A Theory of Bayesian Decision Making

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Abstract

This paper presents a complete, choice-based, axiomatic Bayesian decision theory. It introduces a new choice set consisting of information-contingent plans for choosing actions and bets and subjective expected utility model with effect-dependent utility functions and action-dependent subjective probabilities which, in conjunction with the updating of the probabilities using Bayes' rule, gives rise to a unique prior and a set of action-dependent posterior probabilities representing the decision maker's prior and posterior beliefs.

Keywords: Bayesian decision making, subjective probabilities, prior distributions, beliefs

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1 Introduction

A choice-based theory of Bayesian decision making blends five key ideas. First, the patterns revealed by choice are the sole evidence by which the underlying theoretical concepts may be refuted.[1] Second, the decision-maker's evaluation of the objects of choice – payoffs contingent on the realization of events – reflects his tastes as well as his beliefs regarding the likelihoods of the relevant events. Third, the decision maker's beliefs, both prior and posterior, are measurable cognitive phenomena representable by probabilities. Forth, new information affects the decision maker's preferences, or choice behavior, through its effect on the decision maker's beliefs rather than his tastes. Fifth, the posterior probabilities representing the decision maker's posterior beliefs are obtained by updating the prior probabilities representing his prior beliefs using Bayes' rule. By themselves these ideas do not imply that Bayesian decision makers are expected utility maximizers.

In the wake of the seminal work of Savage (1954), it is commonplace to depict the choice set as the set of all mappings from a set of states, representing the resolutions of uncertainty, to a set of consequences. The objects of choice have the interpretation of alternative courses of action and are referred to as acts. The two most commonly used specifications of the choice set in the literature are those of Savage (1954), in whose formulation the set of states is infinite and the set of consequences arbitrary, and Anscombe and Aumann (1963), in whose specification the set of states is finite and the set of consequences are lotteries with

[1] This is an application of the revealed-preference methodology.

finite sets of arbitrary prizes.

The literature abounds with axiomatic theories specifying preference relations on these choice sets whose representations involve unique subjective probabilities, interpreted as the Bayesian prior.[2] However, in all the models that invoke Savage's analytical framework, the uniqueness of the prior probabilities is due to the use of a convention maintaining that *constant acts are constant-utility acts*. This convention lacks choice-theoretic meaning and as a result is not refutable in the context of the revealed-preference methodology.

To grasp this claim, let S the set of states, C be the set of consequences and F the set of acts. Decision makers are characterized by their preference relations on F. In Savage's subjective expected utility theory, the structure of a preference relation, $\succcurlyeq$, on F, depicted axiomatically, allows its representation by an expected utility functional, that is, for all $f \in F$,

$$f \mapsto \int_S u\left(f(s)\right) d\pi\left(s\right), \tag{1}$$

where u is a real-valued (utility) function defined on the consequences and π is a finitely additive, nonatomic probability measure on S. Moreover, the utility function u is unique up to positive linear transformation, and, given u, the subjective probability measure π is unique.

The uniqueness of π, however, is predicated on the implicit assumption that constant

[2] Prominent among these theories are the expected utility models of Savage (1954), Anscombe and Aumann (1963), and Wakker (1989), as well as the probability sophisticated choice models of Machina and Schmeidler (1992, 1995).

acts are constant utility acts (that is, the utility function is state-independent). As already mentioned, this assumption is not implied by the axioms and is, therefore, devoid of behavioral content. In fact, there are infinitely many prior probability measures consistent with a decision maker's prior preferences. Put differently, even if a decision maker's beliefs constitute a psychological phenomenon quantifiable by a probability measure and his choice behavior is consistent with the axiomatic structure of expected utility theory, the proposition that the subjective probabilities ascribed to him by Savage's model represent the decision maker's beliefs is untestable. To prove this assertion, let γ be a strictly positive, bounded, real-valued function on S, and let $E(\gamma) = \int_S \gamma(s)\, d\pi(s)$. Then the prior preference relation, depicted by the representation (1), is also represented by

$$f \mapsto \int_S \hat{u}(f(s), s)\, d\hat{\pi}(s), \tag{2}$$

where $\hat{u}(\cdot, s) = u(\cdot)/\gamma(s)$ and $\hat{\pi}(s) = \pi(s)\gamma(s)/E(\gamma)$, for all $s \in S$.[3]

The fact that the uniqueness of the subjective probabilities in Savage's theory, and in other theories that invoke Savage's (1954) analytical framework, is not a choice-based property of the model means that these subjective probabilities do not constitute a behavioral foundations of Bayesian statistics.

The popularity of Savage's notion of subjective probabilities among economists and de-

[3] This point has been recognized by Drèze (1987); Schervish, Seidenfeldt, and Kadane (1990); Karni (1996, 2003); Karni and Schmeidler (1993); and Nau (1995). Note also that, even if a decision maker is Bayesian (that is, updates his preferences using Bayes' rule), neither his prior nor his posterior beliefs, as defined by the representing probabilities, are unique.

cision theorists is due, in part, to the elegance of the representation it affords, both in terms of its mathematical formulation and the linguistic ease of describing its ingredients.[4] However, this elegance comes at a cost. To attain the separation of subjective probabilities from utilities it is necessary to assume that the preference relation exhibits state independence, which entails substantial loss of generality and limits range of applications of the model. For instance, Anscombe and Aumann (1963) impose state-independence to decompose the terms of a separately additive representation into a product of utility and probability.[5] The imposition of substantive restrictions to attain mathematical elegance is not good scholarship. Furthermore, it has recently been shown, in the context of insurance in the presence of moral hazard, that even the representation of the agent's prior preferences is correct, failure to ascribe to him his true prior probabilities and utilities, may result in attributing to the agent the wrong posterior preferences. In such case, when new information necessitates changing the insurance policy, the insurer may offer the agent a policy that is individually rational and yet incentive incompatible.[6] A more meaningful notion of subjective probability, one that is a measurement of subjective beliefs when these beliefs have structure that allows their representation by probability measure, is developed in this paper.

Building on Karni (2006), this paper introduces a new analytical framework that in-

[4]Quite often in the literature, the term subjective probability is used interchangeably with the term beliefs. Yet, as just demonstarted, this usage is hardly justified if not outright misleading. It does, however, serves the purpose of lending the theory with intuitive meaning that readers apparently find compelling.

[5]The analogous axioms in Savage's model are P3 and P4 (see Hill (2008)).

[6]See Karni (2008).

cludes actions, effects, bets and a set of observations, or signals, that the decision maker may receive before choosing his action and bet. Bets are real-valued mapping on the set of effects and strategies are maps from the set of observations to the set of action-bet pairs. Decision makers in this model are characterized by preference relations on the set of all strategies. The axiomatic structure of these preference relations lends the notion of constant utility bets choice-theoretic meaning. In other words, unlike models that use Savage's choice set, in this model constant utility bets leave their unique signature, or trace, in the pattern of choice permitting their identification. This makes it possible to define unique family of action-dependent, joint subjective probability distributions on the product set of effects and observations. Moreover, the prior probabilities are the unconditional marginal probabilities on the set of effects and the posterior probabilities are the distributions on the effects conditional on the observations. To my knowledge, this is the only complete choice-based Bayesian decision theory available.

The model presented here accommodates effect-dependent preferences, lending itself to natural interpretations in the context of medical decision making and the analysis of life insurance, health insurance, as well as standard portfolio and property insurance problems. The fact that the probabilities are action dependent means that the model furnishes an axiomatic foundation for the behavior of the principal and agent depicted in the parametrized distribution formulation of agency theory introduced by Mirrlees (1974, 1976).

The pioneering attempt to extend the subjective expected utility model to include moral hazard and state-dependent preferences is due to Drèze (1961,1987). Invoking the analytical framework of Anscombe and Aumann (1963), he departed from their "reversal of order" axiom, assuming instead that decision makers may strictly prefer knowing the outcome of a lottery before the state of nature becomes known. According to Drèze, this suggests that the decision maker believes that he can influence the probabilities of the states. How this influence is produced is not made explicit. The representation entails the maximization of subjective expected utility over a convex set of subjective probability measures.[7]

Section 2 introduces the theory and the main representation theorem. Section 3 discusses dynamic consistency and the uniqueness of the probabilities. Concluding remarks appear in section 4. Section 5 contains the proof of the main theorem.

2 The Theory

2.1 The analytical framework

Let Θ be a finite set of *effects*, X a finite set of *observations* or *signals,* and A a connected separable topological space whose elements are referred to as *actions.* Actions correspond to initiatives (e.g., time and effort) that decision makers may take to influence the likely realization of effects.

[7] The model in this paper differs from that of Drèze in several important respects, including the specification of the means by which a decision maker thinks he may influence the likelihood of the alternative effects. For more deatils see Karni (2006).

A *bet* is a real-valued mapping on Θ interpreted as monetary payoffs contingent on the realization of the effects. Let B denote the set of all bets on Θ and assume that it is endowed with the $\mathbb{R}^{|\Theta|}$ topology. Denote by $(b_{-\theta}r)$ the bet obtained from $b \in B$ by replacing the θ coordinate of b (that is, $b(\theta)$) with r. Effects are analogous to Savage's (1954) states in the sense that they resolve the uncertainty associated with the payoff of the bets. Unlike states, however, the likely realization of effects may conceivably be affected by the decision maker's actions.[8]

Observations may be obtained before the choice of bets and actions, in which case they affect these choices. For example, Upon obtaining the result of a blood test indicating the cholesterol level in his blood, a decision maker may adopt an exercise and diet regimen to reduce the risk of heart attack and, at the same time, take out health insurance and life insurance policies. In this instance the possible blood-test results correspond to observations, the diet and exercise regimens correspond to actions, the states of health are effects, and the financial terms of an insurance policy constitute a bet on Θ.[9] Another example concerns a mutation in the genes BRCA1 or BRCA2. Women with this mutation are at significantly increased risk of developing breast and ovarian cancers. Preventive (prophylactic) oophorectomy (or the surgical removal of the ovaries) can be performed to reduce both risks. Women can undergo genetic testing to determine whether they have this genetic mutation

[8] It is sufficient, for my purpose, that the decision maker believes that he may affect the likely realization of the effects by his choice of action.

[9] Clearly, the information afforded by the new observation is conditioned by the existing regimen. The decision problem is how to modify the existing regimen in light of the new information.

and, as a result, are predisposed to developing breast cancer and ovarian cancer and use this information to decide whether or not to undergo prophylactic oophorectomy. In this case the probability of the existence of the mutation (the signal) is independent of the action (the prophylactic oophorectomy). Yet undergoing surgery affects the probability of developing breast or ovarian cancer.

To model this "dynamic" aspect of the decision making process, I assume that a decision maker formulates a strategy, or contingent plan, specifying the action-bet pairs to be implemented contingent on the observations. Formally, denote by o the event "no new information" and let $\bar{X} = X \cup \{o\}$, then *strategy* is a function $I : \bar{X} \to A \times B$ that has the interpretation of a set of instructions specifying, for each $x \in \bar{X}$, an action-bet pair to be implemented if x is observed.[10] Let $\mathcal{I}$ be the set of all strategies.

A decision maker is characterized by a preference relation $\succcurlyeq$ on $\mathcal{I}$. The strict preference relation, $\succ$, and the indifference relation, $\sim$, are the asymmetric and symmetric parts of $\succcurlyeq$, respectively. Denote by $I_{-x}(a,b) \in \mathcal{I}$ the strategy in which the x coordinate of I is replaced by (a,b). An observation, x, is *essential* if $I_{-x}(a,b) \succ I_{-x}(a',b')$ for some $(a,b),(a',b') \in A \times B$ and $I \in \mathcal{I}$. I assume throughout that all elements of $\bar{X}$ are essential.

In the terminology of Savage (1954), $\bar{X}$ may be interpreted as a set of states and contingent plans as acts. However, because the decision maker's beliefs about the likelihoods of the

[10] Alternatively stated, o is a non-informative observation (that is, anticipating the representaion below, the subjective probability distribution on of the effects conditional on o is the same as that under the current information).

effects depend on both the actions and the observations, the preferences on action-bet pairs are inherently observation dependent. Thus applying Savage's state-independent axioms, P3 and P4, to $\succcurlyeq$ on $\mathcal{I}$, makes no sense.

2.2 Axioms and additive representation of $\succcurlyeq$ on $\mathcal{I}$

The first axiom is standard:

(A.1) (**Weak order**) $\succcurlyeq$ *is a complete and transitive binary relation.*

A topology on $\mathcal{I}$ is needed to define continuity of the preference relation $\succcurlyeq$. Recall that $\mathcal{I} = (A \times B)^{\bar{X}}$ and let $\mathcal{I}$ be endowed with the product topology.[11]

(A.2) (**Continuity**) *For all* $I \in \mathcal{I}$, *the sets* $\{I' \in \mathcal{I} \mid I' \succcurlyeq I\}$ *and* $\{I' \in \mathcal{I} \mid I \succcurlyeq I'\}$ *are closed.*

The next axiom, coordinate independence, is analogous to but weaker than Savage's (1954) sure thing principle.[12]

(A.3) (**Coordinate independence**) *For all* $x \in \bar{X}$, $I, I' \in \mathcal{I}$, *and* $(a,b), (a',b') \in A \times B$, $I_{-x}(a,b) \succcurlyeq I'_{-x}(a,b)$ *if and only if* $I_{-x}(a',b') \succcurlyeq I'_{-x}(a',b')$.

[11] That is, the topology on $\mathcal{I}$ is the product topology on the Cartesian product $(A \times B)^{|\bar{X}|}$.

[12] See Wakker (1989) for details.

An array of real-valued functions $(v_s)_{s\in S}$ is said to be a *jointly cardinal additive representation* for a binary relation $\succeq$ on a product set $D = \Pi_{s\in S} D_s$ if, for all $d, d' \in D$, $d \succeq d'$ if and only if $\sum_{s\in S} v_s(d_s) \geq \sum_{s\in S} v_s(d'_s)$, and the class of all functions that constitute an additive representation of $\succeq$ consists of those arrays of functions, $(\hat{v}_s)_{s\in S}$, for which $\hat{v}_s = \eta v_s + \zeta_s$, $\eta > 0$ for all $s \in S$. The representation is continuous if the functions $v_s, s \in S$ are continuous.

The following theorem is an application of Theorem III.4.1 in Wakker (1989):[13]

Theorem 1 *Let $\mathcal{I}$ be endowed with the product topology and $\mid \bar{X} \mid \geq 3$. Then a preference relation $\succcurlyeq$ on $\mathcal{I}$ satisfies (A.1)–(A.3) if and only if there exist an array of real-valued functions $\{w(\cdot,\cdot,x) \mid x \in \bar{X}\}$ on $A \times B$ that constitute a jointly cardinal continuous additive representation for $\succcurlyeq$.*

2.3 Independent betting preferences

For every given $x \in \bar{X}$, denote by $\succcurlyeq^x$ the induced preference relation on $A \times B$ defined by $(a,b) \succcurlyeq^x (a',b')$ if and only if $I_{-x}(a,b) \succcurlyeq I_{-x}(a',b')$. The induced strict preference relation, denoted by $\succ^x$, and the induced indifference relation, denoted by $\sim^x$, are the asymmetric and symmetric parts of $\succcurlyeq^x$, respectively.[14] The induced preference relation $\succcurlyeq^o$ is referred to as the *prior* preference relation; the preference relations $\succcurlyeq^x, x \in X$, are the *posterior* preference

[13] To simplify the exposition I state the theorem for the case in which $\bar{X}$ contains at least three essential coordinates. Additive representation when there are only two essential coordinates requires the imposition of the hexagon condition (see Wakker [1989] theorem III.4.1).

[14] For preference relations satisfaying (A.1) - (A.3), these relation are well-defined.

relations. For each $a \in A$ the preference relation $\succcurlyeq^x$ induces a conditional preference relation on B defined as follows: for all $b, b' \in B$, $b \succcurlyeq_a^x b'$ if and only if $(a, b) \succcurlyeq^x (a, b')$. The asymmetric and symmetric part of $\succcurlyeq_a^x$ are denoted by $\succ_a^x$ and $\sim_a^x$, respectively.

An effect, θ, is said to be *nonnull given the observation-action pair* (x, a) if $(b_{-\theta} r) \succ_a^x (b_{-\theta} r')$, for some $b \in B$ and $r, r' \in \mathbb{R}$; it is *null given the observation-action pair* (x, a) otherwise. Given a preference relation, $\succcurlyeq$, denote by $\Theta(a, x)$ the subset of effects that are nonnull given the observation-action pair (x, a). Assume that $\Theta(a, o) = \Theta$, for all $a \in A$.

Two effects, θ and θ', are said to be *elementarily linked* if there are actions $a, a' \in A$ and observations $x, x' \in \bar{X}$ such that $\theta, \theta' \in \Theta(a, x) \cap \Theta(a', x')$. Two effects are said to be *linked* if there exists a sequence of effects $\theta = \theta_0, ..., \theta_n = \theta'$ such that θ_j and θ_{j+1} are elementarily linked, $j = 0, ..., n-1$. The set of effects, Θ, is linked if every pair of its elements is linked.

The next axiom requires that the "intensity of preferences" for monetary payoffs contingent on any given effect be independent of the action and the observation:

(A.4) (**Independent betting preferences**) *For all* $(a, x), (a', x') \in A \times \bar{X}$, $b, b', b'', b''' \in B$, $\theta \in \Theta(a, x) \cap \Theta(a', x')$, *and* $r, r', r'', r''' \in \mathbb{R}$, *if* $(b_{-\theta} r) \succcurlyeq_a^x (b'_{-\theta} r')$, $(b'_{-\theta} r'') \succcurlyeq_a^x (b_{-\theta} r''')$, *and* $(b''_{-\theta} r') \succcurlyeq_{a'}^{x'} (b'''_{-\theta} r)$ *then* $(b''_{-\theta} r'') \succcurlyeq_{a'}^{x'} (b'''_{-\theta} r''')$.

To grasp the meaning of independent betting preferences, think of the preferences $(b_{-\theta}, r) \succcurlyeq_a^x (b'_{-\theta}, r')$ and $(b'_{-\theta}, r'') \succcurlyeq_a^x (b_{-\theta}, r''')$ as indicating that given the action a, the observation x, and the effect θ, the intensity of the preferences of r'' over r''' is sufficiently larger than that of r over r' as to reverse the preference ordering of the effect-contingent payoffs $b_{-\theta}$ and $b'_{-\theta}$. The axiom requires that these intensities not be contradicted when the action is a' instead of a and the observation is x' instead of x.

The idea may be easier to grasp by considering a specific instance in which $(b_{-\theta}, r) \sim_a^x (b'_{-\theta}, r')$, $(b_{-\theta} r'') \sim_a^x (b'_{-\theta} r''')$ and $(b''_{-\theta} r') \sim_{a'}^{x'} (b'''_{-\theta} r)$. The first pair of indifferences indicates that, given a and x, the difference in the payoffs b and b' contingent on the effects other than θ measures the intensity of preferences between the payoffs r and r' and between r'' and r''', contingent on θ. The indifference $(b''_{-\theta} r') \sim_{a'}^{x'} (b'''_{-\theta} r)$ then indicates that given another action-observation pair, a' and x', the intensity of preferences between the payoffs r and r' contingent on θ is measured by the difference in the payoffs the bets b'' and b''' contingent on the effects other than θ. The axiom requires that, in this case, the difference in the payoffs b'' and b''' contingent on the effects other than θ is also a measure of the intensity of the payoffs r'' and r''' contingent on θ. Thus the intensity of preferences between two payoffs given θ is independent of the actions and the observations.

2.4 Belief consistency

To link the decision maker's prior and posterior probabilities, the next axiom asserts that for every given $a \in A$ and $\theta \in \Theta$, the prior probability of θ given a is the sum over X of the joint probability distribution on $X \times \Theta$ conditional on θ and a (that is, the prior is the marginal probability on Θ).

Let $I^{-o}(a,b)$ denote the strategy that assigns the action-bet pair (a,b) to every observation other than o (that is, $I^{-o}(a,b)$ is a strategy such that $I(x)=(a,b)$ for all $x \in X$).

(A.5) (**Belief consistency**) *For every $a \in A$, $I \in \mathcal{I}$ and $b, b' \in B$, $I_{-o}(a,b) \sim I_{-o}(a,b')$ if and only if $I^{-o}(a,b) \sim I^{-o}(a,b')$.*

The interpretation of axiom (A.5) is as follows. The decision maker is indifferent between two strategies that agree on X and, in the event that no new information becomes available, call for the implementation of the alternative action-bet pairs (a,b) or (a,b') if and only if he is indifferent between two strategies that agree on o and call for the implementation of the same action-bet pairs (a,b) or (a,b') regardless of the observation. Put differently, given any action, the preferences on bets conditional on there being no new information is the same as that when new information may not be used to select the bet. Hence, in and of itself, information is worthless.

2.5 Constant utility bets

Constant utility bets are bets whose payoffs offset the direct impact of the effects. Formally

Definition 2 *A bet $\bar{b} \in B$ is a constant utility bet according to $\succcurlyeq$ if, for all $I, I', I'', I''' \in \mathcal{I}$, $a, a', a'', a''' \in A$ and $x, x' \in \bar{X}$, $I_{-x}(a,\bar{b}) \sim I'_{-x}(a',\bar{b})$, $I_{-x}(a'',\bar{b}) \sim I'_{-x}(a''',\bar{b})$ and $I''_{-x'}(a,\bar{b}) \sim I'''_{-x'}(a',\bar{b})$ imply $I''_{-x'}(a'',\bar{b}) \sim I'''_{-x'}(a''',\bar{b})$ and $\cap_{(x,a)\in X\times A}\{b \in B \mid b \sim_a^x \bar{b}\} = \{\bar{b}\}$.*

To render the definition meaningful it is assumed that, given $\bar{b}$, for all $a, a', a'', a''' \in A$ and $x, x' \in \bar{X}$ there are $I, I', I'', I''' \in \mathcal{I}$ such that the indifferences $I_{-x}\left(a, \bar{b}\right) \sim I'_{-x}\left(a', \bar{b}\right)$, $I_{-x}\left(a'', \bar{b}\right) \sim I'_{-x}\left(a''', \bar{b}\right)$ and $I''_{-x'}\left(a, \bar{b}\right) \sim I'''_{-x'}\left(a', \bar{b}\right)$ hold.

As in the interpretation of axiom (A.4), to understand the definition of constant utility bets it is useful to think of the preferences $I_{-x}\left(a, \bar{b}\right) \sim I'_{-x}\left(a', \bar{b}\right)$ and $I_{-x}\left(a'', \bar{b}\right) \sim I'_{-x}\left(a''', \bar{b}\right)$ as indicating that, given $\bar{b}$ and x, the preferential difference between the substrategies I_{-x} and I'_{-x} measure the intensity of preference of a over a' and that of a'' over a'''. The indifference $I''_{-x'}\left(a, \bar{b}\right) \sim I'''_{-x'}\left(a', \bar{b}\right)$ implies that, given $\bar{b}$, and another observation x', the preferential difference between the substrategies $I''_{-x'}$ and $I'''_{-x'}$ is another measure the intensity of preference of a over a'. Then it must be true that it also measure the intensity of preference of a'' over a'''. The requirement that $\cap_{(x,a)\in X\times A}\{b \in B \mid b \sim_a^x \bar{b}\} = \{\bar{b}\}$ implicitly asserts that the set of actions and observations is sufficiently rich so that $\bar{b}$ is well-defined.[15]

To understand why this implies that $\bar{b}$ is a constant utility bet recall that, in general, actions affect decision makers in two ways: directly through their utility cost and indirectly by altering the probabilities of the effects. Moreover, only the indirect impact depends on the observations. The definition indicates that, given $\bar{b}$, the intensity of the preferences over the actions is observations independent. This means that the indirect influence of the actions is neutralized, which can happen only if the utility associated with $\bar{b}$ is invariable across the

[15] Anticipating the main result, the assumed richness of the set of observations and acts is such that $\bar{b}$ the unique element in its equivalence class in B whose conditional expected utility is invariante with respect to observations and actions.

effects.

Let $B^{cu}(\succcurlyeq)$ be a subset of all constant utility bets according to $\succcurlyeq$. In general, this set may be empty. This is the case if the range of the utility of the monetary payoffs across effects do not overlap. Here I am concerned with the case in which $B^{cu}(\succcurlyeq)$ is nonempty. The set $B^{cu}(\succcurlyeq)$ is said to be *inclusive* if for every $(x,a) \in X \times A$ and $b \in B$ there is $\bar{b} \in B^{cu}(\succcurlyeq)$ such that $b \sim_a^x \bar{b}$.[16]

The next axiom requires that the trade-offs between the actions and the substrategies that figure in definition 2 are independent of the constant utility bets.

(A.6) (**Trade-off independence**) For all $I, I' \in \mathcal{I}$, $x \in \bar{X}$, $a, a' \in A$ and $\bar{b}, \bar{b}' \in B^{cu}(\succcurlyeq)$, $I_{-x}\left(a, \bar{b}\right) \succcurlyeq I'_{-x}\left(a', \bar{b}\right)$ if and only if $I_{-x}\left(a, \bar{b}'\right) \succcurlyeq I'_{-x}\left(a', \bar{b}'\right)$.

Finally, it is also required that the direct effect (that is, cost) of actions, measured by the preferential difference between $\bar{b}$ and $\bar{b}'$ in B^{cu}, be independent of the observation.

(A.7) (**Conditional monotonicity**) For all $\bar{b}, \bar{b}' \in B^{cu}(\succcurlyeq)$, $x, x' \in \bar{X}$, and $a, a' \in A$, $\left(a, \bar{b}\right) \succcurlyeq^x \left(a', \bar{b}'\right)$ if and only if $\left(a, \bar{b}\right) \succcurlyeq^{x'} \left(a', \bar{b}'\right)$.

[16] Inclusiveness of $B^{cu}(\succcurlyeq)$ simplifies the exposition. For existence and uniquness of the probabilities in the main result below it is enough that for every given x and a, $B^{cu}(\succcurlyeq)$ contains at least two bets.

2.6 The main representation theorem

For each $I \in \mathcal{I}$ let $\left(a_{I(x)}, b_{I(x)}\right)$ denote the action-bet pair corresponding to the x coordinate of I – that is, $I(x) = \left(a_{I(x)}, b_{I(x)}\right)$.

Theorem 3 *Let $\succcurlyeq$ be a preference relation on $\mathcal{I}$ and suppose that $B^{cu}(\succcurlyeq)$ is inclusive. Then $\succcurlyeq$ satisfies (A.1)–(A.7) if and only if there exist array of continuous, jointly cardinal, real-valued, functions $\{u(\cdot,\theta) \mid \theta \in \Theta\}$ on $\mathbb{R}$, $v \in \mathbb{R}^A$ and, a for every $a \in A$, there is a unique joint probability measure $\pi(\cdot,\cdot \mid a)$ on $\bar{X} \times \Theta$ such that $\succcurlyeq$ on $\mathcal{I}$ is represented by*

$$I \mapsto \sum_{x \in \bar{X}} \mu(x) \left[\sum_{\theta \in \Theta} \pi\left(\theta \mid x, a_{I(x)}\right) u\left(b_{I(x)}(\theta), \theta\right) + v\left(a_{I(x)}\right) \right] \tag{3}$$

where $\mu(x) = \sum_{\theta\in\Theta} \pi(x,\theta \mid a)$ for all $x \in \bar{X}$ is independent of a, $\pi(\theta \mid x, a) = \pi(x,\theta \mid a)/\mu(x)$ for all $x \in X$ and for each $a \in A$, $\pi(\theta \mid o, a) = \frac{1}{1-\mu(o)} \sum_{x \in X} \pi(x,\theta \mid a)$. Moreover, for every $\bar{b} \in B^{cu}(\succcurlyeq)$, $u\left(\bar{b}(\theta), \theta\right) = u\left(\bar{b}\right)$ for all $\theta \in \Theta$.

Although the joint probability distributions $\pi(\cdot,\cdot \mid a), a \in A$ depend on the actions, the distribution μ is independent of a. This is consistent with the formulation of the decision problem according to which the choice of actions is contingent on the observations. In other words, if new information in the form of an observation becomes available, it precedes the choice of action. Consequently, the dependence of the joint probability distributions $\pi(\cdot,\cdot \mid a)$ on a captures solely the decision maker's beliefs about his ability to influence the likelihood of the effects by his choice of action.[17]

[17] If an action-effect pair are already "in effect" when new information arrives, they constitute a default course of action. In such instnace, the interpretation of the decision at hand is possible choice of new action

3 Dynamic Consistency and Uniqueness of the Probabilities

3.1 Conditional preferences and dynamic consistency

The specification of the decision problem implies that, before the decision maker chooses an action-bet pair, either no informative signal arrives (that is, the observation is o) or new informative signal arrives in the form of an observation $x \in X$. One way or another, given the information at his disposal, the decision maker must choose among action-bet pairs. Let $\left(\hat{\succcurlyeq}^{x}\right)_{x\in\bar{X}}$ be binary relations on $A \times B$ depicting the decision maker's choice behavior conditional on observing x. I refer to $\left(\hat{\succcurlyeq}^{x}\right)_{x\in\bar{X}}$ by the name *ex-post preference relations.*

Dynamic consistency requires that at each $x \in \bar{X}$, the decision maker implements his plan of action envisioned for that contingency by the original strategy. Formally,

Definition 4 *A preference relation $\succcurlyeq$ on $\mathcal{I}$ is dynamically consistent with the ex-post preference relations $\left(\hat{\succcurlyeq}^{x}\right)_{x\in\bar{X}}$ on $A \times B$ if the posterior preference relations $(\succcurlyeq^{x})_{x\in\bar{X}}$ satisfy $\succcurlyeq^{x} = \hat{\succcurlyeq}^{x}$ for all $x \in \bar{X}$.*

The following is an immediate implication of Theorem 3.

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and bet. For example, a modification of the diet regiment coupled with a possible change of life insurance policy.

Corollary 5 *Let $\succcurlyeq$ be preference relation on $\mathcal{I}$ satisfying (A.1)–(A.7) and suppose that $B^{cu}(\succcurlyeq)$ is inclusive. Then $\succcurlyeq$ is dynamically consistent with the ex-post preference relations $\left(\hat{\succcurlyeq}^{x}\right)_{x\in\bar{X}}$ on $A \times B$ if and only if, for all $x \in \bar{X}$, $\hat{\succcurlyeq}^{x}$ is represented by*

$$(a,b) \mapsto \sum_{\theta\in\Theta} \pi(\theta \mid x,a)\, u(b(\theta),\theta) + v(a), \tag{4}$$

where $\{u(\cdot,\theta) \mid \theta \in \Theta\}$ and $\{\pi(\cdot \mid x,a) \mid x \in \bar{X}, a \in A\}$ are the utility functions and conditional subjective probabilities that appear in the representation (3).

For every $a \in A$ the subjective action-contingent prior on Θ is $\pi(\cdot \mid o,a)$ and the subjective action-contingent posteriors on Θ are $\pi(\cdot \mid x,a)$, $x \in X$. The subjective action-dependent prior is the marginal distribution on Θ induced by the distribution on $X \times \Theta$, and the subjective action-dependent posteriors are obtained from the action-contingent joint distribution on $X \times \Theta$ by conditioning on the observation.

3.2 Bayesian decision makers and the uniqueness of the subjective probabilities

A Bayesian decision maker is characterized by a prior preference relation and corresponding prior subjective probabilities, a set of posterior preference relations and corresponding posterior probabilities, and the condition that the posterior preferences are obtained from the prior preference solely by updating his subjective prior probabilities using Bayes' rule. It is worth underscoring that the transition from the prior preference relation to the posterior preference relations is affected by a change in the probabilities, leaving intact the other functions (e.g., the utility functions) that figure in the representation. This conception of Bayesianism captures the more general notion that a preference relation incorporates the decision maker's beliefs, which are represented by the subjective probabilities, and tastes, which are represented by a utility function, and that *information affects the decision maker's beliefs but not his tastes.* In other words, observations affect the probabilities but not the utilities and, consequently, the expected utility associated with constant utility bets must be observation independent. Clearly, the preference relation $\succcurlyeq$ on $\mathcal{I}$ depicted in Theorem 3 is Bayesian. The question is, are there utility functions outside the class of positive linear transformation of the utility function in Theorem 3 and corresponding joint probability distributions on $\bar{X} \times \Theta$ that represent the the preference relation in the sense of (3)?

To answer this question, let $\gamma : \bar{X} \times \Theta \to \mathbb{R}_{++}$ be a function and define $\tilde{\pi}(\theta, x \mid a) = \pi(\theta, x \mid a)\gamma(x, \theta)/\Gamma_\gamma(a)$ for all $(\theta, x) \in \Theta \times X$ and $a \in A$, where $\Gamma_\gamma(a) = \sum_{x \in \bar{X}} \sum_{\theta \in \Theta} \gamma(x, \theta)\pi(\theta, x \mid a)$. Then the representation (3) may be written as

$$I \mapsto \sum_{x \in \bar{X}} \sum_{\theta \in \Theta} \pi\left(\theta, x \mid a_{I(x)}\right)\left[u\left(b_{I(x)}(\theta), \theta\right) + v\left(a_{I(x)}\right)\right] \tag{5}$$

or, equivalently, as

$$I \mapsto \sum_{x \in \bar{X}} \Gamma_\gamma\left(a_{I(x)}\right) \sum_{\theta \in \Theta} \tilde{\pi}\left(\theta, x \mid a_{I(x)}\right)\left[\tilde{u}\left(b_{I(x)}(\theta), \theta, x\right) + v\left(a_{I(x)}\right)\right], \tag{6}$$

where $\tilde{u}(b(\theta), \theta, x) = u(b(\theta), \theta)/\gamma(x, \theta)$.

Suppose that the preference relation $\succcurlyeq$ on $\mathcal{I}$ displays dynamic consistency. For every

$x \in \bar{X}$, let $\succcurlyeq^x$ and $\tilde{\succcurlyeq}^x$ denote the conditional preference relations on $A \times B$ induced by the representations (5) and (6), respectively. Then Bayesianism requires that $\succcurlyeq^x$ be represented by

$$U(a, b \mid x) = \sum_{\theta \in \Theta} \frac{\pi(\theta, x \mid a)}{\sum_{\theta' \in \Theta} \pi(\theta', x \mid a)} \left[u(b(\theta), \theta) + v(a)\right] \tag{7}$$

and $\tilde{\succcurlyeq}^x$ be represented by

$$\tilde{U}(a, b \mid x) = \sum_{\theta \in \Theta} \frac{\tilde{\pi}(\theta, x \mid a)}{\sum_{\theta' \in \Theta} \tilde{\pi}(\theta', x \mid a)} \left[\tilde{u}(b(\theta), \theta, x) + v(a)\right]. \tag{8}$$

But (8) is may be written as

$$\frac{\mu(x)}{\Gamma_\gamma(a, x)} \sum_{\theta \in \Theta} \frac{\pi(\theta, x \mid a)}{\sum_{\theta' \in \Theta} \pi(\theta', x \mid a)} \left[u(b(\theta), \theta) + v(a)\right], \tag{9}$$

where $\mu(x) = \sum_{\theta' \in \Theta} \pi(\theta', x \mid a)$ and $\Gamma_\gamma(a, x) = \sum_{\theta' \in \Theta} \pi(\theta', x \mid a) \gamma(x, \theta')$. Thus, for all $(a, b) \in A \times B$ and $x \in \bar{X}$,

$$\tilde{U}(a, b \mid x) = \frac{\mu(x)}{\Gamma_\gamma(a, x)} U(a, b \mid x). \tag{10}$$

Hence, $(a, b) \succcurlyeq^x (a', b')$ if and only if $U(a, b \mid x) \geq U(a', b' \mid x)$ and $(a, b) \tilde{\succcurlyeq}^x (a', b')$ if and only if $U(a, b \mid x) / \Gamma_\gamma(a, x) \geq U(a', b' \mid x) / \Gamma_\gamma(a', x)$. Consequently, if there is $x \in \bar{X}$ such that $\Gamma_\gamma(a, x) \neq \Gamma_\gamma(a', x)$ for some $a, a' \in A$ then $\tilde{\succcurlyeq}^x \neq \succcurlyeq^x$. Formally, we have just proved the following theorem.

Theorem 6 *Let $\succcurlyeq$ be a dynamically consistent preference relation on $\mathcal{I}$ satisfying (A.1)–(A.7) and suppose that $B^{cu}(\succcurlyeq)$ is inclusive. Then the only utility function and the joint probability distributions, $\{\pi(\theta, x \mid a)\}_{a \in A}$ that represent $\succcurlyeq$ on $\mathcal{I}$ are those that appear in (3) if and only if $\Gamma_\gamma(a, x) \neq \Gamma_\gamma(a', x)$ for some $x \in \bar{X}$ and $a, a' \in A$.*

The necessary and sufficient condition for the uniqueness of the probabilities in Theorem (6) may be stated as follows: For every γ, there is $x \in \bar{X}$ such that the mean of γ taken with respect to the conditional probabilities $\pi(\theta \mid a, x)$ is not independent of a. This, in turn requires that, for all $\theta \in \Theta$, and there is $x \in \bar{X}$ such that $\pi(\theta \mid a, x) \neq \pi(\theta \mid a', x)$ for some $a, a' \in A$.

4 Concluding Remarks

4.1 Effect-independent preferences and effect-independent utility functions

The choice-based Bayesian decision theory presented in this paper includes, as a special case, effect-independent preferences. In particular, following Karni (2006), effect independent preferences is captured by the following axiom:

(A.8) **Effect-independent betting preferences** *For all* $x \in \bar{X}$, $a \in A$, $b, b', b'', b''' \in B$, $\theta, \theta' \in \Theta$, *and* $r, r', r''r''' \in \mathbb{R}$, *if* $(b'_{-\theta}, r) \succcurlyeq_a^x (a, (b_{-\theta}, r'))$, $(b_{-\theta}, r'') \succcurlyeq_a^x (b'_{-\theta}, r''')$, *and* $(b''_{-\theta'}, r') \succcurlyeq_a^x (b'''_{-\theta'}, r)$ *then* $(b''_{-\theta'}, r'') \succcurlyeq_a^x (b'''_{-\theta'}, r''')$.

The interpretation of this axiom is analogous to that of action-independent betting preferences. The preferences $(b'_{-\theta}, r') \succcurlyeq_a^x (b_{-\theta}, r)$ and $(b_{-\theta}, r'') \succcurlyeq_a^x (b'_{-\theta}, r''')$ indicate that, for every given (a, x), the "intensity" of the preference for r'' over r''' given the effect θ is sufficiently greater than that of r over r' as to reverse the order of preference between the payoffs $b'_{-\theta}$ and $b_{-\theta}$. Effect independence requires that these intensities not be contradicted by the preferences between the same payoffs given any other effect θ.

Adding axiom (A.8) to the hypothesis of Theorem 3 implies that the utility function that figures in the representation takes the form $u(b(\theta), \theta) = t(\theta) u(b(\theta)) + s(\theta)$, where $t(\theta) > 0$. In other words, even if the preference relation exhibits *effect-independence* over bets, the *utility function* may still display effect dependence, in the form of the additive and multiplicative coefficient. Thus, effects may impact the decision maker's well-being without necessarily affecting his risk preferences.

Let B^c be the subset of constant bets (that is, trivial bets with the same payoff regardless of the effect that obtains). If the set of constant utility bets coincides with the set of constant bets (that is, $B^c = B^{cu}(\succcurlyeq)$), then the utility function is effect independent (that is, $u(b(\theta), \theta) = u(b(\theta))$ for all $\theta \in \Theta$). The implicit assumption that the set of constant utility bets coincides with the set of constant bets is the convention invoked by the standard subjective utility models. Unlike in those models, however, in the theory of this paper, this assumption is a testable hypothesis.

4.2 The value of information

When facing a choice, decision makers can act on the basis of the information at hand or seek additional, costly, information before choosing a course of action. A decision maker may seek to obtain the result of a medical test before adopting a health-enhancing life style and taking out life insurance. In the same vein, an investor contemplating the purchase of mineral extraction rights may seek to obtain, at cost, the result of a geological survey before deciding how much such rights are worth to him.

The problem can stated as follows: Denote by $A' \subset A$ the feasible set of actions and $B' \subset B$ a feasible set of bets. Suppose that a decision maker must choose an action- bet pairs in $A' \times B'$. He may choose on the basis of his current information (that is, based on the observation o) or he may pay ϑ to purchase a signal in X before choosing the action-bet pair.

The question of whether it is worth purchasing the signal involves the following comparison. Let (a_x^*, b_x^*) be a maximal element of $A' \times B'$ given the observation $x \in \bar{X}$. Formally, for every given $x \in \bar{X}$, and $(a_x^*, b_x^*) \in A' \times B'$, $(a_x^*, b_x^*) \succcurlyeq^x (a, b)$ for all $(a, b) \in A' \times B'$. Acting with the current information means choosing (a_o^*, b_o^*) to obtain the payoff

$$\sum_{\theta \in \Theta} \pi\left(\theta \mid o, a_o^*\right) u\left(b_o^*\left(\theta\right), \theta\right) + v\left(a_o^*\right). \tag{11}$$

If, instead, the decision is made following the purchase of information the payoff is given by

$$\sum_{x \in X} \frac{\mu\left(x\right)}{1 - \mu\left(o\right)} \left[\sum_{\theta \in \Theta} \pi\left(\theta \mid x, a_x^*\right) u\left(b_x^*\left(\theta\right) - \vartheta, \theta\right) + v\left(a_x^*\right)\right]. \tag{12}$$

The maximal willingness to pay for new information, $\bar{\vartheta}$, is the value of ϑ that equates the payoffs (11) and (12). New information is sought if and only if $\vartheta \leq \bar{\vartheta}$. As the discussion makes clear, the willingness to pay depends on the feasible set of action-bet pairs. In particular, if the feasible set of bets consists of constant utility bets then the maximal willingness to pay is zero.

5 Proof of Theorem 3

For expository convenience, I write B^{cu} instead of $B^{cu}(\succcurlyeq)$.

(Sufficiency) Suppose that $\succcurlyeq$ on $\mathcal{I}$ satisfies (A.1)–(A.7) and B^{cu} is inclusive. By Theorem 1, $\succcurlyeq$ is represented by

$$I \mapsto \sum_{x \in \bar{X}} w\left(a_{I(x)}, b_{I(x)}, x\right). \tag{13}$$

where $w(.,.,x), x \in \bar{X}$ are jointly cardinal, continuous, real-valued functions.

Since $\succcurlyeq$ satisfies (A.4), Lemmas 5 and 6 in Karni (2006) and Theorem III.4.1 in Wakker (1989) imply that for every $(a,x) \in A \times \bar{X}$ such that $\Theta(a,x)$ contains at least two effects, there exist array of functions $\{v_{(a,x)}(\cdot;\theta) : \mathbb{R} \to \mathbb{R} \mid \theta \in \Theta\}$ that constitute a jointly cardinal, continuous additive representation of $\succcurlyeq_a^x$ on B. Moreover, by the proof of Lemma 6 in Karni (2006), $\succcurlyeq$ satisfies (A.1)–(A.4) if and only if, for every $(a,x), (a',x') \in A \times \bar{X}$ such that $\Theta(a,x) \cap \Theta(a',x') \neq \varnothing$ and $\theta \in \Theta(a,x) \cap \Theta(a',x')$, there exist $\beta_{((a',x'),(a,x),\theta)} > 0$ and $\alpha_{((a',x'),(a,x),\theta)}$ such that $v_{(a',x')}(\cdot,\theta) = \beta_{((a',x'),(a,x),\theta)} v_{(a,x)}(\cdot,\theta) + \alpha_{((a',x'),(a,x),\theta)}$.[18]

Fix $\hat{a} \in A$ and define $u(\cdot,\theta) = v_{(\hat{a},o)}(\cdot,\theta)$, $\lambda(a,x;\theta) = \beta_{((a,x),(\hat{a},o),\theta)}$ and $\alpha(a,x,\theta) = \alpha_{((a,x),(\hat{a},o),\theta)}$ for all $a \in A$, $x \in \bar{X}$, and $\theta \in \Theta$. For every given $(a,x) \in A \times \bar{X}$, $w(a,b,x)$

[18] By definition, for all (a,x) and θ, $\beta_{((a,x),(a,x),\theta)} = 1$ and $\alpha_{((a,x),(a,x),\theta)} = 0$.

represents $\succsim_a^x$ on B. Hence

$$w(a,b,x) = H\left(\sum_{\theta\in\Theta}\left(\lambda(a,x,\theta)\,u(b(\theta);\theta) + \alpha(a,x,\theta)\right), a, x\right), \tag{14}$$

where H is a continuous, increasing function.

Consider next the restriction of $\succsim$ to $(A \times B^{cu})^{\bar{X}}$.

Lemma 7 *There exist a function $U : A \times B^{cu} \to \mathbb{R}$, $\xi \in \mathbb{R}_{++}^{|\bar{X}|}$ and $\zeta \in \mathbb{R}^{|\bar{X}|}$ such that, for all $\left(a, \bar{b}, x\right) \in A \times B^{cu} \times \bar{X}$,*

$$w\left(a,\bar{b},x\right) = \xi(x)\,U\left(\bar{b},a\right) + \zeta(x). \tag{15}$$

Proof: Let $I, I', I'', I''' \in \mathcal{I}$, $a, a', a'', a''' \in A$ and $\bar{b}$ be as in definition 2. Then, for all $x, x' \in \bar{X}$, $I_{-x}\left(a,\bar{b}\right) \sim I'_{-x}\left(a',\bar{b}\right)$, $I_{-x}\left(a'',\bar{b}\right) \sim I'_{-x}\left(a''',\bar{b}\right)$, $I''_{-x'}\left(a,\bar{b}\right) \sim I'''_{-x'}\left(a',\bar{b}\right)$ and $I''_{-x'}\left(a'',\bar{b}\right) \sim I'''_{-x'}\left(a''',\bar{b}\right)$. By the representation (13), $I_{-x}\left(a,\bar{b}\right) \sim I'_{-x}\left(a',\bar{b}\right)$ implies that

$$\sum_{y\in\bar{X}-\{x\}} w\left(a_{I(y)}, b_{I(y)}, y\right) + w\left(a,\bar{b},x\right) = \sum_{y\in\bar{X}-\{x\}} w\left(a_{I'(y)}, b_{I'(y)}, y\right) + w\left(a',\bar{b},x\right). \tag{16}$$

Similarly, $I_{-x}\left(a'',\bar{b}\right) \sim I'_{-x}\left(a''',\bar{b}\right)$ implies that

$$\sum_{y\in\bar{X}-\{x\}} w\left(a_{I(y)}, b_{I(y)}, y\right) + w\left(a'',\bar{b},x\right) = \sum_{y\in\bar{X}-\{x\}} w\left(a_{I'(y)}, b_{I'(y)}, y\right) + w\left(a''',\bar{b},x\right), \tag{17}$$

$I''_{-x'}\left(a,\bar{b}\right) \sim I'''_{-x'}\left(a',\bar{b}\right)$ implies that

$$\sum_{y\in\bar{X}-\{x'\}} w\left(a_{I''(y)}, b_{I''(y)}, y\right) + w\left(a,\bar{b},x'\right) = \sum_{y\in\bar{X}-\{x'\}} w\left(a_{I'''(y)}, b_{I'''(y)}, y\right) + w\left(a',\bar{b},x'\right), \tag{18}$$

and $I''_{-x'}\left(a'',\bar{b}\right) \sim I'''_{-x'}\left(a''',\bar{b}\right)$ implies that

$$\sum_{y\in\bar{X}-\{x'\}} w\left(a_{I''(y)}, b_{I''(y)}, y\right) + w\left(a'',\bar{b},x'\right) = \sum_{y\in\bar{X}-\{x'\}} w\left(a_{I'''(y)}, b_{I'''(y)}, y\right) + w\left(a''',\bar{b},x'\right). \tag{19}$$

But (16) and (17) imply that

$$w\left(a,\bar{b},x\right)-w\left(a',\bar{b},x\right)=w\left(a'',\bar{b},x\right)-w\left(a''',\bar{b},x\right). \tag{20}$$

and (18) and (19) imply that

$$w\left(a,\bar{b},x'\right)-w\left(a',\bar{b},x'\right)=w\left(a'',\bar{b},x'\right)-w\left(a''',\bar{b},x'\right). \tag{21}$$

Define a function $\phi_{\left(x,x',\bar{b}\right)}$ as follows: $w\left(\cdot,\bar{b},x\right)=\phi_{\left(x,x',\bar{b}\right)}\circ w\left(\cdot,\bar{b},x'\right)$. Axiom (A.7) with $\bar{b}=\bar{b}'$ imply that it is monotonic increasing. Then $\phi_{\left(x,x',\bar{b}\right)}$ is continuous. Moreover, equations (20) and (21) in conjunction with Lemma 4.4 in Wakker (1987) imply that $\phi_{\left(x,x',\bar{b}\right)}$ is affine.

Let $\beta_{\left(x,o,\bar{b}\right)}$ and $\delta_{\left(x,o,\bar{b}\right)}$ denote, respectively, the multiplicative additive coefficients corresponding to $\phi_{\left(x,o,\bar{b}\right)}$. Observe that $I_{-o}\left(a,\bar{b}\right)\sim I'_{-o}\left(a',\bar{b}\right)$ and $I_{-o}\left(a,\bar{b}'\right)\sim I'_{-o}\left(a',\bar{b}'\right)$ in conjunction with axiom (A.6) imply that

$$\beta_{\left(x,o,\bar{b}\right)}\left[w\left(a,\bar{b},o\right)-w\left(a',\bar{b},o\right)\right]=\beta_{\left(x,o,\bar{b}'\right)}\left[w\left(a,\bar{b}',o\right)-w\left(a',\bar{b}',o\right)\right] \tag{22}$$

for all $\bar{b},\bar{b}'\in B^{cu}$. Thus, for all $x\in\bar{X}$ and $\bar{b},\bar{b}'\in B^{cu}$, $\beta_{\left(x,o,\bar{b}\right)}=\beta_{\left(x,o,\bar{b}'\right)}:=\xi\left(x\right)>0$, where the inequality follows from the monotonicity of $\phi_{\left(x,o,\bar{b}\right)}$.

Let $a,a'\in A$ and $\bar{b},\bar{b}'\in B^{cu}$ satisfy $\left(a,\bar{b}\right)\sim_o\left(a',\bar{b}'\right)$. By axiom (A.7) $\left(a,\bar{b}\right)\sim_x\left(a',\bar{b}'\right)$ if and only if $\left(a,\bar{b}\right)\sim_o\left(a',\bar{b}'\right)$. By the representation this equivalence implies that

$$w\left(a,\bar{b},o\right)=w\left(a',\bar{b}',o\right). \tag{23}$$

if and only if,

$$\xi\left(x\right)w\left(a,\bar{b},o\right)+\delta_{\left(x,o,\bar{b}\right)}=\xi\left(x\right)w\left(a',\bar{b}',o\right)+\delta_{\left(x,o,\bar{b}'\right)}. \tag{24}$$

Thus $\delta_{\left(x,o,\bar{b}\right)} = \delta_{\left(x,o,\bar{b}'\right)}$.

By this argument and continuity (A.2) the conclusion can be extended to B^{cu}. Let $\delta_{\left(x,o,\bar{b}\right)} := \zeta\left(x\right)$ for all $\bar{b} \in B^{cu}$.

For every given $\bar{b} \in B^{cu}$ and all $a \in A$, define $U\left(\bar{b}, a\right) = w\left(a, \bar{b}, o\right)$. Then, for all $x \in \bar{X}$,

$$w\left(a, \bar{b}, x\right) = \xi\left(x\right) U\left(\bar{b}, a\right) + \zeta\left(x\right),\ \xi\left(x\right) > 0. \tag{25}$$

This completes the proof of Lemma 7. ♣

Equations (14) and (15) imply that for every $x \in \bar{X}$, $\bar{b} \in B^{cu}$ and $a \in A$,

$$\xi\left(x\right) U\left(\bar{b}, a\right) + \zeta\left(x\right) = H\left(\sum_{\theta\in\Theta} \lambda\left(a, x, \theta\right) u\left(\bar{b}\left(\theta\right), \theta\right) + \hat{\alpha}\left(a, x\right), a, x\right). \tag{26}$$

Lemma 8 *The identity (26) holds if and only if* $u\left(\bar{b}\left(\theta\right), \theta\right) = u\left(\bar{b}\right)$ *for all* $\theta \in \Theta$, $\sum_{\theta\in\Theta} \frac{\lambda(a,x,\theta)}{\xi(x)} = \varphi\left(a\right)$, $\frac{\hat{\alpha}(a,x)}{\xi(x)} = v\left(a\right)$ *for all* $a \in A$,

$$H\left(\sum_{\theta\in\Theta} \lambda\left(a, x, \theta\right) u\left(\bar{b}\left(\theta\right), \theta\right) + \hat{\alpha}\left(a, x\right), a, x\right) = \xi\left(x\right)\left[u\left(\bar{b}\right) + v\left(a\right)\right] + \zeta\left(x\right), \tag{27}$$

and there is $\kappa\left(a\right) > 0$ *such that*

$$\kappa\left(a\right) \sum_{\theta\in\Theta} \frac{\lambda\left(a, x, \theta\right)}{\xi\left(x\right)} u\left(\bar{b}\left(\theta\right), \theta\right) + \frac{\hat{\alpha}\left(a, x\right)}{\xi\left(x\right)} = U\left(\bar{b}, a\right). \tag{28}$$

Proof: (Sufficiency) Let $u\left(\bar{b}\left(\theta\right), \theta\right) := u\left(\bar{b}\right)$ for all $\theta \in \Theta$, $\sum_{\theta\in\Theta} \frac{\lambda(a,x,\theta)}{\xi(x)} := \varphi\left(a\right)$ and $c\left(a\right) := \kappa\left(a\right) \varphi\left(a\right)$ for all $a \in A$ and suppose that (28) holds.

But axiom (A.6) and the representation imply that, for all $\bar{b}, \bar{b}' \in B^{cu}$,

$$c\left(a\right) u\left(\bar{b}\right) + v\left(a\right) = c\left(a'\right) u\left(\bar{b}\right) + v\left(a'\right)$$

if and only if

$$c(a)\, u\left(\bar{b}'\right) + v(a) = c(a')\, u\left(\bar{b}'\right) + v(a').$$

Hence, $c(a) = c(a') = c$ for all $a, a' \in A$.

Normalize u so that $c = 1$. Then equation (26) follows from equation (27).

(Necessity) Multiply and divide the first argument of H by $\xi(x) > 0$. Equation (26) may be written as follows:

$$\xi(x)\, U\left(\bar{b}, a\right) + \zeta(x) = H\left(\xi(x)\left[\sum_{\theta\in\Theta} \frac{\lambda(a,x,\theta)}{\xi(x)} u\left(\bar{b}(\theta), \theta\right) + \frac{\hat{\alpha}(a,x)}{\xi(x)}\right], a, x\right). \tag{29}$$

Define $V\left(a, \bar{b}, x\right) = \sum_{\theta\in\Theta} \frac{\lambda(a,x,\theta)}{\xi(x)} u\left(\bar{b}(\theta), \theta\right) + \frac{\hat{\alpha}(a,x)}{\xi(x)}$ then, for every given $(a, x) \in A \times X$ and all $\bar{b}, \bar{b}' \in B^{cu}$,

$$U\left(\bar{b}', a\right) - U\left(\bar{b}, a\right) = \left[H\left(\xi(x)\, V\left(a, \bar{b}', x\right), a, x\right) - H\left(\xi(x)\, V\left(a, \bar{b}, x\right), a, x\right)\right] / \xi(x). \tag{30}$$

Hence $H(\cdot, a, x)$ is a linear function whose intercept is $\zeta(x)$ and the slope

$$\left[U\left(\bar{b}', a\right) - U\left(\bar{b}, a\right)\right] / \left[V\left(a, \bar{b}', x\right) - V\left(a, \bar{b}, x\right)\right] := \kappa(a),$$

is independent of x. Thus

$$\xi(x)\, U\left(\bar{b}, a\right) + \zeta(x) = \kappa(a)\, \xi(x)\left[\sum_{\theta\in\Theta} \frac{\lambda(a,x,\theta)}{\xi(x)} u\left(\bar{b}(\theta), \theta\right) + \frac{\hat{\alpha}(a,x)}{\xi(x)}\right] + \zeta(x). \tag{31}$$

Hence

$$U\left(\bar{b}, a\right) / \kappa(a) = \sum_{\theta\in\Theta} \frac{\lambda(a,x,\theta)}{\xi(x)} u\left(\bar{b}(\theta), \theta\right) + \frac{\hat{\alpha}(a,x)}{\xi(x)} \tag{32}$$

is independent of x. However, because $\succsim_a^x \neq \succsim_a^{x'}$ for all a and some $x, x' \in \bar{X}$, in general, $\lambda(a,x,\theta)/\xi(x)$ is not independent of θ. Moreover, because $\hat{\alpha}(a,x)/\xi(x)$ is independent of b, the first term on the right-hand side of (32) must be independent of x. For this to be true $u\left(\bar{b}(\theta),\theta\right)$ must be independent of θ and $\sum_{\theta\in\Theta}\lambda(a,x,\theta)/\xi(x) := \varphi(a)$ be independent of x. Moreover, because the first term on the right-hand side of (32) is independent of x, $\hat{\alpha}(a,x)/\xi(x)$ must also be independent of x. Finally, by definition, $\bar{b}$ the unique element in its equivalence class that has the property that $u\left(\bar{b}(\theta),\theta\right)$ is independent of θ.

Define $v(a) := \hat{\alpha}(a,x)/\xi(x)$, $u\left(\bar{b}(\theta),\theta\right) = u\left(\bar{b}\right)$, for all $\theta\in\Theta$, and $U\left(\bar{b},a\right) = u\left(\bar{b}\right) + v(a)$ and $\kappa(a)\varphi(a) = 1$. Thus

$$U\left(\bar{b},a\right) = \kappa(a)\sum_{\theta\in\Theta}\frac{\lambda(a,x,\theta)}{\xi(x)}u\left(\bar{b}(\theta);\theta\right) + \frac{\hat{\alpha}(a,x)}{\xi(x)}. \tag{33}$$

This completes the proof of Lemma 8. ♣

Note that

$$U\left(\bar{b},a\right) = \sum_{\theta\in\Theta}\frac{\lambda(a,x,\theta)}{\xi(x)\varphi(a)}u\left(\bar{b}(\theta);\theta\right) + \frac{\hat{\alpha}(a,x)}{\xi(x)}. \tag{34}$$

But, by Lemma 8, $\sum_{\theta\in\Theta}\lambda(a,x,\theta) = \xi(x)\varphi(a)$. Hence, by the inclusivity of B^{cu}, the representation (13) is equivalent to

$$I \mapsto \sum_{x\in\bar{X}}\left[\sum_{\theta\in\Theta}\frac{\lambda\left(a_{I(x)},x,\theta\right)}{\sum_{\theta\in\Theta}\lambda\left(a_{I(x)},x,\theta\right)}u\left(b_{I(x)}(\theta);\theta\right) + \frac{\hat{\alpha}\left(a_{I(x)},x\right)}{\xi(x)}\right] \tag{35}$$

For all $x\in X, a\in A$ and $\theta\in\Theta$, define the joint subjective probability distribution on $\Theta\times\bar{X}$ by

$$\pi(x,\theta \mid a) = \frac{\lambda(a,x,\theta)}{\sum_{x'\in\bar{X}}\sum_{\theta'\in\Theta}\lambda(a,x',\theta')}. \tag{36}$$

Since $\sum_{\theta\in\Theta}\lambda(a,x,\theta)=\xi(x)\varphi(a)$, for all $x\in\bar{X}$,

$$\sum_{\theta\in\Theta}\pi(x,\theta\mid a)=\frac{\xi(x)\varphi(a)}{\sum_{x'\in\bar{X}}\xi(x')\varphi(a)}=\frac{\xi(x)}{\sum_{x'\in\bar{X}}\xi(x')}. \tag{37}$$

Define the subjective probability of $x\in\bar{X}$ as follows:

$$\mu(x)=\frac{\xi(x)}{\sum_{x'\in\bar{X}}\xi(x')}. \tag{38}$$

Then the subjective probability of x is given by the marginal distribution on X induced by the joint distributions $\pi(\cdot,\cdot\mid a)$ on $X\times\Theta$ and is independent of a.

Define the subjective posterior on Θ distribution by

$$\pi(\theta\mid x,a)=\frac{\pi(x,\theta\mid a)}{\mu(x)}=\frac{\lambda(a,x,\theta)}{\sum_{\theta\in\Theta}\lambda(a,x,\theta)}, \tag{39}$$

and define the subjective prior on Θ by:

$$\pi(\theta\mid o,a)=\frac{\lambda(a,o,\theta)}{\sum_{\theta\in\Theta}\lambda(a,o,\theta)}. \tag{40}$$

Substitute in (35) to obtain the representation (3),

$$I\mapsto\sum_{x\in\bar{X}}\mu(x)\left[\sum_{\theta\in\Theta}\pi\left(\theta\mid x,a_{I(x)}\right)u\left(b_{I(x)}(\theta),\theta\right)+v\left(a_{I(x)}\right)\right]. \tag{41}$$

Let $a\in A$, $I\in\mathcal{I}$ *and* $b,b'\in B$, satisfy $I_{-o}(a,b)\sim I_{-o}(a,b')$. Then, by (41),

$$\sum_{\theta\in\Theta}\pi(\theta\mid o,a)u(b(\theta),\theta)=\sum_{\theta\in\Theta}\pi(\theta\mid o,a)u(b'(\theta),\theta) \tag{42}$$

and, by axiom (A.5) and (41)

$$\sum_{x\in X}\frac{\mu(x)}{1-\mu(0)}\sum_{\theta\in\Theta}\pi(\theta\mid x,a)u(b(\theta),\theta)=\sum_{x\in X}\frac{\mu(x)}{1-\mu(0)}\sum_{\theta\in\Theta}\pi(\theta\mid x,a)u(b'(\theta),\theta). \tag{43}$$

Thus

$$\sum_{\theta\in\Theta}\left[u\left(b\left(\theta\right),\theta\right)-u\left(b'\left(\theta\right),\theta\right)\right]\left[\pi\left(\theta\mid o,a\right)-\sum_{x\in X}\frac{\mu\left(x\right)}{1-\mu\left(0\right)}\pi\left(\theta\mid x,a\right)\right]=0. \tag{44}$$

This implies that $\pi\left(\theta\mid o,a\right)=\sum_{x\in X}\mu\left(x\right)\pi\left(\theta\mid x,a\right)/\left[1-\mu\left(0\right)\right]$.

(If $\pi\left(\theta\mid o,a\right)>\sum_{x\in X}\mu\left(x\right)\pi\left(\theta\mid x,a\right)/\left[1-\mu\left(0\right)\right]$ for some θ and $\mu\left(o\right)\pi\left(\theta'\mid o,a\right)<\sum_{x\in X}\mu\left(x\right)\pi\left(\theta'\mid x,a\right)/\left[1-\mu\left(0\right)\right]$ for some θ', let $\hat{b},\hat{b}'\in B$ be such that $\hat{b}\left(\theta\right)>b\left(\theta\right)$ and $\hat{b}\left(\hat{\theta}\right)=b\left(\hat{\theta}\right)$ for all $\hat{\theta}\in\Theta-\{\theta\}$, $\hat{b}'\left(\theta'\right)>b'\left(\theta'\right)$ and $\hat{b}'\left(\hat{\theta}\right)=b'\left(\hat{\theta}\right)$ for all $\hat{\theta}\in\Theta-\{\theta'\}$ and $I_{-o}\left(a,\hat{b}\right)\sim I_{-o}\left(a,\hat{b}'\right)$. Then

$$\sum_{\theta\in\Theta}\left[u\left(\hat{b}\left(\theta\right),\theta\right)-u\left(\hat{b}'\left(\theta\right),\theta\right)\right]\left[\pi\left(\theta\mid o,a\right)-\sum_{x\in X}\frac{\mu\left(x\right)}{1-\mu\left(0\right)}\pi\left(\theta\mid x,a\right)\right]>0. \tag{45}$$

But this contradicts (A.5).)

(Necessity) The necessity of (A.1), (A.2) and (A.3) follows from Theorem 1. To see the necessity of (A.4), suppose that $I_{-x}\left(a,b_{-\theta}r\right)\succcurlyeq I_{-x}\left(a,b'_{-\theta}r'\right)$, $I_{-x}\left(a,b'_{-\theta}r''\right)\succcurlyeq I_{-x}\left(a,b_{-\theta}r'''\right)$, and $I_{-x'}\left(a',b''_{-\theta}r'\right)\succcurlyeq I_{-x'}\left(a',b'''_{-\theta}r\right)$. By representation (14)

$$\sum_{\theta'\in\Theta-\{\theta\}}\lambda\left(a,x,\theta'\right)u\left(b\left(\theta'\right),\theta'\right)+\lambda\left(a,x,\theta\right)u\left(r,\theta\right)\geq\sum_{\theta'\in\Theta-\{\theta\}}\lambda\left(a,x,\theta'\right)u\left(b'\left(\theta'\right),\theta'\right)+\lambda\left(a,x,\theta\right)u\left(r',\theta\right) \tag{46}$$

$$\sum_{\theta'\in\Theta-\{\theta\}}\lambda\left(a,x,\theta'\right)u\left(b'\left(\theta'\right),\theta'\right)+\lambda\left(a,x,\theta\right)u\left(r'',\theta\right)\geq\sum_{\theta'\in\Theta-\{\theta\}}\lambda\left(a,x,\theta'\right)u\left(b\left(\theta'\right),\theta'\right)+\lambda\left(a,x,\theta\right)u\left(r''',\theta\right) \tag{47}$$

and

$$\sum_{\theta'\in\Theta-\{\theta\}} \lambda\left(a',x',\theta'\right) u\left(b''\left(\theta'\right),\theta'\right)+\lambda\left(a',x',\theta\right) u\left(r',\theta\right) \geq \sum_{\theta'\in\Theta-\{\theta\}} \lambda\left(a',x',\theta'\right) u\left(b'''\left(\theta'\right),\theta'\right)+\lambda\left(a',x',\theta\right) u\left(r,\theta\right) \tag{48}$$

But (46) and (47) imply that

$$u\left(r'',\theta\right)-u\left(r''',\theta\right) \geq \frac{\sum_{\theta'\in\Theta-\{\theta\}} \lambda\left(a,x,\theta'\right)\left[u\left(b\left(\theta'\right),\theta'\right)-u\left(b'\left(\theta'\right),\theta'\right)\right]}{\lambda\left(a,x,\theta\right)} \geq u\left(r',\theta\right)-u\left(r,\theta\right). \tag{49}$$

Inequality (48) implies

$$u\left(r',\theta\right)-u\left(r,\theta\right) \geq \frac{\sum_{\theta'\in\Theta-\{\theta\}} \lambda\left(a',x',\theta'\right)\left[u\left(b'''\left(\theta'\right),\theta'\right)-u\left(b''\left(\theta'\right),\theta'\right)\right]}{\lambda\left(a',x',\theta\right)} \tag{50}$$

But (49) and (50) imply that

$$u\left(r'',\theta\right)-u\left(r''',\theta\right) \geq \frac{\sum_{\theta'\in\Theta-\{\theta\}} \lambda\left(a',x',\theta'\right)\left[u\left(b'''\left(\theta'\right),\theta'\right)-u\left(b''\left(\theta'\right),\theta'\right)\right]}{\lambda\left(a',x',\theta\right)}. \tag{51}$$

Hence

$$\sum_{\theta'\in\Theta-\{\theta\}} \lambda\left(a',x',\theta'\right)\left[u\left(b''\left(\theta'\right),\theta'\right)-u\left(b'''\left(\theta'\right),\theta'\right)\right]+\lambda\left(a',x',\theta\right)\left[u\left(r'',\theta\right)-u\left(r''',\theta\right)\right] \geq 0 \tag{52}$$

Thus, $I_{-x'}\left(a',b''_{-\theta}r''\right) \succcurlyeq I_{-x'}\left(a',b'''_{-\theta}r'''\right).$

Next I show that if $\bar{b}\in B$ satisfies $u\left(\bar{b}\left(\theta\right),\theta\right)=u\left(\bar{b}\right)$ for all $\theta\in\Theta$ then $\bar{b}\in B^{cu}$. Suppose that representation (**??**) holds and let $I,I',I'',I'''\in\mathcal{I}$, $a,a',a'',a'''\in A$ and $x,x'\in\bar{X}$, such that $I_{-x}\left(a,\bar{b}\right)\sim I'_{-x}\left(a',\bar{b}\right)$, $I'_{-x}\left(a'',\bar{b}\right)\sim I_{-x}\left(a''',\bar{b}\right)$ and $I''_{-x'}\left(a',\bar{b}\right)\sim I'''_{-x'}\left(a,\bar{b}\right)$. Then the representation (13) implies that

$$\sum_{\hat{x}\in\bar{X}-\{x\}} w\left(a_{I(\hat{x})},b_{I(\hat{x})},\hat{x}\right)+\mu\left(x\right)\left[u\left(\bar{b}\right)+v\left(a\right)\right]=\sum_{\hat{x}\in\bar{X}-\{x\}} w\left(a_{I'(\hat{x})},b_{I'(\hat{x})},\hat{x}\right)+\mu\left(x\right)\left[u\left(\bar{b}\right)+v\left(a'\right)\right] \tag{53}$$

$$\sum_{\hat{x}\in\bar{X}-\{x\}} w\left(a_{I'(\hat{x})}, b_{I'(\hat{x})}, \hat{x}\right)+\mu\left(x\right)\left[u\left(\bar{b}\right)+v\left(a''\right)\right] = \sum_{\hat{x}\in\bar{X}-\{x\}} w\left(a_{I(\hat{x})}, b_{I(\hat{x})}, \hat{x}\right)+\mu\left(x\right)\left[u\left(\bar{b}\right)+v\left(a'''\right)\right] \tag{54}$$

and

$$\sum_{\hat{x}\in\bar{X}-\{x'\}} w\left(a_{I''(\hat{x})}, b_{I''(\hat{x})}, \hat{x}\right)+\mu\left(x'\right)\left[u\left(\bar{b}\right)+v\left(a'\right)\right] = \sum_{\hat{x}\in\bar{X}-\{x'\}} w\left(a_{I'''(\hat{x})}, b_{I'''(\hat{x})}, \hat{x}\right)+\mu\left(x'\right)\left[u\left(\bar{b}\right)+v\left(a\right)\right]. \tag{55}$$

But (53) and (54) imply that

$$v\left(a\right)-v\left(a'\right)=v\left(a''\right)-v\left(a'''\right). \tag{56}$$

Equality (**??**) implies

$$\frac{\sum_{\hat{x}\in\bar{X}-\{x'\}}\left[w\left(a_{I''(\hat{x})}, b_{I''(\hat{x})}, \hat{x}\right)-w\left(a_{I'''(\hat{x})}, b_{I'''(\hat{x})}, \hat{x}\right)\right]}{\mu\left(x'\right)}=v\left(a\right)-v\left(a'\right). \tag{57}$$

Thus

$$\sum_{\hat{x}\in\bar{X}-\{x'\}} w\left(a_{I''(\hat{x})}, b_{I''(\hat{x})}, \hat{x}\right)+u\left(\bar{b}\right)+v\left(a'''\right) = \sum_{\hat{x}\in\bar{X}-\{x'\}} w\left(a_{I'''(\hat{x})}, b_{I'''(\hat{x})}, \hat{x}\right)+u\left(\bar{b}\right)+v\left(a''\right) \tag{58}$$

Hence $I''_{-x'}\left(a''', \bar{b}\right)\sim I'''_{-x'}\left(a''', \bar{b}\right)$ and $\bar{b}\in B^{cu}$.

To show the necessity of (A.5) let $a\in A$, $I\in\mathcal{I}$ and $b, b'\in B$, by the representation $I_{-o}\left(a,b\right)\sim I_{-o}\left(a,b'\right)$ if and only if

$$\sum_{\theta\in\Theta}\pi\left(\theta\mid o,a\right)u\left(b\left(\theta\right),\theta\right)=\sum_{\theta\in\Theta}\pi\left(\theta\mid o,a\right)u\left(b'\left(\theta\right),\theta\right). \tag{59}$$

But $\pi\left(\theta\mid o,a\right)=\sum_{x\in X}\mu\left(x\right)\pi\left(\theta\mid x,a\right)/\left[1-\mu\left(0\right)\right]$. Thus (59) holds if and only if

$$\sum_{x\in X}\mu\left(x\right)\sum_{\theta\in\Theta}\pi\left(\theta\mid x,a\right)u\left(b\left(\theta\right),\theta\right)=\sum_{x\in X}\mu\left(x\right)\sum_{\theta\in\Theta}\pi\left(\theta\mid x,a\right)u\left(b'\left(\theta\right),\theta\right). \tag{60}$$

But (60) is valid if and only if $I^{-o}(a,b) \sim I^{-o}(a,b')$.

For all I and x, let $K(I,x) = \sum_{y\in X-\{x\}} \mu(y) \left[\sum_{\theta\in\Theta} \pi(\theta \mid x,a)\, u\left(b_{I(y)}(\theta)\right) + v\left(a_{I(y)}\right)\right]$. To show the necessity of (A.6) Then $I_{-x}\left(a,\bar{b}\right) \succcurlyeq I'_{-x}\left(a',\bar{b}\right)$ if and only if

$$K(I,x) + u\left(\bar{b}\right) + v(a) \geq K(I',x) + u\left(\bar{b}\right) + v(a') \tag{61}$$

if and only if

$$K(I,x) + u\left(\bar{b}'\right) + v(a) \geq K(I',x) + u\left(\bar{b}'\right) + v(a') \tag{62}$$

if and only if $I_{-x}\left(a,\bar{b}'\right) \succcurlyeq I'_{-x}\left(a',\bar{b}'\right)$.

To show that axiom (A.7) is implied, not that $I_{-x}\left(a,\bar{b}\right) \succcurlyeq I'_{-x}\left(a',\bar{b}'\right)$ if and only if

$$K(I,x) + u\left(\bar{b}\right) + v(a) \geq K(I,x) + u\left(\bar{b}'\right) + v(a') \tag{63}$$

if and only if

$$K(I,x') + u\left(\bar{b}\right) + v(a) \geq K(I,x') + u\left(\bar{b}'\right) + v(a') \tag{64}$$

if and only if $I_{-x'}\left(a,\bar{b}\right) \succcurlyeq I'_{-x'}\left(a',\bar{b}'\right)$. ■

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