TEL-AVIV UNIVERSITY

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School of Mathematical Sciences

ONE SIDED LIPSCHITZ STABILITY AND ITS APPLICATIONS FOR NONLINEAR CONSERVATION LAWS

Thesis submitted for the degree "Doctor of Philosophy"

by

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Submitted to the Senate of Tel-Aviv University

June 1993

This work was carried out under the supervision of

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This work is dedicated to my parents

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1. INTRODUCTION

In this work we are interested in various aspects of the stability of entropy solutions of scalar nonlinear hyperbolic conservation laws,

$$u_t + \operatorname{div}_{\vec{x}} \vec{f}(u) = 0 \quad , \quad \vec{x} \in \Re^n , \ t > 0 , \ \vec{f} : \Re \to \Re^n \quad ,$$

with respect to the one-sided Lipschitz seminorm,

$$\|w(x)\|_{Lip^+} \equiv \operatorname*{ess\,sup}_{x \neq y} \left(\frac{w(x) - w(y)}{x - y} \right)^+ \quad , \qquad (\cdot)^+ \equiv \max(\cdot, 0) \quad .$$

Such Lip^+-stability was first established by O.A. Oleinik [O2] for one-dimensional conservation laws,

$$u_t + f(u)_x = 0 \quad , \quad x \in \Re , \ t > 0 \quad ,$$

whose flux, $f(u)$, is convex. This stability asserts that the increasing part of the entropy solution is Lipschitz continuous for all $t > 0$, even if the increasing part of the initial data was not, and the corresponding Lipschitz constant decreases like t^{-1}. This Lip^+-stability identifies, in the one-dimensional convex case, the entropy solutions and, therefore, may serve as an alternative, easy to check, admissibility criterion.

Part I of our work is dedicated to the Lip^+-stability of conservation laws and their approximate solutions.

In §2.1 we prove the sharper forms of Oleinik's Lip^+-stability estimate, as obtained by D. Hoff [H2] and E. Tadmor [T4], in the one-dimensional convex case (Theorem 2.1) and give Hoff's proof [H2] that Lip^+-stability is equivalent to the entropy criteria and, therefore, guarantees uniqueness for the initial value problem (Theorem 2.2). Later, in §5, we propose an innovative proof for this equivalence.

§2.2 is dedicated for the nonconvex case. It is known that Lip^+-stability fails to serve as an entropy condition in this case, since there exist weak non-admissible solutions which are Lip^+-stable. Furthermore, if $f(u)$ has (at least) two inflection points, the entropy solutions need not be Lip^+-stable; namely, there always exist initial data for which the corresponding entropy solution develops increasing jump discontinuities. We demonstrate these facts by two examples, due to Hoff [H2], and address the only question which remains open in this context: the Lip^+-stability of entropy solutions in case the flux has one inflection point.

In §2.3 we consider the multi-dimensional case. We show that Lip^+-stability does not hold even if all components of the flux, $\vec{f}(u)$, are convex. We reproduce Hoff's result regarding

Lip^+-stability for conservation laws of which flux components are all either convex or concave and proportional to each other, modulo linear terms (Theorem 2.3). A direct consequence of this theorem is that multi-dimensional conservation laws with quadratic fluxes are Lip^+-stable (Corollary 2.1).

In §3 we consider a different aspect of Lip^+-stability, namely – compactness. In §3.1 we prove that Lip^+-stability of solution operators with finite speed of propagation, implies their compactness (Theorem 3.1). In §3.2 we comment on the Lip^+-stability of approximate solutions to nonlinear convex conservation laws. Such Lip^+-stability implies, in view of Theorem 3.1, the compactness of the family of approximate solutions and, therefore, thanks to uniqueness (Theorem 2.2), the convergence of the whole sequence to the entropy solution. This convergence argument, however, lacks convergence rate estimates. This leads to the second part of our work.

During the recent decade there was a great deal of research activity in the field of approximate solutions of time-dependent partial differential equations. A large part of this research concentrated on nonlinear equations and particularly on hyperbolic conservation laws. The available convergence rate estimates were the $O(\varepsilon^{\frac{1}{2}})$ L_1-convergence rate for the viscous solution (N.N. Kuznetsov [K2]) and for difference schemes (B.J. Lucier [L3]; R. Sanders [S1]). However, these *global* error estimates, sharp as they are, are not satisfying since we are usually interested in practice in the local value of the entropy solution away from shocks (whose poor resolution is the reason for this weaker than expected L_1-convergence rate).

The first to address the question of *local* convergence rate of approximations to hyperbolic conservation laws was E. Tadmor [T4]; he showed that exact values of the entropy solution may be recovered, within an error as close to $O(\varepsilon)$ as the local smoothness permits, by post-processing the corresponding viscous regularization, provided that the initial data are Lip^+-bounded. The two main ingredients of his approach were the Lip^+-stability of the entropy and viscosity solutions and the use of a Holmgren-like dual backward transport equation in order to estimate the $W^{-1,1}$-size of the error in terms of the $W^{-1,1}$-size of the initial and truncation errors. This error estimate in the weak $W^{-1,1}$-norm implies both the $O(\varepsilon^{\frac{1}{2}})$ L_1-convergence rate and the above mentioned $O(\varepsilon)$ local convergence rate.

Part II of our work is dedicated for convergence rate estimates for approximate solutions of convex conservation laws and reflects a joint work with H. Nessyahu, [NT3], [NTT].

We begin, in §4, with a brief description of the work of Tadmor, [T4]. We proceed, in §5, with our main results which generalize the results described in the previous section by dealing with general families of approximate solutions, $\{u^\varepsilon(x,t)\}_{\varepsilon\downarrow 0}$, and allowing possibly

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Lip^+-unbounded initial data. We define Lip^+-stability for approximate solutions and derive convergence rate estimates for such Lip^+-stable approximations (Theorems 5.1 and 5.2). We further show that Lip^+-stability and $W^{-1,1}$-consistency imply convergence in $W^{-1,1}$ whose rate is $O(\tilde{\varepsilon})$, where $\tilde{\varepsilon} = \varepsilon$ in case of Lip^+-bounded initial data (in agreement with the results given in §4) and $\tilde{\varepsilon} = \varepsilon|\ln\varepsilon|$ in case of Lip^+-unbounded initial data (Corollary 5.1). These $W^{-1,1}$-error estimates may be translated into a variety of global $W^{s,p}$-error estimates and local L_∞-error estimates (Corollary 5.2). One of the consequences of these estimates is that by post-processing the approximate solution, pointwise values of the exact solution and its derivatives may be recovered with an error as close to $O(\tilde{\varepsilon})$ as the local smoothness permits.

In the following sections we demonstrate these results for various types of approximate solutions. We start in §6, by dealing with viscous parabolic regularizations of the hyperbolic conservation law.

In §7 we describe a larger class of parabolic regularizations – the so-called pseudo-viscosity approximations. These parabolic approximations are characterized by a gradient-dependent viscosity coefficient [NR], [MN]. We impose conditions on the viscosity coefficient which enable us to apply the convergence rate estimates of §5.

§8 is devoted to the regularized Chapman-Enskog expansion [R], [ST], which is another viscous approximation of the hyperbolic conservation law, where the viscosity coefficient acts on the approximate solution by means of convolution (rather than multiplication, as in the viscous or pseudo-viscous approximations).

In §9 we demonstrate our convergence rate analysis for the spectral viscosity method [T3]. In this method, one approximates the solution of periodic conservation laws by a sequence of trigonometric polynomials of increasing degrees.

Finally, in §10, we apply our analysis to Godunov type schemes, in the case of Lip^+-bounded initial data. In §10.1 we show how the question of consistency for Godunov type schemes can be answered solely in terms of the behavior of the associated projection operator. Namely, we prove that $W^{-1,1}$-consistent projections guarantee the $W^{-1,1}$-convergence of the corresponding Godunov scheme, provided that the latter is Lip^+-stable. In §10.2 we apply these convergence rate estimates to a variety of scalar Godunov type schemes on a uniform grid as well as variable mesh size ones.

In **Part III** of our work we show how the Lip^+-stability of convex conservation laws may be translated into higher order regularity. The results of this part are mainly summarized in [TT] and [T6].

In §11 we deal with the piecewise smoothness of entropy solutions to convex conservation laws. It is well known that such entropy solutions consist of at most countable number of C^1-

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smooth regions. We obtain in §11.2 new upper bounds on the higher order derivatives of the entropy solution in any one of its C^1-smooth regions. These bounds enable us to quantify the *high order* piecewise smoothness of the entropy solution. To this end we introduce in §11.3 an appropriate new C^N-semi norm, localized to the smooth part of the entropy solution, and we show that the entropy solution is stable with respect to this seminorm.

In §11.4 we address the question regarding the *number* of C^1-smooth pieces, i.e, the size of the set of shock discontinuities. This question has been the subject of many studies by O.A. Oleinik [O1]–[O3], C.M. Dafermos [D1], D.G. Schaeffer [S2] and others. We give a final answer to that question by showing that the number of future shocks *equals* the number of decreasing inflection points of the initial speed, $f'(u(x,0))$, thus providing an elementary and simple tool for an a-priori determination of the size of the set of shock discontinuities.

Loosely speaking, the above described results of §11, combined, imply that in the case of such generic initial data, the entropy solution consists of a finite number of smooth pieces, each of which is as smooth as the data permits. It is this type of piecewise smoothness which is assumed, sometime implicitly, in many finite-dimensional computations of such discontinuous problems.

This high order regularity for the exact solution raised the question of whether analogous results may be established for numerical approximations. This question is related to the convergence rate analysis of §5, since the high order regularity of the approximate solution may be used in order to improve our local error estimates. In §12 we restrict our attention to the Lax-Friedrichs approximation governed by scalar convex conservation laws, subject to increasing C^n-smooth initial data. The exact solution amounts to a smooth rarefaction wave. We quantify the high order regularity of the approximation and obtain estimates which agree with the analogous ones in the differential case.

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PART I: Lip^+-STABILITY

2. Lip^+-STABILITY FOR NONLINEAR CONSERVATION LAWS

2.1. One-dimensional convex conservation laws

Consider the scalar nonlinear conservation law

$$(2.1.1) \qquad u_t + f(u)_x = 0 \ ,$$

subject to the initial data

$$(2.1.2) \qquad u(x,0) = u_0(x) \quad , \quad u_0 \in L_0^\infty \ .$$

Here and henceforth, L_0^∞ denotes the space of compactly supported and uniformly bounded functions.[1]

We deal here with convex conservation laws, i.e., the flux $f(u)$ is strictly convex,

$$(2.1.3) \qquad f'' \geq \alpha > 0 \ .$$

The results which we survey here apply equally to the concave case as well.

Oleinik has shown [O2] that when f is convex, (2.1.3), and C^2-smooth, then the entropy solutions of (2.1.1) satisfy

$$(2.1.4) \qquad u_x \leq \frac{1}{\alpha t}$$

and furthermore, that weak solutions of (2.1.1) which satisfy (2.1.4) are uniquely determined by their initial value and are, consequently, admissible entropy solutions. Her proof is by means of finite difference methods.

The one sided Lipschitz estimate (2.1.4) may be easily obtained by recalling the structure of entropy solutions of (2.1.1) in the convex case [D1], [L1]: They are continuous except on the closed union of an at most countable set of Lipschitz continuous shock curves; the solution remains constant along straight characteristics of the form

$$(2.1.5) \qquad x(t) = x_0 + a(u_0(x_0))t \quad , \quad a = f' \ ,$$

and the shock curves, $x = \xi(t)$, satisfy Lax entropy condition

$$(2.1.6) \qquad a(u(\xi(t) - 0, t)) > \dot{\xi}(t) > a(u(\xi(t) + 0, t)) \ .$$

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[1]The results of this work applies equally to the periodical case.

Let (x, t) be a point in the region of continuity, R. Since R is open, we can find x^- and x^+ such that

(2.1.7)
$$x^- < x < x^+ \quad \text{and} \quad [x^-, x^+] \times \{t\} \subset R .$$

Denoting by $x_0^\pm$ the two initial points of the characteristics on which $x^\pm$ lay, we have

(2.1.8)
$$x_0^\pm = x^\pm - a(u(x^\pm, t))t \quad \text{and} \quad x_0^+ > x_0^- .$$

We conclude by (2.1.7) and (2.8) that

$$0 < \frac{x_0^+ - x_0^-}{x^+ - x^-} = 1 - \frac{a(u(x^+, t)) - a(u(x^-, t))}{x^+ - x^-} \cdot t .$$

Letting $x^\pm \to x$ we get

(2.1.9)
$$a(u)_x \le \frac{1}{t} .$$

Since, on the other hand, Lax entropy condition, (2.1.6), implies that $a(u)_x < 0$ (in the sense of distributions) along the shocks, (2.1.9) is satisfied everywhere in the weak sense.

The one sided Lipschitz estimate for the speed $a(u)$, (2.1.9), is sharper than Oleinik's condition, in virtue of (2.1.3). This estimate, due to Hoff [H2], was optimized by Tadmor in [T4] in the following manner:

Theorem 2.1. (Lip^+-Stability). *Assume that $f \in C^1$ is convex, (2.1.3), and let $u(x, t)$ be an entropy solution of (2.1.1). Then*

(2.1.10)
$$\|a(u(\cdot, t))\|_{Lip^+} \le \frac{1}{\|a(u(\cdot, 0))\|_{Lip^+}^{-1} + t} \quad , \quad t \ge 0 ,$$

where $a = f'$ and $\|\cdot\|_{Lip^+}$ denotes the one-sided Lipschitz seminorm

(2.1.11)
$$\|w(x)\|_{Lip^+} \equiv \operatorname*{ess\,sup}_{x \ne y} \left(\frac{w(x) - w(y)}{x - y} \right)^+ \quad , \quad (\cdot)^+ \equiv \max(\cdot, 0) .$$

Proof. We assume that $f \in C^2$; the result for C^1 fluxes follows by means of regularization (consult [CM, Corollary 5.1]).

The entropy solutions of (2.1.1), u, are those which may be realizable as small viscosity solutions of

(2.1.12)
$$u_t^\varepsilon + f(u^\varepsilon)_x = \varepsilon Q(u^\varepsilon)_{xx} \quad , \quad Q' > 0 , \, \varepsilon \downarrow 0 .$$

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We show below that

$$\|a(u^\varepsilon(\cdot,t))\|_{Lip^+} \leq \frac{1}{\|a(u^\varepsilon(\cdot,0))\|_{Lip^+}^{-1} + t} \quad , \quad t \geq 0 , \tag{2.1.13}$$

which implies (2.1.10) in view of the strong L_1-convergence of u^ε to u. To this end we denote

$$w^\varepsilon(x,t) = Q(u^\varepsilon)_x = Q'(u^\varepsilon)u_x^\varepsilon .$$

Multiplying (2.1.12) by $Q'(u^\varepsilon)$ and differentiating with respect to x we get that w^ε satisfies

$$w_t^\varepsilon + \left[a(u^\varepsilon) - \varepsilon\frac{Q''(u^\varepsilon)}{Q'(u^\varepsilon)}w^\varepsilon \right] w_x^\varepsilon + \frac{a'(u^\varepsilon)}{Q'(u^\varepsilon)}(w^\varepsilon)^2 = \varepsilon Q'(u^\varepsilon)w_{xx}^\varepsilon . \tag{2.1.14}$$

Since $w^\varepsilon(x,t)$ is smooth and tends to zero as $|x| \to \infty$, it attains its maximal value, say in $(x(t),t)$. Since in those points $w_x^\varepsilon = 0$ and $w_{xx}^\varepsilon \leq 0$, we conclude by (2.1.14) that

$$W^\varepsilon(t) \equiv \max_x w^\varepsilon(x,t) = \|Q(u^\varepsilon(\cdot,t))\|_{Lip^+} \tag{2.1.15}$$

is dominated by the Ricatti inequality

$$\frac{d}{dt}W^\varepsilon + \beta(W^\varepsilon)^2 \leq 0 \quad , \quad \beta \equiv \min\frac{a'(\cdot)}{Q'(\cdot)} > 0 . \tag{2.1.16}$$

Hence,

$$\|Q(u^\varepsilon(\cdot,t))\|_{Lip^+} \leq \frac{1}{\|Q(u^\varepsilon(\cdot,0))\|_{Lip^+}^{-1} + \beta t} \quad , \quad t \geq 0 . \tag{2.1.17}$$

Choosing the regularization coefficient $Q(u) = a(u)$, β equals 1 and we obtain (2.1.13). □

Remark. Taking $Q(u) = u$ we get, in view of (2.1.15)–(2.1.17) and (2.1.3), that

$$\|u(\cdot,t)\|_{Lip^+} \leq \frac{1}{\|u(\cdot,0)\|_{Lip^+}^{-1} + \alpha t} \quad , \quad t \geq 0 . \tag{2.1.18}$$

Next, we show that Lip^+-stability, (2.1.10), identifies the entropy solutions in the class of weak solutions of (2.1.1) and, therefore, may serve as an admissibility criterion for convex conservation laws.

Theorem 2.2. *Assume that $f \in C^1$ is convex, (2.1.3). Then all weak solutions of (2.1.1) satisfying (2.1.10) are uniquely determined by their initial value.*

Proof. (Hoff [H2]) Let u and v be two weak solutions of (2.1.1) subject to the same Cauchy data, u_0, both satisfying the Lip^+-decay estimate (2.1.10). We need to show that $e \equiv u - v$ is identically zero for all $t > 0$.

Let $0 < t_1 < t_2$ and ϕ be a smooth function for which $\mathrm{supp}\phi \cap \{t_1 \le t \le t_2\}$ is bounded. Then, since u and v are weak solutions of (2.1.1), we have

$$\int_{\Re} u(x,t)\phi(x,t)dx \Big|_{t_1}^{t_2} = \int_{t_1}^{t_2} \int_{\Re} (u\phi_t + f(u)\phi_x)dxdt \ ,$$

and

$$\int_{\Re} v(x,t)\phi(x,t)dx \Big|_{t_1}^{t_2} = \int_{t_1}^{t_2} \int_{\Re} (v\phi_t + f(v)\phi_x)dxdt \ .$$

Subtracting, we get that

$$(2.1.19) \quad \int_{\Re} e(x,t)\phi(x,t)dx \Big|_{t_1}^{t_2} = \int_{t_1}^{t_2} \int_{\Re} e \cdot \{\phi_t + f[u,v]\phi_x\}dxdt \quad , \quad f[u,v] = \frac{f(u) - f(v)}{u - v} \ .$$

Now, fix $\eta(x) \in C_0^\infty(\Re)$ and let

$$\eta^\pm(x) = \frac{1}{2}\left[\eta(x) \pm \int_{-\infty}^x |\eta'(s)|ds\right]$$

denote its increasing and decreasing monotone parts. Clearly, $\eta = \eta^+ + \eta^-$ and $(\eta^-)' \le 0 \le (\eta^+)'$. Next, we denote by $\phi_{\varepsilon\delta}^\pm$ the solutions of the following backward linear transport equations,

$$(2.1.20) \qquad (\phi_{\varepsilon\delta}^+)_t + [\psi_\varepsilon * a(u)](\phi_{\varepsilon\delta}^+)_x = 0 \quad , \quad (\phi_{\varepsilon\delta}^+)(x,t_2) = \psi_\delta * \eta^+(x) \ ;$$

$$(2.1.21) \qquad (\phi_{\varepsilon\delta}^-)_t + [\psi_\varepsilon * a(v)](\phi_{\varepsilon\delta}^-)_x = 0 \quad , \quad (\phi_{\varepsilon\delta}^-)(x,t_2) = \psi_\delta * \eta^-(x) \ ,$$

where $\varepsilon, \delta > 0$ and ψ_ε, ψ_δ denote non-negative smooth unit mass mollifiers. Finally, let $\phi_{\varepsilon\delta} = \phi_{\varepsilon\delta}^+ + \phi_{\varepsilon\delta}^-$. Since $\phi_{\varepsilon\delta}$ is smooth and $\mathrm{supp}\phi_{\varepsilon\delta} \cap \{t_1 \le t \le t_2\}$ is bounded, we may set $\phi = \phi_{\varepsilon\delta}$ in (2.1.19) and get, using (2.1.20)–(2.1.21),

$$(2.1.22) \qquad \int_{\Re} e(x,t_2)(\psi_\delta * \eta)(x)dx - \int_{\Re} e(x,t_1)\phi_{\varepsilon\delta}(x,t_1)dx =$$

$$\int_{t_1}^{t_2} \int_{\Re} e \cdot \{f[u,v] - \psi_\varepsilon * a(u)\}(\phi_{\varepsilon\delta}^+)_x dxdt + \int_{t_1}^{t_2} \int_{\Re} e \cdot \{f[u,v] - \psi_\varepsilon * a(v)\}(\phi_{\varepsilon\delta}^-)_x dxdt \ .$$

Now, we turn to estimate the derivatives $\mu^\pm = \mu_{\varepsilon\delta}^\pm \equiv (\phi_{\varepsilon\delta}^\pm)_x$. Denoting $w^+ = u$ and $w^- = v$, we differentiate (2.1.20) and (2.1.21) with respect to x to obtain that

$$\mu_t^\pm + [\psi_\varepsilon * a(w^\pm)]\mu_x^\pm = -[\psi_\varepsilon * a(w^\pm)_x]\mu^\pm \ .$$

Therefore, if $x(t)$ is the characteristic curve

$$\frac{dx}{dt} = \psi_\varepsilon * a(w^\pm(x(t),t)) \quad , \quad x(t_2) = y \ ,$$

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then

(2.1.23) $$\mu^{\pm}(x(t),t) = \mu^{\pm}(y,t_2) \cdot \exp\left[\int_t^{t_2} \left(\psi_\varepsilon * a(w^{\pm})_x\right)(x(s),s)ds\right] .$$

But, since we assumed that u and v satisfy (2.1.10), we have that $a(w^{\pm})_x \le t^{-1}$ and, therefore, the exponential in (2.1.23) is bounded by t_2/t. We conclude that

(2.1.24) $$|\mu^{\pm}(x,t)| \le \frac{t_2}{t}\|\eta'\|_{L_\infty} \qquad \forall t > 0 .$$

In addition, since
$$\text{sgn}\left[\mu^{\pm}(y,t_2)\right] = \text{sgn}\left[\left(\psi_\delta * (\eta^{\pm})'\right)(y)\right] = \pm 1 ,$$
we obtain, in view of (2.1.23), that

(2.1.25) $$\mu^- \le 0 \le \mu^+ .$$

We may now estimate the integrals on the right hand side of (2.1.22). Denoting $f_\theta = \psi_\theta * f$, so that f_θ is sufficiently smooth, the first integrand on the right hand side of (2.1.22) may be rewritten as

$$e\mu^+ \left\{ (f[u,v] - f_\theta[u,v]) + (f_\theta[u,v] - f_\theta[u,u]) + (f_\theta[u,u] - f'(u)) + (a(u) - \psi_\varepsilon * a(u)) \right\} .$$

The first and third terms in the curly braces tend to 0 as $\theta \downarrow 0$; we, therefore, abbreviate them as $o_\theta(1)$. The second term is non-positive, in view of (2.1.25) and convexity:

$$e\mu^+ (f_\theta[u,v] - f_\theta[u,u]) = e\mu^+(v-u)f_\theta[u,v,u] = -e^2\mu^+ \cdot \frac{f_\theta''(\xi)}{2} \le 0 \quad , \quad \xi \in \text{conv}\{u,v\} .$$

Hence, using (2.1.24), the first integral on the right hand side of (2.1.22) is upper-bounded by
$$\|e(x,t)\|_{L_\infty}\frac{t_2}{t_1}\|(\eta^+)'\|_{L_\infty}\left(o_\theta(1) + \|a(u) - \psi_\varepsilon * a(u)\|_{L_1(B)}\right) ,$$
for an appropriate bounded domain $B \subset \Re \times [t_1, t_2]$.

The second integral on the right hand side of (2.1.22) is treated similarly. Finally, we observe that the maximum principle for (2.1.20) and (2.1.21), implies that

$$\|\phi_{\varepsilon\delta}(x,t)\|_{L_\infty} \le \|\phi_{\varepsilon\delta}^+(x,t)\|_{L_\infty} + \|\phi_{\varepsilon\delta}^-(x,t)\|_{L_\infty} \le \|\eta^+(x)\|_{L_\infty} + \|\eta^-(x)\|_{L_\infty} .$$

(2.1.22) thus becomes

$$\int_\Re e(x,t_2)(\psi_\delta * \eta)(x)dx \le \|e(\cdot,t_1)\|_{L_1}\left(\|\eta^+\|_{L_\infty} + \|\eta^-\|_{L_\infty}\right) +$$

$$\|e(x,t)\|_{L_\infty}\frac{t_2}{t_1}\left[\|(\eta^+)'\|_{L_\infty} + \|(\eta^-)'\|_{L_\infty}\right] .$$

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$$\cdot \left\{ o_\theta(1) + \|a(u) - \psi_\varepsilon * a(u)\|_{L_1(B)} + \|a(v) - \psi_\varepsilon * a(v)\|_{L_1(B)} \right\} \quad .$$

By letting θ, ε, t_1 and δ go to zero, in that order, we get that $\int_\Re e(x, t_2)\eta(x)dx \leq 0$ for all $\eta \in C_0^\infty(\Re)$. This proves that $e(\cdot, t_2) = 0$ a.e. for all $t_2 > 0$, i.e. $u = v$. □

Remarks.

1. The Lip^+-stability estimate, (2.1.10), was used in the course of the proof in obtaining (2.1.24). However, even the weaker one-sided Lipschitz boundedness

$$\|u(\cdot, t)\|_{Lip^+} < \infty \qquad \forall t > 0$$

is sufficient in order to guarantee uniqueness (since any finite bound on the right hand side of (2.1.24) will do).

2. The method of the above proof of uniqueness makes use of the adjoint equation and goes back to Holmgren (e.g. [J, §3.5]). In §5 we propose an innovative proof of that uniqueness which makes use of a different approach.

3. Theorems 2.1 and 2.2 require the flux to be C^1-smooth. This requirement is necessary in view of the following example.

Example 2.1. [H2] Consider the Riemann problem

$$u_t + f(u)_x = 0 \quad , \quad f(u) = |u| \in C^0 \setminus C^1 \quad ; \quad u(x, 0) = \mathrm{sgn}(x) \ .$$

Its entropy solution is given by

$$u(x, t) = \begin{cases} -1 & \frac{x}{t} < -1 \\ 0 & -1 < \frac{x}{t} < 1 \\ 1 & 1 < \frac{x}{t} \end{cases} \quad .$$

Thus, $a(u(x, t)) = \mathrm{sgn}(u(x, t))$ is not defined on a set of positive measure. Furthermore, any arbitrarily assigned value to $a(0)$ results with $a(u(x, t))$ which violates (2.1.10).

2.2. One-dimensional non-convex conservation laws

Theorem 2.1 implies that Lip^+-stability, (2.1.10), guarantees uniqueness whenever the flux is convex (or concave). The following example shows that whenever f is neither convex nor concave (i.e., f has at least one inflection point) the Lip^+-stability property fails to identify the entropy solutions.

Example 2.2. [H2] Assume that the flux f has an inflection point. Therefore, we may find values $u_\ell < u_r$ and $\underline{u}, \overline{u} \in (u_\ell, u_r)$, such that

$$(2.2.1) \qquad\qquad\qquad a(u_\ell) = a(u_r)$$

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and the point $(\underline{u}, f(\underline{u}))$ lays strictly below the line joining $(u_\ell, f(u_\ell))$ and $(u_r, f(u_r))$, while $(\overline{u}, f(\overline{u}))$ lays strictly above it (see Figure 2.1).
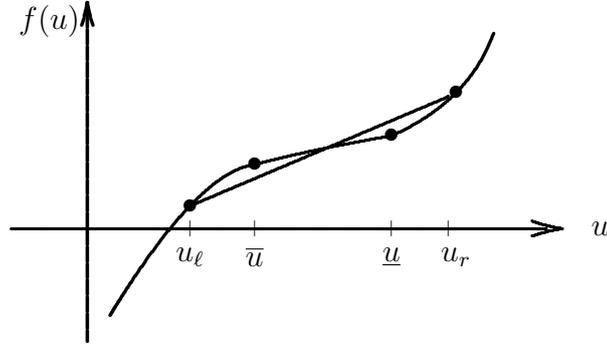
Figure 2.1

Consider the initial data

(2.2.2)
$$u_0(x) = \begin{cases} u_\ell & x < 0 \\ u_r & x > 0 \end{cases} .$$

The function
$$u(x,t) = \begin{cases} u_\ell & x < st \\ u_r & x > st \end{cases} , \quad s = f[u_\ell, u_r] ,$$

is a weak solution of (2.1.1) and (2.2.2) since it satisfies the Rankine-Hugoniot condition, but it violates Oleinik's E-condition and, therefore, differs from the entropy solution. However, like the entropy solution, it satisfies the Lip^+-stability estimate (2.1.10) since, by (2.2.1), $a(u(x,t))$ remains constant for all (x,t), hence $\|a(u(\cdot,t))\|_{Lip^+} = 0$ for all $t \geq 0$.

Example 2.2 showed that whenever the flux f has an inflection point, there exist weak solutions, other than the entropy solutions, which are Lip^+-stable. In the following example we show that if the flux f has at least two inflection points, there always exist initial data for which the corresponding entropy solution is not Lip^+-stable.

Example 2.3. [H2] Consider (2.1.1) augmented with the initial data

$$u_0(x) = \begin{cases} u_1 & x < -1 \\ u_2 & -1 < x < 1 \\ u_3 & 1 < x \end{cases} ,$$

where f is the function depicted in Figure 2.2 and u_1, u_2, u_3 are as indicated.

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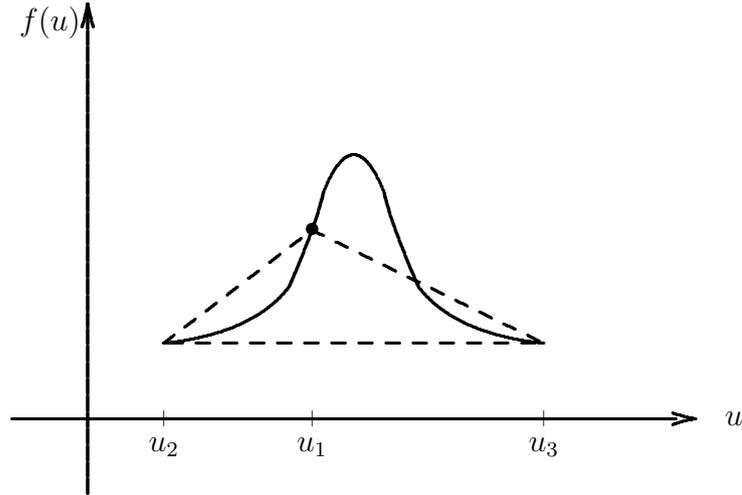
Figure 2.2

The solution of this problem contains two approaching shocks (see Figure 2.3) which intersect at, say, (x_1, t_1).
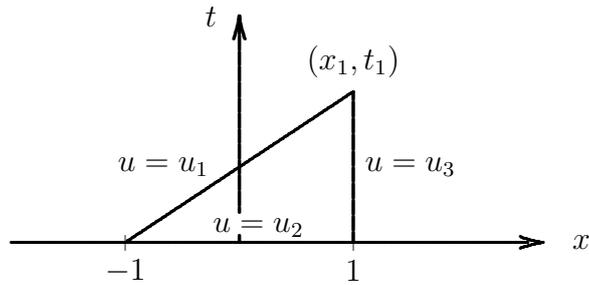
Figure 2.3

Hence, the solution at $t = t_1$ consists of two constant states, u_1 and u_3. As f is nonconvex on $[u_1, u_3]$, the solution for $t > t_1$ contains a rarefaction wave inside which

$$a(u(x, t)) = \frac{x - x_1}{t - t_1} \ .$$

Hence, inside that rarefaction wave $a(u)_x = (t - t_1)^{-1} > t^{-1}$ and (2.1.10) is thus violated.

In view of Example 2.3, the only question which remains to be answered concerns conservation laws whose flux has only one inflection point: are the entropy solutions of such conservation laws Lip^+-stable in the sense of (2.1.10)? This question, though interesting for its own sake, is of minor significance since, in view of Example 2.2, such Lip^+-stability can not serve as an admissibility criterion.

We mention in this context a related result of Dafermos [D2, Theorem 5.1]: He considered

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C^2-smooth fluxes, f, having one inflection point and normalized so that

$$f(0) = f'(0) = f''(0) = 0 \quad , \quad uf''(u) < 0 \quad \forall u \neq 0 .$$

He further assumed that there is ρ, $0 < \rho < 1$, such that

$$\rho a(\tilde{u}) \leq a(u) \qquad \forall u \neq 0 ,$$

where $\tilde{u} \neq u$ is the unique value associated with every $u \neq 0$, defined by the implicit relation

$$a(u) = \frac{f(\tilde{u}) - f(u)}{\tilde{u} - u} \quad .$$

Under these assumptions, he established the following estimate for $u(x,t)$, the entropy solution of (2.1.1)–(2.1.2), as $t \to \infty$:

$$a(u(x,t)) = \frac{x}{t} + O(t^{\rho-1}) \quad , \quad x \in \text{supp } u(\cdot,t) .$$

2.3. Multi-dimensional conservation laws

Here we deal with multi-dimensional conservation laws,

(2.3.1)
$$u_t + \text{div}_{\vec{x}} \vec{f}(u) = 0 \quad , \quad \vec{x} \in \Re^n , \ t > 0 , \ \vec{f} : \Re \to \Re^n .$$

Definition 2.1. *A function $\vec{f} : \Re \to \Re^n$, $n \geq 1$, is called isotropic if for any $\vec{b} \in \Re^n$, the scalar function $F(u) = \vec{b} \cdot \vec{f}(u)$ is either convex, concave or affine.*

The analogous of Theorems 2.1 and 2.2 for isotropic multi-dimensional conservation laws is as follows:

Theorem 2.3. *Assume that $\vec{f} : \Re \to \Re^n$ is C^1-smooth and isotropic and let $\vec{a} = \vec{f}'$. Then $u(\vec{x},t)$ is an entropy solution of (2.3.1) iff it is Lip^+-stable in the sense of (2.1.10).*

Remark. The Lip^+-seminorm is defined in the multi-dimensional case as

(2.3.2)
$$\|\vec{w}(\vec{x})\|_{Lip^+} \equiv \text{ess} \sup_{\vec{x}} (\text{div}_{\vec{x}} \vec{w})^+ ,$$

where the derivatives are taken in the weak sense. This definition coincides with (2.1.11) when $n = 1$.

Proof. One can easily verify that $\vec{f} : \Re \to \Re^n$ is isotropic iff there exist a unit vector $\vec{\gamma} \in \Re^n$, a convex scalar function $F(u)$ and n affine functions $l_i(u)$, $1 \leq i \leq n$, such that

$$f_i(u) = \gamma_i F(u) + l_i(u) \quad , \quad 1 \leq i \leq n .$$

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We now propose the change of variables $\vec{x} \mapsto \vec{y} = \Gamma^{-1}\vec{x}$, where Γ is an orthogonal matrix and $\Gamma_{\cdot,1} = \vec{\gamma}$. With this choice of spatial variables, (2.3.1) reduces to the "essentially one-dimensional" conservation law

$$(2.3.3) \qquad \frac{\partial}{\partial t}u + \frac{\partial}{\partial y_1}F(u) + \sum_{i=1}^{n}\beta_i \frac{\partial}{\partial y_i}u = 0 ,$$

where β_i are constants which depend on the constants l_i' and $\Gamma_{i,j}$. Furthermore, one can easily verify that

$$(2.3.4) \qquad \|\vec{a}(u(\cdot,t))\|_{Lip^+} = \operatorname*{ess\,sup}_{\vec{y}} \left(\frac{\partial}{\partial y_1}F'(u(\vec{y},t)) \right)^+ .$$

In order to get rid of the additional linear term in (2.3.3) we make a further change of variables, $\vec{y} \mapsto \vec{z} = \vec{y} - \vec{\beta}t$. Hence, (2.3.3) is translated into

$$(2.3.5) \qquad \frac{\partial}{\partial t}u + \frac{\partial}{\partial z_1}F(u) = 0$$

and

$$(2.3.6) \qquad \|\vec{a}(u(\cdot,t))\|_{Lip^+} = \operatorname*{ess\,sup}_{\vec{z}} \left(\frac{\partial}{\partial z_1}F'(u(\vec{z},t)) \right)^+ .$$

Theorems 2.1 and 2.2 may be now applied to the one-dimensional conservation law (2.3.5) in order to conclude the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Corollary 2.1. *Consider the multi-dimensional conservation law (2.3.1) where all components of the flux are quadratic functions of u. Then u is an entropy solution iff it is Lip^+-stable, (2.1.10).*

The restriction that f be isotropic is, therefore, a severe one, since it restricts the problem to be essentially one-dimensional. However, Lip^+-stability may fail if the flux is not isotropic, even if all its components are strictly convex, as demonstrated by the next proposition and the following example.

Proposition 2.1. *Consider the conservation law (2.3.1) and assume that there exists a unit vector, $\vec{b} \in \Re^n$, such that the scalar function $F(u) = \vec{b} \cdot \vec{f}(u)$ has two inflection points. Then (2.3.1) admits entropy solutions which violate (2.1.10).*

Proof. Let Γ be an orthogonal matrix such that $\Gamma_{1,\cdot} = \vec{b}$ and let $\vec{y} = \Gamma\vec{x}$. Suppose now that (2.3.1) is subject to an initial condition which remains constant along straight manifolds of the form $\vec{b} \cdot \vec{x} = y_1 = \text{Const}$,

$$u_0(\vec{x}) = \hat{u}_0(\vec{y}) = U_0(y_1) .$$

Then, as can be easily shown by a similarity argument, the resulting entropy solution depends solely on y_1 and t,

$$u(\vec{x}, t) = \hat{u}(\vec{y}, t) = U(y_1, t) \ ,$$

and

$$u_t + \mathrm{div}_{\vec{x}}\vec{f}(u) = \frac{\partial}{\partial t}u + \sum_{i=1}^{n}\frac{\partial}{\partial x_i}f_i(u) = \frac{\partial}{\partial t}\hat{u} + \sum_{i,j=1}^{n}\Gamma_{j,i}\frac{\partial}{\partial y_j}f_i(\hat{u}) =$$

$$\frac{\partial}{\partial t}U + \sum_{i=1}^{n}\Gamma_{1,i}\frac{\partial}{\partial y_1}f_i(U) = \frac{\partial}{\partial t}U + \frac{\partial}{\partial y_1}F(U) = 0 \ .$$

Since F is assumed to have (at least) two inflection points, we conclude, in virtue of Example 2.3, that there exist initial conditions, $U_0(y_1)$, for which the entropy solutions $U(y_1, t)$ are not Lip^+-stable, i.e.,

$$(2.3.7) \qquad \frac{\partial}{\partial y_1}F'(U) > \frac{1}{t}$$

for some $t > 0$. Rewriting (2.3.7) in terms of $(\vec{x}, t)$ yields that for that value of t,

$$\frac{\partial}{\partial y_1}F'(U) = \sum_{i=1}^{n}\frac{\partial}{\partial x_i}f_i'(u) = \mathrm{div}_{\vec{x}}\vec{a}(u) > \frac{1}{t} \ .$$

Therefore, these entropy solutions violate (2.1.10). $\qquad\square$

We now construct an example of an entropy solution to a two-dimensional conservation law with strictly convex fluxes, which is Lip^+-unstable.

Example 2.4. Consider the two-dimensional conservation law

$$(2.3.8) \qquad u_t + f(u)_x + g(u)_y = 0$$

with the strictly convex fluxes

$$(2.3.9) \qquad f(u) = \frac{u^4}{4} + \frac{u^2}{2} \quad , \quad g(u) = u^2 \ .$$

Since $F(u) = \frac{1}{\sqrt{2}}f(u) - \frac{1}{\sqrt{2}}g(u) = \frac{1}{\sqrt{2}}(\frac{u^4}{4} - \frac{u^2}{2})$ has two inflection points $(u = \pm\frac{1}{\sqrt{3}})$, Proposition 2.1 implies the existence of entropy solutions to (2.3.8)–(2.3.9) which are not Lip^+-stable. Indeed, by taking the initial condition

$$(2.3.10) \qquad u(x, y, 0) = u_0(x, y) = \begin{cases} -\frac{1}{\sqrt{3}} & x - y \le -\sqrt{2} \\ -1 & -\sqrt{2} < x - y < \sqrt{2} \\ 1 & \sqrt{2} \le x - y \end{cases} \ ,$$

the two-dimensional problem (2.3.8)–(2.3.10) translates into the one-dimensional problem

$$U_t + F(U)_\xi = 0 \quad , \quad F(U) = \frac{1}{\sqrt{2}}\left(\frac{U^4}{4} - \frac{U^2}{2}\right) \ ,$$

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$$U(\xi, 0) = \begin{cases} -\frac{1}{\sqrt{3}} & \xi \leq -1 \\ -1 & -1 < \xi < 1 \\ 1 & 1 \leq \xi \end{cases} \quad ,$$

where $\xi = \frac{1}{\sqrt{2}}(x - y)$. Figure 2.4 depicts the flux $F(U)$ and the three constant states of which the initial condition consists.
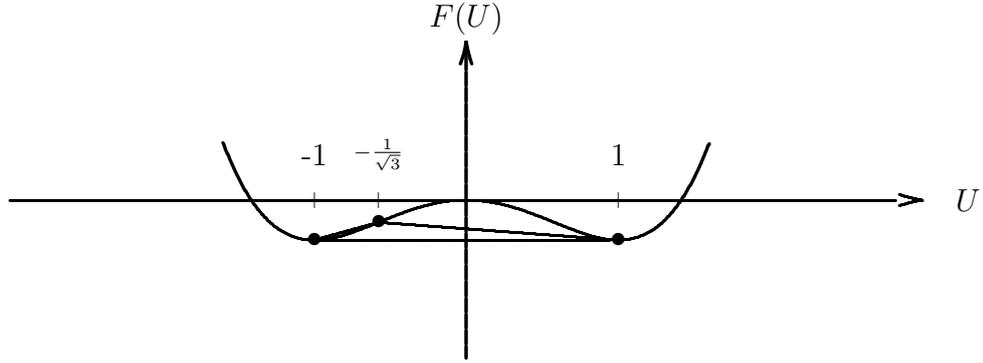
Figure 2.4

In view of Example 2.3 a rarefaction is formed at $(x_1, t_1) = \left(1, \frac{13\sqrt{2}}{36(1 - \frac{1}{\sqrt{3}})}\right)$ and, therefore, $F'(u)_\xi = f'(u)_x + g'(u)_y$ becomes infinite at that point, thus (2.1.10) is violated.

In fact, whenever the multi-dimensional flux has two components, $f_i(u)$ and $f_j(u)$, such that f_i''/f_j'' is not monotone, an example, such as the above, of a Lip^+-unstable entropy solution, may be constructed.

3. COMPACTNESS VIA Lip^+-STABILITY

3.1. Compactness of Lip^+-stable solution operators

The significance of Lip^+-decay in phenomena governed by hyperbolic equations is that it ensures the uniform BV-boundedness of the solution for every $t \geq t_0 > 0$, even if the initial condition is merely L_0^∞. Hence, the solution operator of such problems maps L_0^∞ into BV and is therefore, by Helly's theorem, compact.

Lemma 3.1. *Let $\Omega \subset \Re$ be a bounded domain and let $v \in L_0^\infty(\Omega)$. Then*

$$(3.1.1) \qquad \|v\|_{BV(\Omega)} \leq 2|\Omega| \|v\|_{Lip^+} .$$

Proof. The proof is given for smooth functions v and may be easily generalized by a standard regularization procedure.

$$(3.1.2) \qquad \|v\|_{BV(\Omega)} = \int_\Omega |v'(x)| dx = \int_\Omega (v'(x))^+ dx - \int_\Omega (v'(x))^- dx .$$

But, since v is compactly supported on Ω,

$$(3.1.3) \qquad \int_\Omega (v'(x))^+ dx + \int_\Omega (v'(x))^- dx = \int_\Omega v'(x) dx = 0 .$$

Hence, by (3.1.2) and (3.1.3),

$$\|v\|_{BV(\Omega)} = 2 \int_\Omega (v'(x))^+ dx \leq 2|\Omega| \|v\|_{Lip^+} .$$

$\square$

The result of this Lemma is as follows:

Theorem 3.1. *Let $u(x,t)$ be the solution of the initial value problem $u_t = Lu$, and let $S(t)$ denote the corresponding solution operator. Assume:*
  (a) Finite speed of propagation;
  (b) Lip^+-decay in the sense that

$$(3.1.4) \qquad \|u(\cdot, t)\|_{Lip^+} \leq \frac{C}{t}$$

for some positive constant $C > 0$. Then $S(t)$, $t > 0$, maps L_0^∞ into BV and is therefore compact.

Proof. Due to the compact support at $t = 0$ and the finite speed of propagation, $u(\cdot, t)$ is compactly supported for any $t > 0$ on, say, Ω_t. Hence, we may apply Lemma 3.1 to $u(\cdot, t)$ and conclude by (3.1.4) that

$$\|u(\cdot, t)\|_{BV} = \|u(\cdot, t)\|_{BV(\Omega_t)} \leq 2|\Omega_t| \|u(\cdot, t)\|_{Lip^+} \leq 2|\Omega_t| \frac{C}{t} < \infty \qquad \forall t > 0 .$$

$\square$

We recall that in the case of scalar one-dimensional conservation laws, (2.1.1), the size of the support behaves like $O(1 + \sqrt{t})$, [L2], and therefore we have even BV-decay:

Theorem 3.2. *Let $u(x, t)$ be the entropy solution of (2.1.1)–(2.1.3). Then*

$$(3.1.5) \qquad \|u(\cdot, t)\|_{BV} \leq \text{Const} \cdot \frac{1 + \sqrt{t}}{t} \ .$$

Remark. In the periodical case, since the period does not grow in time, the analogous of (3.1.5) for the total variation of the solution per period is

$$\|u(\cdot, t)\|_{BV} \leq \frac{\text{Const}}{t} \ .$$

It follows that as $t \to \infty$, $u(\cdot, t)$ tends to a constant which, by conservation, equals the average value of the initial data.

3.2. Lip^+-stability of approximate solutions

Here we comment briefly about the applications of Lip^+-stability to the study of approximate solutions of (2.1.1).

Let $\{u^\varepsilon(x, t)\}_{\varepsilon > 0}$ be a family of approximate solutions of (2.1.1). Studying such conservative approximations,

$$(3.2.1) \qquad \int_\Re u^\varepsilon(x, t) dx = \int_\Re u_0(x) dx \qquad \forall t \geq 0 \ ,$$

consistent with (2.1.1)–(2.1.2) in the sense that

$$u^\varepsilon(\cdot, 0) - u_0(\cdot) \underset{\varepsilon \to 0}{\longrightarrow} 0$$

and

$$u_t^\varepsilon + f(u^\varepsilon)_x \underset{\varepsilon \to 0}{\longrightarrow} 0$$

in some appropriate norm, one aims at having both compactness and an entropy condition. The compactness of the family of approximate solutions implies the existence of a convergent subsequence, $\{u^{\varepsilon_i}(x, t)\}$, $\varepsilon_i \downarrow 0$, whose limit, $u(x, t)$, is a weak solution of (2.1.1)–(2.1.2). The entropy condition guarantees that this limit is the admissible entropy solution of (2.1.1)–(2.1.2) and, thanks to uniqueness, the convergence of the whole sequence follows, $u^\varepsilon(x, t) \underset{\varepsilon \to 0}{\longrightarrow} u(x, t) \ $.

Lip^+-stability (in a sense which will be clarified in the following sections) of the family of approximate solutions, ensures both compactness and an entropy condition for that family: If $u_0 \in BV$ then Lip^+-stability implies, along the lines of Lemma 3.1 and Theorem 3.1,

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the uniform BV-boundedness of $\{u^\varepsilon(\cdot, t)\}_{\varepsilon > 0}$. This implies L^1_{loc}-compactness by a standard argument which involves Helly's theorem, the diagonal process and the L_1-Lipschitz continuity of $u^\varepsilon(\cdot, t)$. Therefore, the L_1-limit solution, $u(x, t)$, is a weak solution of (2.1.1)–(2.1.2) for which, due to Lip^+-stability, $\|u(\cdot, t)\|_{Lip^+} < \infty$ for all $t > 0$. This implies, in view of Theorem 2.2 and Remark 1 thereafter, that $u(x, t)$ is the entropy solution of (2.1.1)–(2.1.2).

The above convergence argument lacks convergence rate estimates. In the proceeding sections we show how Lip^+-stability may be used in order to obtain convergence rate estimates as well – both global and local ones.

PART II: CONVERGENCE RATE ESTIMATES FOR APPROXIMATE SOLUTIONS

4. LOCAL ERROR ESTIMATES FOR PARABOLIC APPROXIMATIONS

The first to address the question of *local* error estimates for approximate solutions of hyperbolic conservation laws, was Tadmor in [T4]. He obtained both global and local convergence rate estimates for viscous parabolic approximate solutions, (2.1.12), in the case of rarefaction-free initial data. The two main ingredients in his approach were the Holmgren principle and the Lip^+-stability of both the approximate solution and the exact one.

Let $u(x,t)$ be the entropy solution of the scalar convex conservation law

$$(4.1) \qquad u_t + f(u)_x = 0 \quad , \quad f'' \geq \alpha > 0 \ ,$$

subject to the Lip^+-*bounded* initial data

$$(4.2) \qquad u(x,0) = u_0(x) \quad , \quad u_0 \in L_0^\infty \ , \ \|u_0\|_{Lip^+} < \infty \ .$$

Let $\{u^\varepsilon(x,t)\}_{\varepsilon>0}$ be a family of viscous parabolic approximate solutions,

$$(4.3) \qquad u_t^\varepsilon + f(u^\varepsilon)_x = \varepsilon Q(u^\varepsilon)_{xx} \quad , \quad Q' \geq q > 0 \ , \ \varepsilon \downarrow 0 \ .$$

Subtracting (4.1) from (4.3) we arrive at the equation which governs the error,

$$(4.4) \qquad \frac{\partial}{\partial t}[e^\varepsilon(x,t)] + \frac{\partial}{\partial x}[\bar{a}^\varepsilon(x,t)e^\varepsilon(x,t)] = \varepsilon\frac{\partial^2}{\partial x^2}[Q(u^\varepsilon(x,t))] \quad , \quad e^\varepsilon = u^\varepsilon - u \ ,$$

where $\bar{a}^\varepsilon(x,t)$ denotes the mean-value

$$(4.5) \qquad \bar{a}^\varepsilon(x,t) = \int_0^1 a\left(\xi u^\varepsilon(x,t) + (1-\xi)u(x,t)\right) d\xi \ .$$

Proposition 4.1. (Lip^+-Stability). *The averaged velocity* $\bar{a}^\varepsilon(x,t)$, *given in (4.5), satisfies the one-sided Lipschitz condition*

$$(4.6) \qquad \|\bar{a}^\varepsilon(x,t)\|_{Lip^+} \leq \frac{\eta L_0^+}{1 + tL_0^+} \ ,$$

where the constants L_0^+ *and* η *are given by*

$$(4.7) \qquad L_0^+ = \max\{\|a(u(\cdot,0))\|_{Lip^+}, \|a(u^\varepsilon(\cdot,0))\|_{Lip^+}\} < \infty \ ,$$

and

$$(4.8) \qquad \eta = \frac{\max a' \cdot \max Q'}{\alpha q} \ .$$

Proof. Estimate (2.1.17) implies that

$$(4.9) \qquad \|u^\varepsilon(\cdot, t)\|_{Lip^+} \le \frac{1}{q} \cdot \frac{\|Q(u^\varepsilon(\cdot, 0))\|_{Lip^+}}{1 + \beta t \|Q(u^\varepsilon(\cdot, 0))\|_{Lip^+}} \quad , \quad t \ge 0 \, ,$$

where β and q are given, respectively, in (2.1.16) and (4.3). On the other hand, (2.1.10) and (2.1.3) imply that

$$(4.10) \qquad \|u(\cdot, t)\|_{Lip^+} \le \frac{1}{\alpha} \cdot \frac{\|a(u(\cdot, 0))\|_{Lip^+}}{1 + t \|a(u(\cdot, 0))\|_{Lip^+}} \quad , \quad t \ge 0 \, .$$

Inserting the last two inequalities into (4.5) and using the monotonicity of $a(\cdot)$, we arrive after little rearrangement at (4.6). □

Now, we consider the dual problem of (4.4), given by the backward linear transport equation

$$(4.11) \qquad \frac{\partial}{\partial t}[\phi^\varepsilon(x, t)] + \bar{a}^\varepsilon(x, t)\frac{\partial}{\partial x}[\phi^\varepsilon(x, t)] = 0 \quad , \quad t \le T \, ,$$

with ε-independent and C_0^1 data prescribed at $t = T$,

$$(4.12) \qquad \phi^\varepsilon(x, T) = \phi(x) \, .$$

We recall that if

$$\|\bar{a}^\varepsilon(\cdot, t)\|_{Lip^+} \le m(t) \in L_1[t_0, T] \, ,$$

then equation (4.10) is well posed in $W^{1,\infty}$ and the following estimates hold for all $t \in [t_0, T]$ (consult [T4, Theorem 2.2]): [2]

$$(4.13) \qquad \|\phi^\varepsilon(\cdot, t)\|_{L_\infty} \le \|\phi\|_{L_\infty} \, ;$$

$$(4.14) \qquad \|\phi^\varepsilon(\cdot, t)\|_{Lip} \le \|\phi\|_{Lip} \cdot e^{M(t)} \quad , \quad M(t) = \int_t^T m(\tau)d\tau \, .$$

Hence, in view of (4.6) and (4.7), the dual problem (4.11)–(4.12) has a unique Lipschitz continuous solution, $\phi^\varepsilon(x, t)$, $0 \le t \le T$, and, as implied by (4.6) and (4.14),

$$(4.15) \qquad \|\phi^\varepsilon(\cdot, t)\|_{Lip} \le \left(\frac{1 + TL_0^+}{1 + tL_0^+}\right)^\eta \|\phi\|_{Lip} \quad , \quad 0 \le t \le T \, .$$

We may now use the dual problem in order to estimate the error e^ε. Integrating (4.4) against $\phi^\varepsilon(\cdot, t)$, (4.11) against $e^\varepsilon(\cdot, t)$ and adding the two equations result in

$$(4.16) \qquad \frac{d}{dt}\left(e^\varepsilon(\cdot, t), \phi^\varepsilon(\cdot, t)\right) = \varepsilon\left(\frac{\partial^2}{\partial x^2}[Q(u^\varepsilon(\cdot, t))] \, , \, \phi^\varepsilon(\cdot, t)\right) \, ,$$

---

[2]The notation $\|\cdot\|_{Lip}$ stands for the Lipschitz (or $W^{1,\infty}$) seminorm: $\|w(x)\|_{Lip} \equiv \operatorname{ess\,sup}_{x \ne y} \left|\frac{w(x)-w(y)}{x-y}\right| \, .$

where $(\cdot, \cdot)$ denotes the usual $L_2(\Re_x)$ inner product. The right hand side of (4.16) may be bounded, using (4.15) and the TVD property of (4.3), by

$$(4.17) \qquad \varepsilon \left| \left(\frac{\partial^2}{\partial x^2} [Q(u^\varepsilon(\cdot, t))] \ , \ \phi^\varepsilon(\cdot, t) \right) \right| \leq \varepsilon K_0 \left(\frac{1 + TL_0^+}{1 + tL_0^+} \right)^\eta \|\phi\|_{Lip} \quad , \quad 0 \leq t \leq T \ ,$$

where $K_0 = \max Q' \cdot \sup_{\varepsilon > 0} \|u^\varepsilon(\cdot, 0)\|_{BV}$. Integrating (4.16) over $[0, T]$ yields, using (4.17), the following error estimate [T4, Theorem 3.3]:

$$|(e^\varepsilon(\cdot, T) \ , \ \phi(\cdot))| \leq K_T \|\phi\|_{Lip} \left(\varepsilon + \|e^\varepsilon(\cdot, 0)\|_{W^{-1,1}} \right) \ ,$$

where the constant K_T is given by

$$K_T = \frac{K_0}{L_0^+} \cdot \begin{cases} (1 + TL_0^+) \cdot \ln(1 + TL_0^+) & \eta = 1 \\ \\ \frac{1}{\eta - 1}(1 + TL_0^+)^\eta & \eta > 1 \end{cases} \ ,$$

and the $W^{-1,1}$-norm denotes the L_1-norm of the primitive, i.e.,

$$(4.18) \qquad \|w(\cdot, t)\|_{W^{-1,1}} = \left\| \int_{-\infty}^x w(\xi, t) d\xi \right\|_{L_1(\Re_x)} \ .$$

Note that this norm is defined only when $\int_\Re w(x, t) dx = 0$. Indeed, in our present case $\int_\Re e^\varepsilon(x, t) dx = 0$ for all $t \geq 0$, since the viscous parabolic approximation, (4.3), is conservative, (3.2.1).

Finally, using the straightforward identity

$$(4.19) \qquad \|w(x)\|_{W^{-1,1}} = \sup_{\phi \in C_0^1(\Re)} \frac{|(w, \phi)|}{\|\phi\|_{Lip}} \ ,$$

we may summarize the above results as follows [T4, Theorem 3.4]:

Theorem 4.1. ($W^{-1,1}$-Convergence Rate). *Let $u(x, t)$ and $u^\varepsilon(x, t)$ be the entropy solution and the corresponding viscosity solution of (4.1) and (4.3), respectively. Assume that:*

(a) The initial viscosity data, $u^\varepsilon(\cdot, 0)$, are consistent with the initial entropy data, $u(\cdot, 0)$, in the sense that

$$\|e^\varepsilon(\cdot, 0)\|_{W^{-1,1}} \leq \text{Const} \cdot \varepsilon \ ;$$

(b) The increasing part of the viscosity and entropy initial data is Lipschitz, i.e., (4.7) holds.

Then, for any $T > 0$, there exists a constant $K = K_T$, such that

$$(4.20) \qquad \|e^\varepsilon(\cdot, T)\|_{W^{-1,1}} \leq K_T \cdot \varepsilon \ .$$

We conclude this section by showing (along the lines of [T4] and [NT2]) how the basic $W^{-1,1}$-error estimate, (4.20), may be translated into various global, as well as local, error estimates. In the next sections of this part of our work, we shall concentrate on obtaining error estimates for various approximations, in the $W^{-1,1}$-norm.

We begin by noting that the Lip^+-stability of $u^\varepsilon(\cdot, t)$ implies (in view of Lemma 3.1) that it is BV-bounded, $\|u^\varepsilon(\cdot, t)\|_{BV} \leq \text{Const}_t$ (although u^ε is not compactly supported, as assumed in Lemma 3.1, we may conclude its BV-boundedness due to the exponential decay of its tail). Since $u(\cdot, t)$ is also BV-bounded we conclude that

$$(4.21) \qquad \|e^\varepsilon(\cdot, T)\|_{BV} \leq C_T \ ,$$

for some positive constant C_T.

We now invoke the Sobolev inequality (e.g., [F, Theorem 9.3]) which states that for all $1 \leq p \leq \infty$ and $-1 \leq s \leq \frac{1}{p}$,

$$\|w\|_{W^{s,p}} \leq \text{Const} \cdot \|w\|_{BV}^{1-\nu} \cdot \|w\|_{W^{-1,1}}^\nu \quad , \quad \nu = \frac{1 - sp}{2p} \ .$$

Using this inequality in order to interpolate between the BV-regularity of the error, (4.21), and the $W^{-1,1}$-error estimate, (4.20), we get the following global $W^{s,p}$-error estimates:

$$(4.22) \qquad \|u^\varepsilon(\cdot, T) - u(\cdot, T)\|_{W^{s,p}} \leq O(\varepsilon^{\frac{1-sp}{2p}}) \quad , \quad -1 \leq s \leq \frac{1}{p} \ , \quad 1 \leq p \leq \infty \ .$$

Error estimate (4.22) with $(s, p) = (0, 1)$ yields an $O(\sqrt{\varepsilon})$ convergence rate in L_1, which is familiar from the setup of monotone difference schemes [K2], [S1].

Uniform convergence (which corresponds to $(s, p) = (0, \infty)$) fails in this case due to possible shock discontinuities. Therefore, instead of uniform L_∞-convergence rate, we seek local convergence rate estimates away from the shock discontinuities of the entropy solution.

To this end, we rewrite error estimate (4.20), using identity (4.19), as follows:

$$(4.23) \qquad |(u^\varepsilon(\cdot, T) - u(\cdot, T)) * \phi| \leq K_T \cdot \varepsilon \|\phi\|_{Lip} \ ,$$

where $\phi = \phi(x)$ is any $C_0^1(-1, 1)$-unit mass mollifier. Replacing ϕ in (4.23) by $\phi_\delta(x) = \frac{1}{\delta}\phi(\frac{x}{\delta})$, we get that

$$(4.24) \qquad |(u^\varepsilon(\cdot, T) * \phi_\delta)(x) - (u(\cdot, T) * \phi_\delta)(x)| \leq K_T \cdot \frac{\varepsilon}{\delta^2} \|\phi\|_{Lip} \ .$$

If, in addition, $\phi(x)$ is chosen so that

$$(4.25) \qquad \int_{-1}^1 x^k \phi(x) dx = 0 \quad \text{for } k = 1, 2, ..., p - 1 \ ,$$

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then the following error estimate, based on Taylor's expansion, is straighforward:

$$(4.26) \qquad |(u(\cdot, T) * \phi_\delta)(x) - u(x, T)| \leq \frac{\delta^p}{p!} \|\phi\|_{L_1} \cdot \left\| \frac{\partial^p}{\partial x^p} u(\cdot, T) \right\|_{L_\infty(x-\delta, x+\delta)} .$$

We therefore conclude, using (4.24) and (4.26) and taking $\delta \sim \varepsilon^{\frac{1}{p+2}}$, that if ϕ is any unit mass $C_0^1(-1, 1)$-mollifier, satisfying (4.25), the following error estimate holds:

$$(4.27a) \qquad |(u^\varepsilon(\cdot, T) * \phi_\delta)(x) - u(x, T)| \leq \text{Const}_{x,T} \cdot \varepsilon^{\frac{p}{p+2}} ,$$

where

$$(4.27b) \qquad \text{Const}_{x,T} = \text{Const}_T \cdot \left(1 + \frac{1}{p!} \cdot \left\| \frac{\partial^p}{\partial x^p} u(\cdot, T) \right\|_{L_\infty(x-\delta, x+\delta)} \right) .$$

Error estimate (4.27) shows that by post-processing the approximate solution, $u^\varepsilon(\cdot, t)$, we can recover the pointwise values of $u(x, t)$ with an error as close to $O(\varepsilon)$ as the local smoothness of $u(\cdot, t)$ permits.

Taking $p = 1$ in (4.27) we get that

$$(4.28) \quad |(u^\varepsilon(\cdot, T) * \phi_\delta)(x) - u(x, T)| \leq \text{Const}_T \cdot \left(1 + \|u_x(\cdot, T)\|_{L_\infty(x-\delta, x+\delta)} \right) \cdot \sqrt[3]{\varepsilon} , \quad \delta \sim \sqrt[3]{\varepsilon} .$$

This pointwise convergence rate of order $O(\sqrt[3]{\varepsilon})$ holds even without post-processing the approximate solution. In order to show that, we consider the difference

$$u^\varepsilon(x, T) - (u^\varepsilon(\cdot, T) * \phi_\delta)(x) = \int_{-\delta}^{\delta} [u^\varepsilon(x, T) - u^\varepsilon(x - \xi, T)] \phi_\delta(\xi) d\xi =$$

$$= \int_{-\delta}^{\delta} \left[\frac{u^\varepsilon(x, T) - u^\varepsilon(x - \xi, T)}{\xi} \right] \cdot \xi \phi_\delta(\xi) d\xi .$$

Choosing a positive C_0^1-unit mass mollifier, $\phi(x)$, supported on $(0, 1)$, we conclude in view of the Lip^+-stability of u^ε, (4.9), and the above inequality, that

$$u^\varepsilon(x, T) - (u^\varepsilon(\cdot, T) * \phi_\delta)(x) \leq \text{Const}_T \cdot \delta .$$

Similarly, choosing $\phi(x)$ to be a positive C_0^1-unit mass mollifier supported on $(-1, 0)$, leads to

$$u^\varepsilon(x, T) - (u^\varepsilon(\cdot, T) * \phi_\delta)(x) \geq \text{Const}_T \cdot \delta .$$

Each of the last two inequalities (with $\delta \sim \sqrt[3]{\varepsilon}$), together with (4.28), show that the approximate solution itself converges pointwise to the exact solution with an $O(\sqrt[3]{\varepsilon})$ local convergence rate:

$$(4.29a) \qquad |u^\varepsilon(x, T) - u(x, T)| \leq \text{Const}_{x,T} \cdot \sqrt[3]{\varepsilon} ,$$

$$(4.29b) \qquad \text{Const}_{x,T} = \text{Const}_T \cdot \left(1 + \|u_x(\cdot, T)\|_{L_\infty(x-\sqrt[3]{\varepsilon}, x+\sqrt[3]{\varepsilon})} \right) .$$

5. CONVERGENCE RATE OF APPROXIMATE SOLUTIONS

In this section we obtain convergence rate estimates for general families of approximate solutions, $\{u^\varepsilon(x,t)\}_{\varepsilon>0}$, to the hyperbolic conservation law, (4.1). We allow possibly Lip^+-unbounded initial data,

$$(5.1) \qquad u(x,0) = u_0(x) \quad , \quad u_0 \in L_\infty \cap BV \ , \ \|u_0(x)\|_{Lip^+} \leq \infty \ ,$$

and thus extend the results of the previous section.

Our convergence analysis (mainly summarized in [NT3]) is presented here for the case of compactly supported initial data, but it applies to the periodic case as well.

Since the entropy solutions of (4.1) are characterized by their Lip^+-stability, (2.1.10), we seek the convergence rate of conservative approximations, (3.2.1), which mimic this one sided Lipschitz stability of the exact entropy solution. This leads to

Definition 5.1. *A family $\{u^\varepsilon(x,t)\}_{\varepsilon>0}$ of approximate solutions of the conservation law (4.1) is strongly Lip^+-stable if*

$$(5.2) \qquad \|a(u^\varepsilon(\cdot,t))\|_{Lip^+} \leq \frac{1}{\|a(u^\varepsilon(\cdot,0))\|_{Lip^+}^{-1} + t} \quad , \quad \varepsilon > 0 \ .$$

Our first convergence rate result is the content of the following theorem:

Theorem 5.1. *Let $\{u^\varepsilon(x,t)\}_{\varepsilon>0}$ be a family of conservative and strongly Lip^+-stable approximations to the entropy solution of (4.1)+(5.1), $u(x,t)$. Then,*

(a) If $\|u_0\|_{Lip^+} < \infty$, the following error estimate holds (K_1 and K_2 denote constants which depend on T):

$$(5.3) \qquad \|u^\varepsilon(\cdot,T) - u(\cdot,T)\|_{W^{-1,1}} \leq$$

$$K_1\|u^\varepsilon(\cdot,0) - u(\cdot,0)\|_{W^{-1,1}} + K_2\|u_t^\varepsilon + f(u^\varepsilon)_x\|_{L_\infty([0,T],W^{-1,1}(\Re_x))} \ ;$$

(b) If $\|u_0\|_{Lip^+} = \infty$ and the approximate solutions are also L_1-stable, the following error estimate holds:

$$(5.4) \qquad \|u^\varepsilon(\cdot,T) - u(\cdot,T)\|_{W^{-1,1}} \leq O\left(\frac{1}{\varepsilon}\right) \|u^\varepsilon(\cdot,0) - u(\cdot,0)\|_{W^{-1,1}} +$$

$$O(\varepsilon)\|u^\varepsilon(\cdot,0)\|_{BV} + O(\varepsilon)\|u(\cdot,0)\|_{BV} + O\left(|\ln\varepsilon|\right)\|u_t^\varepsilon + f(u^\varepsilon)_x\|_{L_\infty([0,T],W^{-1,1}(\Re_x))} \ .$$

Remark. An approximate solution operator, $S^\varepsilon(t)$, is considered L_1-stable, if for any two initial conditions, u_0 and v_0,

$$(5.5) \qquad \|S^\varepsilon(t)u_0 - S^\varepsilon(t)v_0\|_{L_1(\Re_x)} \leq \text{Const}_t \|S^\varepsilon(0)u_0 - S^\varepsilon(0)v_0\|_{L_1(\Re_x)} \quad , \quad t > 0 \ .$$

The first $W^{-1,1}$-error estimate, (5.3), for the case of rarefaction free initial data, holds even if the family of approximate solutions is merely Lip^+-bounded,

$$(5.6) \qquad \|u^\varepsilon(\cdot, t)\|_{Lip^+} \leq \text{Const}_t \quad, \quad \varepsilon > 0 \ ,$$

and does not satisfy the strong Lip^+-stability requirement (5.2). This stronger Lip^+-stability is necessary for convergence in the case of initial rarefactions.

As a counter-example we mention the Roe scheme (consult [B]): When $\|u_0\|_{Lip^+} < \infty$ this scheme remains Lip^+-bounded, (5.6), and converges to the exact entropy solution. However, it is not strongly Lip^+-stable and, therefore, it fails to converge to the entropy solution in case of Lip^+-unbounded initial data (as demonstrated by the steady state solution obtained by this scheme for $u_0(x) = \text{sgn}(x)$).

The strong Lip^+-stability of the approximation, (5.2), is indeed one of the main ingredients in establishing convergence when initial rarefactions are present. Unfortunately, many well-known approximations of (4.1) fail to satisfy this restricted condition. However, these approximations are still Lip^+-stable in a weaker sense than that of Definition 5.1. This weak Lip^+-stability proves sufficient in order to establish the same convergence rates as in Theorem 5.1.

Definition 5.2. *Let $\{u^\varepsilon(x, t)\}_{\varepsilon > 0}$ be a family of approximate solutions of (4.1) and let*

$$W^\varepsilon(t) \equiv \|a(u^\varepsilon(\cdot, t))\|_{Lip^+} \ .$$

Then this family is ε-weakly Lip^+-stable if there exists a constant M such that whenever

$$W^\varepsilon(0) \leq \frac{M}{\varepsilon} \ ,$$

the following estimates hold for every $T > 0$:

$$(5.7) \qquad e^{\int_0^T W^\varepsilon(t)dt} \leq O\left(\frac{1}{\varepsilon}\right) \ ;$$

$$(5.8) \qquad \int_0^T e^{\int_t^T W^\varepsilon(\tau)d\tau} dt \leq O\left(|\ln \varepsilon|\right) \ .$$

Note that any strongly Lip^+-stable family of approximate solutions is also weakly Lip^+-stable (for any value of the constant M). We henceforth refer by Lip^+-stability to either weak or strong Lip^+-stability. The following theorem asserts that the convergence rate estimates, given in Theorem 5.1 for strongly Lip^+-stable approximations, hold also for ε-weakly Lip^+-stable ones.

Theorem 5.2. *Let $\{u^\varepsilon(x,t)\}_{\varepsilon>0}$ be a family of conservative and Lip^+-stable approximations to the entropy solution of (4.1), $u(x,t)$. Then,*

(a) If $\|u_0\|_{Lip^+} < \infty$, error estimate (5.3) holds;

(b) If $\|u_0\|_{Lip^+} = \infty$ and the approximate solutions are also L_1-stable, error estimate (5.4) holds.

In order to have convergence, the stability of the family of approximate solutions is not sufficient. The second crucial ingredient is consistency.

Definition 5.3. *The family $\{u^\varepsilon(x,t)\}_{\varepsilon>0}$ of approximate solutions is $W^{-1,1}$-consistent with (4.1)+(5.1) if*

$$(5.9) \qquad \|u^\varepsilon(\cdot,0) - u_0(\cdot)\|_{W^{-1,1}} \leq \begin{cases} \text{Const} \cdot \varepsilon & \text{if } \|u_0\|_{Lip^+} < \infty \\ \text{Const} \cdot \varepsilon^2 |\ln \varepsilon| & \text{if } \|u_0\|_{Lip^+} = \infty \end{cases}$$

and

$$(5.10) \qquad \|u^\varepsilon_t + f(u^\varepsilon)_x\|_{L_\infty([0,T], W^{-1,1}(\Re_x))} \leq \text{Const}_T \cdot \varepsilon .$$

In view of Theorem 5.2 and Definition 5.3, we may now conclude the following convergence rate estimates.

Corollary 5.1. ($W^{-1,1}$-*Error Estimates*). *If the family $\{u^\varepsilon(x,t)\}_{\varepsilon>0}$ of approximate solutions is conservative, $W^{-1,1}$-consistent with (4.1)+(5.1), L_1-stable and Lip^+-stable, then for every $T > 0$ there exists a constant C_T such that*

$$(5.11a) \qquad \|u^\varepsilon(\cdot,T) - u(\cdot,T)\|_{W^{-1,1}} \leq C_T \cdot \tilde\varepsilon ,$$

where

$$(5.11b) \qquad \tilde\varepsilon = \begin{cases} \varepsilon & \text{if } \|u_0\|_{Lip^+} < \infty \\ \varepsilon |\ln \varepsilon| & \text{if } \|u_0\|_{Lip^+} = \infty \end{cases} .$$

Remarks.

1. Error estimate (5.11) suggests that whenever initial rarefactions are present, the convergence rate in $W^{-1,1}$ is nearly $O(\varepsilon)$. The $|\ln \varepsilon|$ term, which somewhat slows the rate of convergence, is a consequence of the initial rarefaction (as we show later on).

2. Error estimate (5.11) relates to that of Harabetian in [H3]. He has shown an $O(\varepsilon|\ln\varepsilon|)$ convergence rate in L_1 for the viscous parabolic regularizations, (4.3), when the exact entropy solution amounts to a pure rarefaction wave.

The $W^{-1,1}$ error estimate (5.11) may be translated, as shown in the previous section, into various global, as well as local, error estimates which we summarize as follows:

Corollary 5.2. (Global and Local Error Estimates). *Let $\{u^\varepsilon(x,t)\}_{\varepsilon>0}$ be a family of conservative, $W^{-1,1}$-consistent, L_1-stable and Lip^+-stable approximate solutions of the conservation law (4.1)+(5.1). Then the following error estimates hold ($\tilde{\varepsilon}$ is as in (5.11b)):*

$$(E1) \qquad \|u^\varepsilon(\cdot,T) - u(\cdot,T)\|_{W^{s,p}} \leq C_T \cdot \tilde{\varepsilon}^{\frac{1-sp}{2p}} \quad , \quad -1 \leq s \leq \frac{1}{p} \quad , \quad 1 \leq p \leq \infty \quad ;$$

$$(E2) \qquad |(u^\varepsilon(\cdot,T) * \phi_\delta)(x) - u(x,T)| \leq \mathrm{Const}_{x,T} \cdot \tilde{\varepsilon}^{\frac{p}{p+2}} \quad , \quad \delta \sim \tilde{\varepsilon}^{\frac{1}{p+2}} \quad ,$$

where

$$\mathrm{Const}_{x,T} = \mathrm{Const}_T \cdot \left(1 + \frac{1}{p!} \cdot \left\|\frac{\partial^p}{\partial x^p}u(\cdot,T)\right\|_{L_\infty(x-\delta,x+\delta)}\right)$$

and $\phi_\delta(x) = \frac{1}{\delta}\phi\left(\frac{x}{\delta}\right)$ is any unit mass $C_0^1(-1,1)$-mollifier, satisfying

$$\int_{-1}^{1} x^k \phi(x)dx = 0 \quad \text{for } k = 1, 2, ..., p-1 \quad ;$$

$$(E3) \qquad |u^\varepsilon(x,T) - u(x,T)| \leq \mathrm{Const}_{x,T} \cdot \sqrt[3]{\tilde{\varepsilon}} \quad ,$$

where

$$\mathrm{Const}_{x,T} = \mathrm{Const}_T \cdot \left(1 + \|u_x(\cdot,T)\|_{L_\infty(x-\sqrt[3]{\tilde{\varepsilon}},x+\sqrt[3]{\tilde{\varepsilon}})}\right) \quad .$$

Remark. A similar treatment enables the recovery of the derivatives of $u(x,t)$ as well, consult [T4, §4].

We finally turn to prove our main results, given in Theorems 5.1 and 5.2. Since Theorem 5.1 deals with strongly Lip^+-stable approximations, which are, as noted before, weakly Lip^+-stable as well, it suffices to prove Theorem 5.2.

Proof of Theorem 5.2. We deal with conservative approximations to (4.1) which take the following form

$$(5.12) \qquad \frac{\partial}{\partial t}[u^\varepsilon(x,t)] + \frac{\partial}{\partial x}[f(u^\varepsilon(x,t))] = r^\varepsilon(x,t) \quad , \quad t > 0 \quad , \quad \varepsilon \downarrow 0 \,,$$

where $r^\varepsilon(x,t)$ is the truncation error of the approximation, and we need to estimate, in $W^{-1,1}$, the error

$$e^\varepsilon(x,t) \equiv u^\varepsilon(x,t) - u(x,t) \,.$$

Step 1. We first assume that both the exact entropy solution, $u(x,t)$, and its approximation, $u^\varepsilon(x,t)$, have a Lip^+-bounded initial data, i.e.,

$$(5.13) \qquad L_0^+ = \max\{\|a(u(\cdot,0))\|_{Lip^+}, \|a(u^\varepsilon(\cdot,0))\|_{Lip^+}\} < \infty .$$

Subtracting (4.1) from (5.12) we arrive at the equation which governs the error $e^\varepsilon(x,t)$,

$$(5.14) \qquad \frac{\partial}{\partial t}[e^\varepsilon(x,t)] + \frac{\partial}{\partial x}[\bar{a}^\varepsilon(x,t)e^\varepsilon(x,t)] = r^\varepsilon(x,t) \quad , \quad t > 0 ,$$

where

$$\bar{a}^\varepsilon(x,t) = \int_0^1 a\left(\xi u^\varepsilon(x,t) + (1-\xi)u(x,t)\right) d\xi .$$

Note that the monotonicity of $a(\cdot)$ implies that

$$(5.15) \qquad \min\{a(u), a(u^\varepsilon)\} \leq \bar{a}^\varepsilon(x,t) \leq \max\{a(u), a(u^\varepsilon)\} .$$

Integration of (5.14) with respect to x yields

$$(5.16) \qquad \frac{\partial}{\partial t}[E^\varepsilon(x,t)] + \bar{a}^\varepsilon(x,t)\frac{\partial}{\partial x}[E^\varepsilon(x,t)] = R^\varepsilon(x,t) \quad , \quad t > 0 ,$$

where

$$E^\varepsilon(x,t) = \int_{-\infty}^x e^\varepsilon(\xi,t)d\xi \quad , \quad R^\varepsilon(x,t) = \int_{-\infty}^x r^\varepsilon(\xi,t)d\xi .$$

Integration of (5.16) over $\Re$ against $\text{sgn}(E^\varepsilon)$ and rearranging, yield that

$$(5.17) \qquad \frac{d}{dt}\|E^\varepsilon(\cdot,t)\|_{L_1} \leq \int_x \bar{a}^\varepsilon(x,t)\left(-\frac{\partial}{\partial x}|E^\varepsilon(x,t)|\right)dx + \|R^\varepsilon(\cdot,t)\|_{L_1} .$$

The main effort henceforth is concentrated on upper bounding the integral on the right hand side of (5.17). To this end we suggest to divide the real line into intervals,

$$\Re = \cup_n I_n(t) \quad , \quad I_n(t) = [x_n(t), x_{n+1}(t)) ,$$

in such a way that neither $\text{sgn}(e^\varepsilon)$ nor $\text{sgn}(E^\varepsilon)$ changes within the interior of these intervals (the implicit assumption of piecewise smoothness of the solution, as in [L2], may be removed by considering a further vanishing parabolic regularization which is omitted). We use this division to define the following function:

$$(5.18) \qquad \hat{a}^\varepsilon(x,t) = \begin{cases} a(u(x,t)) & \text{if } x \in I_n(t) \text{ and } E^\varepsilon(x,t) \geq 0\big|_{I_n(t)} \\ \\ a(u^\varepsilon(x,t)) & \text{if } x \in I_n(t) \text{ and } E^\varepsilon(x,t) \leq 0\big|_{I_n(t)} \end{cases} .$$

We now claim (and prove later on) that

$$(5.19) \qquad \int_x \bar{a}^\varepsilon(x,t)\left(-\frac{\partial}{\partial x}|E^\varepsilon(x,t)|\right)dx \leq \int_x \hat{a}^\varepsilon(x,t)\left(-\frac{\partial}{\partial x}|E^\varepsilon(x,t)|\right)dx .$$

Integration by parts of the right hand side of (5.19) yields

$$(5.20) \qquad \int_x \bar{a}^\varepsilon(x,t)\left(-\frac{\partial}{\partial x}|E^\varepsilon(x,t)|\right)dx \le \int_x \frac{\partial}{\partial x}[\hat{a}^\varepsilon(x,t)] \cdot |E^\varepsilon(x,t)|dx \quad .$$

The following inequality (whose proof is postponed) provides us an upper bound for the integral on the right hand side of (5.20):

$$(5.21a) \qquad \int_x \frac{\partial}{\partial x}[\hat{a}^\varepsilon(x,t)] \cdot |E^\varepsilon(x,t)|dx \le L^\varepsilon(t)\|E^\varepsilon(\cdot,t)\|_{L_1} \quad,$$

where

$$(5.21b) \qquad L^\varepsilon(t) = \max\left\{\frac{L_0^+}{1+tL_0^+} \ , \ W^\varepsilon(t) = \|a(u^\varepsilon(\cdot,t))\|_{Lip^+}\right\} \ .$$

Inserting (5.20) and (5.21a) into (5.17), we arrive at the inequality

$$(5.22) \qquad \frac{d}{dt}\|e^\varepsilon(\cdot,t)\|_{W^{-1,1}} \le L^\varepsilon(t)\|e^\varepsilon(\cdot,t)\|_{W^{-1,1}} + \|r^\varepsilon(\cdot,t)\|_{W^{-1,1}} \ ,$$

which implies that

$$(5.23) \qquad \|e^\varepsilon(\cdot,T)\|_{W^{-1,1}} \le e^{\int_0^T L^\varepsilon(t)dt} \cdot \|e^\varepsilon(\cdot,0)\|_{W^{-1,1}} + \int_0^T e^{\int_t^T L^\varepsilon(\tau)d\tau}\|r^\varepsilon(\cdot,t)\|_{W^{-1,1}}dt \ .$$

Since, by the definition of L_0^+ in (5.13), $W^\varepsilon(0) \le L_0^+$, we conclude, in view of Lip^+-stability (see Definition 5.2), that

$$(5.24) \qquad e^{\int_0^T W^\varepsilon(t)dt} \le \mathrm{Const}_1 \quad , \quad \mathrm{Const}_1 \sim \frac{L_0^+}{M} \quad ,$$

and

$$(5.25) \qquad \int_0^T e^{\int_t^T W^\varepsilon(\tau)d\tau}dt \le \mathrm{Const}_2 \quad , \quad \mathrm{Const}_2 \sim |\ln M - \ln L_0^+| \ .$$

Using (5.24), (5.25) and (5.21b) in (5.23), proves the desired error estimate (5.3).

Finally, in order to conclude Step 1, we return to justify (5.19) and (5.21):

First, we prove (5.19) by showing that the inequality holds in each interval $I_n(t)$, i.e,

$$(5.26) \qquad \int_{I_n(t)} \bar{a}^\varepsilon(x,t)\left(-\frac{\partial}{\partial x}|E^\varepsilon(x,t)|\right)dx \le \int_{I_n(t)} \hat{a}^\varepsilon(x,t)\left(-\frac{\partial}{\partial x}|E^\varepsilon(x,t)|\right)dx \quad .$$

Suppose that $E^\varepsilon(\cdot,t) \ge 0$ in $I_n(t)$. Then by definition (5.18),

$$(5.27) \qquad \hat{a}^\varepsilon(x,t) = a(u(x,t)) \quad \forall x \in I_n(t) \ .$$

There are two possibilities to consider. If $e^\varepsilon(x,t) \geq 0$ in $I_n(t)$ then by (5.15)

(5.28) $\quad \bar{a}^\varepsilon(x,t) \geq a(u(x,t)) \quad , \quad -\dfrac{\partial}{\partial x}|E^\varepsilon(x,t)| = -\mathrm{sgn}(E^\varepsilon(x,t)) \cdot e^\varepsilon(x,t) \leq 0 \quad \forall x \in I_n(t)$.

Therefore, (5.26) follows in this case by (5.27) and (5.28). If, on the other hand, $e^\varepsilon(x,t) \leq 0$ in $I_n(t)$, then

(5.29) $\qquad\qquad \bar{a}^\varepsilon(x,t) \leq a(u(x,t)) \quad , \quad -\dfrac{\partial}{\partial x}|E^\varepsilon(x,t)| \geq 0 \qquad \forall x \in I_n(t)$

and (5.26) follows in this case as well. The case $E^\varepsilon(\cdot,t)\mid_{I_n(t)} \leq 0$ is treated similarly. This concludes the proof of (5.19).

Next, we prove inequality (5.21). In view of definitions (5.18) and (5.21b), we conclude, using the Lip^+-stability of the exact solution,

$$\|a(u(\cdot,t))\|_{Lip^+} \leq \frac{L_0^+}{1 + tL_0^+} \quad ,$$

that $\frac{\partial}{\partial x}[\hat{a}^\varepsilon(x,t)]$ satisfies the following inequality in the sense of distributions:

(5.30) $\qquad \dfrac{\partial}{\partial x}[\hat{a}^\varepsilon(x,t)] \leq L^\varepsilon(t) + \sum[\hat{a}^\varepsilon(x_n(t)+0,t) - \hat{a}^\varepsilon(x_n(t)-0,t)]\delta(x - x_n(t)) \quad ,$

the sum being taken over all division points $x_n(t)$ where $\hat{a}^\varepsilon(\cdot,t)$ experiences a jump discontinuity, namely where $\mathrm{sgn}(E^\varepsilon(\cdot,t))$ changes. But, $E^\varepsilon(\cdot,t)$ – being a continuous primitive function – vanishes at these points. Hence, integration of (5.30) against $|E^\varepsilon(x,t)|$ proves (5.21a) and completes Step 1.

Step 2. Now we turn to the case of initial rarefactions and prove error estimate (5.4). To this end we introduce the function $\psi_\delta(\cdot) = \frac{1}{\delta}\psi(\frac{\cdot}{\delta})$, $\delta > 0$, which is the dilated mollifier of

(5.31) $\qquad\qquad\qquad\qquad \psi(x) = \begin{cases} 1 & |x| \leq \frac{1}{2} \\ 0 & |x| > \frac{1}{2} \end{cases}$.

Clearly

(5.32) $\qquad\qquad\qquad\qquad \|\psi_\delta * w - w\|_{L_1} \leq O(\delta)\|w\|_{BV}$,

and

(5.33) $\qquad\qquad\qquad\qquad \|\psi_\delta * w\|_{Lip^+} \leq O\left(\dfrac{1}{\delta}\right) \qquad \delta \downarrow 0$.

With this in mind we return to the conservation law (4.1) and its approximate solution (5.12) and define a new pair of solutions, u_δ and u_δ^ε, corresponding to the mollified initial data:

(5.34) $\qquad \dfrac{\partial}{\partial t}[u_\delta(x,t)] + \dfrac{\partial}{\partial x}[f(u_\delta(x,t))] = 0 \quad , \quad u_\delta(\cdot,0) = \psi_\delta * u(\cdot,0)$;

$$(5.35) \qquad \frac{\partial}{\partial t}[u_\delta^\varepsilon(x,t)] + \frac{\partial}{\partial x}[f(u_\delta^\varepsilon(x,t)] = r_\delta^\varepsilon(x,t) \quad , \quad u_\delta^\varepsilon(\cdot,0) = \psi_\delta * u^\varepsilon(\cdot,0) .$$

We are now able to estimate the $W^{-1,1}$-error in (5.4) by decomposing it as follows:

$$(5.36) \qquad \|u^\varepsilon(\cdot,T) - u(\cdot,T)\|_{W^{-1,1}} \le$$

$$\|u^\varepsilon(\cdot,T) - u_\delta^\varepsilon(\cdot,T)\|_{W^{-1,1}} + \|u_\delta^\varepsilon(\cdot,T) - u_\delta(\cdot,T)\|_{W^{-1,1}} + \|u_\delta(\cdot,T) - u(\cdot,T)\|_{W^{-1,1}} .$$

Since for compactly supported functions, $\|w\|_{W^{-1,1}} \le |\text{supp}\{w\}| \cdot \|w\|_{L_1}$, we may bound the first term on the right hand side of (5.36), using (5.5), (5.35) and (5.32), as follows (Ω_T denotes the compact support[3] at $t = T$):

$$(5.37) \qquad \|u^\varepsilon(\cdot,T) - u_\delta^\varepsilon(\cdot,T)\|_{W^{-1,1}} \le |\Omega_T| \cdot \|u^\varepsilon(\cdot,T) - u_\delta^\varepsilon(\cdot,T)\|_{L_1} \le$$

$$\le |\Omega_T| \cdot C_T \|u^\varepsilon(\cdot,0) - u_\delta^\varepsilon(\cdot,0)\|_{L_1} \le |\Omega_T| \cdot C_T \cdot O(\delta)\|u^\varepsilon(\cdot,0)\|_{BV} = O(\delta)\|u^\varepsilon(\cdot,0)\|_{BV} .$$

Similarly, the last term on the right hand side of (5.36), may be bounded by

$$(5.38) \qquad \|u_\delta(\cdot,T) - u(\cdot,T)\|_{W^{-1,1}} \le O(\delta)\|u(\cdot,0)\|_{BV} .$$

Hence, it remains only to deal with the term $\|u_\delta^\varepsilon(\cdot,T) - u_\delta(\cdot,T)\|_{W^{-1,1}}$. This requires δ to be appropriately chosen so that

$$(5.39) \qquad W_\delta^\varepsilon(0) \le \frac{M}{\varepsilon} \quad , \quad W_\delta^\varepsilon(t) = \|a(u_\delta^\varepsilon(\cdot,t))\|_{Lip^+}$$

and, consequently, the Lip^+-stability estimates – (5.7) and (5.8) – hold. If D denotes the largest positive jump in $u^\varepsilon(\cdot,0)$ then the choice $\delta = 2D \max[a'(u^\varepsilon(\cdot,0))]\varepsilon/M$ will do for (5.39). By doing so, we may conclude the ε-weak Lip^+-stability estimates, (5.7) and (5.8), for $W_\delta^\varepsilon(t)$:

$$e^{\int_0^T W_\delta^\varepsilon(t)dt} \le O\left(\frac{1}{\varepsilon}\right) \quad ; \quad \int_0^T e^{\int_t^T W_\delta^\varepsilon(\tau)d\tau}dt \le O(|\ln \varepsilon|) .$$

These estimates, together with error estimate (5.23) for $e_\delta^\varepsilon = u_\delta^\varepsilon - u_\delta$, imply that

$$(5.40) \qquad \|u_\delta^\varepsilon(\cdot,T) - u_\delta(\cdot,T)\|_{W^{-1,1}} \le$$

$$O\left(\frac{1}{\varepsilon}\right)\|u_\delta^\varepsilon(\cdot,0) - u_\delta(\cdot,0)\|_{W^{-1,1}} + O(|\ln \varepsilon|)\|r_\delta^\varepsilon\|_{L_\infty([0,T],W^{-1,1}(\Re_x))} .$$

Since $\|\psi_\delta * w\|_{W^{-1,1}} \le \|w\|_{W^{-1,1}}$, estimate (5.40) implies that

$$(5.41) \qquad \|u_\delta^\varepsilon(\cdot,T) - u_\delta(\cdot,T)\|_{W^{-1,1}} \le$$

---

[3]Note that in case $u^\varepsilon(\cdot,T)$ is not compactly supported, the exponential decay which characterizes the tail of various viscosity-like approximations will suffice for our estimates.

$$O\left(\frac{1}{\varepsilon}\right)\|u^\varepsilon(\cdot,0)-u(\cdot,0)\|_{W^{-1,1}}+O(|\ln\varepsilon|)\cdot\|r^\varepsilon\|_{L_\infty([0,T],W^{-1,1}(\Re_x))}\ .$$

Therefore, since $\delta=O(\varepsilon)$, (5.4) follows from (5.36), (5.37), (5.38) and (5.41) and the proof is thus concluded. $\qquad\square$

Remark. Note that if the approximate solution smoothens the initial data so that

$$\|u^\varepsilon(\cdot,0)\|_{Lip^+}\leq O\left(\frac{1}{\varepsilon}\right)\ ,$$

e.g. – the SV-method, there is no need to mollify the initial data of the approximation, as we did in (5.35). Hence, in this case, the error term (5.37) does not exist and, therefore, error estimate (5.4) holds even if the approximate solution is not L_1-stable.

Epilogue. Theorem 5.1 implies two straightforward uniqueness and stability results, interesting for their own sake.

One immediate consequence of Theorem 5.1 is the uniqueness Theorem 2.2, stating that weak solutions of the conservation law which are Lip^+-stable are uniquely determined by their initial value: Let u be the entropy solution of (4.1)+(5.1) and v be another weak solution of (4.1)+(5.1) which is also Lip^+-stable in the sense of (2.1.10). Setting $u^\varepsilon=v$, $\varepsilon>0$, we have

$$u^\varepsilon(\cdot,0)-u(\cdot,0)=0\quad\text{and}\quad u^\varepsilon_t+f(u^\varepsilon)_x=0\qquad\forall\varepsilon>0\ .$$

Hence, error estimate (5.4) implies that

$$\|v(\cdot,T)-u(\cdot,T)\|_{W^{-1,1}}=\|u^\varepsilon(\cdot,T)-u(\cdot,T)\|_{W^{-1,1}}\leq O(\varepsilon)\|u_0\|_{BV}\qquad\forall\varepsilon>0\ .$$

Letting $\varepsilon\downarrow0$, we conclude that $u=v$.

Another immediate consequence of Theorem 5.1 is the following $W^{-1,1}$-stability result:

Theorem 5.3. *Let u and v denote two entropy solutions of the conservation law (4.1), subject to the $L_\infty\cap BV$ initial data u_0 and v_0, respectively. Then*

(5.42) $$\|v(\cdot,t)-u(\cdot,t)\|_{W^{-1,1}}\leq \mathrm{Const}_t\cdot\|v_0-u_0\|^\eta_{W^{-1,1}}\ ,$$

where $\eta=1$ if u_0 and v_0 are Lip^+-bounded and $\eta=\frac{1}{2}$ otherwise.

Proof. We set $u^\varepsilon=v$ for all $\varepsilon>0$ and use error estimates (5.3) and (5.4), given in Theorem 5.1.

In case that both u_0 and v_0 are Lip^+-bounded, error estimate (5.3) holds without the truncation error term on the right hand side. Therefore, (5.42) follows with $\mathrm{Const}_t = K_1$ and $\eta = 1$.

If either of the initial conditions is Lip^+-unbounded, we use error estimate (5.4) in order to conclude that

$$\|v(\cdot, t) - u(\cdot, t)\|_{W^{-1,1}} \leq O\left(\frac{1}{\varepsilon}\right) \|v_0 - u_0\|_{W^{-1,1}} + O(\varepsilon)\left(\|v_0\|_{BV} + \|u_0\|_{BV}\right)$$

for all $\varepsilon > 0$. Taking $\varepsilon = \|v_0 - u_0\|_{W^{-1,1}}^{\frac{1}{2}}$, proves (5.42) with $\eta = \frac{1}{2}$. $\qquad\square$

6. VISCOUS PARABOLIC REGULARIZATIONS

In the following sections, §6–§10, we demonstrate our convergence rate estimates for various types of approximations.

Here we consider viscous parabolic regularizations to (4.1)+(5.1) of the form

$$(6.1a) \qquad \frac{\partial}{\partial t}[u^\varepsilon(x,t)] + \frac{\partial}{\partial x}[f(u^\varepsilon(x,t))] = \varepsilon \frac{\partial^2}{\partial x^2}[Q(u^\varepsilon(x,t))] \quad , \quad Q' \geq 0 \quad , \quad \varepsilon \downarrow 0 \, ,$$

$$(6.1b) \qquad\qquad\qquad\qquad u^\varepsilon(x,0) = u_0(x) \, .$$

These regularizations are:

- Conservative;

- L_∞-bounded, $\|u^\varepsilon(\cdot,t)\|_{L_\infty} \leq \|u_0\|_{L_\infty}$;

- L_1-contractive and therefore, thanks to translation invariance of (6.1a), BV-bounded (see Theorem 7.1, later on, for a proof of L_1-contraction in a more general setting);

- $W^{-1,1}$-consistent in the sense of Definition 5.2, since $u^\varepsilon(\cdot,0) = u_0(\cdot)$ and

$$\|u_t^\varepsilon + f(u^\varepsilon)_x\|_{W^{-1,1}} = \|\varepsilon Q(u^\varepsilon)_x\|_{L_1} \leq \varepsilon \cdot \max_{|u^\varepsilon| \leq \|u_0\|_{L_\infty}} |Q'(u^\varepsilon)| \cdot \|u^\varepsilon(\cdot,t)\|_{BV} \leq O(\varepsilon) \ ;$$

- Lip^+-stable (Theorem 6.1).

In view of the above, error estimates (E1)–(E3) given in Corollary 5.2, apply to this family of approximate solutions.

We are, therefore, left only with the task of proving Lip^+-stability; this is done in the following theorem and lemma.

Theorem 6.1. (Lip^+-Stability). *The (possibly degenerate) parabolic regularization (6.1) is strongly Lip^+-stable if*

$$(6.2) \qquad\qquad\qquad\qquad \left(\frac{Q'}{a'}\right)'' \leq 0$$

and is ε-weakly Lip^+-stable otherwise.

Remarks.

1. Since the most natural choice (already presented by Von-Neumann, Lax and Wendroff, [RM]) of a regularization coefficient which satisfies (6.2) is $Q(u) = a(u)$, we refer henceforth to such regularizations as "speed-like".

2. The most common choice of a regularization coefficient is $Q(u) = u$. For this special choice of $Q(u)$, the speed-like condition (6.2) reads $\left(\frac{1}{a'}\right)'' \leq 0$, consult [LFX].

Proof. Let us first assume that Q' is strictly positive so that the solution u^ε is smooth. Multiplying (6.1a) by $a'(u^\varepsilon(x,t))$ we get

$$(6.3) \qquad \frac{\partial}{\partial t}[a(u^\varepsilon)] + a(u^\varepsilon)\frac{\partial}{\partial x}[a(u^\varepsilon)] = \varepsilon a'(u^\varepsilon)\frac{\partial^2}{\partial x^2}[Q(u^\varepsilon)] .$$

By denoting

$$w^\varepsilon = w^\varepsilon(x,t) = \frac{\partial a(u^\varepsilon)}{\partial x} = a'(u^\varepsilon)\frac{\partial u^\varepsilon}{\partial x} ,$$

the right hand side of (6.3) may be rewritten as follows:

$$(6.4) \qquad \varepsilon a'(u^\varepsilon)\frac{\partial^2}{\partial x^2}[Q(u^\varepsilon)] = \varepsilon\left[Q'(u^\varepsilon)\frac{\partial w^\varepsilon}{\partial x} + \left(\frac{Q'(u^\varepsilon)}{a'(u^\varepsilon)}\right)'(w^\varepsilon)^2\right] .$$

Differentiation of (6.3) with respect to x and using identity (6.4) yields

$$(6.5) \qquad \frac{\partial w^\varepsilon}{\partial t} + (w^\varepsilon)^2 + a(u^\varepsilon)\frac{\partial w^\varepsilon}{\partial x} =$$

$$\varepsilon\left[Q'(u^\varepsilon)\frac{\partial^2 w^\varepsilon}{\partial x^2} + \frac{Q''(u^\varepsilon)}{a'(u^\varepsilon)}w^\varepsilon\frac{\partial w^\varepsilon}{\partial x} + 2\left(\frac{Q'(u^\varepsilon)}{a'(u^\varepsilon)}\right)'w^\varepsilon\frac{\partial w^\varepsilon}{\partial x} + \left(\frac{Q'(u^\varepsilon)}{a'(u^\varepsilon)}\right)''\frac{(w^\varepsilon)^3}{a'(u^\varepsilon)}\right] .$$

Since u^ε is smooth and compactly supported, $w^\varepsilon(\cdot,t)$ attains its maximal value, say in $x = x(t)$, and

$$(6.6) \qquad w^\varepsilon(x(t),t) \geq 0 \ , \quad \frac{\partial w^\varepsilon}{\partial x}(x(t),t) = 0 \ , \quad \frac{\partial^2 w^\varepsilon}{\partial x^2}(x(t),t) \leq 0 \ .$$

Hence, denoting

$$W^\varepsilon(t) = w^\varepsilon(x(t),t) = \|a(u^\varepsilon(\cdot,t))\|_{Lip^+} ,$$

we conclude by (6.5), (6.6) and the positivity of a' and Q', that

$$(6.7) \qquad \frac{dW^\varepsilon}{dt} + (W^\varepsilon)^2 \leq \varepsilon K(W^\varepsilon)^3 ,$$

where

$$(6.8) \qquad K \equiv \frac{1}{\alpha}\max_{|u^\varepsilon|\leq\|u_0\|_{L^\infty}}\left[\left(\frac{Q'(u^\varepsilon)}{a'(u^\varepsilon)}\right)''\right]^+ .$$

In view of Lemma 6.1 below, inequality (6.7) implies ε-weak Lip^+-stability. In case condition (6.2) holds, $K = 0$ and inequality (6.7) amounts to Ricatti's inequality

$$\frac{dW^\varepsilon}{dt} + (W^\varepsilon)^2 \leq 0 \,,$$

which implies strong Lip^+-stability.

If $Q' \geq 0$, equation (6.1a) is degenerate and, therefore, admits non-smooth solutions. This case may be treated, as in [VH], by introducing a further regularization. We replace $Q(\cdot)$ by the strictly monotone regularization term $Q_\delta(\cdot) = Q(\cdot) + \delta a(\cdot)$. Note that with this choice of Q_δ the value of K, (6.8), does not change. Hence, the corresponding solution, u_δ^ε, satisfies inequality (6.7) and by letting $\delta \downarrow 0$, we obtain the same inequality for the limit solution, u^ε. □

Lemma 6.1. *Let $y^\varepsilon(t)$ denote the solution of*

$$(6.9) \qquad \frac{dy^\varepsilon}{dt} + (y^\varepsilon)^2 = \varepsilon K (y^\varepsilon)^3 \quad , \quad K > 0 \quad , \quad t > 0 \,,$$

where

$$(6.10) \qquad y^\varepsilon(t = 0) = \frac{c^\varepsilon}{\varepsilon K}$$

and c^ε satisfies

$$(6.11) \qquad 0 < \underline{c} \leq c^\varepsilon \leq \overline{c} < 1 \quad , \quad \varepsilon \downarrow 0 \,.$$

Then, for any $T > 0$,

$$(6.12) \qquad e^{\int_0^T y^\varepsilon(t)dt} \leq O\left(\frac{1}{\varepsilon}\right)$$

and

$$(6.13) \qquad \int_0^T e^{\int_t^T y^\varepsilon(\tau)d\tau} dt \leq O(|\ln \varepsilon|) \,.$$

Remark. Lemma 6.1 and inequality (6.7) imply that the approximate solutions $u^\varepsilon(x,t)$ are ε-weakly Lip^+-stable with any constant $M < 1/K$ (consult Definition 5.2).

Proof. By rescaling ε we may assume that $K = 1$. Since $y^\varepsilon(t)$ is the solution of a perturbated Ricatti's equation, (6.9), we denote by $y(t)$ the solution of the regular Ricatti's equation,

$$(6.14a) \qquad \frac{dy}{dt} + y^2 = 0 \,,$$

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subject to the same initial condition,

$$(6.14b) \qquad\qquad y(0) = y^\varepsilon(0) = \frac{c^\varepsilon}{\varepsilon} \quad , \quad 0 < \underline{c} \le c^\varepsilon \le \overline{c} < 1 \; .$$

The solution of (6.14) is

$$(6.15) \qquad\qquad y(t) = \left(t + \frac{1}{y^\varepsilon(0)} \right)^{-1} \; ,$$

while the solution of (6.9)–(6.10) is given implicitly by

$$(6.16) \qquad\qquad y^\varepsilon(t) = \left(t + D^\varepsilon + \varepsilon \ln\left(\frac{y^\varepsilon}{\frac{1}{\varepsilon} - y^\varepsilon} \right) \right)^{-1} \; ,$$

with

$$(6.17) \qquad\qquad D^\varepsilon = \frac{1}{y^\varepsilon(0)} - \varepsilon \ln\left(\frac{y^\varepsilon(0)}{\frac{1}{\varepsilon} - y^\varepsilon(0)} \right) \; .$$

First, we note that (6.10) and (6.11) imply that $y^\varepsilon(t)$ is monotonically decreasing. Hence

$$(6.18) \qquad\qquad y^\varepsilon(t) \le y^\varepsilon(0) \qquad \forall t \ge 0 \; .$$

Furthermore, since by (6.9) and (6.14a)

$$y^\varepsilon(t) \ge y(t) \qquad \forall t \ge 0 \; ,$$

it follows, using (6.15) and monotonicity, (6.18), that

$$(6.19) \qquad y^\varepsilon(t) \ge y^\varepsilon(T) \ge y(T) = \left(T + \frac{1}{y^\varepsilon(0)} \right)^{-1} \qquad \forall t \in [0, T] \; .$$

With the upper and lower bounds on $y^\varepsilon(t)$, (6.18) and (6.19), we may estimate the terms in (6.16) and (6.17). We start with the last term in the brackets in (6.16). Using (6.18) and (6.14b) it may be upper-bounded as follows, for all $t \ge 0$:

$$(6.20) \quad \varepsilon \ln\left(\frac{y^\varepsilon}{\frac{1}{\varepsilon} - y^\varepsilon} \right) = -\varepsilon \ln\left(\frac{1}{\varepsilon y^\varepsilon} - 1 \right) \le -\varepsilon \ln\left(\frac{1}{\varepsilon y^\varepsilon(0)} - 1 \right) \le -\varepsilon \ln\left(\frac{1}{\overline{c}} - 1 \right) = O(\varepsilon) \; .$$

On the other hand, using (6.19) together with (6.14b), we get a lower bound for this term:

$$(6.21) \qquad \varepsilon \ln\left(\frac{y^\varepsilon}{\frac{1}{\varepsilon} - y^\varepsilon} \right) \ge -\varepsilon \ln\left(\frac{T + \frac{1}{y^\varepsilon(0)}}{\varepsilon} - 1 \right) = O\left(\varepsilon |\ln \varepsilon| \right) \quad , \quad 0 \le t \le T \; .$$

Next, we estimate the constant D^ε, given in (6.17). Using (6.14b), (6.20) and (6.21) we obtain the following bounds:

$$(6.22) \qquad\qquad D^\varepsilon \le \frac{\varepsilon}{\underline{c}} + \varepsilon \ln\left(\frac{T + \frac{1}{y^\varepsilon(0)}}{\varepsilon} - 1 \right) = O\left(\varepsilon |\ln \varepsilon| \right) \; ;$$

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(6.23)
$$D^\varepsilon \geq \frac{\varepsilon}{c} + \varepsilon \ln\left(\frac{1}{c} - 1\right) = O(\varepsilon) \ .$$

Hence we conclude by (6.16) and (6.20)–(6.23) that

(6.24)
$$y^\varepsilon(t) = \left[t + O(\varepsilon) + O\left(\varepsilon|\ln\varepsilon|\right)\right]^{-1}, \quad 0 \leq t \leq T \ .$$

With (6.24), estimates (6.12) and (6.13) may easily be verified. Indeed,

$$e^{\int_0^T y^\varepsilon(t)dt} = \left|\frac{T + O(\varepsilon) + O\left(\varepsilon|\ln\varepsilon|\right)}{O(\varepsilon) + O\left(\varepsilon|\ln\varepsilon|\right)}\right| \leq O\left(\frac{1}{\varepsilon}\right)$$

and

$$\int_0^T e^{\int_t^T y^\varepsilon(\tau)d\tau} dt = \int_0^T \frac{T + O(\varepsilon) + O\left(\varepsilon|\ln\varepsilon|\right)}{t + O(\varepsilon) + O\left(\varepsilon|\ln\varepsilon|\right)} dt \leq O\left(|\ln\varepsilon|\right) \ ,$$

and the proof is thus completed. □

7. PSEUDO-VISCOSITY APPROXIMATIONS

One of the methods for the approximation of phenomena governed by hyperbolic conservation laws is considering parabolic regularizations with a gradient dependent viscosity. These so-called pseudo-viscosity approximations take the form

$$(7.1) \qquad u_t^\varepsilon + f(u^\varepsilon)_x = \varepsilon Q(u^\varepsilon, p^\varepsilon)_x \quad , \quad p^\varepsilon := u_x^\varepsilon \quad , \quad \varepsilon \downarrow 0 \ ,$$

$$(7.2) \qquad u^\varepsilon(x, 0) = u_0(x) \ ,$$

where

$$(7.3) \qquad \frac{\partial Q}{\partial p^\varepsilon} \geq 0 \ .$$

Note that this class of parabolic regularizations is larger than the class of viscous parabolic approximations, (6.1).

First, we note that these conservative approximations satisfy the maximum principle and, therefore, the solution remains uniformly bounded by $\|u_0\|_{L_\infty}$.

Next, we show that the solution operator of (7.1) is L_1-contractive and therefore, thanks to translation invariance, the solution u^ε remains BV-bounded.

Theorem 7.1. (L_1-Contraction). *Let u^ε and v^ε be two solutions of (7.1), (7.3). Then*

$$(7.4) \qquad \|u^\varepsilon(\cdot, t) - v^\varepsilon(\cdot, t)\|_{L_1} \leq \|u^\varepsilon(\cdot, 0) - v^\varepsilon(\cdot, 0)\|_{L_1} \quad , \quad t > 0 \ .$$

Proof. Let $u^\varepsilon(x, t)$ and $v^\varepsilon(x, t)$ be two solutions of (7.1). We assume that the regularization (7.1) is uniformly parabolic, $Q_p \geq \delta > 0$, hence u^ε and v^ε are smooth. L_1-contraction for the degenerate case, $Q_p \geq 0$, easily follows by adding the term δp to the pseudo-viscosity coefficient $Q(u, p)$ and letting $\delta \downarrow 0$.

As in [L2], we divide the real line into intervals, $\Re = \cup_n I_n(t)$, $I_n(t) = [x_n(t), x_{n+1}(t))$, so that

$$(7.5) \qquad (-1)^n [u^\varepsilon(\cdot, t) - v^\varepsilon(\cdot, t)]\big|_{I_n(t)} \geq 0$$

and consequently

$$(7.6) \qquad u^\varepsilon(x_n(t), t) = v^\varepsilon(x_n(t), t) \ .$$

Using (7.5) and (7.6) we conclude that

$$(7.7) \qquad \frac{d}{dt}\|u^\varepsilon(\cdot,t) - v^\varepsilon(\cdot,t)\|_{L_1(\Re)} =$$

$$= \frac{d}{dt}\sum_n (-1)^n \int_{x_n(t)}^{x_{n+1}(t)} [u^\varepsilon(x,t) - v^\varepsilon(x,t)]dx = \sum_n (-1)^n \int_{x_n(t)}^{x_{n+1}(t)} [u_t^\varepsilon(x,t) - v_t^\varepsilon(x,t)]dx \ .$$

Using (7.1) and carrying out the integral on the right hand side of (7.7), we find that

$$(7.8) \qquad \frac{d}{dt}\|u^\varepsilon(\cdot,t) - v^\varepsilon(\cdot,t)\|_{L_1(\Re)} =$$

$$= \sum_n (-1)^n \Big[- f(u^\varepsilon) + f(v^\varepsilon) \Big]_{x_n(t)}^{x_{n+1}(t)} + \varepsilon \sum_n (-1)^n \Big[Q(u^\varepsilon, u_x^\varepsilon) - Q(v^\varepsilon, v_x^\varepsilon) \Big]_{x_n(t)}^{x_{n+1}(t)} \ .$$

The first term on the right hand side of (7.8) vanishes in view of (7.6). Equality (7.6) also implies that the second term may be written as

$$(7.9) \qquad \varepsilon \sum_n \left[Q_p(u^\varepsilon, w^\varepsilon) \cdot \frac{\partial}{\partial x}\Big[(-1)^n (u^\varepsilon(x,t) - v^\varepsilon(x,t)) \Big] \right]_{x_n(t)}^{x_{n+1}(t)} \ ,$$

where w^ε is a mid-value between u_x^ε and v_x^ε . Since, in light of (7.5),

$$\frac{\partial}{\partial x}\Big[(-1)^n (u^\varepsilon(x,t) - v^\varepsilon(x,t)) \Big] \Big|_{x=x_{n+1}(t)} \leq 0$$

and

$$\frac{\partial}{\partial x}\Big[(-1)^n (u^\varepsilon(x,t) - v^\varepsilon(x,t)) \Big] \Big|_{x=x_n(t)} \geq 0 \ ,$$

and since $Q_p > 0$, we conclude that (7.9) is nonpositive. Therefore, by (7.8),

$$\frac{d}{dt}\|u^\varepsilon(\cdot,t) - v^\varepsilon(\cdot,t)\|_{L_1(\Re)} \leq 0$$

and inequality (7.4) follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

Finally, we address the question of Lip^+-stability. We show that under suitable assumptions on the pseudo-viscosity coefficient, $Q(u,p)$, the solution of (7.1) is weakly Lip^+-stable.

Theorem 7.2. (Lip^+-Stability). *Let Ω denote the domain in $\Re^2$,*

$$\Omega = [\inf u_0 \ , \ \sup u_0] \times [0, \infty) \ .$$

Assume that the following hold for all $(u,p) \in \Omega$ (M_1 and M_2 denote some constants):

$$(A1) \qquad |Q_p(u,p)| \ , \ |Q_{up}(u,p)| \leq M_1 \ ;$$

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$(A2)$
$$Q_{uu}(u,p) \le M_2 \cdot p \quad ;$$

$(A3)$
$$Q_{pp}(u,p) \le 0 .$$

Then the solution of (7.1)–(7.3) is ε-weakly Lip^+-stable.

Proof. As in the proof of Theorem 7.1, we first deal with the uniformly parabolic case, $Q_p \ge \delta > 0$. Let us denote

$$w^\varepsilon(x,t) = \frac{\partial}{\partial x}[a(u^\varepsilon(x,t))] \quad , \quad W^\varepsilon(t) = \max_x w^\varepsilon(x,t) = \|a(u^\varepsilon(\cdot,t))\|_{Lip^+} .$$

In view of Lemma 6.1, it suffices to show that there exists a constant $K > 0$, such that

(7.10)
$$\frac{d}{dt}W^\varepsilon(t) + (W^\varepsilon(t))^2 \le \varepsilon K(W^\varepsilon(t))^3 \quad , \quad t > 0 .$$

Multiplying (7.1) by $a'(u^\varepsilon)$ and differentiating with respect to x, we find that $w = w^\varepsilon(x,t)$ satisfies

$$w_t + w^2 + aw_x = \varepsilon \cdot \left[Q_{uu}\frac{w^2}{a'} + 2Q_{up}\frac{w}{a'}(w_x + A'w^2) + Q_{pp}\frac{(w_x + A'w^2)^2}{a'} + \right.$$

$$\left. + Q_u w_x + Q_p \cdot \left(w_{xx} + 2A'ww_x + A''\frac{w^3}{a'} \right) \right] ,$$

where $a = a(u^\varepsilon)$ and $A = A(u^\varepsilon) = 1/a'(u^\varepsilon)$.

Let $(x(t),t)$ be a positive local maximum of w. Then $w > 0$ in that point and, since $a' \ge \alpha > 0$, (4.1), also $p^\varepsilon = u_x^\varepsilon > 0$ there. Furthermore, $w_x = 0$ and $w_{xx} \le 0$ in that point. Therefore, in view of (7.3) and assumptions (A1)–(A.3), the above inequality implies that

$$w_t + w^2 \le \varepsilon K w^3$$

in $(x(t),t)$, for some constant K which depends on M_1, M_2, α and the uniform bounds on A' and A''. Therefore, (7.10) holds and that concludes the proof for the non-degenerate case.

In the degenerate case, we replace $Q(u,p)$ by $Q_\delta(u,p) = Q(u,p) + \delta p$ so that the resulting pseudo-viscous approximation will be uniformly parabolic, $\partial Q_\delta/\partial p \ge \delta > 0$, and admit a smooth solution, u_δ^ε. Note that Q_δ, $\delta \downarrow 0$, still satisfies conditions (A1)–(A3) with constants, say, $M_1 + 1$ and M_2. Therefore, inequality (7.10), with K independent of δ, holds for u_δ^ε, $\delta \downarrow 0$, and consequently it holds for u^ε as well. $\qquad\square$

Remark. Theorem 7.2 implies, in particular, the (ε-weak) Lip^+-stability of viscous parabolic regularizations, (6.1), stated earlier in Theorem 6.1. These regularizations are identified by viscosity coefficients of the form

$$(7.11) \qquad Q(u, p) = q(u) \cdot p \quad , \quad q(u) \geq 0 \ .$$

Such coefficients satisfy assumptions (A1)–(A3), provided that $q(\cdot)$ is sufficiently smooth.

We therefore conclude, in light of Theorems 7.1 and 7.2, that Theorem 5.2 applies to approximation (7.1) under assumptions (7.3) and (A.1)–(A.3). Hence, if in addition, approximation (7.1) is $W^{-1,1}$-consistent with (4.1), i.e.,

$$\|u_t^\varepsilon + f(u^\varepsilon)_x\|_{W^{-1,1}(\Re_x)} \leq O(\varepsilon) \ ,$$

or simply,

$$(7.12) \qquad \|Q\left(u^\varepsilon, u_x^\varepsilon\right)\|_{L_1(\Re_x)} \leq \text{Const} \ ,$$

Corollary 5.2 may be applied and error estimates (E1)–(E3) hold. We propose below a condition on $Q(u, p)$ which guarantees $W^{-1,1}$-consistency, (7.12).

Proposition 7.1. *If there exists a constant $C > 0$, such that*

$$(7.13) \qquad |Q(u, p)| \leq C|p| \qquad \forall (u, p) \in [\inf u_0 \ , \ \sup u_0] \times \Re \ ,$$

then equation (7.1) is $W^{-1,1}$-consistent with (4.1).

Proof. Condition (7.13) implies that

$$\|Q\left(u^\varepsilon, u_x^\varepsilon\right)\|_{L_1(\Re_x)} \leq C\|u_x^\varepsilon\|_{L_1} = C\|u^\varepsilon\|_{BV} \leq C\|u_0\|_{BV} \ .$$

Therefore, (7.12) holds and the proof is concluded. $\qquad\qquad\square$

An example of a family of pseudo-viscosity coefficients which satisfy all the above requirements, i.e., (7.3), (A1)–(A3) and (7.13), is the following:

$$(7.14) \quad Q(u, p) = Q^{q(u), \beta}(u, p) = q(u)\left[(1 + |p|)^\beta - 1\right]\text{sgn}(p) \quad , \quad q(u) \geq 0 \quad , \quad 0 < \beta \leq 1 \ .$$

Note that by letting β go to zero we obtain $Q \equiv 0$, which corresponds to the inviscid hyperbolic conservation law, while increasing β increases the amount of viscosity until we obtain, when $\beta = 1$, the standard viscous parabolic coefficient, (7.11).

A special class of pseudo-viscosity approximations, (7.1), where $Q = Q(p)$,

$$(7.15) \qquad u_t^\varepsilon + f(u^\varepsilon)_x = \varepsilon Q(p^\varepsilon)_x \quad , \quad Q' \geq 0 \quad , \quad \varepsilon \downarrow 0 \,,$$

was introduced by von Neumann and Richtmeyer in [NR]. In [MN] it is shown, by means of compensated compactness, that under further assumptions on the pseudo-viscosity coefficient, there exists a subsequence of weak solutions of (7.15), subject to the initial data (7.2), which converges in L_{loc}^p to the corresponding entropy solution of (4.1), provided that $u_0 \in W^{2,\infty}$.

One of the additional restrictions assumed on Q in [MN] is that it acts only on shock-waves and does not smear out rarefactions. Namely,

$$(7.16) \qquad Q'(p) = 0 \quad \forall p \geq 0 \quad \text{and} \quad Q'(p) > 0 \quad \forall p < 0 \,.$$

Note that restriction (7.16) guarantees Lip^+-stability, since conditions (7.3) and (A1)–(A3) are clearly satisfied in this case.

An example of a family of such pseudo-viscosity coefficients which lead to $W^{-1,1}$-consistent approximations (in view of Proposition 7.1) is

$$(7.17) \qquad Q^\beta(p) = \left[Q^{q(u)\equiv 1,\beta}(u,p) \right]^- = 1 - (1 - p^-)^\beta \quad , \quad 0 < \beta \leq 1 \,,$$

$Q^{q(u),\beta}(u,p)$ being defined in (7.14). The choice which corresponds to $\beta = 1$, $Q^1(p) = p^-$, activates the regular parabolic regularization only on shock-waves and leaves rarefactions untouched.

8. THE REGULARIZED CHAPMAN-ENSKOG EXPANSION

In this section we discuss the regularized Chapman-Enskog expansion for hydrodynamics, proposed by Rosenau [R]. This so-called R-C-E approximation is studied in [ST], where it is shown that it shares many of the properties of the viscosity approximation, e.g. existence of traveling waves, monotonicity, L_1-contraction and Lip^+-stability.

Let us briefly recall the main results of [ST]. The R-C-E approximation is presented in the form

$$(8.1) \qquad u_t^\varepsilon + f(u^\varepsilon)_x = \varepsilon \left[Q_{m\varepsilon} * u_x^\varepsilon \right]_x \quad , \quad \varepsilon \downarrow 0 \ ,$$

$$(8.2) \qquad u^\varepsilon(\cdot, 0) = u_0(\cdot) \ ,$$

with the choice of unit-mass viscosity kernel

$$(8.3) \qquad Q(x) = \frac{1}{2} e^{-|x|} \quad , \quad Q_{m\varepsilon}(x) = \frac{1}{m\varepsilon} Q\left(\frac{x}{m\varepsilon}\right) \ .$$

This is a pseudo-local dissipative approximation of the conservation law, where the viscosity coefficient is being activated by means of convolution rather than multiplication (compare (8.1) to (6.1a)).

When $m \to 0$, $Q_{m\varepsilon}$ tends to the Dirac measure and the R-C-E approximation, (8.1), turns into the viscous parabolic approximation

$$u_t^\varepsilon + f(u^\varepsilon)_x = \varepsilon u_{xx}^\varepsilon \ .$$

Equation (8.1) may be rewritten in the equivalent form

$$(8.4) \qquad u_t^\varepsilon + f(u^\varepsilon)_x = -\frac{1}{m^2\varepsilon} \left[u^\varepsilon - Q_{m\varepsilon} * u^\varepsilon \right] \ .$$

The solution of (8.4) remains as smooth as its initial data [ST, Theorem 2.1] and, therefore, if the initial data are discontinuous, weak solutions must be admitted. Since such solutions are not uniquely determined by the initial data, (8.4) is augmented with a Kružkov-like [K1] entropy condition [ST, (4.1)],

$$(8.5) \qquad \frac{\partial}{\partial t} |u^\varepsilon - c| + \frac{\partial}{\partial x} \{ \mathrm{sgn}(u^\varepsilon - c)[f(u^\varepsilon) - f(c)] \} \leq$$

$$-\frac{1}{m^2\varepsilon} \{ |u^\varepsilon - c| - \mathrm{sgn}(u^\varepsilon - c)[Q_\varepsilon * (u^\varepsilon - c)] \} \ ,$$

for all $c \in \Re$. In particular, by substituting $c = +\sup |u^\varepsilon|$ or $c = -\sup |u^\varepsilon|$, we obtain from (8.5) that u^ε is, respectively, a supersolution or a subsolution of (8.4) and therefore a weak

solution. Hence, u^ε is considered an entropy solution of (8.4) if it satisfies inequality (8.5) in the sense of distributions for all $c \in \Re$.

The above inequality, (8.5), implies L_1-contraction,

$$\|u^\varepsilon(\cdot, t) - v^\varepsilon(\cdot, t)\|_{L_1} \leq \|u^\varepsilon(\cdot, 0) - v^\varepsilon(\cdot, 0)\|_{L_1} ,$$

and hence BV-boundedness,

$$\|u^\varepsilon(\cdot, t)\|_{BV} \leq \|u_0\|_{BV} .$$

Since, by (8.1),

$$\|u_t^\varepsilon + f(u^\varepsilon)_x\|_{W^{-1,1}} \leq \varepsilon\|Q_{m\varepsilon} * u_x^\varepsilon\|_{L_1} \leq \varepsilon\|Q_{m\varepsilon}\|_{L_1}\|u_0\|_{BV} \leq O(\varepsilon) ,$$

we also have $W^{-1,1}$-consistency.

Finally, we deal with the question of Lip^+-stability. Adding the smoothing viscosity term $\delta u_{xx}^{\varepsilon,\delta}$ to (8.4) and differentiating with respect to x, we get that $w \equiv u_x^{\varepsilon,\delta}$ satisfies

$$w_t + a'(u^{\varepsilon,\delta}) \cdot w^2 + a(u^{\varepsilon,\delta}) \cdot w_x = -\frac{1}{m^2\varepsilon}[w - Q_{m\varepsilon} * w] + \delta w_{xx} .$$

Letting $\delta \downarrow 0$, we get that $W(t) \equiv \max_x w(x, t)$ is governed by the Ricatti differential inequality

$$(8.6) \qquad\qquad\qquad W'(t) + \alpha W^2(t) \leq 0 .$$

Restricting our attention to Burgers' equation, $a(u) = u$, the R-C-E approximation turns to be strongly Lip^+-stable, in virtue of (8.6).

Therefore, we conclude, in view of Theorem 5.1, that the R-C-E approximation converges to the entropy solution of Burgers' equation and error estimates (E1)–(E3) hold. This extends, for Burgers' equation, the convergence rate result of [ST, Corollary 5.2] which was restricted to $u_0 \in C^1$.

9. THE SPECTRAL VISCOSITY METHOD

The method of Spectral Viscosity (SV) is used for the approximate solution of (4.1), when the initial data, (5.1), is 2π-periodic. The family of approximate solutions, $\{u_N(x,t)\}$, constructed by this method, consists of trigonometric polynomials, $u_N(x,t) = \sum_{k=-N}^{N} \hat{u}_k(t)e^{ikx}$, which approximate the spectral projection of the exact entropy solution, $P_N u$.

This method takes the following conservative form (consult [MT], [T3], [T5]):

$$(9.1) \qquad \frac{\partial}{\partial t}u_N(x,t) + \frac{\partial}{\partial x}P_N(f(u_N(x,t))) = \varepsilon_N \frac{\partial}{\partial x}Q_N(x,t) * \frac{\partial}{\partial x}u_N(x,t) \ ,$$

$$(9.2) \qquad u_N(\cdot,0) = P_N u_0(\cdot) \ .$$

The right hand side of (9.1) consists of a vanishing viscosity amplitude of size $\varepsilon_N \downarrow 0$ and a viscosity kernel, $Q_N(x,t) = \sum_{|k|=m_N}^{N} \hat{Q}_k(t)e^{ikx}$, activated only on high wave numbers, $|k| \geq m_N >> 1$. As in [T5] we deal with real viscosity kernels with increasing Fourier coefficients, $\hat{Q}_k \equiv \hat{Q}_{|k|}$, which satisfy

$$(9.3) \qquad 1 - \left(\frac{m_N}{|k|}\right)^{2q} \leq \hat{Q}_k(t) \leq 1 \quad , \quad |k| \geq m_N \quad , \quad q = \text{Const} > 1.5 \ ,$$

and the spectral viscosity parameters, ε_N and m_N, behave asymptotically as

$$(9.4) \qquad \varepsilon_N \sim \frac{1}{N^\theta \log N} \quad , \quad m_N \sim N^{\frac{\theta}{2q}} \quad , \quad 0 < \theta < 1 \ .$$

The use of the projection P_N on the initial data is problematic since even if u_0 has a bounded variation, $\|P_N u_0\|_{BV}$ may grow as much as $O(\log N)$. This may be avoided by taking, for instance, the spectrally accurate de la Vallee Poussin projection,

$$(9.5) \qquad u_N(x,0) = VP_N u_0 \equiv \sum_{k=-N}^{N} \sigma_k \hat{u}_{0k} e^{ikx} \quad , \quad \sigma_k = \begin{cases} 1 & |k| \leq \frac{N}{2} \\ 2 - \frac{2k}{N} & |k| > \frac{N}{2} \end{cases} \ ,$$

which satisfies

$$\|u_N(\cdot,0)\|_{BV} = \|VP_N u_0\|_{BV} \leq 3\|u_0\|_{BV} \ .$$

This, according to the total-variation boundedness of the SV method (consult [T5, Corollary 2.3]), implies that

$$(9.6) \qquad \|u_N(\cdot,t)\|_{BV} \leq \text{Const}_T \quad , \quad t \in [0,T] \ .$$

Hence, we hereafter assume (9.5). At the end of this section we will deal with the case described in (9.2) of employing the regular spectral projection on the initial data.

The SV method smoothens the initial data by smearing its discontinuities: Since $u_0(x) = \sum_{k=-\infty}^{\infty} \hat{u}_{0k} e^{ikx} \in BV$, it follows that $\hat{u}_{0k} = O\left(\frac{1}{k}\right)$. Hence

$$\left| \frac{\partial}{\partial x}[u_N(\cdot, 0)] \right| = \left| \sum_{k=-N}^{N} ik\sigma_k \hat{u}_{0k} e^{ikx} \right| \leq \sum_{k=-N}^{N} |k| \cdot |\hat{u}_{0k}| \leq O(N) \ ,$$

and therefore

$$(9.7) \qquad\qquad \|u_N(\cdot, 0)\|_{Lip^+} \leq O(N) < \infty \ .$$

We now turn to deal with the Lip^+-stability of this approximation. To this end we rewrite (9.1), as in [T5, (2.4a)], in the following form,

$$(9.8a) \qquad\qquad \frac{\partial}{\partial t} u_N(x, t) + \frac{\partial}{\partial x} f(u_N(x, t)) =$$

$$= \varepsilon_N \frac{\partial^2}{\partial x^2} u_N(x, t) - \varepsilon_N \frac{\partial}{\partial x} R_N(x, t) * \frac{\partial}{\partial x} u_N(x, t) + E_N \ ,$$

where $E_N = \frac{\partial}{\partial x}(I - P_N)f(u_N)$ is a spectrally small error term and

$$(9.8b) \qquad R_N(x, t) = \sum_{k=-N}^{N} \hat{R}_k(t) e^{ikx} \ , \quad \hat{R}_k(t) = \begin{cases} 1 & |k| < m_N \\ 1 - \hat{Q}_k(t) & |k| \geq m_N \end{cases} \ .$$

Multiplying (9.8a) by $a'(u_N)$ and differentiating with respect to x yields for $w = \frac{\partial}{\partial x} a(u_N)$:

$$(9.9) \qquad w_t + a(u_N)w_x + w^2 = \varepsilon_N \left[w_{xx} + 2A'(u_N)ww_x + A''(u_N)\frac{w^3}{a'(u_N)} \right] -$$

$$-\varepsilon_N \left[a''(u_N)A(u_N)\left(\frac{\partial}{\partial x}R_N * \frac{\partial}{\partial x}u_N \right)w + a'(u_N)\left(\frac{\partial^2}{\partial x^2}R_N * \frac{\partial}{\partial x}u_N \right) \right] +$$

$$+a''(u_N)A(u_N)wE_N + a'(u_N)\frac{\partial}{\partial x}E_N \ .$$

Here, as in §7, $A(\cdot) = 1/a'(\cdot)$. As before, we find that $W(t) = \max_x w(x, t)$ is governed by

$$(9.10) \qquad\qquad \frac{d}{dt}W(t) + (W(t))^2 \leq \varepsilon_N K(W(t))^3 + \beta_N W(t) + \gamma_N \ ,$$

where

$$(9.11) \qquad\qquad K \equiv \max_{|u| \leq \|u_N\|_{L_\infty}} \left(\frac{A''(u)}{a'(u)} \right)^+ \ ,$$

$$(9.12) \qquad \beta_N = M_1 \cdot \left[\varepsilon_N \left\| \frac{\partial}{\partial x}R_N * \frac{\partial}{\partial x}u_N \right\|_{L_\infty} + \|E_N\|_{L_\infty} \right] \ ; \quad M_1 \equiv \max_{|u| \leq \|u_N\|_{L_\infty}} |a''(u)A(u)|$$

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and

$$(9.13) \qquad \gamma_N = M_2 \cdot \left[\varepsilon_N \| \frac{\partial^2}{\partial x^2} R_N * \frac{\partial}{\partial x} u_N \|_{L_\infty} + \| \frac{\partial}{\partial x} E_N \|_{L_\infty} \right] \; ; \; M_2 \equiv \max_{|u| \leq \|u_N\|_{L_\infty}} a'(u) \; .$$

We now use estimates, obtained in [T5], in order to estimate β_N and γ_N. First, we recall that [T5, Lemma 3.1] supplies us with a uniform bound for the spatial derivatives of R_N:

$$(9.14)_s \qquad \| \frac{\partial^s}{\partial x^s} R_N(\cdot, t) \|_{L_\infty} \leq \text{Const} \cdot m_N^{s+1} \log N \; , \quad 0 \leq s \leq 2q - 1 \; .$$

Using $(9.14)_1$, $(9.14)_2$ and the BV-boundedness (9.6), we conclude that

$$(9.15) \qquad \| \frac{\partial}{\partial x} R_N * \frac{\partial}{\partial x} u_N \|_{L_\infty} \leq \| \frac{\partial}{\partial x} R_N \|_{L_\infty} \cdot \|u_N\|_{BV} \leq \text{Const} \cdot m_N^2 \log N$$

and

$$(9.16) \qquad \| \frac{\partial^2}{\partial x^2} R_N * \frac{\partial}{\partial x} u_N \|_{L_\infty} \leq \| \frac{\partial^2}{\partial x^2} R_N \|_{L_\infty} \cdot \|u_N\|_{BV} \leq \text{Const} \cdot m_N^3 \log N \; .$$

Since $\|E_N\|_{L_\infty}$ and $\|\frac{\partial}{\partial x} E_N\|_{L_\infty}$ are spectrally small, hence negligible, we conclude by (9.12), (9.13), (9.15), (9.16), (9.4) and (9.3) that

$$(9.17) \qquad \beta_N \sim N^{\theta(\frac{1}{q} - 1)} \downarrow 0 \quad , \quad \gamma_N \sim N^{\theta(\frac{3}{2q} - 1)} \downarrow 0 \; .$$

We may now state and prove the following weak Lip^+-stability result:

Theorem 9.1. (Lip^+-Stability). *Consider the SV method (9.1), (9.3)–(9.5), approximating the conservation law (4.1)+(5.1). Assume that $a = f'$ satisfies $\left(\frac{1}{a'} \right)'' \leq 0$. Then the approximate solutions are ε-weakly Lip^+-stable, with $\varepsilon = \frac{1}{N}$.*

Proof. Our assumption on $a(\cdot)$ implies that K, given in (9.11), equals zero. Hence, (9.10) reads in this case:

$$(9.18) \qquad \frac{d}{dt} W(t) \leq -(W(t))^2 + \beta_N W(t) + \gamma_N \; .$$

Solving (9.18) we get that

$$(9.19) \qquad W(t) \leq w_+ + \frac{w_+ - w_-}{\eta e^{(w_+ - w_-)t} - 1} \; ,$$

where

$$(9.20) \qquad w_\pm = \frac{\beta_N \pm \sqrt{\beta_N^2 + 4\gamma_N}}{2} \quad , \quad \eta = \frac{W(t=0) - w_-}{W(t=0) - w_+} \; .$$

Note that $w_\pm$ and η depend on N. Furthermore, by (9.20), (9.17) and (9.3) it follows that

(9.21)
$$w_\pm = O\left(\sqrt{\gamma}_N\right) \sim N^{\theta(\frac{3}{4q} - \frac{1}{2})} \xrightarrow[N \to \infty]{} 0 \ .$$

Also, since by (9.7)

(9.22)
$$W(t = 0) \sim N \ ,$$

we conclude by (9.20) and (9.21) that

(9.23)
$$\eta \xrightarrow[N \to \infty]{} 1 \ .$$

We claim that the weak Lip^+-stability conditions, (5.7)–(5.8), hold here with $\varepsilon = \frac{1}{N}$. Namely,

(9.24)
$$e^{\int_0^T W(t)dt} \leq O(N)$$

and

(9.25)
$$\int_0^T e^{\int_t^T W(\tau)d\tau}dt \leq O(\log N) \ .$$

In order to prove these two estimates we integrate (9.19) and find that

(9.26)
$$\int_t^T W(\tau)d\tau \leq w_-(T - t) + \log\left[\frac{\eta e^{(w_+ - w_-)T} - 1}{\eta e^{(w_+ - w_-)t} - 1}\right] \ .$$

Hence

$$\exp\left[\int_0^T W(\tau)d\tau\right] \leq e^{w_- T} \cdot \frac{\eta e^{(w_+ - w_-)T} - 1}{\eta - 1} = e^{w_+ T} + e^{w_- T} \cdot \frac{e^{(w_+ - w_-)T} - 1}{\eta - 1} \ .$$

But since

(9.27)
$$\eta - 1 = \frac{w_+ - w_-}{W(t = 0) - w_+} \ ,$$

we conclude that

$$\exp\left[\int_0^T W(\tau)d\tau\right] \leq e^{w_+ T} + e^{w_- T} \cdot \frac{e^{(w_+ - w_-)T} - 1}{w_+ - w_-}(W(t = 0) - w_+)$$

and (9.24) follows by using (9.21) and (9.22).

As for (9.25), inequality (9.26) implies (note that $w_- \leq 0$):

(9.28)
$$\int_0^T \exp\left[\int_t^T W(\tau)d\tau\right] dt \leq$$

$$\leq -T \left(\eta e^{(w_+ - w_-)T} - 1 \right) + \frac{\eta e^{(w_+ - w_-)T} - 1}{w_+ - w_-} \log \left(\frac{\eta e^{(w_+ - w_-)T} - 1}{\eta - 1} \right) .$$

First, we observe that (9.21) and (9.23) imply that

(9.29) $$\eta e^{(w_+ - w_-)T} - 1 \xrightarrow[N \to \infty]{} 0 .$$

Now, in order to estimate the second term on the right hand side of (9.28) we deal with each of its two multiplicands. Using (9.27), (9.21) and (9.22) we find that

(9.30) $$\frac{\eta e^{(w_+ - w_-)T} - 1}{w_+ - w_-} = \frac{e^{(w_+ - w_-)T}}{W(t=0) - w_+} + \frac{e^{(w_+ - w_-)T} - 1}{w_+ - w_-} \xrightarrow[N \to \infty]{} 0 + T = T .$$

Furthermore, by (9.27), (9.21) and (9.22),

(9.31) $$\frac{\eta e^{(w_+ - w_-)T} - 1}{\eta - 1} = e^{(w_+ - w_-)T} + \frac{e^{(w_+ - w_-)T} - 1}{w_+ - w_-} \cdot (W(t=0) - w_+) \sim N .$$

Hence, (9.28)–(9.31) prove (9.25) and the proof is thus concluded. $\square$

Corollary 9.1. ($W^{-1,1}$-Convergence Rate). *Consider the SV method (9.1), (9.3)–(9.5), approximating the conservation law (4.1)+(5.1). Then*

(9.32) $$\|u_N(\cdot, T) - u(\cdot, T)\|_{W^{-1,1}} \leq \begin{cases} C_T \varepsilon & \text{if } \|u_0\|_{Lip^+} < \infty \\ C_T \varepsilon |\ln \varepsilon| & \text{if } \|u_0\|_{Lip^+} = \infty \text{ and } \left(\frac{1}{a'} \right)'' \leq 0 \end{cases} ,$$

with $\varepsilon = N^{-\theta}$.

Proof. The case of Lip^+-bounded initial data is straightforward and we, therefore, concentrate on the case that $\|u_0\|_{Lip^+} = \infty$ and $\left(\frac{1}{a'} \right)'' \leq 0$. Since, by Theorem 9.1, we have $\frac{1}{N}$-weak Lip^+-stability in that case, and since $\frac{1}{N} < \varepsilon = N^{-\theta}$, u_N are also ε-weakly Lip^+-stable. Hence, it remains to show ε-$W^{-1,1}$-consistency.

$W^{-1,1}$-consistency with (4.1),

$$\left\| \frac{\partial}{\partial t} u_N + \frac{\partial}{\partial x} f(u_N) \right\|_{L_\infty([0,T], W^{-1,1}(\Re_x))} \leq K_T N^{-\theta} ,$$

has already been shown in [T5, (3.9b)]. As for $W^{-1,1}$-consistency with the initial condition, we claim that

(9.33) $$\|u_N(\cdot, 0) - u(\cdot, 0)\|_{W^{-1,1}} = \|V P_N U_0 - U_0\|_{L_1} \leq K_0 N^{-2} \log N ,$$

where $U_0(x) = \int_{-\pi}^x u_0(\xi) d\xi$. In order to prove (9.33), we recall that (consult [Q, (2.12), (2.14), (2.15)])

(9.34) $$\|P_N U_0 - U_0\|_{L_1} \leq \text{Const} \cdot \log N \cdot N^{-m} \|U_0^{(m)}\|_{L_1} , \quad m \geq 0 .$$

Taking $m = 2$ in (9.34) we find that the initial error allowed by $W^{-1,1}$-consistency, is exhausted in this case:

$$(9.35) \qquad \|P_N U_0 - U_0\|_{L_1} \le \text{Const} \cdot N^{-2} \log N \|u_0\|_{BV} \ .$$

We leave the reader to verify that

$$(9.36) \qquad \|V P_N U_0 - P_N U_0\|_{L_1} \le \text{Const} \cdot N^{-2} \log N \ .$$

Hence, (9.33) follows from (9.35), (9.36) and the proof is thus completed. $\qquad\square$

The case of Lip^+-unbounded initial data when the flux fails to satisfy $(\frac{1}{a'})'' \le 0$ is problematic since the cubic term on the right hand side of (9.10) does not vanish. Still, one can prove (along the lines of the proof of Lemma 6.1) weak Lip^+-stability of order $\varepsilon_N = N^{-\theta} \log N$, provided that

$$W(t = 0) \le \frac{\bar{c}}{\varepsilon_N K}$$

for some $\bar{c} < 1$. Alas, this condition does not hold in our case (consult (9.4) and (9.22)). We, therefore, suggest to overcome this problem by considering a speed-like SV method,

$$(9.37) \qquad \frac{\partial}{\partial t} u_N(x,t) + \frac{\partial}{\partial x} P_N f(u_N(x,t)) = \varepsilon_N \frac{\partial}{\partial x} Q_N(x,t) * \frac{\partial}{\partial x} a(u_N(x,t))$$

with (9.3)–(9.5) as before. This method, still conservative, differs from the regular SV method, (9.1), only in the spectral viscosity term on the right hand side, where u_N was replaced by $a(u_N)$.

Remark on an a-priori L_∞ bound.

The question of uniform L_∞-boundedness of this modified SV method may be tackled along the lines of [S3]. However, we suggest here a simple argument which enables us to circumvent that question:

Since the initial data are always assumed bounded, (5.1), the exact entropy solution of (4.1)+(5.1) will not be affected if we change the flux f outside the interval $I_0 \equiv [\min u_0, \max u_0]$. Therefore, we choose to smoothly extend f from I_0 to $\Re$, so that $f, a = f', a', a''$, etc. remain uniformly bouded on $\Re$. By doing so we may conclude that $f^{(i)}(u_N)$, and by convexity, $A^{(i)}(u_N)$ as well, $i \ge 0$, are all uniformly bounded even if u_N is not. Since our estimates depend only on $\|f^{(i)}(u_N)\|_{L_\infty}$ and $\|A^{(i)}(u_N)\|_{L_\infty}$ and never on the L_∞-bound of u_N itself, this argument is sufficient for our needs and no a-priori L_∞-bound is required.

We would like to comment that L_∞-boundedness proofs for approximate solutions of (4.1)+(5.1) may be sometimes tedious (as in our present case). Hence, it is sometimes customary to assume an a-priori L_∞-bound, based, for instance, on numerical evidence. The

above, to the best of our knowledge, innovative extension argument, may be applied to such approximations as well, so that assumptions, not fully justified, may be avoided.

The convergence rate estimates for this modified SV method are given in the following theorem.

Theorem 9.2. ($W^{-1,1}$-Convergence Rate for the Modified SV Method). *Consider the modified SV method (9.37), (9.3)–(9.5), approximating the conservation law (4.1)+(5.1). Then u_N converges to the exact entropy solution $u(x,t)$, as $N \to \infty$, and for every $T > 0$ there exists a constant C_T such that*

$$(9.38a) \qquad \|u_N(\cdot, T) - u(\cdot, T)\|_{W^{-1,1}} \le C_T \cdot \tilde{\varepsilon} \ ,$$

where

$$(9.38b) \qquad \tilde{\varepsilon} = \begin{cases} \varepsilon & \text{if } \|u_0\|_{Lip^+} < \infty \\ \varepsilon |\ln \varepsilon| & \text{if } \|u_0\|_{Lip^+} = \infty \end{cases} \quad \text{and} \quad \varepsilon = N^{-\theta} \ .$$

Proof. We first note that (9.37) is still L_1-stable (consult the proof of [T5, Lemma 2.2]) and hence (9.6) still holds. Therefore, (9.37) describes a family of conservative, L_1-stable and BV-bounded approximate solutions of (4.1)+(5.1).

Next, we address the question of weak Lip^+-stability. We rewrite (9.37) as

$$\frac{\partial}{\partial t} u_N + \frac{\partial}{\partial x} f(u_N) = \varepsilon_N \frac{\partial^2}{\partial x^2} a(u_N) - \varepsilon_N \frac{\partial}{\partial x} R_N * \frac{\partial}{\partial x} a(u_N) + E_N \ ,$$

where R_N and E_N are as in (9.8). Multiplying by $a'(u_N)$ and differentiating with respect to x, we find that $w = a(u_N)_x$ satisfies (compare to (9.9)):

$$w_t + a(u_N)w_x + w^2 = \varepsilon_N \left[a'(u_N)w_{xx} + a''(u_N)A(u_N)ww_x \right] -$$

$$-\varepsilon_N \left[a''(u_N)A(u_N) \left(\frac{\partial}{\partial x} R_N * \frac{\partial}{\partial x} a(u_N) \right) w + a'(u_N) \left(\frac{\partial^2}{\partial x^2} R_N * \frac{\partial}{\partial x} a(u_N) \right) \right] +$$

$$+a''(u_N)A(u_N)wE_N + a'(u_N)\frac{\partial}{\partial x} E_N \ .$$

We conclude that $W(t) = \max_x w(x,t)$ satisfies

$$\frac{d}{dt} W(t) + (W(t))^2 \le \beta_N W(t) + \gamma_N \ ,$$

where β_N and γ_N are not the same as in (9.12), (9.13) but still satisfy (9.17) (since $\|a(u_N)\|_{BV}$ remains uniformly bounded). This, according to the proof of Theorem 9.1, implies the $\frac{1}{N}$-weak Lip^+-stability of (9.37).

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Hence, by Theorem 5.2, error estimates (5.3)–(5.4) hold with $\varepsilon = \frac{1}{N}$. Since it is easy to verify that our modified SV method is also $W^{-1,1}$-consistent of order $N^{-\theta}$, error estimate (9.38) follows. $\qquad\qquad\square$

Before concluding this section we consider the case of P_N projecting the initial data, (9.2). We recall that the resulting approximation, u_N, may not be bounded in BV and in fact $\|u_N\|_{BV}$ may grow as much as $O(\log N)$. We note that this slightly changes our convergence rate results, stated in Corollary 9.1 and Theorem 9.2, so that (9.32) and (9.38) hold with $\varepsilon = N^{-\theta} \log N$, rather than $\varepsilon = N^{-\theta}$.

The first effect of replacing VP_N by P_N is that estimate (9.17) changes to

$$\beta_N \sim N^{\theta(\frac{1}{q}-1)} \log N \downarrow 0 \quad , \quad \gamma_N \sim N^{\theta(\frac{3}{2q}-1)} \log N \downarrow 0 \ ,$$

(consult (9.12), (9.13), (9.15) and (9.16)). This, however, does not change the final result of ε-weak-Lip^+-stability with $\varepsilon = \frac{1}{N}$. Hence, by Theorem 5.2, error estimate (5.3)–(5.4) still hold with $\varepsilon = \frac{1}{N}$. In view of (9.35) it remains only to consider the $W^{-1,1}$-consistency of $u_N(x,t)$ with (4.1). Ignoring the spectrally small discretization error $E_N = \frac{\partial}{\partial x}(I - P_N)f(u_N)$, we obtain from (9.1) (the proof for (9.37) is similar) that

$$\|\frac{\partial}{\partial t}u_N(\cdot,t) + \frac{\partial}{\partial x}f(u_N(\cdot,t))\|_{W^{-1,1}} \leq \varepsilon_N \|Q_N(\cdot,t) * \frac{\partial}{\partial x}u_N(\cdot,t)\|_{L_1} \ .$$

Using (9.4), Young inequality and the fact that $\|Q_N(\cdot,t)\|_{L_1}$ does not exceed $O(\log N)$ (consult [T5, (3.9b)]), we get

$$(9.39) \qquad \|\frac{\partial}{\partial t}u_N(\cdot,t) + \frac{\partial}{\partial x}f(u_N(\cdot,t))\|_{W^{-1,1}} \leq \varepsilon_N \|Q_N(\cdot,t)\|_{L_1} \|u_N(\cdot,t)\|_{BV} \leq$$

$$\leq \text{Const} \cdot \frac{1}{N^\theta \log N}(\log N)^2 = O(N^{-\theta} \log N) \ .$$

Hence, the order of $W^{-1,1}$-consistency reduced from $O(N^{-\theta})$ to $O(N^{-\theta} \log N)$. Therefore, (5.3)–(5.4), (9.35) and (9.39) imply an $O(N^{-\theta} \log N)$ convergence rate in $W^{-1,1}$.

10. FINITE DIFFERENCE APPROXIMATIONS

10.1. Convergence rate estimates

We study in this section the convergence rate of finite difference approximations of Godunov type to the scalar convex conservation law, (4.1), subject to the Lip^+-bounded initial condition

$$(10.1.1) \qquad u(x, t = 0) = u_0(x) \quad , \quad u_0 \in L_0^\infty \quad , \quad \|u_0(x)\|_{Lip^+} < \infty \quad .$$

Godunov type schemes form a special class of transport projection methods for the approximate solution of nonlinear hyperbolic conservation laws. This class of schemes takes the following form:

$$(10.1.2) \qquad v^{\Delta x}(\cdot, t) = \begin{cases} E(t - t^{n-1})v^{\Delta x}(\cdot, t^{n-1}) & t^{n-1} < t < t^n \\ \\ P(\{I_j^n\})v^{\Delta x}(\cdot, t^n - 0) & t = t^n = n\Delta t \end{cases} \qquad n \geq 1 \quad ,$$

where the initialization step is:

$$(10.1.3) \qquad v^{\Delta x}(\cdot, t^0 = 0) = P(\{I_j^0\})u_0(\cdot) \quad .$$

These schemes are composed of the following four ingredients:

(i) The possibly variable size grid cells, $I_j^n \equiv [x_{j-\frac{1}{2}}^n, x_{j+\frac{1}{2}}^n)$, where the grid is regular in the sense that:

$$(10.1.4) \qquad \Delta x \equiv \Delta x_{\min} \leq |I_j^n| \leq \Delta x_{\max} \quad ; \quad \frac{\Delta x_{\max}}{\Delta x_{\min}} \leq \text{Const} \quad ;$$

(ii) A conservative piecewise polynomial grid projection, $P = P(\{I_j^n\})$,

$$(10.1.5) \qquad \int_x Pw(x)dx = \int_x w(x)dx \quad ;$$

(iii) The exact entropy solution operator associated with (4.1), $E = E(t)$;
(iv) The time step Δt, which is restricted by the CFL condition:

$$(10.1.6) \qquad \lambda \max_{x,t} |f'(v^{\Delta x}(x, t))| \leq 1 \quad , \quad \lambda = \frac{\Delta t}{\Delta x} \quad .$$

In view of Corollary 5.1, Lip^+-consistency, (5.6), and $W^{-1,1}$-stability, (5.9)–(5.10), imply convergence of first order in $W^{-1,1}$, as stated in the following theorem.

Theorem 10.1. *Assume that the Godunov type scheme is Lip^+-stable,*

$$(10.1.7) \qquad \|v^{\Delta x}(\cdot, t)\|_{Lip^+} \leq C \quad , \quad t \geq 0 \quad ,$$

and $W^{-1,1}$-consistent with the conservation law, (4.1), and the Lip^+-bounded initial condition, (10.1.1), in the sense that there exists $\varepsilon = \varepsilon(\Delta x)$ such that $\varepsilon(\Delta x) \downarrow 0$ for $\Delta x \downarrow 0$ and

$$(10.1.8) \quad \|v^{\Delta x}(x,0) - u_0(x)\|_{W^{-1,1}} + \|v^{\Delta x}(x,t)_t + f(v^{\Delta x}(x,t))_x\|_{L_\infty([0,T],W^{-1,1}(\Re_x))} \le O(\varepsilon) .$$

Then the following error estimate holds:

$$(10.1.9) \qquad\qquad \|v^{\Delta x}(\cdot,t) - u(\cdot,t)\|_{W^{-1,1}} \le O(\varepsilon) .$$

The parameter ε is a function of the smallest scale, Δx. If $\varepsilon(\Delta x) = O(\Delta x^k)$, the corresponding scheme will be kth order accurate in $W^{-1,1}$, in view error estimate (10.1.9). This error estimate implies error estimates (E1)–(E3), given in Corollary 5.2; in particular, by the local error estimate (E2), we have local kth order accuracy for the post-processed approximation, wherever the solution is infinitely smooth.

However, our analysis presented here is limited to $W^{-1,1}$-first order accuracy, i.e. $\varepsilon = \Delta x$. A more delicate analysis will, hopefully, demonstrate (10.1.8) with $\varepsilon = O(\Delta x^k)$, $k > 1$, for higher-order schemes.

In view of the above we henceforth use the notation Δx instead of ε. Therefore, our $W^{-1,1}$-consistency requirement, (10.1.8), now reads

$$(10.1.11) \quad \|v^{\Delta x}(x,0) - u_0(x)\|_{W^{-1,1}} + \|v^{\Delta x}(x,t)_t + f(v^{\Delta x}(x,t))_x\|_{L_\infty([0,T],W^{-1,1}(\Re_x))} \le O(\Delta x) .$$

The convergence Theorem 10.1 requires to verify the $W^{-1,1}$-consistency and Lip^+-stability of the scheme in question. We begin by reducing the question of $W^{-1,1}$-consistency to the level of a mere approximation problem, namely, measuring in $W^{-1,1}$ the distance between the exact solution and its grid projection. Thus, our first theorem below enables us to avoid the delicate bookkeeping of error accumulation due to the dynamic transport part of the scheme.

Theorem 10.2. ($W^{-1,1}$-Consistency). *The Godunov type approximation (10.1.2)–(10.1.3) satisfies the following truncation error estimate:*

$$(10.1.12) \quad \|v_t^{\Delta x} + f(v^{\Delta x})_x\|_{L_\infty([0,T],W^{-1,1}(\Re_x))} \le O\left(\frac{1}{\Delta t}\right) \max_{0 < t^n \le T} \|(P-I)v^{\Delta x}(\cdot, t^n - 0)\|_{W^{-1,1}} .$$

Remark. We emphasize that this theorem applies to both fixed and variable grid schemes.

Proof. Let N denote the number of time steps in $[0, T]$, i.e., $T = t^N = N\Delta t$. Then for every $\phi \in C_0^1(\Re \times [0, T])$

$$(10.1.13a) \qquad (F^{\Delta x}, \phi)_{x,t} = \sum_{n=1}^{N} \left[\int_{t^{n-1}}^{t^n} (v_t^{\Delta x}, \phi) dt + \int_{t^{n-1}}^{t^n} (f(v^{\Delta x})_x, \phi) dt \right] \quad,$$

where

$$(10.1.13b) \qquad F^{\Delta x}(x, t) \equiv v_t^{\Delta x} + f(v^{\Delta x})_x \ ,$$

and $(\cdot, \cdot)$ denotes the $L_2(\Re_x)$ inner product. Integration by parts of (10.1.13a) gives that

$$(10.1.14) \qquad (F^{\Delta x}, \phi)_{x,t} = \sum_{n=1}^{N} \left[(v^{\Delta x}, \phi) \Big|_{t^{n-1}}^{t^n} - \int_{t^{n-1}}^{t^n} \left((v^{\Delta x}, \phi_t) + (f(v^{\Delta x}), \phi_x) \right) dt \right] \quad.$$

But since $v^{\Delta x}$ is a weak solution in the strip $\Re \times (t^{n-1}, t^n)$, as definition (10.1.2) implies, then

$$(10.1.15) \qquad \int_{t^{n-1}}^{t^n} \left((v^{\Delta x}, \phi_t) + (f(v^{\Delta x}), \phi_x) \right) dt = (v^{\Delta x}, \phi) \Big|_{t^{n-1}+0}^{t^n-0} \quad.$$

Therefore, by (10.1.14) and (10.1.15),

$$(F^{\Delta x}, \phi)_{x,t} = \sum_{n=1}^{N} \left[(v^{\Delta x}, \phi) \Big|_{t^{n-1}}^{t^n} - (v^{\Delta x}, \phi) \Big|_{t^{n-1}+0}^{t^n-0} \right] \quad,$$

and since, by (10.1.2), $v^{\Delta x}(\cdot, t^{n-1} + 0) = v^{\Delta x}(\cdot, t^{n-1})$, we get that

$$(F^{\Delta x}, \phi)_{x,t} = \sum_{n=1}^{N} (v^{\Delta x}, \phi) \Big|_{t^n-0}^{t^n} = \sum_{n=1}^{N} ((P-I)v^{\Delta x}(\cdot, t^n - 0), \phi(\cdot, t^n)) \quad.$$

Therefore, by Hölder inequality,

$$|(F^{\Delta x}, \phi)_{x,t}| \le \max_{1 \le n \le N} \|(P-I)v^{\Delta x}(\cdot, t^n - 0)\|_{W^{-1,1}} \sum_{n=1}^{N} \|\phi(\cdot, t^n)\|_{Lip} \quad,$$

or, by the rectangular quadrature rule,

$$(10.1.16) \quad |(F^{\Delta x}, \phi)_{x,t}| \le \max_{1 \le n \le N} \|(P-I)v^{\Delta x}(\cdot, t^n - 0)\|_{W^{-1,1}} \cdot O\left(\frac{1}{\Delta t}\right) \int_0^T \|\phi(\cdot, t)\|_{Lip} dt \quad.$$

We conclude, in view of (10.1.16), that

$$(10.1.17) \qquad \int_0^T \|F^{\Delta x}(\cdot, t)\|_{W^{-1,1}} \|\phi(\cdot, t)\|_{Lip} dt \le$$

$$O\left(\frac{1}{\Delta t}\right) \max_{1 \le n \le N} \|(P-I)v^{\Delta x}(\cdot, t^n - 0)\|_{W^{-1,1}} \int_0^T \|\phi(\cdot, t)\|_{Lip} dt \quad,$$

which, with definition (10.1.13b), proves (10.1.12). □

Next, we turn to the question of Lip^+-stability. We note that the Lip^+-seminorm $\|\cdot\|_{Lip^+}$, (2.1.11), does not suit discontinuous piecewise polynomial functions and hence we replace it by a discrete analogous, defined as

$$(10.1.18) \qquad \|v^{\Delta x}(\cdot, t^n)\|_{DLip^+} \equiv \max_x \left(\frac{v^{\Delta x}(x + \Delta x, t^n) - v^{\Delta x}(x, t^n)}{\Delta x} \right)^+ .$$

The infinite divided difference in (2.1.11) is replaced here by differences divided by the (*finite*) smallest scale of the underlying grid, Δx. We show that, instead of (10.1.7), the weaker Lip^+-stability condition

$$(10.1.19) \qquad \|v^{\Delta x}(\cdot, t^n)\|_{DLip^+} \leq C ,$$

suffices in order to obtain our convergence rate results.

To this end, we employ a compactly supported non-negative unit mass mollifier,

$$\psi_\delta(x) = \frac{1}{\delta}\psi(\frac{x}{\delta}) \quad , \quad \int_x \psi_\delta(x)dx = \int_x \psi(x)dx = 1 .$$

We first show that $W^{-1,1}$-consistency of order $O(\Delta x)$ remains invariant under a mollification with ψ_δ where $\delta = O(\Delta x)$.

Theorem 10.3. *Assume $v^{\Delta x}(x,t)$ has a bounded variation and is $W^{-1,1}$-consistent with (4.1) of order $O(\Delta x)$,*

$$(10.1.20) \qquad \|F^{\Delta x}(x,t)\|_{W^{-1,1}} \leq O(\Delta x) \quad , \quad F^{\Delta x}(x,t) \equiv v_t^{\Delta x} + f(v^{\Delta x})_x .$$

Then $v^{\Delta x, \delta} \equiv \psi_\delta * v^{\Delta x}$ is $W^{-1,1}$-consistent with (4.1) of order $O(\Delta x) + O(\delta)$.

Proof. We begin by stating the following two straightforward facts:

$$(10.1.21) \qquad \|\psi_\delta * F\|_{W^{-1,1}} \leq \|F\|_{W^{-1,1}} ;$$

$$(10.1.22) \qquad \|\psi_\delta * w - w\|_{L_1} \leq O(\delta) \cdot \|w\|_{BV} .$$

Next, we decompose the truncation error of $v^{\Delta x, \delta}$ as follows:

$$(10.1.23) \quad \|v_t^{\Delta x, \delta} + f(v^{\Delta x, \delta})_x\|_{W^{-1,1}} \leq \|\psi_\delta * F^{\Delta x}\|_{W^{-1,1}} + \|\psi_\delta * f(v^{\Delta x})_x - f(v^{\Delta x, \delta})_x\|_{W^{-1,1}} .$$

The first term on the right hand side of (10.1.23) is of order $O(\Delta x)$, as implied by (10.1.20) and (10.1.21). Using definition (4.18) and inequality (10.1.22), we get that the second term on the right hand side of (10.1.23) is of order $O(\delta)$ and, therefore, conclude the proof:

$$\|\psi_\delta * f(v^{\Delta x})_x - f(v^{\Delta x, \delta})_x\|_{W^{-1,1}} = \|\psi_\delta * f(v^{\Delta x}) - f(\psi_\delta * v^{\Delta x})\|_{L_1} \leq$$

$$\leq \|\psi_\delta * f(v^{\Delta x}) - f(v^{\Delta x})\|_{L_1} + \|f(v^{\Delta x}) - f(\psi_\delta * v^{\Delta x})\|_{L_1} \leq$$

$$\leq \|\psi_\delta * f(v^{\Delta x}) - f(v^{\Delta x})\|_{L_1} + \|a\|_{L_\infty} \|v^{\Delta x} - \psi_\delta * v^{\Delta x}\|_{L_1} = O(\delta) \quad .$$

$\square$

Finally, we combine Theorems 10.2 and 10.3 to achieve our main convergence rate estimate for Godunov type schemes.

Theorem 10.4. ($W^{-1,1}$-Convergence Rate). *Assume that the Godunov type approximation (10.1.2)–(10.1.3) is discrete Lip^+-stable, (10.1.19), and $W^{-1,1}$-consistent in the sense that*

(10.1.24) $$\|(P - I)w\|_{W^{-1,1}} \leq O(\Delta x^2)\|w\|_{BV} \quad .$$

Then the following error estimate holds:

(10.1.25) $$\|v^{\Delta x}(\cdot, t) - u(\cdot, t)\|_{W^{-1,1}} \leq O(\Delta x) \quad .$$

Proof. Let us denote $\tilde{v}^{\Delta x}(\cdot, t) \equiv \psi_{\Delta x} * v^{\Delta x}(\cdot, t)$, where $\psi_{\Delta x}$ is the dilated mollifier of

$$\psi(x) = \begin{cases} 1 & |x| \leq \frac{1}{2} \\ 0 & |x| > \frac{1}{2} \end{cases} \quad .$$

This choice of mollifier satisfies the following $W^{-1,1}$-error estimate (the proof of which is postponed to the end of this section):

(10.1.26) $$\|\psi_{\Delta x} * w - w\|_{W^{-1,1}} \leq O(\Delta x^2)\|w\|_{BV} \quad .$$

We show below that $\tilde{v}^{\Delta x}$ satisfies Lip^+-stability, (10.1.7), and $W^{-1,1}$-consistency, (10.1.11), in order to use Theorem 10.1.

We start with the Lip^+-stability question. The definitions of the regular and discrete Lip^+-seminorms, (2.1.11) and (10.1.18), imply that $\|\tilde{v}^{\Delta x}(\cdot, t^n)\|_{Lip^+} = \|v^{\Delta x}(\cdot, t^n)\|_{DLip^+}$. As $v^{\Delta x}$ is assumed to be discrete Lip^+-stable we conclude that at each time level t^n we have

(10.1.27) $$\|\tilde{v}^{\Delta x}(\cdot, t^n)\|_{Lip^+} = D_n \leq C \quad .$$

This, together with the fact that the intermediate exact solution operator decreases the Lip^+-seminorm, (2.1.18), imply Lip^+-boundedness for all $t \geq 0$:

(10.1.28) $$\|\tilde{v}^{\Delta x}(\cdot, t)\|_{Lip^+} \leq C \qquad \forall t \geq 0 \quad .$$

Namely, the mollified approximation $\tilde{v}^{\Delta x}$ is Lip^+-stable.

$v^{\Delta x}(\cdot, t)$, being compactly supported and Lip^+-bounded, has a bounded variation (Lemma 3.1). Turning to the question of $W^{-1,1}$-consistency we, therefore, conclude from assumption (10.1.24) together with the truncation error estimate (10.1.12) that $v^{\Delta x}$ is $W^{-1,1}$-consistent with (4.1) of order $O(\Delta x)$; in view of Theorem 10.3, so is $\tilde{v}^{\Delta x}$, i.e.,

$$\|\tilde{v}_t^{\Delta x} + f(\tilde{v}^{\Delta x})_x\| \le O(\Delta x) \quad .$$

Furthermore, $\tilde{v}^{\Delta x}$ is also $W^{-1,1}$-consistent with the initial condition (10.1.1), since by (10.1.26), (10.1.3) and (10.1.24):

$$\|\tilde{v}^{\Delta x}(\cdot, 0) - u(\cdot, 0)\|_{W^{-1,1}} \le \|\tilde{v}^{\Delta x}(\cdot, 0) - v^{\Delta x}(\cdot, 0)\|_{W^{-1,1}} + \|v^{\Delta x}(\cdot, 0) - u_0(\cdot)\|_{W^{-1,1}} \le O(\Delta x^2) \quad .$$

Therefore, Theorem 10.1 holds and by (10.1.9) we get that

$$(10.1.29) \qquad\qquad \|\tilde{v}^{\Delta x}(\cdot, T) - u(\cdot, T)\|_{W^{-1,1}} \le O(\Delta x) \quad .$$

In addition, we have by (10.1.26),

$$(10.1.30) \qquad\qquad \|\tilde{v}^{\Delta x}(\cdot, T) - v^{\Delta x}(\cdot, T)\|_{W^{-1,1}} \le O(\Delta x^2) \quad .$$

Combining (10.1.29) and (10.1.30) we end up with

$$(10.1.31) \qquad\qquad \|v^{\Delta x}(\cdot, T) - u(\cdot, T)\|_{W^{-1,1}} \le O(\Delta x) \quad ,$$

and the proof is thus completed. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We close this section by proving (10.1.26). To this end, we need the following basic $W^{-1,1}$-estimate, which proves essential also in the next section.

Lemma 10.1. *Let u and v be two Δx-grid functions. Assume there exist constants, K and L, such that:*
(i) $\|u - v\|_{L_1} \le K\Delta x$;
(ii) the distance between two successive zeroes of $\int^x (u - v)$ is $L\Delta x$ at the most.
Then $\|u - v\|_{W^{-1,1}} \le LK\Delta x^2$.

Proof. Let z_j denote the zeroes of $\int^x (u - v)$ and $L_j = [z_j, z_{j+1}]$. Then

$$\|u - v\|_{W^{-1,1}} = \int_x \left| \int_{-\infty}^x u - v \right| dx = \sum_j \int_{L_j} \left| \int_{z_j}^x u - v \right| dx \le \sum_j \int_{L_j} \left(\int_{L_j} |u - v| \right) dx =$$

$$= \sum_j |L_j| \cdot \int_{L_j} |u - v| \le L\Delta x \cdot \sum_j \int_{L_j} |u - v| = L\Delta x \|u - v\|_{L_1} \le LK\Delta x^2 \quad .$$

$$\square$$

Let $A = A(I_j^n)$ denote the cell averaging operator,

$$(10.1.32) \qquad Aw(x) \equiv \frac{1}{|I_j^n|} \int_{I_j^n} w(\xi)d\xi \qquad \forall x \in I_j^n \quad .$$

In view of Lemma 10.1, this averaging operator satisfies

$$(10.1.33) \qquad \|(A - I)w\|_{W^{-1,1}} \leq O(\Delta x^2)\|w\|_{BV} \quad ,$$

provided that the mesh is regular, (10.1.4).

With this definition of A, we decompose the $W^{-1,1}$-mollification error in (10.1.26) into three simpler error terms,

$$(10.1.34) \qquad \|\psi_{\Delta x} * w - w\|_{W^{-1,1}} \leq$$

$$\|\psi_{\Delta x} * (w - Aw)\|_{W^{-1,1}} + \|\psi_{\Delta x} * (Aw) - Aw\|_{W^{-1,1}} + \|Aw - w\|_{W^{-1,1}} = T_1 + T_2 + T_3 \quad ,$$

where A denotes the fixed Δx-grid averaging operator. In view of (10.1.21) and (10.1.33) we have that

$$(10.1.35) \qquad T_1 \leq T_3 \leq O(\Delta x^2)\|w\|_{BV}.$$

As for T_2, since Aw is piecewise constant, $Aw(x) = \sum_j w_j \chi_{I_j}(x)$ (w_j being the averaged values of w in the cell I_j), $\psi_{\Delta x} * (Aw)$ is a continuous linear interpolant of Aw at $\{x_j\}$ – the centers of the fixed grid cells. It can be easily verified that the two functions, Aw and $\psi_{\Delta x} * (Aw)$, satisfy conditions (i)–(ii) of Lemma 10.1 with $K = \frac{1}{4}\|w\|_{BV}$ and $L = 1$. Therefore

$$(10.1.36) \qquad T_2 \leq O(\Delta x^2)\|w\|_{BV} \quad .$$

Error estimate (10.1.26) now follows from (10.1.34)–(10.1.36).

10.2. Examples

In this section we demonstrate our results for a variety of Godunov type schemes. The Godunov scheme is a Godunov type scheme par excellence and is identified by the choice of projection $P = A$, where $A = A(I_j^n)$ is the cell averaging operator, (10.1.32). We denote the cell averaged values of the approximation and their differences by:

$$(10.2.1) \qquad v_j^n = Av^{\Delta x}(\cdot, t^n - 0)\Big|_{I_j^n} \qquad ; \qquad \Delta v_{j+\frac{1}{2}}^n = v_{j+1}^n - v_j^n \quad .$$

Using these notations we may introduce a different discrete Lip^+-seminorm (compare to definition (10.1.18)),

$$(10.2.2) \qquad \|v^{\Delta x}(\cdot, t^n)\|_{lip^+} \equiv \max_j \left(\frac{\Delta v^n_{j+\frac{1}{2}}}{\Delta x} \right)^+ \quad ,$$

which we refer to as the lip^+-seminorm of the cell averages. The need for this additional discrete Lip^+-seminorm will be clarified in the course of the discussion.

1. E-SCHEMES – ON A FIXED MESH

We begin by dealing with piecewise constant Godunov type approximations where the grid cells are fixed:

$$I_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}) \qquad ; \qquad x_{j\pm\frac{1}{2}} = (j \pm \frac{1}{2})\Delta x \quad .$$

The simplest choice of a projection in this case is $P = A$. There are two schemes which take precisely this form: The Godunov and the staggered Lax-Friedrichs (LxF) schemes (in the latter, the mesh moves in each time step, by $\frac{\Delta x}{2}$, to the right or to the left, alternately). In view of error estimate (10.1.33), these schemes are $W^{-1,1}$-consistent, in the sense of (10.1.24).

Since the discrete Lip^+-seminorm, $\|\cdot\|_{DLip^+}$, and the cell averages lip^+-seminorm, $\|\cdot\|_{lip^+}$, coincide in the case of piecewise constant grid functions, the discrete Lip^+-stability condition (10.1.18) reads in this case:

$$(10.2.3) \qquad \|v^{\Delta x}(\cdot, t^n)\|_{lip^+} \leq C \qquad , \qquad n \geq 0 \quad .$$

A proof of the (discrete) Lip^+-stability of Godunov and LxF schemes can be found in [GL] and [T1]. Hence, our convergence rate estimates are easily obtained for these schemes by Theorem 10.4.

Godunov and LxF schemes are members of the family of essentially three point schemes. This family consists of schemes which admit the following viscosity form [T2]:

$$(10.2.4) \qquad v^{n+1}_j = v^n_j - \frac{\lambda}{2}[f(v^n_{j+1}) - f(v^n_{j-1})] + \frac{1}{2}[Q^n_{j+\frac{1}{2}}\Delta v^n_{j+\frac{1}{2}} - Q^n_{j-\frac{1}{2}}\Delta v^n_{j-\frac{1}{2}}] \quad .$$

The Godunov and LxF schemes are identified by the viscosity coefficients:

$$Q^{G,n}_{j+\frac{1}{2}} = \lambda \max_v \left[\frac{f(v^n_{j+1}) + f(v^n_j) - 2f(v)}{\Delta v^n_{j+\frac{1}{2}}} \right] \qquad , \qquad Q^{LxF,n}_{j+\frac{1}{2}} = 1 \quad .$$

To extend our discussion to this family of schemes, we present them in terms of a projection operator, $P = MA$. With this choice of projection we modify the cell averages by an appropriate operator M tailored to the specific essentially three point scheme in question. In the following proposition we prove $W^{-1,1}$-consistency for these schemes:

Proposition 10.1. *The modifying operators, M, which correspond to fixed mesh essentially three point BV schemes, (10.2.4), satisfy*

$$(10.2.5) \qquad \|(M - I)Av^{\Delta x}\|_{W^{-1,1}} \leq O(\Delta x^2) \ ,$$

provided that the viscosity coefficients are uniformly bounded,

$$(10.2.6) \qquad 0 \leq Q^n_{j+\frac{1}{2}} \leq C \ .$$

Proof. M is the operator which generates the grid values of the scheme, given in (10.2.4), from the cell averages of Godunov scheme,

$$v^{n+1}_j = MAv^{\Delta x}(\cdot, t^{n+1} - 0)\big|_{I_j} \ .$$

On the other hand, since Godunov scheme uses the exact solver, its averaged value on I^{n+1}_j is given by

$$v^{G,n+1}_j = Av^{\Delta x}(\cdot, t^{n+1} - 0)\big|_{I_j} \ .$$

Hence, in view of (10.2.4), the difference which we need to estimate in $W^{-1,1}$ is a piecewise constant grid function,

$$(10.2.7) \qquad w(x) \equiv (M - I)Av^{\Delta x}(x, t^{n+1}) = \sum_j w^{n+1}_j \chi_{I_j}(x) \ ,$$

where w^{n+1}_j depend upon the difference between the viscosity coefficients,

$$(10.2.8) \qquad w^{n+1}_j = \frac{1}{2}[(Q^n_{j+\frac{1}{2}} - Q^{G,n}_{j+\frac{1}{2}})\Delta v^n_{j+\frac{1}{2}} - (Q^n_{j-\frac{1}{2}} - Q^{G,n}_{j-\frac{1}{2}})\Delta v^n_{j-\frac{1}{2}}] \ .$$

The primitive, $W(x) = \int_{-\infty}^x w(\xi)d\xi$, is piecewise linear and is given by

$$(10.2.9) \qquad W(x) = \sum_{i=-\infty}^{j-1} w^{n+1}_i \Delta x + (x - x_{j-\frac{1}{2}})w^{n+1}_j =$$

$$= \frac{\Delta x}{2}(Q^n_{j-\frac{1}{2}} - Q^{G,n}_{j-\frac{1}{2}})\Delta v^n_{j-\frac{1}{2}} + (x - x_{j-\frac{1}{2}})w^{n+1}_j \qquad \forall x \in I_j \ .$$

Since by (10.2.6)

$$(10.2.10) \qquad |Q^n_{j+\frac{1}{2}} - Q^{G,n}_{j+\frac{1}{2}}| \leq C \ ,$$

it follows that w_j^{n+1}, given in (10.2.8), may be bounded as follows:

$$(10.2.11) \qquad |w_j^{n+1}| \leq \frac{C}{2}\left(|\Delta v_{j+\frac{1}{2}}^n| + |\Delta v_{j-\frac{1}{2}}^n|\right) \ .$$

Therefore, by (10.2.9)–(10.2.11),

$$(10.2.12) \qquad |W(x)| \leq \frac{C}{2}|\Delta v_{j-\frac{1}{2}}^n|\Delta x + \frac{C}{2}(x - x_{j-\frac{1}{2}})\left(|\Delta v_{j+\frac{1}{2}}^n| + |\Delta v_{j-\frac{1}{2}}^n|\right) \qquad \forall x \in I_j \ .$$

Equipped with (10.2.12) we conclude, by carrying out the integration, that

$$\|w(x)\|_{W^{-1,1}} = \|W(x)\|_{L_1} = \sum_j \int_{I_j} |W(\xi)|d\xi \leq C\Delta x^2 \sum_j |\Delta v_{j+\frac{1}{2}}^n| \leq$$

$$\leq C\Delta x^2\|v^{\Delta x}(\cdot, t^n)\|_{BV} = O(\Delta x^2) \ ,$$

which proves (10.2.5). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proposition 10.1 and error estimate (10.1.33) imply that essentially three point schemes with bounded viscosity coefficients, (10.2.6), are $W^{-1,1}$-consistent of (at least) order $O(\Delta x)$. Hence, all our error estimates follow for such Lip^+-stable (hence BV) schemes. Two more examples of Lip^+-stable members of this family are Roe and Engquist-Osher schemes (e.g. [B], [NT2]).

Remark. The Godunov and LxF schemes are the two extreme members of the well known family of E-schemes. This family consists of all essentially three point schemes, (10.2.4), for which $Q_{j+\frac{1}{2}}^{G,n} \leq Q_{j+\frac{1}{2}}^n \leq Q_{j+\frac{1}{2}}^{LxF,n}$. These schemes are known to be of first order resolution (consult [O4]).

Let us consider now, as an additional example, E-schemes with a constant viscosity coefficient,

$$(10.2.13) \qquad v_j^{n+1} = v_j^n - \frac{\lambda}{2}[f(v_{j+1}^n) - f(v_{j-1}^n)] + \frac{1}{2}[Q\Delta v_{j+\frac{1}{2}}^n - Q\Delta v_{j-\frac{1}{2}}^n] \ ,$$

where

$$(10.2.14) \qquad \lambda \max_{j,n}|a(v_j^n)| \leq Q \leq \frac{1}{2} \ .$$

Since such schemes are (as we show below) Lip^+-stable, (10.2.3), and $W^{-1,1}$-consistent as well (in virtue of (10.2.14) and Proposition 10.1), they satisfy our convergence rate estimates.

The proof of Lip^+-stability for (10.2.13) is similar to the Lip^+-stability proofs for the previously mentioned examples of essentially three point schemes. Differencing equality (10.2.13) we get that

$$(10.2.15) \qquad \Delta v_{j+\frac{1}{2}}^{n+1} = \Delta v_{j+\frac{1}{2}}^n - \frac{\lambda}{2}[\Delta f_{j+\frac{3}{2}}^n - \Delta f_{j-\frac{1}{2}}^n] + \frac{Q}{2}[\Delta v_{j+\frac{3}{2}}^n - 2\Delta v_{j+\frac{1}{2}}^n + \Delta v_{j-\frac{1}{2}}^n] \ .$$

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Taylor expansion and the convexity of f imply that

$$(10.2.16) \qquad \Delta f^n_{j-\frac{1}{2}} \le a(v^n_j)\Delta v^n_{j-\frac{1}{2}} \quad , \quad \Delta f^n_{j+\frac{3}{2}} \ge a(v^n_{j+1})\Delta v^n_{j+\frac{3}{2}} \quad .$$

Inserting (10.2.16) into (10.2.15) and rearrangeing yields

$$(10.2.17) \quad \Delta v^{n+1}_{j+\frac{1}{2}} \le (1-Q)\cdot \Delta v^n_{j+\frac{1}{2}} + \frac{1}{2}[Q - \lambda a(v^n_{j+1})]\cdot \Delta v^n_{j+\frac{3}{2}} + \frac{1}{2}[Q + \lambda a(v^n_j)]\cdot \Delta v^n_{j-\frac{1}{2}} \quad .$$

All three coefficients on the right hand side of (10.2.17) are non-negative, in view of (10.2.14), and they add up to

$$S = 1 - \frac{\lambda}{2}(a(v_{j+1}) - a(v_j)) \quad .$$

If $\Delta v^n_{j+\frac{1}{2}} \ge 0$ then, thanks to the monotonicity of a, $a(v_{j+1}) \ge a(v_j)$ and $S \le 1$. Therefore, we conclude by (10.2.17) that

$$\Delta v^{n+1}_{j+\frac{1}{2}} \le \max(\Delta v^n_{j-\frac{1}{2}}, \Delta v^n_{j+\frac{1}{2}}, \Delta v^n_{j+\frac{3}{2}}) \quad .$$

If, on the other hand, $\Delta v^n_{j+\frac{1}{2}} < 0$ then, by omitting the first term on the right hand side of (10.2.17) and using (10.2.14), we get that

$$\Delta v^{n+1}_{j+\frac{1}{2}} \le \max(\Delta v^n_{j-\frac{1}{2}}, \Delta v^n_{j+\frac{3}{2}}) \quad .$$

The last two inequalities imply, after dividing by Δx, that

$$\|v^{\Delta x}(\cdot, t^{n+1})\|_{lip^+} \le \|v^{\Delta x}(\cdot, t^n)\|_{lip^+} \quad ,$$

which implies (10.2.3) with $C = \|u_0\|_{Lip^+}$.

2. GODUNOV SCHEME – ON A VARIABLE MESH

As a prototype example for the use of variable grid we concentrate on Godunov scheme. We briefly recall the variable mesh algorithm advocated in [HH]. The fixed-mesh Godunov scheme is modified to a variable-mesh one, by adjusting the grid to follow the dynamics of the solution: when two neighboring grid values are connected through a shock wave, the mesh algorithm places one of the next step mesh points on the shock's path to enable its perfect resolution. The above choice of mesh points $\{x^n_{j+\frac{1}{2}}\}$ is done so that the mesh regularity condition (10.1.4) will not be violated.

Clearly, this variable-mesh Godunov scheme is $W^{-1,1}$-consistent (consult Theorem 10.2 and error estimate (10.1.33)). The question of discrete Lip^+-stability, however, is more delicate and, therefore, we introduce a further slight modification. The above described mesh algorithm chooses the variable mesh points $x^n_{j+\frac{1}{2}}$ so that $x^n_{j+\frac{1}{2}} \in [x_j, x_{j+1})$, where $\{x_j\}$

is an underlying fixed uniform mesh. Our modification applies when two neighboring grid values are connected through a rarefaction wave; in this case we suggest to choose the next step mesh point as the center of the fixed underlying mesh. By doing so, the evolution procedure coincides with the regular fixed mesh Godunov scheme whenever the solution is increasing. Hence, this modified algorithm describes a Lip^+-stable scheme without affecting the shock resolution of the original variable mesh scheme. Therefore, this modified scheme converges to the exact solution of (4.1) and satisfies all our error estimates.

3. MUSCL SCHEMES

We now turn to MUSCL schemes which employ a piecewise linear reconstruction of the cell averages in order to increase the resolution. These schemes are Godunov type schemes with a projection of the form $P \equiv RA$, [VL], [H1]. The reconstruction $R = R(\{I_j\})$ acts on piecewise constant grid functions by rotating the constant value in each cell I_j around its center, $x_j = j\Delta x$:

$$(10.2.18) \qquad RAv^{\Delta x}(x, t^n - 0) = R\left[\sum_j v_j^n \chi_{I_j}(x)\right] \equiv v_j^n + (x - x_j)s_j^n \qquad \forall x \in I_j \ .$$

The reconstruction is identified by the choice of a limiter function $s(\cdot, \cdot)$ which defines the slopes,

$$(10.2.19) \qquad\qquad s_j^n = s\left(\frac{\Delta v_{j-\frac{1}{2}}^n}{\Delta x}, \frac{\Delta v_{j+\frac{1}{2}}^n}{\Delta x}\right) \ ,$$

and usually constrained to satisfy

$$(10.2.20) \qquad\qquad \min(a, b) \leq s(a, b) = s(b, a) \leq \max(a, b) \ .$$

This choice of projection is conservative, i.e. $AP = A$.

$W^{-1,1}$-consistency of these schemes follows directly from Lemma 10.1 and error estimate (10.1.33), as stated in the following proposition:

Proposition 10.2. *The projection $P = RA$ satisfies*

$$(10.2.21) \qquad\qquad \|(P - I)w\|_{W^{-1,1}} \leq O(\Delta x^2)\|w\|_{BV} \ .$$

The verification of the discrete Lip^+-stability condition, (10.1.19), is rather delicate for this family of schemes. In the following proposition we show the equivalence of the discrete Lip^+-seminorm $- \|\cdot\|_{DLip^+} -$ and the lip^+-seminorm of the cell averages $- \|\cdot\|_{lip^+} -$ for a subclass of limiters.

Proposition 10.3. *If the limiter $s(\cdot, \cdot)$ satisfies*

(10.2.22) $$\text{minmod}(a, b) \leq s(a, b) \leq \max(a, b) \quad,$$

then for every function $w(x)$

(10.2.23) $$\|RAw\|_{lip^+} \leq \|RAw\|_{DLip^+} \leq K \cdot \|RAw\|_{lip^+} \quad,$$

where $1 \leq K \leq 1.5$.

Remarks.

1. The class of limiters defined in (10.2.22) forms a subclass of the one given in (10.2.20). The lower most limiter in the latter – min – is replaced here by the well known minmod limiter,

$$\text{minmod}(a, b) \equiv \frac{1}{2}[\text{sgn}(a) + \text{sgn}(b)] \cdot \min(|a|, |b|) \quad.$$

Minmod based reconstructions are often used in practice, since they yield non-oscillatory schemes, [H1], [OT].

2. Proposition 10.3 enables us, when dealing with Lip^+-stability of MUSCL schemes satisfying (10.2.22), to concentrate on the cell averaged values and check condition (10.2.3) rather then the intricate condition (10.1.19).

Proof. Recalling the definitions of the two seminorms, (10.1.18) and (10.2.2), the left inequality in (10.2.23) is trivial, since

$$\left. \frac{RAw(x + \Delta x) - RAw(x)}{\Delta x} \right|_{x = x_j} = \frac{\Delta w_{j+\frac{1}{2}}}{\Delta x} \quad.$$

As for the second inequality in (10.2.23) we observe that every x can be expressed as $x = x_j + \theta \Delta x$ for some x_j and $|\theta| \leq \frac{1}{2}$ and therefore, by (10.2.18) and (10.2.19),

$$\frac{RAw(x + \Delta x) - RAw(x)}{\Delta x} = \frac{\Delta w_{j+\frac{1}{2}}}{\Delta x} + \theta \left(s \left(\frac{\Delta w_{j+\frac{1}{2}}}{\Delta x}, \frac{\Delta w_{j+\frac{3}{2}}}{\Delta x} \right) - s \left(\frac{\Delta w_{j-\frac{1}{2}}}{\Delta x}, \frac{\Delta w_{j+\frac{1}{2}}}{\Delta x} \right) \right) \quad.$$

Hence, in order to prove (10.2.23) it suffices to show that

$$\frac{\Delta w_{j+\frac{1}{2}}}{\Delta x} + \theta \left(s \left(\frac{\Delta w_{j+\frac{1}{2}}}{\Delta x}, \frac{\Delta w_{j+\frac{3}{2}}}{\Delta x} \right) - s \left(\frac{\Delta w_{j-\frac{1}{2}}}{\Delta x}, \frac{\Delta w_{j+\frac{1}{2}}}{\Delta x} \right) \right) \leq$$

$$\leq K \cdot \max \left(\frac{\Delta w_{j-\frac{1}{2}}}{\Delta x}, \frac{\Delta w_{j+\frac{1}{2}}}{\Delta x}, \frac{\Delta w_{j+\frac{3}{2}}}{\Delta x} \right)^+ \quad,$$

for some constant $K \leq 1.5$. This last inequality may be written in the more abstract form:

(10.2.24) $$I \equiv b + \theta[s(b, c) - s(a, b)] \leq K \cdot \max(a, b, c)^+ \qquad \forall \theta \; : \; |\theta| \leq \frac{1}{2} \quad.$$

We note that due to the symetry of $s(\cdot, \cdot)$ it suffices to deal with $\theta \geq 0$. Therefore, in order to upper-bound I, we have to upper-bound $s(b, c)$ and lower-bound $s(a, b)$.

First we show that if $b \leq 0$, (10.2.24) holds with $K = \frac{1}{2}$. Using the limitation assumption, (10.2.22), we can summarize the upper-bounds for I as follows:

$$(10.2.25) \qquad I \leq \begin{cases} b + \theta(c - 0) \leq \theta c & a \geq 0, c \geq 0 \\ b + \theta(0 - 0) \leq 0 & a \geq 0, c \leq 0 \\ b + \theta(c - b) \leq \theta c & a \leq 0, c \geq 0 \\ b + \theta(0 - b) \leq 0 & a \leq 0, c \leq 0 \end{cases}$$

Since $0 \leq \theta \leq \frac{1}{2}$, (10.2.24) follows from (10.2.25) with $K = \frac{1}{2}$.

Now we turn to the case $b \geq 0$. Using (10.2.22) we arrive at

$$(10.2.26) \qquad I \leq \begin{cases} b + \theta(c - b) \leq (1 - \theta)b + \theta c \leq c & a \geq b, c \geq b \\ b + \theta(b - b) = b & a \geq b, c \leq b \\ b + \theta(c - a^+) \leq b + \theta c \leq 1.5c & a \leq b, c \geq b \\ b + \theta(b - a^+) \leq b + \theta b \leq 1.5b & a \leq b, c \leq b \end{cases}$$

Hence, (10.2.24) holds with $K = 1.5$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Remarks.

1. If $s(\cdot, \cdot) = \max(\cdot, \cdot)$, (10.2.24) holds with $K = 1$, since in the last two cases of (10.2.26), which are the only cases where $K > 1$ may appear, $s(a, b) = b$, $s(b, c) = \max(b, c)$ and therefore

$$I = b + \theta(\max(b, c) - b) \leq \max(b, c) \leq \max(a, b, c)^+ \ .$$

2. If $s(\cdot, \cdot) = \text{minmod}(\cdot, \cdot)$, the estimate $K \leq 1.5$ is sharp since if $b = c > 0, a \leq 0$ and $\theta = \frac{1}{2}$ we have

$$I = b + \theta(b - 0) = 1.5b = 1.5\max(a, b, c)^+ \ .$$

3. The equivalence (10.2.23) does not hold for $s(\cdot, \cdot) = \min(\cdot, \cdot)$: For example, if $b = c = 0$ and $a < 0$ then $I = -\theta a$ which violates (10.2.24). Hence, condition (10.2.22) is indeed necessary.

Example. The Maxmod Scheme.

The upper extreme case of (10.2.22) is the maxmod scheme. This scheme is shown to be Lip^+-stable in [BO].

The reconstruction of this scheme, $R_{\max}$, has the unique feature (among the class of limiters satisfying (10.2.20)) that it avoids increasing discontinuities, hence it yields Lip^+-bounded piecewise linear functions, $\|R_{\max}Aw\|_{Lip^+} < \infty$. Furthermore, all three Lip^+-seminorms, the regular one – (2.1.11), the discrete one – (10.1.18) and the cell averaged

values one – (10.2.2), are equal in this case, i.e.

(10.2.27) $$\|R_{\max}Aw\|_{Lip^+} = \|R_{\max}Aw\|_{DLip^+} = \|R_{\max}Aw\|_{lip^+} \quad .$$

Brenier and Osher [BO] show that the maxmod scheme is Lip^+ monotonically decreasing, namely

$$\|v^{\Delta x}(\cdot, t^{n+1})\|_{Lip^+} < \|v^{\Delta x}(\cdot, t^n)\|_{Lip^+} \qquad \forall n \geq 0 \quad .$$

Therefore, (10.1.7) is met with $C = \|u_0\|_{Lip^+}$.

The maxmod scheme is, to the best of our knowledge, the only MUSCL scheme for which lip^+-stability has been established. Other reconstructions, such as the minmod, may increase the cell averages lip^+-seminorm. However, numerical experiments confirm our strong belief that MUSCL schemes based on such reconstructions are lip^+-bounded, though their lip^+-seminorm is not monotonically decreasing. Given this lip^+-stability together with our proof of $W^{-1,1}$-consistency, we obtain the convergence rate estimate (10.1.25).

4. MUSCL SCHEMES WITH APPROXIMATE EVOLUTION

MUSCL schemes involve the exact evolution for a short time of a piecewise linear initial condition, namely, solving a generalized Riemann problem. This difficulty is intricate to carry out and therefore, simpler alternative projections are sought. We present here two such projections being commonly used in practice.

One way of diffusing the problem of solving a generalized Riemann problem is by replacing the piecewise linear initial condition $v^{\Delta x}(\cdot, t^n) = RAv^{\Delta x}(\cdot, t^n - 0)$ by $v^{\Delta x}(\cdot, t^n) = MRAv^{\Delta x}(\cdot, t^n - 0)$ where the operator M decomposes the reconstructed piecewise linear profile at each time step into a piecewise constant one as follows:

(10.2.28) $$MRAv^{\Delta x}(x, t^n - 0) = \sum_j \left[v^n_{j,-}\chi_{I_{j,-}}(x) + v^n_{j,+}\chi_{I_{j,+}}(x) \right] \quad .$$

Here $v^n_{j,\pm}$ denote the values of the reconstruction in the two end points of I_j, $x_{j-\frac{1}{2}}$ and $x_{j+\frac{1}{2}}$,

$$v^n_{j,\pm} = v^n_j \pm \frac{\Delta x}{2}s^n_j$$

and $I_{j,\pm}$ denote the left and right halves of the interval I_j, i.e.,

$$I_{j,-} = [x_{j-\frac{1}{2}}, x_j) \quad , \quad I_{j,+} = [x_j, x_{j+\frac{1}{2}}) \quad .$$

By this modification, the solution of (4.1) consists of a successive sequence of non-interacting Riemann problems, provided that we half the CFL condition (10.1.6),

(10.2.29) $$\lambda \max_{x,t} |f'(v^{\Delta x}(x,t))| \leq \frac{1}{2} \quad .$$

Let $W(x/t; u_L, u_R)$ denote the Riemann solver of (4.1). Then our modified schemes recast, after integration of the exact solution over a typical cell $I_j \times [t^n, t^{n+1}]$, into the final form

$$(10.2.30) \qquad v_j^{n+1} = v_j^n - \lambda \left[f(W(0+; v_{j,+}^n, v_{j+1,-}^n)) - f(W(0+; v_{j-1,+}^n, v_{j,-}^n)) \right] \quad .$$

These modified schemes fit into our framework of Godunov type schemes with the projection $P = MRA$, where the piecewise constant decomposition operator, M, is given in (10.2.28). With this formulation in mind we observe that our modified schemes are $W^{-1,1}$-consistent. Indeed, the definition of M and Lemma 10.1 imply that

$$\|(M-I)RAv^{\Delta x}\|_{W^{-1,1}} \le O(\Delta x^2)\|RAv^{\Delta x}\|_{BV} \le O(\Delta x^2)\|v^{\Delta x}\|_{BV} \quad ,$$

and, therefore, condition (10.1.24) is met by the modified projection $P = MRA$. Thus, the $W^{-1,1}$-consistency of the original MUSCL schemes is retained. Hence, these modified MUSCL schemes, if Lip^+-stable, satisfy our error estimates.

Another way to avoid the solution of the generalized Riemann problem is replacing the exact evolution operator E by an approximate one, $\tilde{E}$ (compare to (10.1.2)),

$$(10.2.31) \qquad v^{\Delta x}(\cdot, t^{n+1}) = RA\tilde{E}(t^{n+1} - t^n)v^{\Delta x}(\cdot, t^n) \quad .$$

This modification fits into our framework, (10.1.2)–(10.1.3), by rewriting the evolution procedure (10.2.31) as

$$(10.2.32) \qquad v^{\Delta x}(\cdot, t^{n+1}) = PE(t^{n+1} - t^n)v^{\Delta x}(\cdot, t^n) \quad , \qquad P = RMA \quad ,$$

where M takes care of the differences between the averaged values of the exact and approximate evolutions.

In the following proposition we show that our convergence rate estimates are not affected by the use of an approximate evolution, provided that the local truncation error is of second order.

Proposition 10.4. *If the modified MUSCL scheme (10.2.32) is conservative, discrete Lip^+-stable and the operator M, which identifies the approximate evolution $\tilde{E}$, satisfies*

$$(10.2.33) \qquad |(MAE - AE)v^{\Delta x}| \le O(\Delta x^2) \quad ,$$

then error estimate (10.1.25) holds.

Proof. In view of Theorem 10.4, we have only to show that for $w = Ev^{\Delta x}$,

$$(10.2.34) \qquad \|(P-I)w\|_{W^{-1,1}} \le O(\Delta x^2) \quad .$$

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Applying the triangle inequality we may decompose this error term into three different error terms,

(10.2.35) $$\|(P-I)w\|_{W^{-1,1}} = \|(RMA-I)w\|_{W^{-1,1}} \le$$

$$\|(R-I)MAw\|_{W^{-1,1}} + \|(MA-A)w\|_{W^{-1,1}} + \|(A-I)w\|_{W^{-1,1}} = T_1 + T_2 + T_3 \quad.$$

Lemma 10.1 implies that

(10.2.36) $$T_1 = O(\Delta x^2) \quad.$$

As for T_2, we let $g = (MAE - AE)v^{\Delta x}$ and $G = \int^x g$. Since the scheme is conservative, (10.1.5), the averaged value of g over its compact support, which we denote by Ω, is zero. This implies that G is also compactly supported on Ω. Therefore, by (10.2.33),

(10.2.37) $$T_2 = \|(MA-A)Ev^{\Delta x}\|_{W^{-1,1}} = \|g\|_{W^{-1,1}} = \|G\|_{L_1} \le$$

$$|\Omega| \cdot \|G\|_{L_\infty} \le |\Omega| \cdot \|g\|_{L_1} \le |\Omega|^2 \cdot \|g\|_{L_\infty} \le O(\Delta x^2) \quad.$$

Finally, (10.2.34) follows from (10.2.35), (10.2.36), (10.2.37) and (10.1.33). □

Example. Non-Oscillatory Central Difference Scheme

We consider a family of MUSCL-type non-oscillatory central differencing schemes, presented in [NT1]. We briefly recall the construction of these schemes and present them in our notations. The grid in use is a staggered one, namely, the cell size Δx is fixed, but the grid moves in each time step by $\frac{\Delta x}{2}$.

The exact solution of the generalized Riemann problems is averaged on the staggered grid (i.e. use $A = A(\{I_j\})$ or $A = A(\{I_{j+\frac{1}{2}}\})$ every other step). This central Lax-Friedrichs type solver may be written exactly, using (10.1.2)–(10.1.3), as (compare the following formulation to [NT1, (2.11)]):

(10.2.38) $$v_{j+\frac{1}{2}}^{n+1} = \frac{1}{\Delta x}\left[\int_{x_j}^{x_{j+\frac{1}{2}}} v^{\Delta x}(x,t^n)dx + \int_{x_{j+\frac{1}{2}}}^{x_{j+1}} v^{\Delta x}(x,t^n)dx\right] -$$

$$-\frac{1}{\Delta x}\left[\int_{t^n}^{t^{n+1}} f(v^{\Delta x}(x_{j+1},\tau))d\tau - \int_{t^n}^{t^{n+1}} f(v^{\Delta x}(x_j,\tau))d\tau\right] \quad.$$

The time step Δt is restricted by the CFL condition (10.2.29) so that no interaction occurs between two neighboring Riemann problems.

The evaluation of the temporal integrals in (10.2.38) requires the exact solution of the generalized Riemann problems along the lines $x = x_j$. This is being avoided by using the mid-point rule,

(10.2.39a) $$\int_{t^n}^{t^{n+1}} f(v^{\Delta x}(x_j,\tau))d\tau \approx \Delta t \cdot f(v^{\Delta x}(x_j, t^n + \frac{\Delta t}{2})) \quad,$$

where the mid-point value is linearly approximated,

$$(10.2.39b) \qquad v^{\Delta x}(x_j, t^n + \frac{\Delta t}{2}) \approx w_j^{n+\frac{1}{2}} \equiv v_j^n - \frac{\Delta t}{2} a(v_j^n) s_j^n \ .$$

Thus, with $v_{j+\frac{1}{2}}^{n+1}$ in (10.2.38) denoting the *exact* evolution averages, these approximations result in the modified averaged values, $Mv_{j+\frac{1}{2}}^{n+1}$, given by

$$(10.2.40) \qquad Mv_{j+\frac{1}{2}}^{n+1} = \frac{1}{2}\left(v_j^n + v_{j+1}^n\right) + \frac{\Delta x}{8}\left(s_j^n - s_{j+1}^n\right) - \lambda\left(f(w_{j+1}^{n+\frac{1}{2}}) - f(w_j^{n+\frac{1}{2}})\right) \ .$$

With this modification in mind we turn to show the $W^{-1,1}$-consistency of this family of schemes. To this end we show that the modifying operator M, given in (10.2.40), satisfies the consistency condition (10.2.33).

Since the Riemann problems do not interact, the solution $v^{\Delta x}(x_j, \tau)$ is smooth on the line $x_j \times [t^n, t^{n+1}]$. Hence, the mid-point rule local truncation error gives that

$$(10.2.41) \qquad \left| \int_{t^n}^{t^{n+1}} f(v^{\Delta x}(x_j, \tau)) d\tau - \Delta t \cdot f(v^{\Delta x}(x_j, t^n + \frac{\Delta t}{2})) \right| = O(\Delta t^3) \ .$$

Furthermore, by Taylor expansion and (10.2.39b)

$$v^{\Delta x}(x_j, t^n + \frac{\Delta t}{2}) = v^{\Delta x}(x_j, t^n) + \frac{\Delta t}{2} v_t^{\Delta x}(x_j, t^n) + O(\Delta t^2) =$$

$$= v^{\Delta x}(x_j, t^n) - \frac{\Delta t}{2} a(v^{\Delta x}(x_j, t^n)) v_x^{\Delta x}(x_j, t^n) + O(\Delta t^2) =$$

$$= v_j^n - \frac{\Delta t}{2} a(v_j^n) s_j^n + O(\Delta t^2) = w_j^{n+\frac{1}{2}} + O(\Delta t^2) \ ,$$

which implies that

$$(10.2.42) \qquad |v^{\Delta x}(x_j, t^n + \frac{\Delta t}{2}) - w_j^{n+\frac{1}{2}}| = O(\Delta t^2) \ .$$

Comparing (10.2.38) to (10.2.39) and (10.2.40) gives, using (10.2.41) and (10.2.42), that

$$|Mv_{j+\frac{1}{2}}^{n+1} - v_{j+\frac{1}{2}}^{n+1}| = |(MAE - AE)v^{\Delta x}(x, t^n)|\big|_{I_{j+\frac{1}{2}}} \leq O(\Delta t^2) = O(\Delta x^2) \ .$$

Thus, according to Proposition 10.4, the above described family of schemes is $W^{-1,1}$-consistent. Augmented with Lip^+-stability we conclude that non-oscillatory central differencing schemes satisfy our error estimates.

Epilogue. MUSCL schemes are viewed as second-order accurate since for smooth functions, $w \in C^2$, $s_j = w_x(x_j) + O(\Delta x)$. However, local second order accuracy away from discontinuities has not been yet proven. We proved here a weaker result for Lip^+-stable MUSCL

schemes, namely, a local first order accuracy (for the post-processed values) whenever the exact solution is infinitely smooth. The error estimates given in Theorem 10.1, are the optimal ones. The problem is due to the $W^{-1,1}$-norm which proves to be appropriate for first order convergence rate only : It is easy to see that

$$(10.2.43) \qquad \qquad \|(RA - I)w\|_{W^{-1,1}} = O(\Delta x^3)\|w\|_{BV}$$

whenever w is C^1 in the interior of the grid cells I_j. However, if w experiences a discontinuity inside a grid cell, (10.2.43) no longer holds and the weaker error estimate (10.2.21) is then sharp. Comparing the two $W^{-1,1}$-error estimates, (10.1.33) and (10.2.21), shows that the reconstruction R does not improve the $W^{-1,1}$-accuracy in that case. Therefore, when shocks are present, formally second order schemes are only first order accurate in $W^{-1,1}$.

Motivated by this discussion we suggest to surpass this $W^{-1,1}$-first order accuracy barrier by moving the mesh so that no shock will occur in the interior of a grid cell. By doing so, the better error estimate (10.2.43) will hold, and the resulting scheme, if Lip^+-stable, will be second-order accurate in $W^{-1,1}$ and local second order accuracy, for the post-processed grid values, will follow wherever the exact solution is infinitely smooth.

PART III: HIGH ORDER REGULARITY

11. THE PIECEWISE SMOOTHNESS OF ENTROPY SOLUTIONS

11.1. Introduction

In this part of our work we show how the Lip^+-stability of convex conservation laws, namely, the decay of the positive part of the first spatial derivative, implies a higher order stability of the entropy solutions and sheds more light on their behavior and structure.

The structure of entropy solutions of convex conservation laws, (4.1), having piecewise smooth initial data, has been determined by Oleinik [O1]–[O3] and Lax [L1]; more refined information was obtained by Dafermos [D1]. The entropy solutions are continuous except on the union of an at most countable set of Lipschitz continuous shock curves. The complement of the shock set is open, [D1], and from each point (x, t) in this open set one can trace a straight characteristic backward in time to $t = 0$, where the initial condition is given. Since the slope of this characteristic equals $a(u(x, t)) = f'(u(x, t))$, the entropy solution is given by the implicit relation

$$(11.1.1) \qquad\qquad u(x, t) = u_0(x - a(u(x, t))t) \quad .$$

The Implicit Function Theorem implies that if $a, u_0 \in C^N$, $N \geq 1$, then $u \in C^N$ in its region of continuity, since in that region

$$(11.1.2) \qquad\qquad 1 + a'(u)u_0'(x - a(u)t)t > 0 \qquad \forall t \geq 0 \quad ,$$

consult [D1, Theorem 5.1].

In the following sections we *quantify* the regularity of the entropy solution using sharp upper bounds for its high order spatial derivatives in its region of C^1-smoothness, and we determine the size of the complement set of that region, namely — the set of shock discontinuities.

We recall the behavior of the first spatial derivative of the solution, u_x: Being non-negative, it decays like $O(t^{-1})$, as implied by the Lip^+-stability estimates (e.g. (2.1.18)), while otherwise, it decreases unboundedly until it becomes infinite, in a finite time, on the shock curves.

In §11.2 we examine the behavior of the higher order spatial derivatives, $|\partial_x^n u| \equiv |\partial^n u / \partial x^n|$, $2 \leq n \leq N$. We derive sharp estimates for them and show (Theorem 11.1) that their behavior depends on the sign of u_x: There exist constants, Const_n, which depend solely on the initial condition, u_0, such that the following holds.

- Along characteristics where u_x is positive we have

$$(11.1.3) \qquad |\partial_x^n u| \leq \mathrm{Const}_n (u_x)^n \quad ,$$

  and therefore — since u_x decays like $O(t^{-1})$ along those curves, the higher order derivatives decay at a rate which increases with n;

- Along characteristics where u_x is negative we have

$$(11.1.4) \qquad |\partial_x^n u| \leq \mathrm{Const}_n |u_x|^{2n-1} \quad ,$$

  and therefore — since the solution breaks in a finite time, t_c, along these characteristics, $|\partial_x^n u|$ tends to infinity as $t \to t_c$ at a rate which increases with n;

- Finally, along characteristics where $u_x = 0$ we have $|\partial_x^n u| \leq \mathrm{Const}_n t^{n-2}, \quad n > 1$.

These estimates on the spatial high order derivatives can be converted into an appropriate stability estimate on the *piecewise* regularity of the entropy solution. This is carried out in §11.3 in terms of a suitable C^n-seminorm which is localized to the C^1-smoothness part of the entropy solution. Theorem 11.2 shows that the solution operator of the convex conservation law (4.1) is stable with respect to that seminorm. In this context we refer to DeVore & Lucier, [DL], for a different type of high order regularity result which manifests itself in terms of a high order spatial Besov stability estimate.

Finally, for the sake of completeness, we discuss in §11.4 the complement of the C^1-smoothness part of the entropy solution, that is, we determine the size of the set of shocks. Theorem 11.3 asserts that this set is equivalent to the set of negative minima of $a(u_0)'$. Thus, Theorem 11.3 complements Schaeffer's regularity theorem [S2], by realizing the first category set of infinitely smooth initial conditions, $\{u_0\}$, which evolve into entropy solutions with infinitely many shock discontinuities.

In summary we conclude that if $a(u_0)$ has a finite number of decreasing inflection points, then only a finite number of shocks will occur. Hence, if $a, u_0 \in C^N$ and $a(u_0)$ has a finite number of decreasing inflection points, then the corresponding entropy solution consists of finite number of pieces, each of which is C^N-smooth; moreover, the regularity of these pieces is bounded by the initial regularity. It is this type of *piecewise* regularity of the entropy solution which is assumed — sometime implicitly, in many finite-dimensional computations.

11.2. High order regularity estimates

We consider solutions of the single convex conservation law (4.1) where

(11.2.1)
$$u_0(x) \in C^N(\Re) \bigcap W^{N,\infty}(\Re) \quad , \quad N \geq 2$$

and

(11.2.2)
$$a = f' \in C^N[\inf u_0, \sup u_0] \quad .$$

We examine here the behavior of the higher order spatial derivatives $\partial_x^n u = \partial^n u/\partial x^n$, $2 \leq n \leq N$, the existence of which is guaranteed by (11.2.1–11.2.2) everywhere apart from the singular set of shock curves.

Since the solution u is smooth in the open complement of the set of shocks, we may multiply (4.1) by $a'(u)$ to find out that $v := a(u)$ satisfies Burgers' equation in that region,

(11.2.3)
$$v_t + v v_x = 0 \quad .$$

We now differentiate (11.2.3) $n \leq N$ times with respect to x to obtain the equation which governs the evolution of $w^n := \partial_x^n v$ in the smooth region:

$$w_t^n + \partial_x^n(v v_x) = 0 \quad .$$

Leibnitz rule yields

$$w_t^n + \sum_{k=0}^{n} \binom{n}{k} (\partial_x^k v)(\partial_x^{n-k} v_x) = 0 \quad ,$$

or equivalently ,

(11.2.4)
$$w_t^n + v w_x^n = -n w^1 w^n - \sum_{k=2}^{n} \binom{n}{k} w^k w^{n-k+1} \quad , \quad 1 \leq n \leq N \quad .$$

Observe that all the spatial derivatives of v are governed by a first order quasi-linear equation (11.2.4) with the same principal part as the equation for v itself in (11.2.3), hence having the same characteristic geometry. However – unlike equation (11.2.3) which tells us that v remains constant along characteristics, the non-vanishing right hand side of (11.2.4) implies that w^n changes along the characteristics. Let the value of w^n along a characteristic $x(t)$ denoted by $w^n(t) = w^n(x(t), t)$, then (11.2.4) implies that

(11.2.5)
$$\frac{dw^n(t)}{dt} = -n w^1(t) w^n(t) - \sum_{k=2}^{n} \binom{n}{k} w^k(t) w^{n-k+1}(t) \quad , \quad 1 \leq n \leq N \quad .$$

We start by examining the first derivative $w^1 = v_x = a(u)_x$. Since it proves playing a significant role in our analysis we denote it, for convenience, by w. Equation (11.2.5) reduces in that case, $n = 1$, to the well known Ricatti equation

$$\text{(11.2.6)} \qquad\qquad \frac{dw}{dt} = -(w)^2$$

whose solution is :

$$\text{(11.2.7)} \qquad\qquad w(t) = \frac{w(0)}{1 + w(0)t} \quad .$$

We see that if $w(0) > 0$, $w(t)$ remains positive and decays to zero like $O(t^{-1})$; if $w(0) = 0$ then $w(t) = 0$ for all $t > 0$ while if $w(0) < 0$, $w(t)$ remains negative and decreases until it becomes infinite. Note that whenever $u_x \geq 0$, (11.2.7) is in complete agreement with the Lip^+-stability estimate (2.1.10).

We now use (11.2.5) and (11.2.7) in order to estimate $w^n(t)$, arriving at the following.

Proposition 11.1. *For every $2 \leq n \leq N$ and $t \geq 0$ there holds:*

$$\text{(11.2.8a)} \qquad |w^n(t)| \leq C_n(1 + w(0)t)^{-n-1} \qquad \text{if } w(0) > 0 \ ;$$

$$\text{(11.2.8b)} \qquad |w^n(t)| \leq D_n(1 + w(0)t)^{-2n+1} \qquad \text{if } w(0) < 0 \ ;$$

$$\text{(11.2.8c)} \qquad w^n(t) = w^n(0) + P_{n-2}(t) \qquad \text{if } w(0) = 0 \ .$$

Here, the constants C_n and D_n are given recursively by

$$\text{(11.2.9a)} \qquad C_n = |w^n(0)| + \frac{1}{w(0)} \sum_{k=2}^{n-1} \binom{n}{k} C_k C_{n-k+1} \quad , \quad 2 \leq n \leq N$$

$$\text{(11.2.9b)} \qquad D_n = |w^n(0)| + \frac{1}{|w(0)|(n-2)} \sum_{k=2}^{n-1} \binom{n}{k} D_k D_{n-k+1} \quad , \quad 2 \leq n \leq N$$

and $P_{n-2}(t)$ is a polynomial of degree $n - 2$ which vanishes for $t = 0$.

Remarks.

1. Throughout this section we shall use the notations C, C_n, D_n etc. to denote constants which do not depend on t, and P_n to denote polynomials of degree n. Note that these notations can stand for different constants or polynomials in different occurrences.

2. Equality (11.2.7) allows us to rewrite (11.2.8a-b) as

$$(11.2.10) \qquad |w^n(t)| \leq \begin{cases} \tilde{C}_n w(t)^{n+1} & w(0) > 0 \\ \\ \tilde{D}_n |w(t)|^{2n-1} & w(0) < 0 \end{cases} \quad , \qquad t \geq 0 \quad ,$$

where the constants $\tilde{C}_n$ and $\tilde{D}_n$,

$$(11.2.11) \qquad \tilde{C}_n = \frac{C_n}{w(0)^{n+1}} \quad , \quad \tilde{D}_n = \frac{D_n}{|w(0)|^{2n-1}} \quad ,$$

depend solely on the initial condition.

Proof. Equation (11.2.5) may be written for $n \geq 2$ as follows:

$$(11.2.12a) \qquad \frac{dw^n(t)}{dt} = -(n+1)w(t)w^n(t) + q_n(t) \quad ,$$

$$(11.2.12b) \qquad q_n(t) := -\sum_{k=2}^{n-1} \binom{n}{k} w^k(t)w^{n-k+1}(t) \quad .$$

Using (11.2.7), the solution of (11.2.12a) is

$$(11.2.13) \qquad w^n(t) = (1 + w(0)t)^{-n-1} \left[w^n(0) + \int_0^t (1 + w(0)\tau)^{n+1} q_n(\tau)d\tau \right] \quad .$$

We prove (11.2.8) by induction. The case $n = 2$ is immediate since $q_2 = 0$ and therefore, by (11.2.13),

$$(11.2.14) \qquad w^2(t) = (1 + w(0)t)^{-3}w^2(0) \quad .$$

Hence (11.2.8) is proved for $n = 2$ with $C_2 = D_2 = |w^2(0)|$ (in agreement with (11.2.9)) and $P_0(t) \equiv 0$.

We turn now to the proof of (11.2.8) for $2 < n \leq N$, assuming it holds for all $2 \leq k < n$. The proof is separated for three cases according to the sign of $w(0)$.

If $w(0) > 0$ then by (11.2.12b) and induction we get that

$$(11.2.15) \qquad |q_n(t)| \leq \sum_{k=2}^{n-1} \binom{n}{k} |w^k(t)||w^{n-k+1}(t)| \leq$$

$$\sum_{k=2}^{n-1} \binom{n}{k} C_k (1 + w(0)t)^{-k-1} C_{n-k+1} (1 + w(0)t)^{-n+k-2} = \sum_{k=2}^{n-1} \binom{n}{k} C_k C_{n-k+1} (1 + w(0)t)^{-n-3} \quad .$$

Therefore, by (11.2.13) and (11.2.15),

$$(11.2.16) \quad |w^n(t)| \leq (1 + w(0)t)^{-n-1} \left[|w^n(0)| + \sum_{k=2}^{n-1} \binom{n}{k} C_k C_{n-k+1} \int_0^t (1 + w(0)\tau)^{-2}d\tau \right] \quad .$$

Evaluating the integral in (11.2.16) proves (11.2.8a) and (11.2.9a).

Similarly, if $w(0) < 0$ then

$$(11.2.17) \qquad |q_n(t)| \le \sum_{k=2}^{n-1} \binom{n}{k} |w^k(t)||w^{n-k+1}(t)| \le$$

$$\sum_{k=2}^{n-1} \binom{n}{k} C_k (1 + w(0)t)^{-2k+1} C_{n-k+1}(1 + w(0)t)^{-2n+2k-1} = \sum_{k=2}^{n-1} \binom{n}{k} C_k C_{n-k+1}(1 + w(0)t)^{-2n} \quad .$$

Hence, by (11.2.13) and (11.2.17),

$$(11.2.18) \quad |w^n(t)| \le (1 + w(0)t)^{-n-1} \left[|w^n(0)| + \sum_{k=2}^{n-1} \binom{n}{k} C_k C_{n-k+1} \int_0^t (1 + w(0)\tau)^{-n+1} d\tau \right]$$

and (11.2.8b), (11.2.9b) follow by evaluating the integral in (11.2.18).

Finally, if $w(0) = 0$, (11.2.13) implies that

$$(11.2.19a) \qquad w^n(t) = w^n(0) + \int_0^t q_n(\tau) d\tau \quad .$$

But, by (11.2.12b) and the induction assumption,

$$(11.2.19b) \qquad q_n(t) = -\sum_{k=2}^{n-1} \binom{n}{k} (w^k(0) + P_{k-2}(t))(w^{n-k+1}(0) + P_{n-k-1}(t)) = P_{n-3}(t) \quad .$$

Therefore, $\int_0^t q_n(\tau) d\tau$ is a polynomial of degree $n-2$ which vanishes for $t = 0$, hence (11.2.8c) is proved, and that concludes the proof. $\qquad\square$

Example. The estimates offered by Proposition 11.1 are sharp, as demonstrated by Burgers' equation, $u_t + u u_x = 0$, subject to initial condition

$$u(x, 0) = u_0(x) = \begin{cases} \frac{x^2 - 1}{2} & -1 < x < 1 \\ 0 & \text{elsewhere} \end{cases} \quad .$$

Its solution along characteristics $x(t)$ for which $-1 < x(0) < 1$ is given by

$$u(x(t), t) = \frac{1 + x(t)t - \sqrt{1 + 2x(t)t + t^2}}{t^2} \quad .$$

Therefore, for $n \ge 2$ we get that

$$(11.2.20a) \qquad w^n(t) = \frac{\partial^n u(x(t), t)}{\partial x^n} = (-1)^n C_n (1 + 2x(t)t + t^2)^{-n+\frac{1}{2}} t^{n-2} \quad ,$$

where

$$(11.2.20b) \qquad C_n = \prod_{k=1}^{n-1}(2k-1) \ .$$

Let $x(t)$ be the characteristic which starts at $x_0 \in (-1, 1)$. Its speed is $u_0(x_0) = \frac{1}{2}(x_0^2 - 1)$ and therefore

$$(11.2.21) \qquad x(t) = x_0 + \frac{1}{2}(x_0^2 - 1)t \ .$$

For that characteristic $w(0) = u_0'(x_0) = x_0$ and therefore , by (11.2.21),

$$(11.2.22) \qquad (1 + 2x(t)t + t^2)^{\frac{1}{2}} = 1 + x_0 t = 1 + w(0)t \ .$$

Using (11.2.22) in (11.2.20a) gives :

$$(11.2.23) \qquad w^n(t) = (-1)^n C_n (1 + w(0)t)^{-2n+1} t^{n-2} \ .$$

If $x_0 > 0$ then $w(0) > 0$ and therefore, for $t \gg w(0)^{-1}$,

$$|w^n(t)| = C_n(1 + w(0)t)^{-2n+1} t^{n-2} \approx C_n(1 + w(0)t)^{-2n+1}\left(\frac{1 + w(0)t}{w(0)}\right)^{n-2} =$$

$$= \frac{C_n}{w(0)^{n-2}}(1 + w(0)t)^{-n-1} \ .$$

If $x_0 < 0$ then $w(0) < 0$ and the characteristic will not exist beyond the critical time $t_c = 1/|w(0)|$. Therefore, by (11.2.23), when $t \to t_c$

$$|w^n(t)| = C_n(1 + w(0)t)^{-2n+1} t^{n-2} \approx \frac{C_n}{|w(0)|^{n-2}}(1 + w(0)t)^{-2n+1} \ .$$

If $x_0 = 0$ then $w(0) = 0$ and therefore $w^n(t) = (-1)^n C_n t^{n-2}$. Since $w^2(0) = 1$ and $w^n(0) = 0$ for $n > 2$, (11.2.8c) is met with $P_0(t) \equiv 0$ and $P_{n-2}(t) = (-1)^n C_n t^{n-2}$ for $n > 2$.

After establishing estimates for $w^n = \partial_x^n a(u)$ we are ready to translate them into analogous estimates for $\partial_x^n u$. For that matter we observe that w^n has the form (successive chain rule)

$$(11.2.24a) \qquad w^n = \partial_x^n a(u) = a'(u)\partial_x^n u + \sum_i K_i a^{(m_i)}(u) \prod_{j=1}^{m_i} \partial_x^{r_j^i} u \ ,$$

where K_i are positive integer coefficients and

$$(11.2.24b) \qquad m_i \geq 2 \quad ; \quad 1 \leq r_j^i \leq n + 1 - m_i \quad ; \quad \sum_{j=1}^{m_i} r_j^i = n \quad .$$

We denote

$$(11.2.25) \qquad M := \max_{2 \le n \le N} ||a^{(n)}(u)||_{L_\infty} = \max_{2 \le n \le N} ||a^{(n)}(u_0)||_{L_\infty} \quad .$$

With (11.2.24) and (11.2.25) we get, using convexity, (4.1), that for $n \le N$

$$(11.2.26) \qquad |\partial_x^n u| \le \frac{1}{\alpha} \left(|w^n| + \sum_i K_i M \prod_{j=1}^{m_i} |\partial_x^{r_j^i} u| \right) \quad .$$

Note that for Burgers' equation $\alpha = 1$ and $M = 0$ and (11.2.26) holds with an equality.

If we now denote $\partial_x^n u(t) := \partial_x^n u(x(t), t)$, where $x(t)$ is a characteristic curve, we may state the analogous of Proposition 11.1.

Theorem 11.1. *For every $1 \le n \le N$ and $t \ge 0$ there holds*

$$(11.2.27a) \qquad |\partial_x^n u(t)| \le C_n (1 + w(0)t)^{-n} \qquad\qquad \text{if } \partial_x u(t) > 0 \;\; ;$$

$$(11.2.27b) \qquad |\partial_x^n u(t)| \le D_n (1 + w(0)t)^{-2n+1} \qquad\qquad \text{if } \partial_x u(t) < 0 \;\; ;$$

$$(11.2.27c) \qquad \partial_x^n u(t) = \partial_x^n u(0) + P_{n-2}(t) \qquad\qquad \text{if } \partial_x u(t) = 0 \;\; .$$

Here, C_n and D_n are constants which depend on the initial condition and $P_{n-2}(t)$ is a polynomial of degree $n - 2$ which vanishes for $t = 0$.

Proof. Since u remains constant along its characteristics, (11.2.7) implies that

$$(11.2.28) \qquad \partial_x u(t) = \frac{\partial_x u(0)}{1 + w(0)t} \quad .$$

Hence, (11.2.27) holds for $n = 1$ with $C_1 = D_1 = |\partial_x u(0)|$ and $P_{-1}(t) \equiv 0$. (11.2.28) implies (due to convexity) that $\partial_x u(t), \partial_x u(0)$ and $w(0)$ have the same sign.

As for $n \ge 2$, we proceed by induction.

If $\partial_x u(t) > 0$, (11.2.26) and (11.2.8a), together with the induction assumption, imply that

$$(11.2.29) \qquad |\partial_x^n u| \le \frac{1}{\alpha} \left(C_n (1 + w(0)t)^{-n-1} + \sum_i K_i M \prod_{j=1}^{m_i} C_{r_j^i} (1 + w(0)t)^{-r_j^i} \right) \quad .$$

But, by (11.2.24b),

$$(11.2.30) \qquad \sum_i K_i M \prod_{j=1}^{m_i} C_{r_j^i} (1 + w(0)t)^{-r_j^i} = \sum_i C_i (1 + w(0)t)^{-n} = C_n (1 + w(0)t)^{-n} \quad .$$

Hence (11.2.27a) follows from (11.2.29) and (11.2.30).

Similarly, if $\partial_x u(t) < 0$ then (11.2.26), (11.2.8b) and induction imply that

$$(11.2.31) \qquad |\partial_x^n u| \leq \frac{1}{\alpha} \left(C_n (1 + w(0)t)^{-2n+1} + \sum_i K_i M \prod_{j=1}^{m_i} C_{r_j^i} (1 + w(0)t)^{-2r_j^i + 1} \right) \ .$$

Using (11.2.24b) we get that

$$\sum_i K_i M \prod_{j=1}^{m_i} C_{r_j^i} (1 + w(0)t)^{-2r_j^i + 1} = \sum_i C_i (1 + w(0)t)^{-2n + m_i} \ .$$

But $m_i \geq 2$ and, therefore, the first term on the right hand side of (11.2.31) is the dominant one as t tends to the critical time, $t_c = 1/|w(0)|$, hence (11.2.27b) follows.

As for the case $\partial_x u(t) = 0$, since u remains constant along $x(t)$, (11.2.24a) implies that

$$w^n(t) - w^n(0) = a'(u)(\partial_x^n u(t) - \partial_x^n u(0)) + \sum_i K_i a^{(m_i)}(u) \left[\prod_{j=1}^{m_i} \partial_x^{r_j^i} u(t) - \prod_{j=1}^{m_i} \partial_x^{r_j^i} u(0) \right] \ .$$

Using (11.2.8c) we therefore conclude that

$$\partial_x^n u(t) = \partial_x^n u(0) + \frac{1}{a'(u)} \left(P_{n-2}(t) - \sum_i K_i a^{(m_i)}(u) \left[\prod_{j=1}^{m_i} \partial_x^{r_j^i} u(t) - \prod_{j=1}^{m_i} \partial_x^{r_j^i} u(0) \right] \right) \ .$$

But since, by induction, the term in the brackets is a polynomial of degree $n - 2$ and it vanishes at $t = 0$, (11.2.27c) is proved and we are done. $\qquad \square$

Remarks.

1. We call attention that (11.2.27a) is slightly different from (11.2.8a). This difference in the exponent is the reason why (11.2.27a) holds for $n \geq 1$ while (11.2.8a) holds only for $n \geq 2$.

2. Equality (11.2.28) allows us to rewrite (11.2.27a-b) in the form announced in the introduction,

$$(11.2.32) \qquad |\partial_x^n u(t)| \leq \begin{cases} \tilde{C}_n (\partial_x u(t))^n & \partial_x u(t) > 0 \\ \tilde{D}_n |\partial_x u(t)|^{2n-1} & \partial_x u(t) < 0 \end{cases} \ , \qquad t \geq 0 \ ,$$

with constants

$$(11.2.33) \qquad \tilde{C}_n = \frac{C_n}{(\partial_x u(0))^n} \ , \qquad \tilde{D}_n = \frac{D_n}{|\partial_x u(0)|^{2n-1}} \ ,$$

which depend solely on the initial condition.

3. The large time behavior of the second spatial derivative in (planar) rarefaction waves has been studied before by Xin in [X]. Xin considered the scalar viscous conservation law

$$u_t + f(u)_x = \varepsilon u_{xx}$$

subject to the C^2-smooth and bounded initial condition, u_0, satisfying

(11.2.34) $$u_0' > 0$$

and

(11.2.35) $$|u_0''| \le k_0 u_0' \quad , \quad 0 \le k_0 = \text{Const} \ .$$

He showed that in that case there exists a positive constant K such that

(11.2.36) $$|u_{xx}(x,t)| \le K u_x(x,t) \qquad \forall (x,t) \in \Re \times \Re^+ \ .$$

This estimate can be recovered for the inviscid hyperbolic conservation law (4.1) from our analysis. Let us denote

(11.2.37) $$L^+ \equiv \max_{x,t} u_x(x,t) = \max_x u_0'(x) \ .$$

By (11.2.7) and (11.2.14) we get that

(11.2.38) $$w^2(t) = w(t) \frac{w^2(0)}{w(0)(1 + w(0)t)^2} \ .$$

Therefore, since by (11.2.34) and (4.1) $w(t) = a'(u)u_x > 0$, (11.2.38) implies that

(11.2.39) $$|w^2(t)| \le \frac{|w^2(0)|}{w(0)} w(t) \ .$$

As $w(0)$ and $w^2(0)$ are given by (consult (11.2.24a))

(11.2.40) $$w(0) = a'(u_0)u_0' \quad , \quad w^2(0) = a''(u_0)(u_0')^2 + a'(u_0)u_0'' \ ,$$

we get from (11.2.39) that

$$|w^2(t)| \le \left[\frac{|u_0''|}{u_0'} + \frac{|a''(u_0)|u_0'}{a'(u_0)} \right] w(t) \ .$$

Using (4.1), (11.2.25), (11.2.35) and (11.2.37), we conclude that

(11.2.41) $$|w^2(t)| \le K_1 w(t) \quad , \quad K_1 \equiv \left(k_0 + \frac{M L^+}{\alpha} \right) \ .$$

Thus, $v = a(u)$ satisfies inequality (11.2.36) since, by definition, $w(t) = v_x(x(t), t)$ and $w^2(t) = v_{xx}(x(t), t)$. The desired inequality for u easily follows from (4.1), (11.2.25), (11.2.37) and (11.2.41):

$$\frac{|u_{xx}|}{u_x} = \frac{|w^2 - a'' u_x^2|}{w} \leq \frac{|w^2|}{w} + \frac{|a''|}{a'} u_x \leq K \equiv K_1 + \frac{ML^+}{\alpha}$$

Note that (11.2.36) holds even if condition (11.2.34) is replaced by $u_0' \geq 0$, since along characteristics where $u_x = 0$, u_{xx} remains constant (by (11.2.27c)) which must be zero in view of restriction (11.2.35).

Theorem 11.1 tells us the behavior of the high order derivatives of the entropy solution along its characteristics, depending on the sign of the first derivative there: if the first derivative is positive, then according to (11.2.27a) the higher derivatives decay in time; if it is negative – the higher derivatives tend in absolute value, to infinity as the characteristic approaches the shock curve, (11.2.27b); and along characteristics where the first derivative is zero, the higher order derivatives experience a polynomial growth rate indicated in (11.2.27c). Furthermore, the rate of decay or growth increases with the order of the derivative.

11.3. High order piecewise stability estimates

The estimates obtained in §11.2, consult (11.2.10-11) and (11.2.32-33), show how the smoothness of the entropy solution depends on the *distance* from the set of shock discontinuities, where this distance is measured by the size of $\partial_x u(t)$. These estimates involve, apart from $\partial_x u(t)$, also the value of the first derivative of the initial condition, $\partial_x u(0)$.

We now turn to upper bound the higher order derivatives in regularity regions solely in terms of the local value of the first derivative, thus extending the special case of an estimate for the second spatial derivative of planar rarefaction waves in (11.2.36). Moreover, our bound will indicate the dependence of the high order regularity on the *distance* from the singular set of shocks. The distance from the singular set is measured by a lower bound of the first derivative. To quantify this dependence we define for every $L \leq 0$ the following seminorm:

$$(11.3.1) \qquad \|g(x)\|_{C_L^n} \equiv \sup_{x \in D_{g,L}} \left| \frac{d^n g}{dx^n} \right| \quad , \quad D_{g,L} \equiv \left\{ x : \frac{dg}{dx}(x) \geq L \right\} \ .$$

This is a localized version of the regular C^n (or $W^{n,\infty}$) seminorm which may be obtained from $\|\cdot\|_{C_L^n}$ by letting $L \to -\infty$.

We show that the solution operator of (4.1) is stable with respect to this seminorm. As before, we deal first with the "Burgerized" equation, (11.2.3), in the unknown $v = a(u)$.

Proposition 11.2. *For every* $2 \leq n \leq N$ *and* $L < 0$ *there holds*

(11.3.2) $$\|v(\cdot, t)\|_{C_L^n} \leq e^{(n+1)|L|t} \|v(\cdot, 0)\|_{C_L^n} + P_{n-2}(|L|^{-1}) e^{3(n-1)|L|t} \quad,$$

where the coefficients of P_{n-2} *depend on* $\{\|v(\cdot, 0)\|_{C_L^k}\}_{2 \leq k < n}$.

Proof. We recall equation (11.2.12a) which governs the evolution of $w^n(t)$ along a characteristic $(x(t), t)$. Let (x, t) be located on the characteristic $x = x(t)$ and assume that $x(t) \in D_{v(\cdot, t), L}$, i.e.,

(11.3.3) $$v_x(x(t), t) = w(t) \geq L \quad.$$

Since by (11.2.7) $w(t)$ can only decrease along a characteristic, (11.3.3) implies that

(11.3.4a) $$w(\tau) \geq L \quad, \quad 0 \leq \tau \leq t \quad,$$

or equivalently,

(11.3.4b) $$x(\tau) \in D_{v(\cdot, \tau), L} \quad, \quad 0 \leq \tau \leq t \quad.$$

The solution of (11.2.12a) is

(11.3.5) $$w^n(t) = e^{\int_0^t -(n+1)w(\tau)d\tau} w^n(0) + \int_0^t e^{\int_\tau^t -(n+1)w(s)ds} q_n(\tau)d\tau \quad.$$

Therefore, by (11.3.4a) and (11.3.5), we get that in $D_{v(\cdot, t), L}$

(11.3.6) $$|w^n(t)| \leq e^{-(n+1)Lt} \left[|w^n(0)| + \int_0^t e^{(n+1)L\tau} |q_n(\tau)| d\tau \right] \quad.$$

We start by dealing with $n = 2$. Here $q_2 = 0$ and (11.3.6) reads

$$|w^2(t)| \leq e^{3|L|t} |w^2(0)| \quad,$$

hence (11.3.2) follows with $P_0(|L|^{-1}) = 0$.

We proceed by induction assuming (11.3.2) holds for all $2 \leq k < n$,

(11.3.7) $$\|v(\cdot, t)\|_{C_L^k} \leq e^{(k+1)|L|t} \|v(\cdot, 0)\|_{C_L^k} + P_{k-2}(|L|^{-1}) e^{3(k-1)|L|t} \quad,$$

where the coefficients of $P_{k-2}(|L|^{-1})$ depend on $\{\|v(\cdot, 0)\|_{C_L^m}\}_{2 \leq m < k}$. Clearly, since for $k \geq 2$, $3(k-1) \geq k+1$, we may rewrite (11.3.7) as

(11.3.8) $$\|v(\cdot, t)\|_{C_L^k} \leq P_{k-2}(|L|^{-1}) e^{3(k-1)|L|t} \quad, \quad 2 \leq k < n \quad,$$

where the coefficients of $P_{k-2}(|L|^{-1})$ in (11.3.8) depend on $\{\|v(\cdot,0)\|_{C_L^m}\}_{2\le m\le k}$. Using (11.3.6), (11.2.12b) and (11.3.8) we arrive at

$$|w^n(t)| \le e^{-(n+1)Lt}\left[|w^n(0)| + \int_0^t e^{(n+1)L\tau}\sum_{k=2}^{n-1}\binom{n}{k}|w^k(\tau)||w^{n-k+1}(\tau)|d\tau\right] \le$$

$$e^{-(n+1)Lt}\left[|w^n(0)| + \sum_{k=2}^{n-1}\binom{n}{k}\int_0^t e^{(n+1)L\tau}P_{k-2}(|L|^{-1})e^{-3(k-1)L\tau}P_{n-k-1}(|L|^{-1})e^{-3(n-k)L\tau}d\tau\right] =$$

$$e^{-(n+1)Lt}\left[|w^n(0)| + \tilde{P}_{n-3}(|L|^{-1})\int_0^t e^{(-2n+4)L\tau}d\tau\right] \quad,$$

where $\tilde{P}_{n-3}(\cdot)$ abbreviates

$$\tilde{P}_{n-3}(|L|^{-1}) = \sum_{k=2}^{n-1}\binom{n}{k}P_{k-2}(|L|^{-1})P_{n-k-1}(|L|^{-1})$$

which depends on $\{\|v(\cdot,0)\|_{C_L^k}\}_{2\le k<n}$. Evaluating the last integral we arrive at

(11.3.9a) $\qquad |w^n(t)| \le e^{-(n+1)Lt}\left[|w^n(0)| + P_{n-2}(|L|^{-1})e^{(-2n+4)Lt} - P_{n-2}(|L|^{-1})\right] \quad,$

where

(11.3.9b) $\qquad P_{n-2}(|L|^{-1}) = \dfrac{1}{(-2n+4)L}\tilde{P}_{n-3}(|L|^{-1}) = \dfrac{1}{2(n-2)|L|}\tilde{P}_{n-3}(|L|^{-1}) \quad.$

Since $L < 0$ and $n > 2$, $P_{n-2}(|L|^{-1})$ is positive and, therefore, by (11.3.9a) we conclude that

(11.3.10) $\qquad |w^n(t)| \le e^{-(n+1)Lt}\left[|w^n(0)| + P_{n-2}(|L|^{-1})e^{(-2n+4)Lt}\right] =$

$$e^{(n+1)|L|t}|w^n(0)| + P_{n-2}(|L|^{-1})e^{3(n-1)|L|t}$$

which proves (11.3.2). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark. It can be easily shown, in the same manner, that for $L = 0$

$$\|v(\cdot,t)\|_{C_0^n} \le \|v(\cdot,0)\|_{C_0^n} + P_{n-2}(t) \quad,$$

where $P_{n-2}(t)$ depends on $\{\|v(\cdot,0)\|_{C_0^k}\}_{2\le k<n}$. This result is not surprising in view of (11.2.8a) and (11.2.8c).

Finally, we translate the estimates offered by Proposition 11.2 for $v = a(u)$, into analogous estimates for u itself.

Theorem 11.2. (Piecewise Stability). *For every $2 \le n \le N$ and $L < 0$ there holds*

(11.3.11) $\qquad \|u(\cdot,t)\|_{C_L^n} \le e^{(n+1)\hat{L}t}\|u(\cdot,0)\|_{C_L^n} + P_{n-2}(\hat{L}^{-1})e^{3(n-1)\hat{L}t} \quad,$

where $\hat{L} = A|L|$, $A = \|a'(u)\|_{L_\infty}$ and the coefficients of P_{n-2} depend on $\{\|u(\cdot,0)\|_{C_L^k}\}_{2\leq k<n}$.

Proof. The verification of (11.3.11) for $n = 2$ is left to the reader and we proceed by induction. Let (x,t) be a point on the characteristic $x = x(t)$ where $x(t) \in D_{u(\cdot,t),L}$. Therefore, the definition of $v = a(u)$ and the monotonicity of a imply that

$$(11.3.12) \qquad x(t) \in D_{v(\cdot,t),\tilde{L}} \qquad , \qquad \tilde{L} = a'(u(x(t),t))L \ .$$

Furthermore, by (11.2.7) and (11.2.28) we conclude that

$$(11.3.13) \qquad x(\tau) \in D_{u(\cdot,\tau),L} \bigcap D_{v(\cdot,\tau),\tilde{L}} \qquad 0 \leq \tau \leq t \ .$$

This, together with (11.3.12) and (11.3.10), imply that

$$(11.3.14) \qquad |w^n(t)| \leq e^{(n+1)|\tilde{L}|t}|w^n(0)| + P_{n-2}(|\tilde{L}|^{-1})e^{3(n-1)|\tilde{L}|t} \ .$$

Using (11.2.24a) we obtain

$$(11.3.15) \qquad |w^n(0)| \leq a'(u)|\partial_x^n u(0)| + C \ ,$$

where C depends on $\{|\partial_x^k u(0)|\}_{2\leq k<n}$. Therefore, since $n+1 \leq 3(n-1)$, we conclude from (11.3.14) and (11.3.15) that

$$(11.3.16) \qquad |w^n(t)| \leq a'(u)e^{(n+1)|\tilde{L}|t}|\partial_x^n u(0)| + P_{n-2}(|\tilde{L}|^{-1})e^{3(n-1)|\tilde{L}|t} \ .$$

Recalling (11.2.24) and (11.2.25), inequality (11.3.16) implies that

$$|\partial_x^n u(t)| \leq \frac{1}{a'(u)}\left(|w^n(t)| + \sum_i K_i M \prod_{j=1}^{m_i} |\partial_x^{r_j^i} u(t)|\right) \leq$$

$$\leq e^{(n+1)|\tilde{L}|t}|\partial_x^n u(0)| + P_{n-2}(|\tilde{L}|^{-1})e^{3(n-1)|\tilde{L}|t} + C\sum_i \prod_{j=1}^{m_i} |\partial_x^{r_j^i} u(t)| \ .$$

By induction we may conclude, as we did in the proof of Proposition 11.2, that

$$(11.3.17) \qquad |\partial_x^n u(t)| \leq e^{(n+1)|\tilde{L}|t}|\partial_x^n u(0)| + P_{n-2}(|\tilde{L}|^{-1})e^{3(n-1)|\tilde{L}|t} \ ,$$

and taking the supremum over $x(t) \in D_{u(\cdot,t),L}$ in (11.3.17) we arrive at (11.3.11). $\qquad \square$

Remarks.

1. In the case of Burgers' equation, $A = 1$ and, therefore, (11.3.11) reduces in that case to the stability estimate (11.3.2).

2. The analogous of (11.3.11) for $L = 0$ is

$$(11.3.18) \qquad \|u(\cdot,t)\|_{C_0^n} \leq \|u(\cdot,0)\|_{C_0^n} + P_{n-2}(t) \ ,$$

where $P_{n-2}(t)$ depends on $\{\|u(\cdot, 0)\|_{C_0^k}\}_{2 \leq k < n}$.

11.4. On the size of the set of shock discontinuities

We show in this section that, generically, the set of shocks is finite and identify the initial conditions for which an infinite number of shock curves is generated.

The first result concerning the size of the shock set was Oleinik's. She has shown (consult [O1]–[O3]) that the shock set is countable at the most. Her result, however, still allows a very complicated structure such as an everywhere dense shock set.

Two proceeding results have simplified the picture: Dafermos [D1] has shown that in case that both the (convex) flux and the initial condition are infinitely smooth, the solution is C^∞ a.e. apart from the shock set which must be closed. Thus, the shock set cannot be everywhere dense but shocks may still accumulate.

Schaeffer [S2] has proved that, generically, the shock set is finite when the initial condition is infinitely smooth. He has shown that if $f \in C^\infty$ there exists a subset, Ω, of the first category in Schwartz space, $S(\Re)$, such that if $u_0 \in S(\Re) \setminus \Omega$ then $u \in C^\infty(\Re \times (0, \infty) \setminus \Gamma)$ where Γ is a finite set of smooth shock curves. He furthermore gives an example of such an initial condition $u_0 \in \Omega$ which evolves , according to Burgers' equation, to an almost everywhere C^∞ function with infinitely many shock curves in a bounded region. However, we are left unable to check whether a given initial condition is in Ω or not.

It seems to be a part of the folklore that if u_0 has a finite number of inflection points, then the corresponding entropy solution of Burgers' equation experiences a finite number of shock discontinuities. In the general case, the function whose inflection points are to be examined is $a(u_0)$.

Theorem 11.3. *Let u be the entropy solution of the convex hyperbolic conservation law, (4.1), subject to the bounded and piecewise C^1 initial condition, u_0, satisfying*

$$(11.4.1) \qquad \lim_{|x| \to \infty} a(u_0)' = 0 \quad .$$

Then the number of disjoint shock curves equals the number of negative minima of $a(u_0)'$.

Remarks.

1. Since u_0 is assumed to be only piecewise C^1 it may be discontinuous and, therefore, will not have a classical derivative. Therefore, we refer by $a(u_0)'$ to the generalized derivative of $a(u_0)$. Hence, in decreasing discontinuities of u_0 , $a(u_0)'$ has a negative (infinite) minimum.

2. If $a(u_0)'$ has a continuum of negative minimal points , namely, $a(u_0)$ linearly decreases along some interval, it is considered as one minimum.

3. Shocks which occur as a consequence of an interaction of two (or more) other shocks, are not counted. We consider only "original" shocks. Obviously, the number of original shocks dominates the number of simultaneous shocks in every $t \geq 0$.

Corollary 11.1. *If $a(u_0)$ has a finite number of decreasing inflection points, then the set of shock discontinuities is finite.*

Theorem 11.3 implies that the set of functions $u_0 \in S(\Re)$ for which $a(u_0)'$ has infinitely many negative minima, is the set $\Omega \subset S(\Re)$ of the first category that Schaeffer refers to in [S2].

Proof. Denote the set of disjoint (original) shock curves by $S = \{X_i(t)\}_{i \in I}$ and the set of points where $a(u_0)'$ has a negative minimum by $M = \{x_j\}_{j \in J}$. We will establish an equivalence between these two sets in order to prove our statement.

For every $X_i(t) \in S$ let t_i^0 denote its creation time $(t_i^0 \geq 0)$ and t_i^∞ its termination time $(t_i^0 < t_i^\infty \leq \infty)$. t_i^∞ is finite if $X_i(t)$ collides with another shock, and infinite otherwise.

According to Lax entropy condition

$$(11.4.2) \qquad a(u(X_i(t_i)-, t_i)) > a(u(X_i(t_i)+, t_i)) \qquad , \qquad t_i^0 < t_i < t_i^\infty \ .$$

We choose one value of t_i in that time interval,

$$(11.4.3) \qquad t_i \in (t_i^0, t_i^\infty) \ ,$$

and denote by x_i^- and x_i^+ the two initial points of the characteristics which impinge upon the shock $X_i(t)$ from both sides at $t = t_i$. (11.4.2) implies that

$$(11.4.4) \qquad a(u_0(x_i^-)) > a(u_0(x_i^+)) \qquad , \qquad x_i^- < x_i^+ \ .$$

A consequence of (11.4.4) is that $a(u_0)'$ must become negative somewhere along the interval $[x_i^-, x_i^+]$. Let x_i denote the point in that interval where $a(u_0)'$ achieves its minimal value. The shock's creation time is determined by this minimum ,

$$(11.4.5) \qquad t_i^0 = -\frac{1}{a(u_0)'(x_i)} \ .$$

On the other hand

$$(11.4.6) \qquad -\frac{1}{a(u_0)'(x_i^\pm)} \geq t_i > t_i^0 \ ,$$

since otherwise , the characteristics which start at $(x_i^\pm, 0)$ would not have lasted until $t = t_i$. (11.4.5) and (11.4.6) imply that

$$(11.4.7) \qquad a(u_0)'(x_i) < a(u_0)'(x_i^\pm)$$

and, therefore, x_i is a negative local minimum of $a(u_0)'$, i.e., $x_i \in M$.

We have thus shown that to each $X_i(t) \in S$ corresponds a $x_i \in M$. This correspondence is one-to-one since if $X_i(t)$ and $X_j(t)$ are two disjoint shocks then our choice of t_i , (11.4.3), implies that

$$[x_i^-, x_i^+] \bigcap [x_j^-, x_j^+] = \emptyset$$

and therefore $x_i \neq x_j$.

Now we show an one-to-one correspondence from M to S to conclude the equivalence of the two sets. Let $x_1, x_2 \in M$ and let ξ be the point where $a(u_0)'$ reaches its maximal value in the interval $[x_1, x_2]$. Let $x_i(t)$ be the characteristic which starts at $(x_i, 0)$, $i = 1, 2$, and $\xi(t)$ be the one which starts at $(\xi, 0)$. The solution along $x_i(t)$ becomes discontinuous at time

$$(11.4.8) \qquad\qquad t_i = -\frac{1}{a(u_0(x_i))'} \qquad , \qquad i = 1, 2 \ .$$

Therefore, each of the points $(x_i(t_i), t_i)$, $i = 1, 2$, is on a shock. By Lagrange mean value theorem and since $a(u_0)'$ has local minima in x_i we conclude that

$$\frac{a(u_0(x_i)) - a(u_0(\xi))}{x_i - \xi} > a(u_0(x_i))' \qquad , \qquad i = 1, 2 \ .$$

or, by (11.4.8),

$$(11.4.9) \qquad\qquad -\frac{x_i - \xi}{a(u_0(x_i)) - a(u_0(\xi))} > t_i \qquad , \qquad i = 1, 2 \ .$$

Since the left hand side of (11.4.9) indicates the time when $x_i(t)$ and $\xi(t)$ were to meet, we conclude that $x_1(t_1) < \xi(t_1)$ and $x_2(t_2) > \xi(t_2)$. Therefore, the points $(x_i(t_i), t_i)$, $i = 1, 2$, lay on two different shocks, the first is on the left side of $\xi(t)$ and the second is on its right side.

Finally, we note that if $a(u_0)'$ has a continuum of negative minimal points, i.e., if $a(u_0)$ is linearly decreasing along some interval, $[x_1, x_2]$, this minimum creates only one shock since the characteristics from that interval will all meet at

$$t = -\frac{1}{a(u_0)'\big|_{[x_1, x_2]}}$$

to start that shock. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

12. HIGH ORDER REGULARITY FOR THE LAX-FRIEDRICHS SCHEME

In the last section we saw how the one-sided Lipschitz condition, (2.1.4) or (2.1.18), which identifies the entropy solutions of (4.1), is generalized to higher order regularity estimates; namely, we have shown that in rarefaction waves (the regions where $u_x \geq 0$), the high order spatial derivatives experience an algebraic growth rate,

$$(12.1) \qquad \left| \frac{\partial^n u}{\partial x^n} \right| \leq O(t^{n-2}) \quad , \quad n \geq 1 .$$

In this section we concentrate on the discrete analogous of (12.1) for the LxF scheme, given (in its staggered version) by

$$(12.2) \qquad v_{j+\frac{1}{2}}(t + \Delta t) = \frac{1}{2}\Big(v_j(t) + v_{j+1}(t)\Big) - \lambda\Big(f(v_{j+1}(t)) - f(v_j(t))\Big) ,$$

and subject to the initial condition

$$(12.3) \qquad v_j(0) = u_0(x_j) .$$

Here $v_j(t) \equiv v^h(x_j, t)$ denotes the approximation value at the grid point $(x_j \equiv jh, t)$. The mesh size and the time step, denoted respectively by h and Δt, are restricted by the CFL condition

$$(12.4) \qquad \lambda \|f'\|_\infty < \frac{1}{2} \quad , \quad \lambda = \frac{\Delta t}{h} ,$$

where $\|\cdot\|_\infty$ denotes henceforth the maximal absolute value in the interval $[\inf u_0 , \sup u_0]$.

Tadmor has studied the large time behavior of the (non-staggered) LxF scheme and showed [T1, Corollary 3.2] that it satisfies a discrete analogous of (2.1.18),

$$(12.5) \qquad D_1^+(t) \leq \frac{1}{(D_1^+(0))^{-1} + \alpha t} \quad , \quad D_1^+(t) \equiv \max_j \left(\frac{\Delta v_j(t)}{h} \right)^+ .$$

Here, Δ denotes the forward difference operator, $\Delta v_j = v_{j+1} - v_j$. Using this result we prove our large time behavior estimate, stating:

Theorem 12.1. *Let $v(x,t) = v^h(x,t)$ be an approximated solution of (4.1)–(4.2) obtained by the LxF scheme (12.2)–(12.3). Then if u_0 and f' are C^n-smooth, $n \geq 2$, and*

$$(12.6) \qquad u_0' \geq 0 ,$$

there exists a constant C_n, depending on α, u_0 and $\|f\|_{C^{n+1}} \equiv \max_{0 \leq k \leq n+1} \|f^{(k)}\|_\infty$, such that

$$(12.7) \qquad D_n(t) \equiv \max_j \left| \frac{\Delta^n v_j(t)}{h^n} \right| \leq P_{n-2}(t) \quad , \quad t \geq 0 ,$$

where $P_{n-2}(t)$ is some polynomial in t of degree $n-2$.

Remark. The notations $P_n(t)$, $Q_{n-3}(t)$, $Q_{-1}(t)$, etc. denote henceforth polynomials in t whose degree equals to their subscript (note that the same notation can stand for different polynomials in different occurrences). By a polynomial with negative degree we refer to a rational function having no singularities for $t \geq 0$ (such as the $P_{-1}(t)$ polynomial on the right hand side of inequality (12.5)).

Theorem 12.1 implies that the large time behavior of the LxF scheme in planar rarefaction waves (namely, solutions of (4.1) with monotonically increasing initial conditions; consult also [X]) agrees with that of the exact solution, (12.1), and the LxF solution operator is stable in planar rarefactions with respect to the discrete $W^{n,\infty}$-seminorm, given in (12.7).

We begin by stating the following straightforward discrete analogous of Leibnitz rule of differentiation.

Lemma 12.1. *Let $\{a_j\}$, $\{b_j\}$ be two sequences and let Δ denote the undivided forward difference operator. Then*

$$\Delta^n(a_j b_j) = \sum_{k=0}^{n} \binom{n}{k} \Delta^k a_j \Delta^{n-k} b_{j+k} .$$

Lemma 12.2. *If $\{a_j^1\}, \{a_j^2\}, ..., \{a_j^k\}$ are k sequences then*

$$\Delta^n(a_j^1 \cdot a_j^2 \cdot ... \cdot a_j^k) = \sum \binom{n}{n_1, n_2, \ldots, n_k} \Delta^{n_1} a_j^1 \cdot \Delta^{n_2} a_{j+n_1}^2 \cdot ... \cdot \Delta^{n_k} a_{j+n_1+...+n_{k-1}}^k ,$$

where the sum goes over all n-tuples of non-negative integers, $(n_1, n_2, ..., n_k)$, for which $\sum_{i=1}^{k} n_i = n$.

Equipped with these lemmas we may turn to the main result on which the proof of Theorem 12.1 relies.

Proposition 12.1. *Let $v^h(x,t)$ be a bounded h-grid function, with grid values $v_j(t)$, such that*

$$(12.8) \qquad \max_j \left| \frac{\Delta^m v_j(t)}{h^m} \right| \leq P_{m-2}(t) \quad , \quad \forall h > 0 , \; 1 \leq m \leq n-1 , \; n \geq 2 .$$

Then, for every C^{n+1}-smooth function, f, there exists a polynomial $Q(t)$ whose degree is given by

$$(12.9) \qquad \deg Q = \begin{cases} -3 & n = 2 \\ n-3 & n \geq 3 \end{cases}$$

and a constant K_n, depending solely on $\|f''\|_\infty$ and $P_{-1}(0)$, such that

$$(12.10) \qquad \Delta^{n+1} f(v_j) = f'(v_j)\Delta^{n+1} v_j + A_j^n(t)\Delta^n v_j + B_j^n(t)\Delta^n v_{j+1} + C_j^n(t) ,$$

where

$$(12.11) \qquad |A_j^n(t)| , \ |B_j^n(t)| \le K_n h$$

and

$$(12.12) \qquad |C_j^n(t)| \le h^{n+1} Q(t) .$$

Proof. We prove (12.9)–(12.12) by induction over n. The proof of the case $n = 2$ is postponed and we deal first with $n \ge 3$. Since f is assumed to be C^{n+1}-smooth, we may use its Taylor expansion around v_j and get

$$(12.13) \qquad \Delta^{n+1} f(v_j) = \Delta^n(\Delta f(v_j)) =$$

$$= \sum_{r=1}^n \frac{1}{r!}\Delta^n(f^{(r)}(v_j)\Delta v_j^r) + \Delta^n\left(f^{(n+1)}(w_{j+\frac{1}{2}})\frac{\Delta v_j^{n+1}}{(n+1)!}\right) ,$$

where $w_{j+\frac{1}{2}}$ is a mid-point between v_j and v_{j+1}. Using Lemma 12.1 for the nth-order difference in the sum in (12.13) we get

$$(12.14) \qquad \Delta^{n+1} f(v_j) =$$

$$= \sum_{r=1}^n \frac{1}{r!}\sum_{k=0}^n \binom{n}{k}\Delta^k f^{(r)}(v_j)\Delta^{n-k}(\Delta v_{j+k}^r) + \Delta^n\left(f^{(n+1)}(w_{j+\frac{1}{2}})\frac{\Delta v_j^{n+1}}{(n+1)!}\right) =$$

$$= \sum_{k=0}^n \sum_{r=1}^n T_{k,r} + \Delta^n\left(f^{(n+1)}(w_{j+\frac{1}{2}})\frac{\Delta v_j^{n+1}}{(n+1)!}\right) .$$

We shall now estimate the terms on the right of (12.14) in order to arrive at (12.9)–(12.12).

We start by dealing with the right most term in (12.14). By assumption (12.8) with $m = 1$, each term in that difference is bounded by

$$(12.15) \qquad \frac{\|f^{(n+1)}\|_\infty}{(n+1)!}h^{n+1}\big(P_{-1}(t)\big)^{n+1}$$

and hence

$$(12.16) \qquad \left|\Delta^n\left(f^{(n+1)}(w_{j+\frac{1}{2}})\frac{\Delta v_j^{n+1}}{(n+1)!}\right)\right| \le \frac{2^n\|f^{(n+1)}\|_\infty}{(n+1)!}h^{n+1}\big(P_{-1}(t)\big)^{n+1} = h^{n+1}Q_{-n-1}(t) .$$

It remains to take care of the terms of the sum in the right hand side of (12.14), $T_{k,r}$.

Clearly,

$$(12.17) \qquad\qquad T_{0,1} = f'(v_j)\Delta^{n+1}v_j$$

which is the first term on the right hand side of (12.10).

Next,

$$(12.18) \qquad\qquad T_{0,2} = \frac{1}{2}f''(v_j)\Delta^n(\Delta v_j \Delta v_j) \ .$$

Here, we decompose the nth-order difference in (12.18) according to Lemma 12.1 in order to get that (recall that we deal now with the case $n \geq 3$)

$$(12.19) \qquad T_{0,2} = \frac{1}{2}f''(v_j)\Big[(\Delta v_j + \Delta v_{j+n})\Delta^{n+1}v_j + n(\Delta^2 v_j \Delta^n v_{j+1} + \Delta^n v_j \Delta^2 v_{j+n-1})+$$

$$+ \sum_{\ell=2}^{n-2}\binom{n}{\ell}\Delta^{\ell+1}v_j\Delta^{n-\ell+1}v_{j+\ell}\Big] \ .$$

The terms on the right hand side of (12.19), apart from the first one, may be bounded, using (12.8), as follows:

$$(12.20) \qquad\qquad |\Delta^2 v_j\Delta^n v_{j+1} + \Delta^n v_j\Delta^2 v_{j+n-1}| \leq$$

$$|\Delta^2 v_j|\Big(|\Delta^{n-1}v_{j+2}| + |\Delta^{n-1}v_{j+1}|\Big) + |\Delta^2 v_{j+n-1}|\Big(|\Delta^{n-1}v_{j+1}| + |\Delta^{n-1}v_j|\Big)$$

$$\leq 4h^{n+1}P_0(t)P_{n-3}(t) \ ;$$

$$(12.21) \qquad\qquad |\Delta^{\ell+1}v_j\Delta^{n-\ell+1}v_{j+\ell}| \leq \Big(|\Delta^\ell v_{j+1}| + |\Delta^\ell v_j|\Big)|\Delta^{n-\ell+1}v_{j+\ell}|$$

$$\leq 2h^{n+1}P_{\ell-2}(t)P_{n-\ell-1}(t) \quad , \quad 2 \leq \ell \leq n-2 \ .$$

Therefore, combining (12.19), (12.20) and (12.21) we conclude that

$$(12.22) \quad T_{0,2} = f''(v_j)\frac{\Delta v_j + \Delta v_{j+n}}{2}(\Delta^n v_{j+1} - \Delta^n v_j) + R_j^n(t) \quad , \quad |R_j^n(t)| \leq h^{n+1}Q_{n-3}(t) \ .$$

To conclude our discussion with the case $k = 0$ we consider the terms

$$(12.23) \qquad T_{0,r} = \frac{1}{r!}f^{(r)}(v_j)\Delta^n(\Delta v_j^r) = \frac{1}{r!}f^{(r)}(v_j)\Delta^{r-1}\Delta^{n-r+1}(\Delta v_j^r) \quad , \quad 3 \leq r \leq n \ .$$

The $(n-r+1)$-order difference may be decomposed according to Lemma 12.2 into

$$(12.24) \quad \Delta^{n-r+1}(\Delta v_j^r) = \sum \binom{n-r+1}{n_1, n_2, \ldots, n_r}\Delta^{n_1+1}v_j \cdot \Delta^{n_2+1}v_{j+n_1} \cdot \ldots \cdot \Delta^{n_r+1}v_{j+n_1+\ldots+n_{r-1}} \ ,$$

where the sum goes over all r-tupples, $(n_1, n_2, ..., n_r)$ such that

$$(12.25) \qquad\qquad n_1 + n_2 + \ldots + n_r = n - r + 1 \quad , \quad n_i \geq 0 \ , \ 1 \leq i \leq r \ .$$

Identity (12.24), combined with (12.25) and (12.8), implies the existence of a polynomial $Q_{n-2r+1}(t)$ for which

(12.26) $$|\Delta^{n-r+1}(\Delta v_j^r)| \leq h^{n+1} Q_{n-2r+1}(t) \quad , \quad 3 \leq r \leq n .$$

Therefore, by (12.23) and (12.26)

(12.27) $$\left| \sum_{r=3}^{n} T_{0,r} \right| \leq h^{n+1} Q_{n-5}(t)$$

for some polynomial $Q_{n-5}(t)$ of degree $n-5$. This concludes our treatment of the case $k=0$.

We proceed with the terms with $k=1$. The first term is

(12.28) $$T_{1,1} = n\Delta f'(v_j) \Delta^n v_{j+1} ,$$

while the rest of the terms are

(12.29) $$T_{1,r} = \frac{n}{r!} \Delta f^{(r)}(v_j) \Delta^{n-1}(\Delta v_{j+1}^r) \quad , \quad 2 \leq r \leq n .$$

We appeal again to Lemma 12.2 by which

(12.30) $$\Delta^{n-1}(\Delta v_{j+1}^r) = \Delta^n v_{j+1} \cdot \sum_{\ell=0}^{r-1} \Delta v_{j+1}^\ell \Delta v_{j+n}^{r-1-\ell} +$$

$$+ \sum \binom{n-1}{n_1, n_2, \ldots, n_r} \Delta^{n_1+1} v_j \cdot \Delta^{n_2+1} v_{j+n_1} \cdot \ldots \cdot \Delta^{n_r+1} v_{j+n_1+\ldots+n_{r-1}} ,$$

where

(12.31) $$n_1 + n_2 + \ldots + n_r = n - 1 \quad , \quad 0 \leq n_i \leq n - 2 , \ 1 \leq i \leq r .$$

Assumption (12.8) (with $m=1$ and $m=n-1$) enables us to bound the first term on the right of (12.30),

(12.32) $$\left| \Delta^n v_{j+1} \cdot \sum_{\ell=0}^{r-1} \Delta v_{j+1}^\ell \Delta v_{j+n}^{r-1-\ell} \right| \leq$$

$$\left(|\Delta^{n-1} v_{j+2}| + |\Delta^{n-1} v_{j+1}| \right) \cdot \sum_{\ell=0}^{r-1} |\Delta v_{j+1}|^\ell |\Delta v_{j+n}|^{r-1-\ell} \leq$$

$$2h^{n-1} P_{n-3}(t) \cdot r h^{r-1} \left(P_{-1}(t) \right)^{r-1} = h^{n+r-2} Q_{n-r-2}(t) .$$

The second sum in (12.30) may be bounded similarly, using (12.31), by $h^{n+r-1} Q_{n-r-1}(t)$. Hence, as $r \geq 2$, we conclude from (12.30) and (12.32) that

(12.33) $$|\Delta^{n-1}(\Delta v_{j+1}^r)| \leq h^n Q_{n-3}(t) .$$

In addition, the smoothness of f and (12.8) imply that

(12.34) $$|\Delta f^{(r)}(v_j)| \le \|f^{(r+1)}\|_\infty h P_{-1}(t) .$$

Combining (12.29), (12.33) and (12.34) we conclude that

(12.35) $$\left| \sum_{r=2}^{n} T_{1,r} \right| \le h^{n+1} Q_{n-4}(t) .$$

We now turn to deal with the rest of the terms in the sum in (12.14),

(12.36) $$T_{k,r} = \frac{1}{r!} \binom{n}{k} \Delta^k f^{(r)}(v_j) \Delta^{n-k}(\Delta v_{j+k}^r) \quad , \quad 2 \le k \le n , \ 1 \le r \le n .$$

Using Lemma 12.2 and assumption (12.8) in a similar fashion as before, we get that

(12.37) $$|\Delta^{n-k}(\Delta v_{j+k}^r)| \le h^{n-k+r} Q_{n-k-r}(t) .$$

If $r \ge k + 1$, (12.37) implies that

(12.38) $$|\Delta^{n-k}(\Delta v_{j+k}^r)| \le h^{n+1} Q_{n-2k-1}(t)$$

and therefore, by (12.36) and (12.38),

(12.39) $$\left| \sum_{k=2}^{n} \sum_{r=k+1}^{n} T_{k,r} \right| \le h^{n+1} Q_{n-5}(t) .$$

The complement case, namely $1 \le r \le k$, needs a more delicate treatment. We observe that in that case, the kth-order difference in (12.36) may be written as

(12.40) $$\Delta^k f^{(r)}(v_j) = \Delta^{r-1} \Delta^{k-r+1} f^{(r)}(v_j) .$$

As $k - r + 1 < n + 1$ and $f^{(r)} \in C^{k-r+1}$ (since $f \in C^{n+1} \subset C^{k+1}$) we may use the induction assumption for $k-r$. Namely, by (12.9) and assumption (12.8) for the lower order differences,

(12.41) $$\Delta^{k-r+1} f^{(r)}(v_j) = f^{(r+1)}(v_j) \Delta^{k-r+1} v_j + R_j^n(t) \quad , \quad |R_j^n(t)| \le h^{k-r+1} Q_{k-r-3}(t) .$$

If $k - r + 1 < n$, i.e., if $(k,r) \ne (n,1)$, we may use (12.8) to bound also the first term on the right of (12.41) and get, using also (12.40), that

(12.42) $$|\Delta^k f^{(r)}(v_j)| \le h^{k-r+1} Q_{k-r-3}(t) \quad , \quad 1 \le r \le k \le n , \ (k,r) \ne (n,1) .$$

If $(k,r) = (n,1)$ then (12.8) does not apply to $\Delta^{k-r+1} v_j = \Delta^n v_j$ and, therefore, we have by (12.40) and (12.41) that

(12.43) $$\Delta^n f'(v_j) = f''(v_j) \Delta^n v_j + R_j^n(t), \quad |(R_j^n(t)| \le h^n Q_{n-4}(t) .$$

Therefore, (12.36), (12.37) and (12.42) imply that

$$(12.44) \qquad |T_{k,r}| \le h^{n+1} Q_{n-2r-3}(t), \quad 1 \le r \le k \le n, \quad (k,r) \ne (n,1),$$

while, by (12.43) and (12.8) with $m = 1$,

$$(12.45) \quad T_{n,1} = \Delta^n f'(v_j) \cdot \Delta v_{j+n} = f''(v_j) \Delta^n v_j \Delta v_{j+n} + R_j^n(t), \quad |R_j^n(t)| \le h^{n+1} Q_{n-5}(t) \ .$$

Estimates (12.44) and (12.45) may be now combined in order to get that

$$(12.46) \qquad \sum_{k=2}^{n} \sum_{r=1}^{k} T_{k,r} = f''(v_j) \Delta^n v_j \Delta v_{j+n} + R_j^n(t), \quad |R_j^n(t)| \le h^{n+1} Q_{n-5}(t) \ .$$

We have thus evaluated or estimated all the terms in the right hand side of (12.14). (12.9)–(12.12) now follow from identity (12.14), together with (12.16), (12.17), (12.22), (12.27), (12.28), (12.35), (12.39) and (12.46). The coefficients of the nth-order differences, $A_j^n(t)$ and $B_j^n(t)$, are equal, according to (12.22), (12.28) and (12.46) to

$$A_j^n(t) = f''(v_j) \frac{\Delta v_{j+n} - \Delta v_j}{2} \quad , \quad B_j^n(t) = n\Delta f'(v_j) + f''(v_j) \frac{\Delta v_j + \Delta v_{j+n}}{2}$$

and may be bounded, using (12.8) (with $m = 1$), by

$$|A_j^n(t)| \le h \|f''\|_\infty P_{-1}(t) \le h \|f''\|_\infty P_{-1}(0) \ ,$$

$$|B_j^n(t)| \le h(n+1) \|f''\|_\infty P_{-1}(t) \le h(n+1) \|f''\|_\infty P_{-1}(0) \ .$$

We now return to prove (12.9)–(12.12) for $n = 2$. The proof for $n \ge 3$ applies, with slight modifications, also to $n = 2$. The only terms which call for special attention in the case $n = 2$ are $T_{0,2}$ (in whose estimation we relied on the assumption that $n \ge 3$ by using (12.8) for $m = 2$; consult (12.20)) and $T_{k,r}$ for $2 \le k \le n$, $1 \le r \le k$ (in the estimate of which we used induction).

$T_{0,2}$, given in (12.18), equals when $n = 2$ to

$$(12.47) \qquad T_{0,2} = \frac{1}{2} f''(v_j) \Delta^2 (\Delta v_j \Delta v_j) =$$

$$= f''(v_j) \left[\frac{\Delta v_{j+1} + \Delta v_{j+2}}{2} \Delta^2 v_{j+1} - \frac{\Delta v_j + \Delta v_{j+1}}{2} \Delta^2 v_j \right] \ ;$$

both coefficients of $\Delta^2 v_j$ and $\Delta^2 v_{j+1}$ in that expression satisfy (12.11).

As for the case $2 \le k \le n$, $1 \le r \le k$ — there are only two such terms when $n = 2$,

$$(12.48) \qquad\qquad T_{2,1} = \Delta^2 f'(v_j) \Delta v_{j+2}$$

and

(12.49)
$$T_{2,2} = \frac{1}{2}\Delta^2 f''(v_j)\Delta v_{j+2}^2 .$$

A linear approximation of f' around v_j and Lemma 12.1 give that

(12.50)
$$\Delta^2 f'(v_j) = \Delta\Big(f''(v_j)\Delta v_j + \frac{1}{2}f^{(3)}(w_{j+\frac{1}{2}})\Delta v_j^2\Big) =$$

$$= f''(v_j)\Delta^2 v_j + \Delta f''(v_j)\Delta v_j + \frac{1}{2}\Delta\Big(f^{(3)}(w_{j+\frac{1}{2}})\Delta v_j^2\Big) = f''(v_j)\Delta^2 v_j + R_j(t) \quad,$$

$$|R_j(t)| \le 2h^2\|f^{(3)}\|_\infty\big(P_{-1}(t)\big)^2 \quad,$$

where $v_j \le w_{j+\frac{1}{2}} \le v_{j+1}$. With (12.48) and (12.50) we conclude that

(12.51)
$$T_{2,1} = f''(v_j)\Delta v_{j+2}\Delta^2 v_j + R_j(t) ,$$

$$|R_j(t)| \le 2h^3\|f^{(3)}\|_\infty\big(P_{-1}(t)\big)^3 = h^3 Q_{-3}(t) .$$

As for $T_{2,2}$,

(12.52)
$$|T_{2,2}| = \frac{1}{2}\Big|\Delta^2 f''(v_j)\Big|\Delta v_{j+2}^2 = \frac{1}{2}\Big|\Delta(f^{(3)}(w_j)\Delta v_j)\Big|\Delta v_{j+2}^2$$

$$\le \frac{1}{2}\|f^{(3)}\|_\infty\big(|\Delta v_j| + |\Delta v_{j+1}|\big)\Delta v_{j+2}^2 \le h^3\|f^{(3)}\|_\infty\big(P_{-1}(t)\big)^3 = h^3 Q_{-3}(t) .$$

We further note that (12.35) may be improved for $n = 2$ since in that case the second sum on the right hand side of (12.30) is empty. This results in the fact that when $n = 2$

(12.53)
$$|T_{1,2}| \le h^3 Q_{-3}(t) .$$

Finally, (12.9)–(12.12) follow for $n = 2$ from (12.16), (12.17), (12.28), (12.47), (12.48), (12.51), (12.52) and (12.53) (note that (12.27) and (12.39) are irrelevant when $n = 2$), and the proof is therefore completed. □

With Proposition 12.1 and the large time behavior estimate (12.5) we may now prove the main theorem:

Proof of Theorem 12.1. Δ^n-differencing of equation (12.2) yields

(12.54)
$$\Delta^n v_{j+\frac{1}{2}}(t + \Delta t) = \frac{1}{2}\Big(\Delta^n v_j(t) + \Delta^n v_{j+1}(t)\Big) - \lambda\Delta^{n+1} f(v_j(t)) \quad, \quad t \ge 0 .$$

We deal first with the case $n = 2$. In view of (12.6), $\Delta v_j(t) \ge 0$ for all j and t, and therefore (12.5) reads

(12.55)
$$D_1(t) \le P_{-1}(t) .$$

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Hence, we may use Proposition 12.1, with $n = 2$, and get that

(12.56)
$$\Delta^3 f(v_j(t)) =$$

$$f'(v_j(t))(\Delta^2 v_{j+1}(t) - \Delta^2 v_j(t)) + O(h)\Delta^2 v_j(t) + O(h)\Delta^2 v_{j+1}(t) + C_j^2(t) \,,$$

where

(12.57)
$$|C_j^2(t)| \leq h^3 Q_{-3}(t) \,.$$

Using (12.56) in (12.54) we arrive at

(12.58)
$$\Delta^2 v_{j+\frac{1}{2}}(t + \Delta t) =$$

$$\left(\frac{1}{2} - \lambda f'(v_j(t)) + O(h)\right)\Delta^2 v_j(t) + \left(\frac{1}{2} + \lambda f'(v_j(t)) + O(h)\right)\Delta^2 v_{j+1}(t) + C_j^2(t) \quad, \quad t \geq 0 \,.$$

The CFL condition (12.4) implies that, for sufficiently small h, the two coefficients of $\Delta^2 v_j$ and $\Delta^2 v_{j+1}$ in (12.58) are non-negative and add up to no more than 1. Then, dividing (12.58) by h^2 and taking the maximum over j yields

(12.59)
$$D_2(t + \Delta t) \leq D_2(t) + hQ_{-3}(t) \quad, \quad t \geq 0 \,.$$

The solution of (12.59) gives that

$$D_2(t) \leq D_2(0) + \int_0^t Q_{-3}(\tau)d\tau \leq C_0 \equiv P_0(t) \,,$$

which proves (12.7) for $n = 2$.

We may proceed by induction for $n > 2$ and assume that (12.7) holds for differences of order $2 \leq m \leq n - 1$. Hence, we may use Proposition 12.1 for the $(n+1)$-order difference in (12.54) and get, similarly, that

(12.60)
$$\Delta^n v_{j+\frac{1}{2}}(t + \Delta t) =$$

$$\left(\frac{1}{2} - \lambda f'(v_j(t)) + O(h)\right)\Delta^n v_j(t) + \left(\frac{1}{2} + \lambda f'(v_j(t)) + O(h)\right)\Delta^n v_{j+1}(t) + C_j^n(t) \quad, \quad t \geq 0 \,,$$

where

(12.61)
$$|C_j^n(t)| \leq h^{n+1} Q_{n-3}(t) \,.$$

As before, we conclude that

(12.62)
$$D_n(t + \Delta t) \leq D_n(t) + hQ_{n-3}(t) \quad, \quad t \geq 0 \,.$$

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Finally, the solution of (12.62) gives

$$D_n(t) \leq P_{n-2}(t) = D_n(0) + \int_0^t Q_{n-3}(\tau)d\tau \ ,$$

which proves (12.7).   □

Remarks.

1. This theorem shows that the solution operator of the genuinely nonlinear LxF scheme is stable with respect to the discrete $W^{n,\infty}$-seminorm in the case of planar rarefaction waves.

2. Unlike the differential case this kind of stability does not hold for general smooth initial conditions. The exact solution operator of (4.1) is stable for such initial conditions with respect to a local $W^{n,\infty}$-seminorm, in the sense that within rarefaction waves (namely, whenever $u_x \geq 0$) inequality (12.1) holds (consult [TT, (3.1)] for a definition of that local seminorm). The difference between the differential case and the discrete one lies in the domain of dependence: while in the differential case the domain of dependence of each point $(x,t) \in \Re \times \Re^+$ is a single point on the initial line $\Re \times \{0\}$, the domain of dependence of a point $(x_j = jh, t^n = n\Delta t)$ in LxF scheme is the interval $[x_j - \frac{t^n}{2\lambda}, \ x_j + \frac{t^n}{2\lambda}]$. Therefore – information of shock waves and rarefaction waves, contained in the initial condition, mixes up and one cannot distinguish the domain of rarefaction in the solution. This explains why (12.7) holds only in planar rarefaction waves where the initial condition is monotonically increasing and the solution, therefore, consists of a pure rarefaction.

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