchable (together with (v_1, v_1') and (v_3, v_4') it forms a maximum matching), but it is not maximally-matchable in G_u.

All maximally-matchable edges of type I can be identified by applying Algorithm 2 on the subgraph G_u (since G_u is a balanced graph with a perfect matching). It remains only to identify the maximally-matchable edges of type II.

Definition 2.6. *A set P of $\ell - 1 \geq 1$ upper edges in G is called an* upper alternating path *(with respect to M) if there exist ℓ distinct indices, $1 \leq i_1, \ldots, i_\ell \leq t$, such that*

$$P = \{(v_{i_1}, v_{i_2}'), (v_{i_2}, v_{i_3}'), \ldots, (v_{i_{\ell-1}}, v_{i_\ell}')\}. \tag{1}$$

If $i_0 > t$ and $\lambda = (v_{i_0}, v_{i_1}')$ is a lower edge in E then $\{\lambda\} \cup P$ is called a left-augmented alternating path. *If $i_{\ell+1} > t$ and $\rho = (v_{i_\ell}, v_{i_{\ell+1}}')$ is a lower edge in E then $P \cup \{\rho\}$ is called a* right-augmented alternating path. *Finally, we let $\mathcal{AP}$ denote the set consisting of all alternating paths of all three sorts — upper, left- and right-augmented.*

For example, $P = \{(v_2, v_3')\}$ is an upper alternating path of length 1 in the graph G of Figure 2, while $P \cup \{(v_3, v_4')\}$ is a corresponding right-augmentation.

Theorem 2.7. *Any maximally-matchable edge of type II with respect to M must be contained in an either left- or right-augmented alternating path in G with respect to M. On the other hand, an edge that is contained in an either left- or right-augmented alternating path is maximally-matchable.*

Proof. Let $e = (v_i, v_{i'}')$ be an upper edge which is a maximally-matchable edge of type II with respect to M. Let M_e be a maximum matching that includes e. Define

$$\mathcal{AP}_e = \{P \in \mathcal{AP} : P \subseteq M_e \text{ and } e \in P\};$$

namely, $\mathcal{AP}_e$ consists of all alternating paths (upper, left- and right-augmented ones) that are contained in M_e and include e. Clearly, $\mathcal{AP}_e$ is nonempty since it includes the upper alternating path of length 1 that consists only of e. Hence, we may select a path $P \in \mathcal{AP}_e$ of maximal length among all paths in $\mathcal{AP}_e$. We claim, and prove below, that P cannot be an upper alternating path. Hence, P must be either left- or right-augmented alternating path. Since P includes e, that will settle the necessary condition in the theorem: Any maximally-matchable edge of type II with respect to M must be contained in an either left- or right-augmented alternating path with respect to M.

Assume that P is an upper alternating path. Denoting its length by $\ell - 1$, P must take the form

$$P = \{(v_{i_1}, v_{i_2}'), (v_{i_2}, v_{i_3}'), \ldots, (v_{i_{\ell-1}}, v_{i_\ell}')\},$$

where $1 \leq i_1, \ldots, i_\ell \leq t$. We claim that the matching M_e does not match neither v_{i_1}' nor v_{i_ℓ}. Assume, towards contradiction, that M_e includes an edge (v_j, v_{i_1}'). Then j cannot be any of the

indices outside the set $\{i_1, \ldots, i_\ell\}$, as that would contradict the assumed maximality of P. In addition, j cannot be any of the indices in $\{i_1, \ldots, i_{\ell-1}\}$ since then M_e would include two edges that are adjacent to v_j. Finally, j cannot be i_ℓ since then e would have been contained in an alternating cycle in G_u (Definition 2.1) and then, by Theorem 2.2, e would have been maximally-matchable in G_u, thus contradicting our assumption that e is a maximally-matchable edge of type II. Therefore, M_e does not match v'_{i_1}. For the same reasons, M_e cannot match v_{i_ℓ}. But then, consider the set of edges

$$M'_e = (M_e \setminus P) \cup \{(v_{i_k}, v'_{i_k}) : 1 \le k \le \ell\}.$$

In view of the above discussion, M'_e is a matching (since $M_e \setminus P$ does not cover any of the nodes v_{i_k} and v'_{i_k}, $1 \le k \le \ell$). But as $|M'_e| = |M_e| + 1$, it contradicts our assumption that M_e is a maximum matching.

Next, we turn to prove the sufficiency of the condition; namely, that an edge that is contained in an either left- or right-augmented alternating path with respect to M must be maximally-matchable in G. Let

$$P = \{(v_{i_0}, v'_{i_1}), (v_{i_1}, v'_{i_2}), (v_{i_2}, v'_{i_3}), \ldots, (v_{i_{\ell-1}}, v'_{i_\ell})\}$$

be a left-augmented path in G (namely, $i_0 > t$ and $i_k \le t$ for all $1 \le k \le \ell$). Then

$$M_0 = P \cup \{(v_j, v'_j) : j \in \{1, \ldots, t\} \setminus \{i_1, \ldots, i_\ell\}\} \tag{2}$$

is a matching of cardinality t. Hence, all edges in P are included in a maximum matching, whence they are all maximally-matchable. (One of those maximally-matchable edges, namely (v_{i_0}, v'_{i_1}), is a lower edge, while all the other upper edges may be maximally-matchable edges of either type I or II). The proof for right-augmented paths is similar. $\qquad\square$

Hence, in order to identify all maximally-matchable edges of type II with respect to M, we have to scan all left- and right-augmented alternating paths and mark all upper edges along them as being maximally-matchable (some of those maximally-matchable edges may be of type I, whence they will be "discovered" twice). To do that, we use again the representation of G as a directed graph $H = (U, F)$.

Definition 2.8. *Let $G = (V, E)$ be a bipartite graph where the two parts of the graph are $V_1 = \{v_1, \ldots, v_{n_1}\}$ and $V_2 = \{v'_1, \ldots, v'_{n_2}\}$, and $n_1 \le n_2$. The corresponding left-to-right directed graph is $H_{LR} = (U, F_{LR})$ where $U = \{u_1, \ldots, u_{n_2}\}$ and $(u_i, u_j) \in F_{LR}$ if and only if $(v_i, v'_j) \in E$. The right-to-left directed graph is $H_{RL} = (U, F_{RL})$ where $U = \{u_1, \ldots, u_{n_2}\}$ and $(u_i, u_j) \in F_{RL}$ if and only if $(v_j, v'_i) \in E$.*

With those definitions, an edge (v_i, v'_j) is on a left-augmented alternating path if and only if the edge (u_i, u_j) is reachable in H_{LR} from one of the nodes u_k, where $t+1 \le k \le n_2$. Hence, all that is needed is to perform a BFS of H_{LR} from those nodes and for every edge (u_i, u_j) that we visit along this scan, mark the corresponding edge (v_i, v'_j) in G as a maximally-matchable edge. Similarly, an edge (v_i, v'_j) is on a right-augmented alternating path if and only if the edge (u_j, u_i) is reachable in H_{RL} from one of the nodes u_k, where $t + 1 \le k \le n_2$. Here, we need to perform a BFS of H_{RL} from those nodes and for every edge (u_j, u_i) that we visit along this scan, mark the corresponding edge (v_i, v'_j) in G as a maximally-matchable edge. The runtime of such a scan is linear. Algorithm 3 summarizes the procedure of finding all maximally-matchable edges in a general bipartite graph.

Algorithm 3 Finding all maximally-matchable edges in a general bipartite graph, given a maximum matching.

Input: A bipartite graph $G = (V, E)$ where $V = V_1 \cup V_2$, $V_1 = \{v_1, \ldots, v_{n_1}\}$, $V_2 = \{v'_1, \ldots, v'_{n_2}\}$, $E \subseteq V_1 \times V_2$, and $\{(v_i, v'_i) : 1 \leq i \leq t\}$ is a given maximum matching in G.

Output: All maximally-matchable edges in E.

1: Mark all lower edges as maximally-matchable.
2: Mark all upper edges as not maximally-matchable.
3: Apply Algorithm 2 on the restriction G_u of G to $\{v_1, v'_1, \ldots, v_t, v'_t\}$, thus marking all upper edges that are maximally-matchable in G_u.
4: Construct the left-to-right directed graph H_{LR}.
5: Add to H_{LR} a new source node u_0 and connect it to each of the nodes u_k, where $t+1 \leq k \leq n_1$.
6: Apply a BFS on the graph H_{LR} starting from u_0. For each edge (u_i, u_j) that the BFS visits, mark the corresponding edge (v_i, v'_j) in G as maximally-matchable.
7: Construct the right-to-left directed graph H_{RL}.
8: Add to H_{RL} a new source node u_0 and connect it to each of the nodes u_k, where $t+1 \leq k \leq n_2$.
9: Apply a BFS on the graph H_{RL} starting from u_0. For each edge (u_i, u_j) that the BFS visits, mark the corresponding edge (v_j, v'_i) in G as maximally-matchable.

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3 The dynamic setting

Going back to the opening example of the matchmaking agency, if one of the maximally-matchable edges materializes and another couple of clients leaves happily the matching game, the agency has to update the set of maximally-matchable edges in the reduced graph. Here we comment on how to cope with such dynamic updates efficiently. Before doing so, we describe another interesting example.

A domino board is a bounded region D in the Euclidean plane which is the union of unit squares ($S_{i,j} := [i, i+1] \times [j, j+1]$ for some $i, j \in \mathbb{N}$) that are connected in the sense that there exists a path between the centers of every two unit squares that is fully contained within the interior of D. Each unit square $S_{i,j}$ is colored white in case $i + j$ is even, or black otherwise. A perfect tiling of the region D is a cover of D by $|D|/2$ non-overlapping dominos ($|D|$ being the area of D), where dominos are shapes formed by the union of two unit squares meeting edge-to-edge.

Now, let us consider the following computerized game: A domino region that has a perfect tiling is presented to the player, who needs to tile it by placing dominos on it, one at a time. If he places a domino in a location that prevents any completion of the tiling, the computer issues an alert and then the player has a chance to try again. The player wins if he was able to complete the tiling successfully with a number of bad moves below some predefined threshold.

Here too, there is an underlying bipartite graph, where each node represents a white or a black square in D. Each square is connected to any of its neighboring squares of the opposite color. A perfect tiling of D is a perfect matching in that graph. The bad moves correspond to edges in that graph that are not maximally-matchable. In each step in the game, the player places one domino; it is then needed to remove the corresponding nodes and adjacent edges from the underlying graph and update the list of maximally-matchable edges in the reduced graph.

Following the two motivating examples, we now formulate the problem in the dynamic setting. Let G be a bipartite graph as discussed in Section 2.2. Assume that we already found a maximum matching in G, say $M := \{(v_1, v'_1), \ldots, (v_t, v'_t)\}$, and identified all maximally-matchable edges in G. In addition, we may assume that each maximally-matchable edge is marked by its type:

In balanced graphs with a perfect matching all maximally-matchable edges are of the same type, but in other graphs we distinguish between maximally-matchable edges that are lower edges, and maximally-matchable edges that are upper edges of type I or of type II. Let (v_i, v'_j) be a maximally-matchable edge in G and let G' be the graph that is obtained from G by removing v_i, v'_j, and all adjacent edges. We wish to find a maximum matching in G' in an efficient manner, in order to proceed and identify in linear time all maximally-matchable edges in G'. We separate the discussion to three cases:

- If (v_i, v'_j) is a maximally-matchable edge of type I, we look in the directed graph H_{LR} for a path from u_j to u_i. Assume that the path is

$$(u_j, u_{j_1}), (u_{j_1}, u_{j_2}), \ldots, (u_{j_{\ell-1}}, u_{j_\ell}), (u_{j_\ell}, u_i) \, .$$

Then it is easy to see that

$$M \setminus \{(v_k, v'_k) : k \in \{i, j, j_1, \ldots, j_\ell\}\} \cup \{(v_j, v'_{j_1}), (v_{j_1}, v'_{j_2}), \ldots, (v_{j_{\ell-1}}, v'_{j_\ell}), (v_{j_\ell}, v'_i)\}$$

is a maximum matching in G'.

- If (v_i, v'_j) is a maximally-matchable edge of type II, then Theorem 2.7 implies that either the edge (u_i, u_j) is contained in a path in H_{LR} that ends with a lower node, or the edge (u_j, u_i) is contained in a path in H_{RL} that ends with a lower node. Hence, we may apply a BFS on H_{LR} starting in u_i in order to look for a path that connects it to a lower node; if none is found, a similar BFS on H_{RL}, starting in u_j, is guaranteed to find a path that reaches a lower node. Once such a path is found, we may reconstruct the corresponding maximum matching M_0 in G that contains the edge (v_i, v'_j) (see Eq. (2)). Clearly, $M_0 \setminus \{(v_i, v'_j)\}$ is a maximum matching in G'.

- Finally, if (v_i, v'_j) is a lower edge then $M \setminus \{(v_i, v'_i)\}$ is a maximum matching in G' in case $i \le t$, or $M \setminus \{(v_j, v'_j)\}$ is, in case $j \le t$.

4 Related work

Costa [5] presented an algorithm for decomposing the edge set of a bipartite graph $G = (V, E)$ into three disjoint partitions, $E = E_1 \cup E_w \cup E_0$, where E_1 contains all maximally-matchable edges that belong to all maximum matchings in G, E_w contains all other maximally-matchable edges, and E_0 consists of the non-maximally-matchable edges. The runtime of her algorithm is $O(nm)$. Carvalho and Cheriyan [6] designed an algorithm with the same time complexity for finding all maximally-matchable edges in general graphs, using well-known results on efficient implementations of Edmonds' maximum-matching algorithm and other results from the matching folklore.

Rabin and Vazirani [15] designed a simple randomized algorithm for finding all maximally-matchable edges in perfectly-matchable general graphs. We outline their algorithm: Let $G = (V, E)$ be the input graph. Its Tutte matrix is defined as the following $n \times n$ matrix,

$$A_{i,j} = \begin{cases} x_{i,j} & (v_i, v_j) \in E \, , \;\; i > j \\ -x_{i,j} & (v_i, v_j) \in E \, , \;\; i < j \\ 0 & \text{otherwise} \end{cases} ,$$

where $x_{i,j}$ are indeterminates. Let $p > n^4$ be any prime, $S(A)$ be a random substitition of all $x_{i,j}$ with elements from $GF(p)$, and $B = S(A)^{-1}$. Then with probability at least e^{-2/n^2}, the matrix B identifies the set of maximally-matchable edges, in the sense that (v_i, v_j) is a maximally-matchable edge if and only if $B_{i,j} \neq 0$. The runtime of this algorithm is determined by matrix inversion time and it equals $\tilde{O}(n^{2.376})$. Cheriyan [3] designed a similar algorithm with the same runtime that applies for any general graphs (i.e., not necessarily ones with a perfect matching).

Our algorithm is deterministic. Its runtime is determined by the time to find a single maximum matching, which is $O(n^{1/2}m)$ (Hopcroft and Karp [12]) or $O((n/\log n)^{1/2}m)$ for dense graphs with $m = \Theta(n^2)$ (Alt et al. [2]). Hence, it improves upon that of [5, 6] by a factor of $n^{1/2}$ (or $(n \log n)^{1/2}$ for dense graphs, where $m = \Theta(n^2)$). In comparison to the randomized algorithms, our algorithm is faster when $m = O(n^r)$ and $r < 1.876$, since then its runtime is $O(n^{r+1/2})$, where $r + 1/2 < 2.376$. In addition, our algorithm is elementary, conceptually simple, and easy to implement. Another advantage of our algorithm in comparison to the above mentioned algorithms is that in cases where there is one "natural" maximum matching, which is known without invoking a costly procedure for finding one, the runtime of our algorithm is only $O(n + m)$. The runtime of all of the above mentioned algorithms remains the same even in such cases. (Examples of cases where one maximum matching is known upfront are described in Section 5.)

Identifying all maximally-matchable edges is equivalent to computing the union of all maximum matchings in the graph. Another line of research concentrated on algorithms for enumerating all maximum matchings in bipartite graphs, e.g. [7, 18, 19]. Naturally, the runtime of such algorithms depends on the number of the counted matchings (perfect matchings in [7, 19] and maximal, maximum, or perfect matchings in [18]). We recall in this context that counting the number of perfect matchings in a bipartite graph is equivalent to computing the permanent of a $\{0, 1\}$-matrix, a problem that is known to be #P-complete [20].

5   Epilogue

This study was motivated by a problem of providing anonymity in databases [8]. Consider a data owner that has a database of records that hold information about individuals in some population. He needs to publish the database for the purpose of analysis and data mining. Typically, such data records contain sensitive information (like medical condition, or income), whence they are published without identifying attributes such as name or SSN. However, even after removing identifying attributes, the data still contains so-called quasi-identifier attributes, e.g. age, gender, or location, that could be used for identification. In particular, an adversary may use combinations of those attributes in order to uniquely link records from the published de-identified database to records in other publicly accessible databases that contain identifying attributes. The model of k-anonymity was proposed as a countermeasure for such linking attacks [16]. That model suggests to generalize the quasi-identifiers attributes (e.g., by replacing an age entry with a range of ages that contains it) until every record in the database has at least $k - 1$ other records which are identical to it, when projected on the quasi-identifiers. By doing so, any linking attack will succeed in locating a record of a target individual in no less than k records in the published database, whence, privacy is respected to some extent. On the other hand, the utility of the data is damaged since the published data is generalized. The goal is then to find, for a given database and a given anonymity parameter k, a corresponding k-anonymization that maintains maximal utility. (That problem is NP-hard [1, 9, 14].)

In order to enhance utility, Gionis et al. [8] suggested to relax the k-anonymity notion in a way that, on one hand, maintains the same level of privacy, and on the other hand, allows anonymizations

with higher utility. k-Anonymity demands to generalize the database entries until it consists of clusters of records, where each cluster contains at least k records that are identical to each other when projected on the quasi-identifiers. In [8] it was recognized that a similar protection against linking attacks can be achieved even without reaching this clustered structure. All that is needed is to generalize the data until the following condition holds: For each individual in the underlying population, his record of quasi-identifiers (which the adversary is assumed to know) is consistent with at least k generalized records in the published database.

This setting may be modeled by a bipartite graph: The nodes on the left part of the graph correspond to the records in a publicly-accessible database (the records that include identifying attributes and exact quasi-identifiers) and the nodes on the right correspond to the records in the published database (those records include generalized quasi-identifiers and also additional sensitive information). An edge connects two nodes if their quasi-identifier data is consistent; namely, if the generalized values on the right generalize the specific values on the left. As the adversary knows that there is a one-to-one mapping between the records in the two databases, he may infer that all edges in the bipartite graph that are not maximally-matchable can be ignored, as they cannot stand for a correct link. Therefore, the data owner needs to make sure that even after the removal of such non-maximally-matchable edges from the bipartite graph, the anonymity constraint is still respected. Since the data owner knows the true perfect matching (because he has the raw data and he performs on it the generalization), he may use Algorithm 2 to identify all non-maximally-matchable edges efficiently and then verify that the generalized version of the database that he created provides the required level of anonymity.

Another example from the realm of privacy and anonymity is the work of Yao et al. [21]. In that work, they consider the case in which multiple views of the same database need to be released and it is needed to check whether their release would violate some privacy constraint (k-anonymity). The algorithm which they suggest needs to identify all edges in a bipartite graph that are part of some complete matching. (A matching in a bipartite graph is called complete if it covers all nodes in the possibly smaller part of the graph.) They suggest to do it using a naïve approach: Testing each edge in the graph whether its removal leaves a graph that still has a complete matching. Such an approach entails invoking the Hopcroft-Karp algorithm for each edge, whence its time complexity is $O(n^{1/2}m^2)$. However, in the case that was studied there, the data owner knows one complete matching in the underlying bipartite graph. Whence, he may use Algorithm 3 in order to perform the same task in $O(n + m)$. The improvement factor in runtime is $O(n^{1/2}m)$. As in such applications the graphs tend to be very large, such an improvement offers a practical solution instead of an impractical one.

An interesting research problem that this study suggests is to extend this approach to the general case, namely to non-bipartite graphs. The question is whether a similar algorithm that uses one known maximum matching can be devised in order to find all maximally-matchable edges efficiently.

A Extending a set of non-adjacent edges to a matching

The discussion in Section 2 implies that any set of k non-adjacent edges in a bipartite graph may be extended to a matching of size at least $t - k$, where t is the size of a maximum matching in G.

Theorem A.1. *Let $G = (V, E)$ be a bipartite graph where the two parts of the graph are $V_1 = \{v_1, \ldots, v_{n_1}\}$ and $V_2 = \{v'_1, \ldots, v'_{n_2}\}$, $n_1 \leq n_2$, and the maximum matchings are of size $t \leq n_1$. Let P be a set of k non-adjacent edges in E. Then there exists a matching of size $t - k$ in G that includes P. More specifically, assume that $P = \{(v_{j_i}, v'_{\ell_i}) : 1 \leq i \leq k\}$ where $A = \{j_1, \ldots, j_k\} \subset [n_1]$ and $A' = \{\ell_1, \ldots, \ell_k\} \subset [n_2]$ are the corresponding subsets of k distinct indices. Then there exists a matching of size $k + t - |A \cup A'|$ in G that includes P.*

Proof. As before, we may assume, without loss of generality, that one of the maximum matchings in G is $M := \{(v_1, v'_1), \ldots, (v_t, v'_t)\}$. Let $B = [t] \setminus (A \cup A')$ denote the set of indices of edges from M that are not adjacent to any of the edges in P. The size of B is at least $t - |A \cup A'|$. Clearly, $P \cup \{(v_i, v'_i) : i \in B\}$ is a matching that includes P. Its size is $k + |B| \geq k + t - |A \cup A'|$. Since $|A \cup A'| \leq |A| + |A'| = 2k$, we infer that P is included in a matching of size at least $t - k$. $\square$

It is easy to see that the lower bound in Theorem A.1 is sharp. To exemplify that, consider the case where M is a perfect matching consisting of t edges, the set of k non-adjacent edges P is such that $|A \cup A'| = 2k$, and $E = M \cup P$. The only way to extend P into a matching is by adding to it edges from M. The maximal number of edges from M that could be added to P is precisely $t - |A \cup A'|$ in this case.

B All edges in a regular bipartite graph are maximally-matchable

As noted earlier, the problem of finding a maximum matching in a d-regular bipartite graph is easier. The maximum matchings in such graphs are perfect. A randomized algorithm due to Goel et al. [10] finds them in time $O(n \log n)$. However, the problem of finding all maximally-matchable edges in such graphs is trivial, since the set of edges in such graphs may be partitioned into a disjoint union of d perfect matchings, and, consequently, all edges in such graphs are maximally-matchable.

Theorem B.1. *All edges in a regular bipartite graph are maximally-matchable.*

Proof. In regular bipartite graphs, $G = (V, E)$, where $V = V_1 \cup V_2$, all nodes have the same degree d. Hence, such graphs are balanced, $|V_1| = |V_2|$. Let X be any subset of V_1 and let $N(X)$ denote the set of all its neighbors in V_2. Clearly, $|N(X)| \geq |X|$ since otherwise at least one of the nodes in $N(X)$ would have to be of degree higher than d. Hence, Hall's marriage theorem [11] implies that G has a perfect matching. Let M_1 be such a matching and consider the graph $G_1 = (V, E \setminus M_1)$. This graph is $(d - 1)$-regular, whence it too has a perfect matching, say M_2. Continuing in that manner we may construct d disjoint perfect matchings, $M_1, \ldots, M_d$, such that $E = \bigcup_{1 \leq i \leq d} M_i$. It follows that all edges in E are maximally-matchable. $\square$

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