A central quaternionic Nullstellensatz

Gil Alon and Elad Paran*

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Abstract

Let I be a proper left ideal in the ring $\mathbb{H}[x_1, \ldots, x_n]$ of polynomials in n central variables over the quaternion algebra $\mathbb{H}$. Then there exists a point $a = (a_1, \ldots, a_n) \in \mathbb{H}^n$ with $a_i a_j = a_j a_i$ for all i, j, such that every polynomial in I vanishes at a. This generalizes a theorem of Jacobson, who proved the case $n = 1$. Moreover, a polynomial $f \in \mathbb{H}[x_1, \ldots, x_n]$ vanishes at all common zeroes of polynomials in I if and only if f belongs to the intersection of all completely prime left ideals that contain I – a notion introduced by Reyes in 2010.

1 Introduction

Let $R = \mathbb{C}[x_1, \ldots, x_n]$ be the ring of complex polynomials in n variables, and let I be a proper ideal in R. By Hilbert's Nullstellensatz, there exists a point $a = (a_1, \ldots, a_n) \in \mathbb{C}^n$ such that all elements of I vanish at a.[1] In the case $n = 1$, this reduces to the statement that every non-constant polynomial in one variable over $\mathbb{C}$ admits a zero – the fundamental theorem of algebra. Thus the Nullstellensatz may be regarded as a higher dimensional generalization of Gauss's celebrated theorem.

Consider now the ring $\mathbb{H}[x]$ of polynomials over Hamilton's quaternion algebra[2] $\mathbb{H} = \mathbb{R} + \boldsymbol{i}\mathbb{R} + \boldsymbol{j}\mathbb{R} + \boldsymbol{k}\mathbb{R}$, in a central[3] variable x. In [Niv41], Niven gives a quaternionic "fundamental theorem of algebra": Every non-constant polynomial in $\mathbb{H}[x]$ admits a zero. Niven attributes this result to Jacobson.

*Corresponding Author.

[1] This is the Bezout form of the Nullstellensatz, also known as the "weak" Nullstellensatz. We discuss the "strong" Nullstellensatz below.

[2] We denote the standard generators of $\mathbb{H}$ by $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$, as opposed to the letters i, j, k which we use for indices.

[3] That is, where the variable x commutes with the coefficients.

It is natural to ask whether Jacobson's theorem extends to higher dimension – is there a "quaternionic Nullstellensatz"? In this work we prove such a theorem. Let $R = \mathbb{H}[x_1, \ldots, x_n]$ be the ring of polynomials in n central variables over $\mathbb{H}$, and let $\mathbb{H}_c^n$ denote the set of points $(a_1, \ldots, a_n) \in \mathbb{H}^n$ satisfying $a_i a_j = a_j a_i$ for all $i \neq j$. We observe (see Proposition 2.2 below) that every point $a \in \mathbb{H}_c^n$ yields a well-defined substitution map $p \mapsto p(a)$ from R to $\mathbb{H}$. We show that the maximal left ideals in R are precisely those generated by $x_1 - a_1, \ldots, x_n - a_n$ for some $a = (a_1, \ldots, a_n) \in \mathbb{H}_c^n$ – i.e. the ideal of polynomials in R which vanish at a. As a consequence, we obtain the following "weak Nullstellensatz" for $\mathbb{H}$:

Theorem 1.1 (Weak Nullstellensatz)**.** *Let I be a proper left ideal in R. Then there exists a point $a = (a_1, \ldots, a_n) \in \mathbb{H}_c^n$ such that all polynomials in I vanish at a.*

The ring $\mathbb{H}[x]$ is a left principal ideal domain [Ore33, p. 483]. In particular, the set of polynomials vanishing at a point $a \in \mathbb{H}$ is a left ideal in $\mathbb{H}[x]$. Thus the case $n = 1$ in Theorem 1.1 above is Jacobson's theorem in [Niv41].

Let I be a proper ideal in $\mathbb{C}[x_1, \ldots, x_n]$. The "strong" Nullstellensatz asserts that a polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$ vanishes at all common zeroes of I if and only if f belongs the radical $\sqrt{I}$ of I – the intersection of all prime ideals that contain I. We prove an analogous result for $\mathbb{H}[x_1, \ldots, x_n]$:

Theorem 1.2 (Strong Nullstellensatz)**.** *Let I be a proper left ideal in R. A polynomial $f \in R$ vanishes at all common zeroes of polynomials in I in $\mathbb{H}_c^n$ if and only if f belongs to the intersection of all completely prime left ideals that contain I.*

Here a left ideal I in a ring R is called **completely prime** if given $a, b \in R$ with $ab \in I$ and $Ib \subseteq I$, it follows that $a \in I$ or $b \in I$.[4] This notion was introduced by Reyes in 2010, who demonstrated in [Rey10] and [Rey12] that, from certain aspects, completely prime one-sided ideals in noncommutative rings are a good analogue of prime ideals in commutative rings. Theorem 1.2 above gives further evidence of that.

Finally, we note that one may ask for a Nullstellensatz for quaternionic polynomials in **non-central** variables, but here already in dimension 1 the "fundamental theorem" fails – for example the polynomial function $X \mapsto X\boldsymbol{i} + \boldsymbol{i}X + \boldsymbol{j}$ admits no zeros in $\mathbb{H}$. Nevertheless, there is a form of Nullstellensatz for such quaternionic polynomial functions, closer in nature to the Real Nullstellensatz, see [AP21].

[4] In commutative rings, this definition obviously coincides with the usual definition of a prime ideal.

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2 Weak Nullstellensatz

Let $R = \mathbb{H}[x_1, \ldots, x_n]$ be the ring of polynomials in n central variables over $\mathbb{H}$. Given a tuple $a = (a_1, a_2, \ldots, a_n) \in \mathbb{H}^n$, we denote the left ideal generated by $x_1 - a_1, \ldots, x_n - a_n$ in R by I_a.

Lemma 2.1. *Let $a = (a_1, a_2, \ldots, a_n) \in \mathbb{H}^n$, and suppose that $a_i a_j \neq a_j a_i$ for some $1 \leq i, j \leq n$. Then $I_a = R$.*

Proof. One directly verifies that

$$(x_i - a_i)(x_j - a_j) - (x_j - a_j)(x_i - a_i) = a_i a_j - a_j a_i$$

hence $a_i a_j - a_j a_i$ is a non-zero element of $\mathbb{H}$ in I_a, therefore $I_a = R$. □

Let $\mathbb{H}_c^n$ denote the set of points $(a_1, \ldots, a_n) \in \mathbb{H}^n$ satisfying the condition $a_i a_j = a_j a_i$ for all $i \neq j$. For a point $a = (a_1, \ldots, a_n) \in \mathbb{H}_c^n$ and a monomial[5] $M = b x_1^{k_1} \cdot \ldots \cdot x_n^{k_n}$ with $b \in \mathbb{H}$, we define the substitution of a in M as $M(a) = b a_1^{k_1} \cdot \ldots \cdot a_n^{k_n}$. We additively expand this to a substitution map $p \mapsto p(a)$ from R to $\mathbb{H}$. We say that $p \in R$ **vanishes** at $a \in \mathbb{H}_c^n$ if $p(a) = 0$. We note that the substitution map is generally not a homomorphism[6].

Proposition 2.2. *Let $a = (a_1, \ldots, a_n) \in \mathbb{H}_c^n$. Then I_a is a proper left ideal in R, and a polynomial $p \in R$ vanishes at a if and only if $p \in I_a$. Moreover, I_a is a maximal left ideal in R.*

Proof. One directly checks that for any monomial $M = b x_1^{k_1} \cdot \ldots \cdot x_n^{k_n}$, the polynomial $M(x_i - a_i)$ vanishes at a for any i. It follows that any polynomial in I_a vanishes at a. In particular, $1 \notin I_a$.

Given a polynomial $p \in R$, we may perform "division with remainder": Repeatedly rewrite each occurrence of x_i as $(x_i - a_i) + a_i$ and open brackets, to express p in the form $q + b$ with $q \in I_a$ and $b \in \mathbb{H}$. Then $b = p(a) - q(a) = p(a)$. Thus $p(a) = 0$ if and only if $p \in I_a$. □

[5] The choice to express monomials with x_1 to the left and x_n to the right is arbitrary, but since the a_i commute, this choice does not matter for substitution.

[6] We note that for $n = 1$, substitution satisfies the following product formula: $(fg)(a) = f(g(a)ag(a)^{-1})(g(a))$ whenever $g(a) \neq 0$, see [LL88, S2].

We note that any point $(a_1, \dots, a_n) \in \mathbb{H}_c^n$ lying outside of $\mathbb{R}^n$ generates a field $\mathbb{R}(a_1, \dots, a_n)$ which is necessarily isomorphic to $\mathbb{C}$. Thus the space $\mathbb{H}_c^n$ is formed by "patching" uncountably many copies of $\mathbb{C}^n$, intersecting at $\mathbb{R}^n$. Note also that in light of Lemma 2.1, one cannot define substitution at tuples in $\mathbb{H}^n$ lying outside of $\mathbb{H}_c^n$ in any meaningful way.

Let $R' = \mathbb{R}[x_1, \dots, x_n]$ be the center of R.

Lemma 2.3. *The extension R/R' is integral. That is, every $f \in R$ satisfies an equality of the form $f^n + g_{n-1}f^{n-1} + \ldots + g_1 f + g_0 = 0$ with $g_0, \dots, g_{n-1} \in R'$.*

Proof. Since R' is commutative, R' is a finitely-accessible ring [Son76, Definition 1.4], hence by [Son76, Theorem 1.3], the extension R/R' is integral. □

The proof of the following "going-down" lemma is essentially the same as for finite extensions of commutative domains.

Lemma 2.4. *Let B/A be an integral extension of rings, where A is a domain contained in the center of B. If M is a maximal left ideal in B, then $M \cap A$ is a maximal ideal in A.*

Proof. Since M is maximal, For any $a \in A \setminus (M \cap A)$ we have $M + Ba = B$, so there exist $m \in M, b \in B$ such that $ab + m = 1$. Since B/A is integral, there exist elements $h_0, \dots, h_{n-1} \in A$ such that $b^n + \sum_{i=0}^{n-1} h_i b^i = 0$. Since $a \in A$, this implies that $(ab)^n + \sum_{i=0}^{n-1} a^{n-i} h_i (ab)^i = 0$. That is, $(1-m)^n + \sum_{i=0}^{n-1} a^{n-i} h_i (1-m)^i = 0$, which implies that $1 + \sum_{i=0}^{n-1} a^{n-i} h_i \in M \cap A$. But this implies that a is invertible modulo $M \cap A$. Thus $A/(M \cap A)$ is a field. □

Lemma 2.5. *The left ideal I generated by $x_1^2 + 1, x_2, \dots, x_n$ in R does not contain $x_1 + \boldsymbol{i}$.*

Proof. Let $\varphi \colon R \to \mathbb{H}[x_1]$ be the $\mathbb{H}[x_1]$-preserving epimorphism given by $\varphi(x_2) = \varphi(x_3) = \ldots = \varphi(x_n) = 0$. Suppose $x_1 + \boldsymbol{i} \in I$, and write $x_1 + \boldsymbol{i} = p(x_1^2 + 1) + p_2 x_2 + \ldots + p_n x_n$. Then $x_1 + \boldsymbol{i} = \varphi(x_1 + \boldsymbol{i}) = \varphi(p)(x_1^2 + 1) = \varphi(p)(x_1 - \boldsymbol{i})(x_1 + \boldsymbol{i})$, hence $1 = \varphi(p)(x_1 - \boldsymbol{i})$. Thus $x_1 - \boldsymbol{i}$ is invertible in $\mathbb{H}[x_1]$, a contradiction. □

Proposition 2.6. *The maximal left ideals in R are those of the form I_a for $a \in \mathbb{H}_c^n$.*

Proof. One direction of the claim is given by Proposition 2.2. For the converse, let M be a maximal left ideal in R and let $P = M \cap R'$. The extension R/R' is integral by Lemma 2.3, hence by Lemma 2.4, P is a maximal ideal in R'. Thus $F := R'/P$ is a finite field extension of $\mathbb{R}$, and P is the kernel of the projection homomorphism $R' \to F$.

If $F \cong \mathbb{R}$, then P is generated by $x_1 - a_1, \ldots, x_n - a_n$ for some $a_1, \ldots, a_n \in \mathbb{R}$. Then $(a_1, \ldots, a_n) \in \mathbb{H}_c^n$, thus by Proposition 2.2, the elements $x_1 - a_1, \ldots, x_n - a_n \in P \subseteq M$ generate a maximal left ideal I in R, hence $M = I$.

If $F \cong \mathbb{C}$, then P is the set of polynomials in R' vanishing at a complex point $(c_1 + d_1\boldsymbol{i}, \ldots, c_n + d_n\boldsymbol{i})$. We may make the real change[7] of variables $x_i \to x_i - c_i$ to assume, without loss of generality, that $c_i = 0$ for all i. We may further replace x_i with $d_i^{-1}x_i$ whenever $d_i \neq 0$ to assume that $d_i = 1$ or $d_i = 0$ for all i. At least one of the d_i is 1, so we assume, without loss of generality that $d_1 = 1$. Finally, for any $i > 1$ with $d_i = 1$, replace x_i with $x_i - x_1$ to assume that $d_i = 0$.[8] Thus P is the set of polynomials vanishing at $(i, 0, \ldots, 0)$, hence $P = \langle x_1^2 + 1, x_2, \ldots, x_n \rangle$. By Lemma 2.5, $x_1^2 + 1, x_2, \ldots, x_n$ do not generate a maximal left ideal in R: Indeed, the left ideal generated by $x_1 + \boldsymbol{i}, x_2, \ldots, x_n$ is larger. Thus M must contain a non-zero element $h \in R$ which is not generated by $x_1^2 + 1, x_2, \ldots, x_n$. By replacing in h every occurrence of $x_2, \ldots, x_n$ with 0 and every occurrence of x_1^2 with -1, we may assume that $h = cx_1 - d$ for some $c, d \in \mathbb{H}$. Since M is a proper ideal we have $c \neq 0$. Multiplying h from the left by c^{-1}, we may assume that $c = 1$. By Proposition 2.2, the left ideal I generated by $x_1 - d, x_2, \ldots, x_n$ is maximal in R, hence $M = I$. □

Theorem 1.1 is now an immediate consequence of Proposition 2.6.

We note that if one initially defines "right substitution" by $x_1^{k_1} \cdot \ldots \cdot x_n^{k_n} b \mapsto a_1^{k_1} \cdot \ldots \cdot a_n^{k_n} b$, then one obtains symmetric results to those given here, where left ideals are replaced by right ideals.

One may ask if Theorem 1.1 generalizes to other division algebras. However, $\mathbb{H}$ is essentially the only noncommutative division algebra for which Jacobson's theorem in [Niv41] holds: A theorem of Baer asserts that if D is a noncommutative division algebra with center C, such that every polynomial in $D[x]$ admits a root in D, then C is a real-closed field and D is the quaternion algebra over C (see the introduction of [Niv41]). We also note that Jacobson's theorem was extended in [GSV08] to the octonion algebra of real Cayley numbers.

3 A going-up theorem

For this section, let B be a right Ore domain. That is, for each nonzero $x, y \in B$ there exist $r, s \in B$ such that $xr = ys \neq 0$. Then [GW04, Theorem

[7]That is, we put $y_i = x_i - c_i$. Clearly, $\mathbb{H}[x_1, \ldots, x_n] = \mathbb{H}[y_1, \ldots, y_n]$.

[8]Here we put $y_i = x_i - x_1$ or $y_i = x_i$ for each i, according to our construction. We have, as before, $\mathbb{H}[x_1, \ldots, x_n] = \mathbb{H}[y_1, \ldots, y_n]$, and any ideal of the form $\langle y_1 - b_1, ..., y_n - b_n \rangle$ for some $(b_1, \ldots, b_n) \in \mathbb{H}_c^n$ is also of the form $\langle x_1 - a_1, \ldots, x_n - a_n \rangle$ for some $(a_1, \ldots, a_n) \in \mathbb{H}_c^n$.

6.8] B admits a classical (skew) field of fractions, whose elements are of the form ab^{-1} with $a, b \in B, b \neq 0$. Let A be a subring of the center of B. Suppose B/A is an integral extension: Every element $0 \neq b \in B$ satisfies an equation of the form $b^n + a_{n-1}b^{n-1} + \ldots + a_0 = 0$ with $a_0, \ldots, a_{n-1} \in A$.

In this section we prove a "going-up" theorem for the extension B/A, connecting completely prime ideals in B and prime ideals in A.

Given a multiplicative subgroup S of A, the localization $B_S = \{bs^{-1} | b \in B, s \in S\}$ is clearly a subring of the fraction field of B.

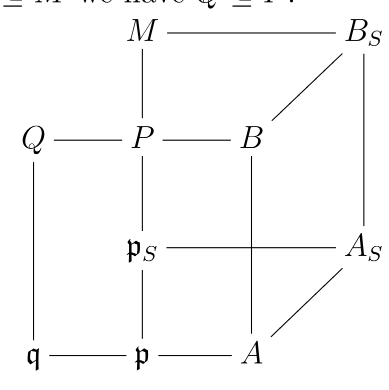
Lemma 3.1. *Let S be a multiplicative subgroup of A and P a completely prime left ideal in B_S. Then $P \cap B$ is a completely prime left ideal in B.*

Proof. Let $a, b \in B$ be such that $ab \in P \cap B$, $(P \cap B)b \subseteq (P \cap B)$. Let us prove that $Pb \subseteq P$: Given an element $ps^{-1} \in P$ with $p \in B, s \in S$, we have $s(ps^{-1}) = p \in P \cap B$, hence $pb \in P \cap B$, therefore $(ps^{-1})b = s^{-1}pb \in P$. Thus $Pb \subseteq P$, hence $a \in P \cap B$ or $b \in P \cap B$. □

For a left ideal $I \subseteq B$, we denote by I_c the contraction $I \cap A$ of I in A. We have the following "going-up" theorem:

Theorem 3.2. *Let Q be a completely prime left ideal in B, let $\mathfrak{q} = Q_c = A \cap Q$ and let $\mathfrak{p}$ be a prime ideal in A with $\mathfrak{q} \subseteq \mathfrak{p}$. Then there exists a completely prime left ideal P in B such that $P_c = \mathfrak{p}$ and $Q \subseteq P$.*

Proof. Put $S = A \setminus \mathfrak{p}$. Then S is a multiplicative subset of A, and $A_S = A_{\mathfrak{p}}$ is a local ring, which we view as a subring of B_S. We have $S \cap Q = \emptyset$ hence Q_S is a proper ideal of B_S. Let M be a maximal left ideal in B_S containing Q_S. Since B/A is integral, it is straightforward to check that B_S/A_S is also integral. By Lemma 2.4 (with (A_S, B_S) instead of (A, B)) we get that $M \cap A_S$ is a maximal ideal in A_S, hence $M \cap A_S$ is the unique maximal ideal $\mathfrak{p}_S$ of A_S. By [Rey10, Corollary 2.10], M is a completely prime left ideal, hence by Lemma 3.1, $P = M \cap B$ is a completely prime left ideal in B, and we have $P_c = P \cap A \subseteq M \cap A \subseteq (M \cap A_S) \cap A = \mathfrak{p}_S \cap A = \mathfrak{p}$. On the other hand, $\mathfrak{p} \subseteq \mathfrak{p}_S \subseteq M$ and $\mathfrak{p} \subseteq A \subseteq B$, hence $\mathfrak{p} \subseteq M \cap B = P$. Thus $P_c = \mathfrak{p}$, and since $Q \subseteq M$ we have $Q \subseteq P$.



□

4 Strong Nullstellensatz

Let $R = \mathbb{H}[x_1, \ldots, x_n]$ and $R' = \mathbb{R}[x_1, \ldots, x_n]$. For a quaternion $z = a + \boldsymbol{i}b + \boldsymbol{j}c + \boldsymbol{k}d$ with $a, b, c, d \in \mathbb{R}$, let $\bar{z} = a - \boldsymbol{i}b - \boldsymbol{j}c - \boldsymbol{k}d$ denote its quaternion conjugate. Then $\bar{z} + z, z\bar{z} = \bar{z}z \in \mathbb{R}$ for all $z \in \mathbb{H}$. For any $f \in R$, let $\bar{f}$ be the polynomial obtained from f by conjugating all its coefficients. Then $f + \bar{f}, f\bar{f} = \bar{f}f \in R'$ for for all $f \in R$.

Proposition 4.1. *The ring R is a left and right Ore domain. That is, for each $a, b \in R$ with $a, b \neq 0$ there exists a non-zero element in R which is divisible from the right by both a and b, and a non-zero element which is divisible from the left by a and b.*

Proof. We have $a\bar{a}\bar{b}b = b\bar{b}\bar{a}a$. □

Proposition 4.1 will allow us to apply Theorem 3.2 in the proof of Proposition 4.4 below.

Lemma 4.2. *Let P be a two-sided ideal P of R, and let $\mathfrak{p} = P \cap R'$. Then the ideal $\mathfrak{p}\mathbb{H} = \mathbb{H}\mathfrak{p} = \mathfrak{p} + \mathfrak{p}\boldsymbol{i} + \mathfrak{p}\boldsymbol{j} + \mathfrak{p}\boldsymbol{k}$ is P.*

Proof. The inclusion $\mathbb{H}\mathfrak{p} \subseteq P$ is clear. For the opposite inclusion, let $u = a + \boldsymbol{i}b + \boldsymbol{j}c + \boldsymbol{k}d \in P$, with $a, b, c, d \in R'$. Direct computation gives:

$$\begin{aligned} a &= \frac{1}{4}(u - \boldsymbol{i}u\boldsymbol{i} - \boldsymbol{j}u\boldsymbol{j} - \boldsymbol{k}u\boldsymbol{k}) \\ b &= \frac{1}{4}(\boldsymbol{j}u\boldsymbol{k} - u\boldsymbol{i} - \boldsymbol{i}u - \boldsymbol{k}u\boldsymbol{j}) \\ c &= \frac{1}{4}(\boldsymbol{k}u\boldsymbol{i} - u\boldsymbol{j} - \boldsymbol{j}u - \boldsymbol{i}u\boldsymbol{k}) \\ d &= \frac{1}{4}(\boldsymbol{i}u\boldsymbol{j} - u\boldsymbol{k} - \boldsymbol{k}u - \boldsymbol{j}u\boldsymbol{i}) \end{aligned}$$

hence $a, b, c, d \in \mathfrak{p}$ and $u \in \mathbb{H}\mathfrak{p} = \mathfrak{p}\mathbb{H}$. □

The following "incomparability lemma" is key to the proof of Theorem 1.2.

Lemma 4.3. *If $P \subseteq Q$ are left ideals in R such that P is completely prime and $Q \cap R' = P \cap R'$, then $Q = P$.*

Proof. Let $\mathfrak{p} = P \cap R'$. For any $a \in Q$ we have $a\bar{a} = \bar{a}a \in Q \cap R' = \mathfrak{p} \subseteq P$.

Thus, for any $a \in Q$ and $b \in P$, we have

$$\bar{a}b + \bar{b}a = (\bar{a} + \bar{b})(a + b) - \bar{a}a - \bar{b}b \in P$$

Since P is a left ideal, we conclude that $\bar{b}a \in P$, so $\bar{P}Q \subseteq P$. Conjugating, we get $\bar{Q}P \subseteq \bar{P}$. Since $\bar{P}$ is evidently a right ideal in R, we have $\bar{Q}PR \subseteq \bar{P}$, where PR is the right R-ideal generated by P. Note that PR is a two-sided ideal. By Lemma 4.2, we have $PR = \mathbb{H}\mathfrak{p}'$ for some ideal $\mathfrak{p}' \supseteq \mathfrak{p}$ of R'. In particular, we have $\bar{Q}\mathfrak{p}' \subseteq \bar{P}$. Conjugating again, keeping in mind that $\mathfrak{p}'$ is invariant under conjugation, we get that $\mathfrak{p}'Q \subseteq P$. We now consider two cases:

- Case 1: $\mathfrak{p}' \neq \mathfrak{p}$. Let $a \in \mathfrak{p}' \setminus \mathfrak{p}$. For any $q \in Q$, we have $aq = qa \in P$, and $Pa = aP \subseteq P$, since a is in the center R' of R. Since P is completely prime, we have $a \in P$ or $q \in P$, but per our choice, $a \notin P$, so $q \in P$. Thus $Q \subseteq P$.
- Case 2: $\mathfrak{p}' = \mathfrak{p}$. Then $PR = \mathbb{H}\mathfrak{p} \subseteq P$, so P is a two-sided ideal. Let $a \in Q$, then as before, $\bar{a}a \in P$. We have $Pa \subseteq P$, and since P is completely prime, we have $a \in P$ or $\bar{a} \in P$. If $\bar{a} \in P$ then $\bar{a} \in Q$, so $a + \bar{a} \in Q \cap R' = \mathfrak{p}$, and in particular, $a + \bar{a} \in P$, hence $a \in P$. So either way, we have $a \in P$. Thus $Q \subseteq P$.

□

We can now show that R satisfies the following "Jacobson property":

Proposition 4.4. *Let P be a completely prime left ideal. Then P is an intersection of maximal left ideals in R.*

Proof. Put $\mathfrak{p} = P \cap R'$. Since P is completely prime, $\mathfrak{p}$ is clearly a prime ideal in R'. Since R' is a Jacobson ring [Eis04, Theorem 4.19], we have $\mathfrak{p} = \bigcap_{\mathfrak{p} \subseteq \mathfrak{m}} \mathfrak{m}$, where the intersection is taken over all maximal ideals in R' that contain $\mathfrak{p}$. By Theorem 3.2 (with $B = R, A = R'$), for each such $\mathfrak{m}$ there exists a maximal left ideal M in R such that $P \subseteq M$ and $M \cap R' = \mathfrak{m}$. Let Q be the intersection of all such M. Then Q is a left ideal in B with $Q \cap R' = P \cap R' = \mathfrak{p}$, hence by Lemma 4.3 we have $P = Q$. □

Definition 4.5. Let A be an associative ring with unity. For a left ideal I in A, we define the **left radical** $\sqrt{I}$ of I as the intersection of all completely prime left ideals that contain I.

Clearly, if A is a commutative ring and I is an ideal in A, then the left radical of I is the classical radical of I.

Given a left ideal I in R, let $Z(I)$ be the set of points in $\mathbb{H}_c^n$ at which all polynomials in I vanish. Given a set of points $Z \subseteq \mathbb{H}_c^n$, let $I(Z)$ be the left ideal of polynomials that vanish at every point of Z. We can now prove the strong Nullstellensatz:

Theorem 4.6. *Let I be a left ideal in R. Then $I(Z(I)) = \sqrt{I}$.*

Proof. The ideal I_a is the ideal of functions vanishing at $a \in \mathbb{H}_c^n$, hence $I(Z(I)) = \bigcap_{a \in Z(I)} I_a$, by definition. Note that $I \subseteq I_a$ if and only if $a \in Z(I)$, also by definition. Thus, by Proposition 2.6, the maximal left ideals that contain I are precisely those of the form I_a, with $a \in Z(I)$. Thus $I(Z(I))$ is the intersection of all maximal left ideals that contain I.

By [Rey10, Corollary 2.10], every left maximal ideal in R is completely prime. By Proposition 4.4, every completely prime left ideal that contains I is an intersection of maximal left ideals that contain I. Hence, taking the intersection of all maximal left ideals containing I yields the same result as taking the intersection of the family of completely prime left ideals containing I. Thus $I(Z(I)) = \sqrt{I}$. □

We note that the classical strong Nullstellensatz for $\mathbb{C}[x_1, \ldots, x_n]$ can be easily deduced as an immediate consequence of the weak Nullstellensatz, using the famous Rabinowitsch trick (see [Lan05, p. 380, proof of Theorem 1.5]). Such a proof does not seem possible for $\mathbb{H}[x_1, \ldots, x_n]$, since substitution is not a homomorphism. Therefore we took a longer route of proof, as presented above.

The definition of the left radical $\sqrt{I}$ given here is an abstract one, a generalization of the abstract definition of the classical radical. One may ask if the left radical $\sqrt{I}$ of an ideal I in $\mathbb{H}[x_1, \ldots, x_n]$ can also be described explicitly as the set of roots of elements of I, as in the commutative case. Below we give an example showing that this is not the case. We shall first need the following lemma:

Lemma 4.7. *Let $R = \mathbb{H}[x]$ and let $p \in R$ be a monic polynomial. The ideal Rp is completely prime if and only if $p = x - a$ for some $a \in \mathbb{H}$.*

Proof. First suppose that $p = x - a$ for $a \in \mathbb{H}$, and that $f, g \in R$ satisfy $fg \in Rp$ and $Rpg \subseteq Rp$. Then $(fg)(a) = 0$ and $(pg)(a) = 0$. If $g \notin Rp$, then $g(a) \neq 0$ and by [LL88, Theorem 2.8] we have $f(a^{g(a)}) = 0$ and $p(a^{g(a)}) = 0$, where $a^{g(a)} = g(a)ag(a)^{-1}$. The equality $p(a^{g(a)}) = 0$ thus implies $a^{g(a)} = a$, hence we have $f(a) = 0$, hence $f \in Rp$. Thus $R(x - a)$ is completely prime.

Conversely, suppose Rp is completely prime, but p is composite. By Jacobson's theorem in [Niv41], every polynomial in $\mathbb{H}[x]$ factors into a product of linear terms. Thus we may write $p = (x - a)f$ with f monic of positive degree. Put $g = (x - \bar{a})(x - a) = (x - a)(x - \bar{a}) \in \mathbb{R}[x]$. Then $(x - \bar{a})p = gf = fg \in Rp$, and $Rpg \subseteq Rp$ since g belongs to the center $\mathbb{R}[x]$ of $\mathbb{H}[x]$. Since Rp is completely prime, we have $f \in Rp$ or $g \in Rp$. The first option cannot hold since $\deg(f) < \deg(p)$, and the second option implies that $\deg(p) = \deg(g) = 2$ and $f = x - \bar{a}$. We have $(x-a)(x-\bar{a})(x-\bar{a}) = (x-\bar{a})(x-a)(x-\bar{a}) = (x-\bar{a})p$, hence $Rp(x-\bar{a}) \subseteq Rp$. Since $p = (x-a)(x-\bar{a}) \in Rp$ and Rp is completely prime, we have $x-a \in Rp$ or $x - \bar{a} \in Rp$, a contradiction. $\square$

Example 1. *Let $f = (x - \boldsymbol{i})(x - \boldsymbol{j})$ in $R = \mathbb{H}[x]$, and let $I = Rf$. Then $\boldsymbol{j}$ is the only zero of f, see [GS08, Example 4.4][9]). Thus by Lemma 4.7 we have $\sqrt{I} = R(x - \boldsymbol{j})$. However, if $(x - \boldsymbol{j})^n \in Rf$ for some $n > 1$, then $(x-\boldsymbol{j})^{n-1}$ vanishes at $\boldsymbol{i}$, a contradiction. (Indeed, using [LL88, Theorem 2.8], one proves inductively that $(x - \boldsymbol{j})^m(\boldsymbol{i}) = (-2\boldsymbol{j})^{m-1}(\boldsymbol{i} - \boldsymbol{j})$ for all $m \in \mathbb{N}$.)*

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[9]We note that in [GS08], polynomials are considered with coefficients on the right, rather than the left, as we do here. Consequentially, the root $\boldsymbol{i}$ in [GS08, Example 4.4] is replaced with the root $\boldsymbol{j}$ here.

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