Asymptotic Analysis of Large Auctions with Risk-Averse Bidders

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September 2008

Abstract

We study private-value auctions with n risk-averse bidders, where n is a large number. We first use asymptotic techniques to calculate explicit approximations of the equilibrium bids and of the seller's revenue in any k-price auction ($k = 1, 2, \dots$), and use these explicit approximations to show that all large k-price auctions with risk-averse bidders are $O(1/n^2)$ revenue equivalent. We then prove that there exist auction mechanisms for which the limiting revenue as $n \longrightarrow \infty$ in the case of risk-averse bidders is strictly below the risk-neutral limit. Therefore, these auction mechanisms are not revenue equivalent to large k-price auctions even in the limit as $n \longrightarrow \infty$. Finally, we formulate a general condition under which the limiting revenue with risk-averse bidders is equal to the risk-neutral limit.

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1 Introduction

Since the pioneering work of Vickrey (1961) who established the revenue equivalence of the classical private-value auctions (first-price, second-price, English, Dutch), a considerable research effort has been devoted to revenue ranking of different auction mechanisms. Vickrey's result was generalized twenty years later by the Revenue Equivalence Theorem (Riley and Samuelson (1981) and Myerson (1981)), according to which the seller's revenue is the same for a large class of private-value auctions with symmetric and risk-neutral bidders. Private-value auctions are, however, generically not revenue equivalent when bidders are asymmetric (Marshall et al. (1994), Maskin and Riley (2000)) or risk-averse (Maskin and Riley (1984), Matthews (1987)).

Many auctions, such as those that appear on the Internet, have a large number of bidders. The standard approach to study large auctions has been to consider their limit as n, the number of bidders, approaches infinity. Using this approach, it has been shown for quite general conditions that as n goes to infinity, the bid approaches the true value, the seller's expected revenue approaches the maximal possible value, and the auction becomes efficient (Wilson (1977), Pesendorfer and Swinkels (1997), Kremer (2002), Swinkels (1999), Swinkels (2001), and Bali and Jackson (2002)). Most of the studies that adopted this approach, however, do not provide the rate of convergence to the limit, i.e., a bound on the difference between the limiting value and the value at a finite n. Therefore, it is not clear how large n should be (5, 10, 100?) in order for the auction "to be large" i.e., in order for the limiting results for $n = \infty$ to be applicable. Results of this type were obtained by Satterthwaite and Williams (1989), who showed that the rate of convergence of the bid to the true value in a double auction is $O(1/m)$, where m is the number of traders on each side of the market, by Rustichini, Satterthwaite and Williams (1994), who showed that the rate of convergence of the bid to the true value in a k-double auction is $O(1/m)$ and the corresponding inefficiency is $O(1/m^2)$, and by Hong and Shum (2004), who calculated the convergence rate in common-value multi-unit first-price auctions.

In the first part of this study we use asymptotic analysis techniques in order to go beyond rate-of-convergence results. To do that, we utilize the existence of the large parameter n to calculate explicitly the $O(1/n)$ leading-order deviation from the limiting value, of the equilibrium bids and of the seller's revenue in large k-price auctions with risk-averse bidders. Since our asymptotic approximations include both the limiting value and the O(1/n) correction, they are $O(1/n^2)$ accurate. Hence, the number of bidders at which they become valid is considerably smaller than the number of bidders at which the limiting values (which are only O(1/n) accurate) become valid. Roughly speaking, if we require a 1% accuracy, than our $O(1/n^2)$ asymptotic approximations are already valid for $n = 10$ bidders, whereas the limiting-value approximations become valid only for $n = 100$ bidders.

Why should we care whether the asymptotic approximation becomes valid at $n = 10$ or at $n = 100$? One practical reason is as follows. Consider, for example, the case where we want to analyze a specific real-life auction with 10 bidders. If we could utilize the asymptotic approximations for large auctions, this would lead to a considerable simplification in the analysis. For an auction with 10 bidders, this can be done with our $O(1/n^2)$

approximation, but not with the $O(1/n)$ accurate limiting-value approximation.

The paper is organized as follows. In Section 2, we introduce the model of symmetric private-value auctions with risk-averse bidders. In Section 3, we calculate asymptotic approximations of the equilibrium bids and of the seller's revenue in large first-price auctions with risk-averse bidders. This calculation shows that the differences in the equilibrium bids and in the seller's revenue between risk-neutral and risk-averse bidders are only $O(1/n^2)$.

We note that one measure of a 'good' asymptotic technique is that it can be used, at least in theory, to calculate as many terms in the expansion as desired. To show that this is the case here, we calculate explicitly the next-order, $O(1/n^2)$ terms in the expressions for the equilibrium bids and the revenue. This calculation shows that the $O(1/n^2)$ affect of risk aversion is proportional to the Arrow-Pratt measure of risk aversion at zero. In addition, this calculation provides an analytic estimate for the value of the constant of the $O(1/n^2)$ error term.

In Section 3.1, we present numerical examples that suggest that the asymptotic approximations derived in this study are quite accurate even for auctions with as little as $n = 6$ bidders. Although we only present a few numerical examples, we note that the parameters of these examples were chosen "at random", and that we observed the same behavior in numerous other examples that we tested.[1]

In Section 4, we calculate asymptotic approximations of the equilibrium bids and of

[1] The fact that an expansion for large n is already valid for $n = 6$ may be surprising to researchers not familiar with asymptotic expansions. However, quite often, this is the case with asymptotic expansions (see e.g., Bender and Orszag (1978)).

the seller's revenue in large symmetric k-price auctions ($k = 3, 4, \dots$) with risk-averse bidders. As in the case of first-price auctions, this calculation shows that the differences in the equilibrium bids and in the seller's revenue between risk-neutral and risk-averse bidders are only $O(1/n^2)$. Since in the risk-neutral case all k-price auctions are revenue equivalent, we conclude that all large k-price auctions ($k = 1, 2, \dots$) with risk-averse or risk-neutral bidders are $O(1/n^2)$ revenue equivalent.

Since the revenue differences among all large k-price auctions with n risk-averse bidders are $O(1/n^2)$, it seems natural to conjecture that this result should extend to all incentive-compatible and individually-rational mechanisms that deliver efficient allocations. This, however, is not the case. Indeed, in Section 5 we prove that the limiting revenue as $n \longrightarrow \infty$ in *generalized all-pay auctions*[2] with risk-averse bidders is strictly below the risk-neutral limit, and in Section 6 we show that this also occurs for *last-price auctions*.[3] Therefore, unlike k-price auctions where risk-aversion only has an $O(1/n^2)$ effect on the revenue, in the case of all-pay and last price auctions risk-aversion has an $O(1)$ effect on the revenue. To the best of our knowledge, these are the first examples of private-value auctions whose limiting revenue is not equal to the risk-neutral limit (i.e. to the maximal value).

The above results raise the question of whether there is a condition that would imply that the limiting revenue with risk-averse bidders is equal to the risk-neutral limit. In

[2] i.e., when the highest bidder wins the object and pays his bid, and the losing bidders pay a fixed fraction of their bids.

[3] i.e., when the highest bidder wins the object and pays the lowest bid.

Section 7 we prove in Theorem 1 that if the equilibrium payment of the winning bidder approaches his type as $n \longrightarrow \infty$ uniformly for all types, then the limiting revenue with risk-averse bidders is equal to the risk-neutral limit. We then show that it is sufficient for this condition to hold only at an $O(1/n)$ neighborhood of the highest type. The Appendix contains proofs omitted from the main body of the paper.

At a first sight, it may seem that the results for k-price auctions in Section 4 are a special case of Theorem 1. This is not the case, however, since in order to apply Theorem 1 to k-price auctions, one needs to prove that the equilibrium payment of the winning bidder in k-price auctions approaches his type as $n \longrightarrow \infty$ uniformly. In addition, the result of Theorem 1 is weaker, since it only shows that the limiting revenue is unaffected by risk-aversion, whereas in Section 4 we show that the $O(1/n)$ correction is also unaffected by risk-aversion.

Finally, we note that this paper differs from our previous studies, in which we used perturbation analysis techniques to analyze auctions with weakly asymmetric bidders (Fibich and Gavious (2003), Fibich, Gavious and Sela, (2004)) and with weakly risk-averse bidders (Fibich, Gavious and Sela, (2006)), in two main aspects:

1. From the economic theory aspect, in those studies we had to assume that the level of risk-aversion (or asymmetry) is small, in order to be able to expand the solution in the small risk-aversion (or asymmetry) parameter. The results of this study are stronger, since we do not make such assumptions.

2. From the mathematical methodology aspect, in our previous studies we used per-

turbation techniques that "essentially" amounts to Taylor expansions in a small parameter that lead to convergent sums. In contrast, in this study we use asymptotic methods (e.g., Laplace method for evaluation of integrals) which typically lead to divergent sums if carried out to all orders (see, e.g., Murray, 1984). To the best of our knowledge, these asymptotic methods have not been used in auction theory so far. It is quite likely that these and other asymptotic methods (WKB, method of steepest descent, etc.) will be useful in other economic problems where a large parameter exist, e.g., multi-unit auctions with many units (Jackson and Kremer, 2004; Jackson and Kremer, 2006).

2 The model

Consider a large number ($n \gg 1$) of bidders who are competing for a single object in an auction mechanism where the highest bidder wins the object. Assume that bidder i's valuation v_i is a private information, and that bidders are symmetric such that for any $i = 1, \ldots, n$, v_i is independently distributed according to a common distribution function $F(v)$ on the interval $[0, 1]$. We denote by $f = F'$ the corresponding density function. We assume that F is twice continuously differentiable and that $f > 0$ in $[0, 1]$. We assume that each bidder's utility is given by a function $U(v - b)$, which is twice continuously differentiable, monotonically increasing, concave, and normalized to have a zero utility at zero, i.e.,

$$U(x) \in \mathcal{C}^2, \quad U(0) = 0, \quad U'(x) > 0, \quad U''(x) < 0. \tag{1}$$

3 First-price auctions

Consider a first-price auction with risk-averse bidders, in which the bidder with the highest bid wins the object and pays his bid. In this case, the inverse equilibrium bids satisfy the ordinary-differential equation[4]

$$v'(b) = \frac{1}{n-1} \frac{F(v(b))}{f(v(b))} \frac{U'(v(b)-b)}{U(v(b)-b)}, \qquad v(0) = 0. \tag{2}$$

Unlike the risk-neutral case, there are no explicit formulae for the equilibrium bids and for the revenue, except for some special cases. Recently, Fibich, Gavious and Sela (2006) obtained explicit approximations of the equilibrium bids for the case of weak risk aversion, by using perturbation methods to expand the solution in the small risk-aversion parameter. In contrast, here we use different mathematical techniques and expand the solution in the large parameter n, without making the assumption that risk aversion is weak.[5]

Proposition 1 *Consider a symmetric first-price auction with n bidders with utility function $U(x)$ that satisfies Assumptions (1). Then, the equilibrium bid for a sufficiently large n is given by*

$$b(v) = v - \frac{1}{n-1} \frac{F(v)}{f(v)} + O\left(\frac{1}{n^2}\right), \tag{3}$$

[4]Under the conditions of Section 2, existence of a symmetric equilibrium follows from Maskin and Riley (1984).

[5]Caserta and de Vries (2002) used extreme value theory to derive an asymptotic expression for the revenue which is equivalent to (4). The result of Caserta and de Vries (2002) holds, however, only in the risk-neutral case, where an explicit expression for the revenue is available.

and the seller's expected revenue is given by

$$R = 1 - \frac{2}{n}\frac{1}{f(1)} + O\left(\frac{1}{n^2}\right). \tag{4}$$

Proof. See Appendix B.

Since the utility function $U(x)$ does not appear in the $O(1/n)$ term, Proposition 1 shows that the differences in the equilibrium bids and in the seller's revenue between first-price auctions with risk-neutral or with risk-averse bidders are at most $O(1/n^2)$. In other words, risk aversion has (at most) an $O(1/n^2)$ effect on the equilibrium bids and on the revenues in symmetric first-price auctions.

The results of Proposition 1 raise several questions:

1. Is the effect of risk-aversion truly $O(1/n^2)$, or is it even smaller?
2. Can we estimate the constants in the $O(1/n^2)$ error terms?

We can answer these questions by calculating explicitly the $O(1/n^2)$ terms:

Proposition 2 *Consider a symmetric first-price auction with n bidders with utility function $U(x)$ that satisfies Assumptions (1). Then, the equilibrium bid for a sufficiently large n is given by*

$$b(v) = v - \frac{1}{n-1}\frac{F(v)}{f(v)} + \frac{1}{(n-1)^2}\left[\frac{F(v)}{f(v)} - \frac{F^2(v)f'(v)}{f^3(v)} - \frac{F^2(v)}{2f^2(v)}\frac{U''(0)}{U'(0)}\right] + O\left(\frac{1}{n^3}\right), \tag{5}$$

and the seller's expected revenue is given by

$$R = 1 - \frac{1}{n}\frac{2}{f(1)} + \frac{1}{n^2}\left[\frac{2}{f(1)} - \frac{3f'(1)}{f^3(1)} - \frac{1}{2f^2(1)}\frac{U''(0)}{U'(0)}\right] + O\left(\frac{1}{n^3}\right). \tag{6}$$

Proof. See Appendix C.

We thus see that risk-aversion has an $O(1/n^2)$ effect on the bid when $U''(0) \neq 0$, but a smaller effect if $U''(0) = 0$. As expected, the bids and revenue increase (decrease) for risk-averse (risk-loving) bidders. Note that *the magnitude of risk-aversion effect is determined, to leading-order, by $-U''(0)/U'(0)$, i.e., by the value of the absolute risk-aversion at zero.*[6]

The observation that risk-aversion has a small effect on the revenue in large first-price auctions has the following intuitive explanation. Since in large first-price auctions the bids are close to the values, one can approximate $U(v-b) \approx (v-b)U'(0)$, which is the risk-neutral case. Similarly, adding the next term in the Taylor expansion gives

$$U(v-b) \approx U'(0)\left[(v-b) + \frac{(v-b)^2}{2}\frac{U''(0)}{U'(0)}\right]. \tag{7}$$

Hence, for large n, the leading-order effect of risk-aversion is proportional to $-U''(0)/U'(0)$.

3.1 Examples

Consider a first-price auction where bidders' valuations are uniformly distributed on $[0, 1]$, i.e., $F(v) = v$. Assume first that each bidder has a CARA utility function $U(x) = 1 - e^{-\lambda x}$ where $\lambda > 0$. In Figure 1 we compare the (exact) equilibrium bid[7] for $\lambda = 2$, with the equilibrium bid in the risk-neutral case, for $n = 2, 4, 6$, and 8 bidders.[8] Already for $n = 6$ bidders, the equilibrium bids in the risk-neutral and risk averse cases are almost

[6]Note that although we assume in (1) that bidders are risk averse (i.e., $U'' < 0$), the results hold also for risk loving bidders (i.e., $U'' > 0$).

[7]i.e., the numerical solution of equation (2).

[8]Observe that as $\lambda \to 0$, $U(x) \sim \lambda x$, i.e., the utility of risk-neutral bidders. Therefore, $\lambda = 2$ corresponds to a significant deviation from risk-neutrality.

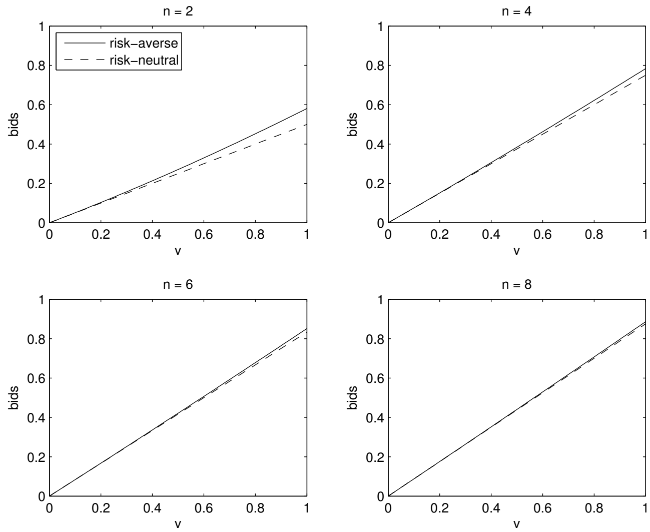

Figure 1: Equilibrium bids in a first-price auction with risk-averse (solid) and risk-neutral (dashes) bidders.

identical. This observation is consistent with Dyer, Kagel and Levin (1989), who found in experiments with six bidders that the actual bids in first-price auctions were very close to the theoretical risk-neutral equilibrium bid.[9]

Next, we consider the revenue in first-price auctions with $n = 6$ risk-averse players,

[9]For $n = 3$, the highest bid was much higher than the risk neutral case.

$U(x)$	R	$\frac{R-R_{\rm rn}}{R_{\rm rn}}$
$x - x^2/2$	0.7220	1.08%
$\ln(1+x)$	0.7209	0.92%
CARA ($\lambda = 1$)	0.7214	0.99%
CARA ($\lambda = 2$)	0.7278	1.89%

Table 1: Expected revenue in a symmetric first-price auction with 6 players with a utility function $U(x)$.

denoted by R. Recall that when $F(v) = v$, the revenue in the risk-neutral case is equal to $R_{\text{rn}} = (n-1)/(n+1)$. Therefore, in the case of six players, $R_{\text{rn}} = 5/7 \approx 0.7143$. In Table 1 we give the value of R and the relative change in the revenue due to risk-aversion for four different utility functions. The first thing to note is that in all four cases the effect of risk-aversion is small (less that 2%), even though the number of players is not really large, and the utility functions are not close to risk-neutrality. The second thing to notice is that in the first three cases the effect of risk-aversion on the revenue is nearly the same ($\approx 1\%$), even though the three utility functions are quite different. The reason for this is that the difference between the revenue in the risk-averse and risk-neutral case is given by, see Proposition 2,

$$R - R_{\text{rn}} \approx -\frac{1}{n^2}\frac{1}{2f^2(1)}\frac{U''(0)}{U'(0)}.$$

Hence, the effect of risk-aversion is proportional to the value of the absolute risk aversion $-U''(0)/U'(0)$. In the first three cases $-U''(0)/U'(0)$ is identical ($= 1$), explaining why they have "the same" effect on the revenue. In the fourth case $-U''(0)/U'(0) = 2$, and indeed, the change in the revenue nearly doubles.

In Table 2, we repeat the simulations of Table 1, but with $n = 2$ players. In this case, the revenue in the risk-neutral case is equal to $R_{\text{rn}} = 1/3 \approx 0.3333$. As expected, the relative effect of risk-aversion is much larger than for $n = 6$ players, showing that risk-aversion cannot be neglected in small first-price auctions. In addition, as before, the effect of risk-aversion depends predominantly on $-U''(0)/U'(0)$, which is why the additional revenues due to risk aversion are roughly the same in the first three cases, but

$U(x)$	R	$\frac{R-R_{\rm rn}}{R_{\rm rn}}$
$x-x^2/2$	0.3592	7.7%
$\ln(1+x)$	0.3508	5.25%
CARA ($\lambda=1$)	0.3541	6.2%
CARA ($\lambda=2$)	0.3741	12.2%

Table 2: Expected revenue in a symmetric first-price auction with 2 players with a utility function $U(x)$, obtained from numerical simulations.

roughly double in the fourth case.

4 k-price auctions

Consider k-price auctions in which the bidder with the highest bid wins the auction and pays the k-th highest bid. The results of Proposition 1 can be generalized to any k-price auction as follows:[10]

Proposition 3 *Consider a symmetric k-price auction ($k=1,2,3,\dots$), with n bidders with a utility function $U(x)$ that satisfies Assumptions (1). Then, the equilibrium bid for a sufficiently large n is*

$$b(v) = v + \frac{k-2}{n-k}\frac{F(v)}{f(v)} + O\left(\frac{1}{n^2}\right), \tag{8}$$

and the seller's expected revenue is given by (4).

Proof. See Appendix D.

[10]For more details on k-price auctions with risk averse bidders, see Monderer and Tennenholtz (2000).

Recall that in the risk-neutral case $U(x) = x$, the equilibrium bids in k-price auctions ($k = 2, 3, \dots$) are given by (Wolfstetter, 1995)

$$b(v) = v + \frac{k-2}{n-k+1} \frac{F(v)}{f(v)}. \tag{9}$$

Comparison with equation (8) shows that in large symmetric k-price auctions, risk aversion only has an $O(1/n^2)$ effect on the equilibrium bids. Proposition 3 also shows that risk aversion only has an $O(1/n^2)$ effect on the revenue in large symmetric k-price auctions.[11] Since all k-price auctions are revenue equivalent in the risk-neutral case, this implies, in particular, that *all large symmetric k-price auctions with risk-averse bidders are $O(1/n^2)$ revenue equivalent.*

5 All-pay auctions

In Proposition 3 we saw that all large k-price auctions with risk-averse bidders are $O(1/n^2)$ revenue equivalent. A natural question, is, therefore, whether this asymptotic revenue equivalence holds for "all" auction mechanisms. To see that this is not the case, let us consider an all-pay auction with risk-averse bidders in which the highest bidder wins the object and all bidders pay their bid. In this case, the limiting value of the revenue is strictly below the risk-neutral limit:

[11] As in the case of first-price auctions (see Section 3), we can calculate explicitly the $O(1/n^2)$ terms in order to see that the leading-order effect of risk-aversion is truly $O(1/n^2)$ and is proportional to $-U''(0)/U'(0)$. Indeed, since $\lim_{n\to\infty} b(v) = v$, see equation (8), this conclusion follows from equation (7).

Proposition 4 *Consider a symmetric all-pay auction with n bidders that have a utility function U that satisfies (1), and let $R = R(n)$ be the expected seller's revenue in equilibrium. Then,*

$$\lim_{n\to\infty} R < \lim_{n\to\infty} R_{\mathrm{rn}}.$$

Proof. This is a special case of Proposition 5.

Therefore, even the limiting revenue of all-pay auctions is not revenue equivalent to that of k-price auctions with risk-averse bidders.

In (Fibich, Gavious, Sela, 2007) it was shown that in the case of all-pay auctions, risk-aversion lowers the equilibrium bids of the low types but increases the bids of the high types, and that, as a result, the seller's revenue may either increase or decrease due to risk-aversion. In the case of large all-pay auctions, however, Proposition 5 shows that risk aversion always lowers the expected revenue.

Example 1 *In Figure 2 we present the numerically calculated revenue as a function of n for an all-pay action with $F(v) = v$ and $U(x) = x - 0.5x^2$. In this case, risk-aversion increases the expected revenue when the number of bidders is small. As n increases, however, this trend reverses and risk-aversion decreases the expected revenue. In particular, as $n \longrightarrow \infty$, the expected revenue in the risk-averse case approaches ≈ 0.74, which is well below the risk-neutral limit of $1 = \lim_{n\to\infty} R_{\mathrm{rn}}$.*[12]

[12] In the case of risk-loving bidders the limiting revenue is above the risk-neutral limit. For example, we find numerically for $U(x) = x + 0.5x^2$ that $lim_{n\to\infty} R \approx 1.26$.

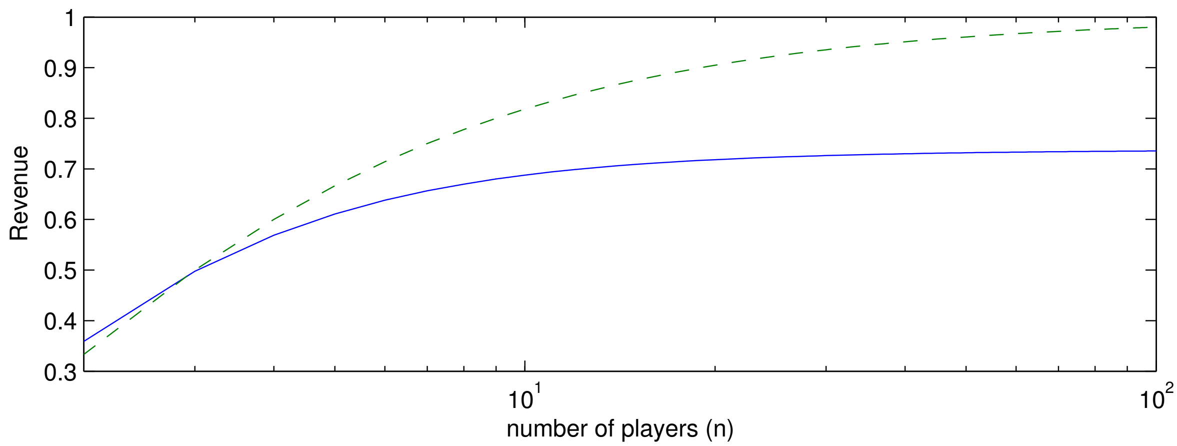

Figure 2: Expected revenue in all-pay auction with risk-averse (solid) and risk-neutral (dashes) players, as a function of the number of players. Data plotted on a semi-logarithmic scale.

5.1 Generalized all-pay auctions

In order to show that there are additional auctions mechanisms for which the limiting revenue is below the risk-neutral limit, let us define *generalized all-pay auctions* as follows. The highest bidder wins the object and pays his bid. The other bidders pay α times their bid, where $0 \leq \alpha \leq 1$. Thus, $\alpha = 1$ corresponds to the standard all-pay auction, and $\alpha = 0$ to first-price auction. We now prove that the limiting revenue in generalized all-pay auctions with risk-averse bidders is below the risk-neutral limit:

Proposition 5 *Consider a generalized all-pay auction where bidders have a utility function U that satisfies (1). Then,*

$$\lim_{n\to\infty} R < \lim_{n\to\infty} R_{\rm rn}, \qquad \textit{for} \ \ 0 < \alpha \leq 1.$$

Proof. See Appendix E.

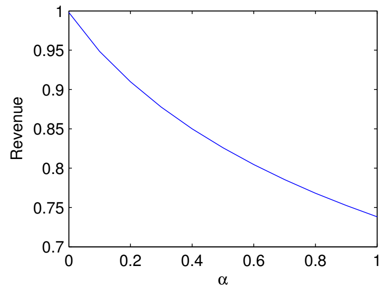

Figure 3: Limiting value of the revenue, as a function of α.

Example 2 *In Figure 3 we present the numerically-calculated limiting value of the expected revenue for a generalized all-pay auction with $F(v) = v$ and $U(x) = x - 0.5x^2$, as a function of α. As expected, the limiting revenue is 1 ($= \lim_{n\to\infty} R_{\rm rn}$) for $\alpha = 0$, but less than 1 for $\alpha > 0$. Moreover, it decreases smoothly from 1 for first-price auctions ($\alpha = 0$), to ≈ 0.74 for "standard" all-pay auctions ($\alpha = 1$).*

6 Last-price auctions

So far, the only case where risk-aversion reduced the limiting revenue was of generalized all-pay auctions, in which the losing bidders pay a fixed portion of their bid. We therefore ask whether risk-aversion can reduce the limiting revenue even when only the winner pays. To see that this is possible, we consider an auction in which the highest-bidder wins the object and pays the lowest bid, i.e., a *last-price auction*.

Example 3 *Consider a last-price auction with n bidders that are risk averse with the CARA utility function $U(x) = 1 - e^{\lambda x}$, where $\lambda > 0$. Assume that bidders values are* *distributed uniformly in* $[0,\ 1]$*. Then,*

$$\lim_{n\to\infty} R < \lim_{n\to\infty} R_{\mathrm{rn}}.$$

Proof. *See Appendix F.* $\square$

Although a last-price auction is a k-price auction with $k = n$, the results of Section 4 do not apply here. Indeed, in a last-price auction $k \longrightarrow \infty$ as $n \longrightarrow \infty$, hence the kth value approaches 0 as $n \longrightarrow \infty$. In contrast, in the k-price auctions of Section 4, k is held fixed as $n \longrightarrow \infty$. Hence, the kth value approaches 1 as $n \longrightarrow \infty$.

7 An asymptotic revenue equivalence theorem

We saw that in the case of risk-averse bidders, all large k-price auctions are $O(1/n^2)$ revenue equivalent to each other, but not to large all-pay auctions or last-price auctions. In particular, the limiting revenue approaches the risk-neutral limit for all k-price auctions, but not for all-pay auctions or last-price auctions. Therefore, a natural question is under which condition the limiting revenue would approach the risk-neutral limit. The following Theorem provides such a condition:

Theorem 1 *Consider any symmetric auction where bidders have a utility function* U *that satisfies (1). Let* $\beta^{win}(v_i, \mathbf{v}_{-i})$ *denote the equilibrium payment of bidder* i *when he wins with type* v_i*, and the other bidders have types* $\mathbf{v}_{-i} = (v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n)$*. Assume that* $\beta^{win}(v_i, \mathbf{v}_{-i}) \longrightarrow v_i$ *uniformly as* $n \longrightarrow \infty$*, i.e., that there exists a series* $\{\epsilon_n\}$*,* *independent of* v_i *and* $\mathbf{v}_{-i}$*, such that* $\lim_{n\to\infty}\epsilon_n = 0$*, and*

$$|v_i - \beta^{win}(v_i, \mathbf{v}_{-i})| \le \epsilon_n, \qquad 0 \le v_i \le 1, \quad 0 \le \mathbf{v}_{-i} \le v_i. \tag{10}$$

Then, the limiting revenue approaches the risk-neutral limit, i.e.,

$$\lim_{n\to\infty} R = \lim_{n\to\infty} R_{\mathrm{rn}}. \tag{11}$$

Proof. See Appendix G. □

Remark. The Opposite direction is not necessarily true, see Example 4 below.

Condition (10) says that the equilibrium payment of a player who wins with value v_i approaches v_i uniformly as the number of bidders goes to infinity. The motivation for this condition is as follows. When the bidder wins and Condition (10) is satisfied, then his utility is $U(v - \beta^{win}) \sim (v - \beta^{win})U'(0)$. Therefore, the utility function can be approximated with $U(x) \sim U'(0)x$, the utility of a risk-neutral bidder.

In principle, there should be a second condition in Theorem 1 that would imply that when the bidder loses, his utility is $U(-\beta^{lose}) \sim -U'(0)\beta^{lose}$, i.e., the utility of a risk-neutral bidder, where β^{lose} is the equilibrium payment of a losing bidder. This second condition is not needed, however, for the following reason. The seller's revenue can be written as

$$R = R^{win} + R^{lose}, \tag{12}$$

where

$$\begin{aligned} R^{win} &= \sum_{i=1}^n \int_0^1 E_{\mathbf{v}_{-i}}[\beta^{win}(v_i, \mathbf{v}_{-i})] F^{n-1}(v) f(v)\, dv, \\ R^{lose} &= \sum_{i=1}^n \int_0^1 E_{\mathbf{v}_{-i}}[\beta^{lose}(v_i, \mathbf{v}_{-i})](1 - F^{n-1}(v)) f(v)\, dv, \end{aligned} \tag{13}$$

are the revenues due to payments of the winning and losing bidders, respectively. When Condition (10) is satisfied, then $\lim_{n\to\infty} R^{lose} = 0$, see equation (34), or equivalently,

$$\lim_{n\to\infty} R = \lim_{n\to\infty} R^{win}. \tag{14}$$

Therefore, even if the payments of the losing bidders are affected by risk-aversion, this has no effect on the limiting revenue.

¿From the proof of Theorem 1 it immediately follows that the pointwise Condition (10) can be replaced with the weaker condition that $E_{\mathbf{v}_{-i}}[v_i - \beta^{win}(v_i, \mathbf{v}_{-i})] \le \epsilon_n$ for $0 \le v_i \le 1$. An even weaker condition ca nbe derived as follows. As noted, the limiting revenue is only due to the contribution of the payments of the winning bidders. Because R^{win} has the multiplicative term $F^{n-1}(v)$ which is exponentially small except in an $O(1/n)$ region near the maximal value, Condition (10) can be relaxed to hold only in this shrinking region:

Corollary 1 *Theorem 1 remains valid if we replace Condition (10) with the weaker condition that for any $C > 0$,*

$$|v_i - \beta^{win}(v_i, \mathbf{v}_{-i})| \le \epsilon_n, \qquad 1 - C/n \le v_i \le 1, \quad 0 \le \mathbf{v}_{-i} \le v_i. \tag{15}$$

Proof. See Appendix H. □

Example 4 *Consider a generalized all-pay auction with $F(v) = v$, $U(x) = x - 0.5x^2$, and $\alpha = 1/n$. Although $\alpha \longrightarrow 0$, the equilibrium bids are highly influenced by risk-aversion. Indeed, the bids are everywhere exponentially small (see Figure 4, left panel), except in an*

$O(1/n)$ *region near* $v = 1$ *where they approach the first-price bids (Figure 4, right panel).*[13] *Therefore, Condition (10) is not satisfied. Nevertheless, the* $O(1/n)$ *small region near the maximal value where Condition (15) holds is sufficient to have the limiting revenue go to 1, the risk-neutral limit. (Figure 5).*[14]

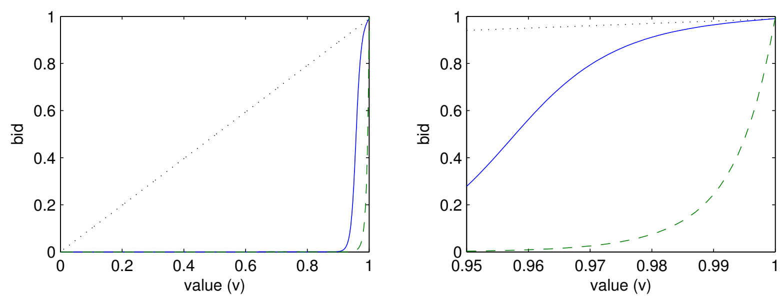

Figure 4: Equilibrium bids in generalized all-pay auction with $\alpha = 1/n$ with risk-averse players (solid line) for $n = 100$. Also plotted are the equilibrium bids in the first-price (dots) and all-pay (dashes) auctions. Right panel is a magnification of the $O(1/n)$ region near the maximal value.

In the case of a generalized all-pay auction with a fixed α, Condition (10) is satisfied at $v = 1$, the maximal value, i.e., $\lim_{n\to\infty} b(1) = 1$, see Lemma 4. However, it is not satisfied in an $O(1/n)$ neighborhood of 1. Indeed, the heart of the proof of Proposition 5 is the key observation that

$$\lim_{n\to\infty} b(1 - 1/n) \neq 1,$$

see equation (29).

[13]The transition from exponentially-small bids to the first-price bids has nothing to do with risk-aversion, as it exists also in the risk-neutral case, see equation (37).

[14]In this case, however risk aversion does affect the $O(1/n)$ correction to the revenue.

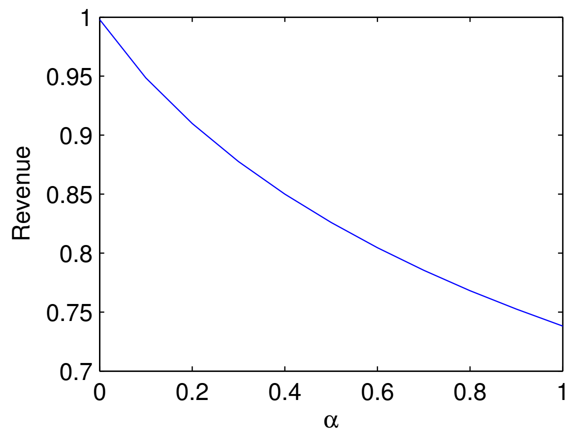

Figure 5: Expected revenue in generalized all-pay auction with $\alpha = 1/n$ with risk-averse players as a function of the number of players. Data plotted on a semi-logarithmic scale.

An obvious weakness of Theorem 1 is that Condition (10) involves the unknown bidding strategies. We note, however, that for all the auction mechanisms that we considered in this study, Condition (10) is satisfied in the risk-averse case if and only if it is satisfied in the risk-neutral case (see Appendix I). Indeed, generically, if for a given auction mechanism Condition (10) is not satisfied in the risk-neutral case, there is " no reason" for Condition (10) to be satisfied in the risk-averse case, hence it is "likely" that this auction mechanism will not be asymptotically revenue equivalent to large k-price auctions in the risk-averse case. Since in the risk-neutral case the bidding strategies are usually known explicitly, it is easy to check whether they satisfy Condition (10). For example, from equation (9) it immediately follows that any k-price auction with $k = n - 1$, $k = n - 2$, $k = n - 3, \ldots$, or with $k = n/2$, $k = n/3, \ldots$, would not satisfy Condition (10) in the risk-neutral case. This suggests that these auction mechanisms are not asymptotically revenue equivalent to large k-price auctions in the risk-averse case, as can be confirmed by a minor modification of the proof for last-price auctions in Appendix F.

Acknowledgments

We thank Aner Sela, Eilon Solan and Steven Schochet for useful discussions. The research of G. Fibich was partially supported by a F.I.R.S.T grant number 1460/04 from the Israel Science Foundation (ISF).

A Auxiliary Lemmas

Lemma 1 *Let $n \gg 1$, let $b(v) = v + (1/n)B_1(v) + (1/n^2)B_2(v) + O(1/n^3)$, and let $v(b) = b + \frac{1}{n}v_1(b) + \frac{1}{n^2}v_2(b) + O(1/n^3)$ be the inverse function of $b(v)$. Then,*

$$B_1(v) = -v_1(v), \qquad B_2(v) = -B_1(v)v_1'(v) - v_2(v). \tag{16}$$

Proof. We substitute the two expansions into the identity $v \equiv v(b(v))$ and expand in $1/n$:

$$\begin{aligned} v &= v(b(v)) = b(v) + \frac{1}{n}v_1(b(v)) + \frac{1}{n^2}v_2(b(v)) + O(1/n^3) \\ &= v + \frac{1}{n}B_1(v) + \frac{1}{n^2}B_2(v) + \frac{1}{n}v_1\left(v + \frac{1}{n}B_1(v)\right) + \frac{1}{n^2}v_2(v) + O(1/n^3) \\ &= v + \frac{1}{n}\left[B_1(v) + v_1(v)\right] + \frac{1}{n^2}\left[B_2(v) + B_1(v)v_1'(v) + v_2(v)\right] + O\left(\frac{1}{n^3}\right). \end{aligned}$$

Balancing the $O(1/n)$ and $O(1/n^2)$ terms proves (16). $\square$

In the following we calculate an asymptotic expansion of the integral $\int_0^v F^n(x)\,dx$ using integration by parts (for an introduction to asymptotic evaluation of integrals using integration by parts, see, e.g., Murray (1984)):

Lemma 2 *Let $F(v)$ be a twice-continuously differentiable, function and let $f = F' > 0$. Then, for a sufficiently large n,*

$$\int_0^v F^n(x)\,dx = \frac{1}{n}\frac{F^{n+1}(v)}{f(v)}\left[1 + O\left(\frac{1}{n}\right)\right]. \tag{17}$$

Proof. Using integration by parts,

$$\int_0^v F^n(x)\,dx = \int_0^v [F^n(x)f(x)]\frac{1}{f(x)}\,dx = \frac{1}{n+1}\frac{F^{n+1}(v)}{f(v)} + \frac{1}{n+1}\int_0^v [F^{n+1}(x)f(x)]\frac{f'(x)}{f^3(x)}\,dx$$
$$= \frac{1}{n+1}\frac{F^{n+1}(v)}{f(v)} + \frac{1}{n+1}\frac{1}{n+2}F^{n+2}(v)\frac{f'(v)}{f^3(v)} - \frac{1}{n+1}\frac{1}{n+2}\int_0^v F^{n+2}(x)\left(\frac{f'(x)}{f^3(x)}\right)'\,dx.$$

Therefore, the result follows.

B Proof of Proposition 1

Since $\lim_{n\to\infty} v(b) = b$, we can look for a solution of (2) of the form

$$v(b) = b + \frac{1}{n-1}v_1(b) + O\left(\frac{1}{n^2}\right).$$

Substitution in (2) gives

$$\begin{aligned} 1 &+ O\left(\frac{1}{n}\right) \\ &= \frac{1}{n-1}\frac{F(b) + (v_1/(n-1))f(b) + O(n^{-2})}{f(b) + (v_1/(n-1))f'(b) + O(n^{-2})} \cdot \frac{U'(0) + (v_1/(n-1))U''(0) + O(n^{-2})}{U(0) + (v_1/(n-1))U'(0) + O(n^{-2})}. \end{aligned}$$

Since $U(0) = 0$ and $U'(0) > 0$, the balance of the leading order terms gives

$$1 = \frac{F(b)}{f(b)} \cdot \frac{U'(0)}{v_1 U'(0)}.$$

Therefore, $v_1(b) = F(b)/f(b)$ and the inverse equilibrium bids are given by

$$v(b) = b + \frac{1}{n-1}\frac{F(b)}{f(b)} + O\left(\frac{1}{n^2}\right).$$

Inverting this relation (see Lemma 1) shows that the equilibrium bids are given by (3).

To calculate the expected revenue, we use (3) to obtain

$$\begin{aligned} R &= \int_0^1 b(v)\,dF^n(v) = b(1) - \int_0^1 b'(v)F^n(v)\,dv \\ &= 1 - \frac{1}{n}\frac{1}{f(1)} + O\left(\frac{1}{n^2}\right) - \int_0^1 [1 + O(1/n)]F^n(v)\,dv. \end{aligned}$$

Therefore, by (17), the result follows.

C Proof of Proposition 2

Since $\lim_{n\to\infty} v(b) = b$, we can look for a solution of the form

$$v(b) = b + \frac{1}{n-1}v_1(b) + \frac{1}{(n-1)^2}v_2(b) + O\left(\frac{1}{n^3}\right). \tag{18}$$

Substituting (18) in (2) and using $U(0) = 0$ and $0 < U'(0) < \infty$ gives

$$\begin{aligned} &1 + \frac{1}{n-1}v_1'(b) + O\left(\frac{1}{n^2}\right) \\ &= \frac{1}{n-1}\frac{F(b) + \frac{v_1}{n-1}f(b) + O(n^{-2})}{f(b) + \frac{v_1}{n-1}f'(b) + O(n^{-2})} \cdot \frac{U'(0) + \frac{v_1}{n-1}U''(0) + O(n^{-2})}{U(0) + (\frac{v_1}{n-1} + \frac{v_2}{(n-1)^2})U'(0) + \frac{v_1^2}{2(n-1)^2}U''(0) + O(n^{-3})} \\ &= \frac{F(b) + \frac{v_1}{n-1}f(b) + O(n^{-2})}{f(b) + \frac{v_1}{n-1}f'(b) + O(n^{-2})} \cdot \frac{U'(0) + \frac{v_1}{n-1}U''(0) + O(n^{-2})}{(v_1 + \frac{v_2}{(n-1)})U'(0) + \frac{v_1^2}{2(n-1)}U''(0) + O(n^{-2})} \\ &= \left(\frac{F(b)}{f(b)} + \frac{v_1}{n-1} + O(n^{-2})\right)\left(1 - \frac{v_1}{n-1}\frac{f'(b)}{f(b)} + O(n^{-2})\right) \times \\ &\quad \left(\frac{1}{v_1} + \frac{1}{n-1}\frac{U''(0)}{U'(0)} + O(n^{-2})\right)\left(1 - \frac{1}{(n-1)}\frac{v_2}{v_1} - \frac{v_1}{2(n-1)}\frac{U''(0)}{U'(0)} + O(n^{-2})\right) \\ &= \frac{F(b)}{f(b)}\frac{1}{v_1} + \frac{1}{n-1}\left[1 - \frac{f'(b)F(b)}{f^2(b)} + \frac{F(b)}{2f(b)}\frac{U''(0)}{U'(0)} - \frac{F(b)}{f(b)}\frac{v_2}{v_1^2}\right] + O(n^{-2}). \end{aligned}$$

Balancing the $O(1)$ terms gives, as before,

$$v_1(b) = \frac{F(b)}{f(b)}. \tag{19}$$

Balancing the $O(\frac{1}{n-1})$ terms gives

$$v_1'(b) = 1 - \frac{f'(b)F(b)}{f^2(b)} + \frac{F(b)}{2f(b)}\frac{U''(0)}{U'(0)} - \frac{F(b)}{f(b)}\frac{v_2}{v_1^2}.$$

Substituting $v_1(b) = F(b)/f(b)$ and $v_1'(b) = 1 - \frac{F(b)f'(b)}{f^2(b)}$ gives

$$v_2(b) = \frac{F^2(b)}{2f^2(b)}\frac{U''(0)}{U'(0)}. \tag{20}$$

Using Lemma 1 and (19,20) to invert the expansion (18) gives

$$b(v) = v + \frac{1}{n-1}B_1(v) + \frac{1}{(n-1)^2}B_2(v) + O\left(\frac{1}{n^3}\right),$$

where

$$B_1(v) = -\frac{F(v)}{f(v)}, \qquad B_2(v) = \frac{F(v)}{f(v)} - \frac{F^2(v)f'(v)}{f^3(v)} - \frac{F^2(v)}{2f^2(v)}\frac{U''(0)}{U'(0)}.$$

This completes the proof of (5).

To calculate the expected revenue, we first use (5) to obtain

$$\begin{aligned}
R &= \int_0^1 b(v)\,dF^n(v) = b(1) - \int_0^1 b'(v)F^n(v)\,dv \\
&= 1 - \frac{1}{n-1}\frac{1}{f(1)} + \frac{1}{(n-1)^2}\left[\frac{1}{f(1)} - \frac{f'(1)}{f^3(1)} - \frac{1}{2f^2(1)}\frac{U''(0)}{U'(0)}\right] \\
&\quad - \int_0^1 \left[1 - \frac{1}{n-1}\left(\frac{F(v)}{f(v)}\right)'\right]F^n(v)\,dv + O\left(\frac{1}{n^3}\right).
\end{aligned}$$

Integration by integration by parts (as in Lemma 2) gives,

$$\int_0^1 F^n(v)\,dv = \frac{1}{n+1}\frac{1}{f(1)} + \frac{1}{n+1}\frac{1}{n+2}\frac{f'(1)}{f^3(1)} + O\left(\frac{1}{n^3}\right),$$

and

$$\int_0^1 \left(\frac{F(v)}{f(v)}\right)' F^n(v)\,dv = \frac{1}{n+1}\frac{1}{f(1)}\left(\frac{F(v)}{f(v)}\right)'_{v=1} + O\left(\frac{1}{n^2}\right).$$

Therefore,

$$
\begin{aligned}
& \int_0^1 \left[1 - \frac{1}{n-1}\left(\frac{F(v)}{f(v)}\right)'\right] F^n(v)\, dv \\
= \;& \frac{1}{n}\frac{1}{1+\frac{1}{n}}\frac{1}{f(1)} + \frac{1}{n^2}\frac{f'(1)}{f^3(1)} - \frac{1}{n^2}\frac{1}{f(1)}\left(1 - \frac{f'(1)}{f^2(1)}\right) + O\left(\frac{1}{n^3}\right) \\
= \;& \frac{1}{n}\frac{1}{f(1)} - \frac{1}{n^2}\frac{2}{f(1)} + \frac{2}{n^2}\frac{f'(1)}{f^3(1)} + O\left(\frac{1}{n^3}\right).
\end{aligned}
$$

Substitution in the expression for R proves (6).

D Proof of Proposition 3

The case $k = 1$ was proved in Proposition 1. When $k = 2$ the result follows immediately, since $b(v) = v$. Therefore, we only need to prove for $k \geq 3$. In that case, the equilibrium strategies in k-price auctions are the solutions of (see Monderer and Tennenholtz (2000))

$$
\int_0^v U(v - b(t)) F^{n-k}(t)(F(v) - F(t))^{k-3} f(t)\, dt = 0. \tag{21}
$$

Defining $m = n - k$ and $t = v - s$, we can rewrite equation (21) as

$$
\begin{aligned}
0 &= \int_0^v U(v - b(t)) F^m(t)(F(v) - F(t))^{k-3} f(t)\, dt \qquad (22) \\
&= \int_0^v e^{m \ln(F(t))} U(v - b(t))(F(v) - F(t))^{k-3} f(t)\, dt \\
&= e^{m \ln F(v)} \int_0^v e^{-m[\ln F(v) - \ln F(v-s)]}\, U(v - b(v-s))(F(v) - F(v-s))^{k-3} f(v-s)\, ds.
\end{aligned}
$$

Since the maximum of $\ln(F(v-s))$ is attained at $s = 0$, we can calculate an asymptotic approximation of this integral using Laplace method (see, e.g., Murray (1984)). To do that, we make the change of variables $x(s) = [\ln F(v) - \ln F(v-s)]$ and expand all the terms in the last integral in a Taylor series in s near $s = 0$.

Expansion of $x(s)$ near $s=0$ gives $x = sf(v)/F(v) + O(s^2)$. Therefore,

$$\frac{dx}{ds} = f(v)/F(v) + O(s), \qquad s = x\frac{F(v)}{f(v)} + O(s^2), \qquad ds = \frac{dx}{f(v)/F(v)}\,[1+O(x)]$$

Let us expand the solution $b(v)$ in a power series in m, i.e.,

$$b(v) = b_0(v) + \frac{1}{m}b_1(v) + O\left(\frac{1}{m^2}\right).$$

Therefore, near $s=0$,

$$b(v-s) = b_0(v) - sb_0'(v) + \frac{1}{m}b_1(v) - \frac{1}{m}sb_1'(v) + O(s^2) + O\left(\frac{1}{m^2}\right).$$

In addition,

$$(F(v) - F(v-s))^{k-3} = (sf(v) + O(s^2))^{k-3} = s^{k-3}f^{k-3}(v)1 + O(s)],$$

and

$$f(v-s) = f(v) + O(s).$$

Substitution all the above in (22) gives

$$\begin{aligned}
0 \;&=\; \int_0^v \Bigg\{ e^{-mx}\, U\left[v - \left(b_0(v) - sb_0'(v) + \frac{b_1(v)}{m} - \frac{sb_1'(v)}{m} + O(s^2) + O\left(\frac{1}{m^2}\right)\right)\right] \times \\
&\qquad\qquad s^{k-3}f^{k-3}(v)\,[1+O(s)]\,[f(v)+O(s)]\Bigg\}\, ds \\
&\sim\; \int_0^\infty \Bigg\{ e^{-mx}\, U\left[v - \left(b_0(v) - x\frac{F(v)}{f(v)}b_0'(v) + \frac{1}{m}b_1(v) - \frac{x}{m}\frac{F(v)}{f(v)}b_1'(v) + O(x^2) + O\left(\frac{1}{m^2}\right)\right)\right] \times \\
&\qquad\qquad x^{k-3}F^{k-3}(v)\,[1+O(x)]\,[f(v)+O(x)]\,\frac{dx}{f(v)/F(v)}\,[1+O(x)]\Bigg\} \\
&=\; F^{k-2}(v)\int_0^\infty \Bigg\{ e^{-mx}\,\Bigg[U(v-b_0(v)) + U'(v-b_0(v))\left(x\frac{F(v)}{f(v)}b_0'(v) - \frac{b_1(v)}{m} + \frac{x}{m}\frac{F(v)}{f(v)}b_1'(v)\right) \\
&\qquad\qquad +O(x^2) + O\left(\frac{1}{m^2}\right)\Bigg]\, x^{k-3}\,[1+O(x)]\Bigg\}\, dx. \qquad (23)
\end{aligned}$$

We recall that for p integer, $\int_0^\infty e^{-mx} x^p \, dx = p!/m^{p+1}$. Therefore, balancing the leading $O(m^{-(k-2)})$ terms gives

$$U(v - b_0(v)) F^{k-2}(v) \int_0^\infty e^{-mx} \, x^{k-3} \, dx = 0.$$

Since $U(z) = 0$ only at $z = 0$, this implies that $b_0(v) \equiv v$. Using this and $U'(0) = 0$, equation (23) reduces to

$$0 = \int_0^\infty \left\{ e^{-mx} \left(x \frac{F(v)}{f(v)} - \frac{1}{m} b_1(v) + \frac{x}{m} \frac{F(v)}{f(v)} b_1'(v) \right) \left[x^{k-3} + O(x^{k-2}) \right] \right\} dx$$

Therefore, balance of the next-order $O(m^{-(k-1)})$ terms gives

$$\frac{F(v)}{f(v)} \int_0^\infty e^{-mx} x^{k-2} \, dx - \frac{1}{m} b_1(v) \int_0^\infty e^{-mx} x^{k-3} \, dx = 0,$$

or

$$\frac{F(v)}{f(v)} \frac{(k-2)!}{m^{k-1}} - \frac{(k-3)!}{m^{k-1}} b_1(v) = 0.$$

Therefore,

$$b_1(v) = (k-2) \frac{F(v)}{f(v)}.$$

Hence, we proved (8).

The seller's expected revenue in a k-price auction is given by

$$R_k = \int_0^1 b(v) dF_k(v),$$

where $b(v)$ is the equilibrium bid in the k price auction and $F_k(v)$ is the distribution of the k-th valuation in order (i.e., k-order statistic of the bidders private valuations). Substituting the asymptotic expansion for the equilibrium bids gives

$$R_k = \int_0^1 \left[v + \frac{k-2}{n-k} \frac{F(v)}{f(v)} \right] dF_k(v) + O(\frac{1}{n^2}).$$

Since the asymptotic expansion for the equilibrium bid is independent of the utility function U until order $O(\frac{1}{n^2})$, the revenue in the risk-averse case is the same as in the risk-neutral case, with $O(\frac{1}{n^2})$ accuracy. By the revenue equivalence theorem, the latter is given by (4).

E Proof of Proposition 5

We first show that the maximal bid $b(1)$ is monotonically increasing in α:

Lemma 3 *Consider a generalized all-pay auction where bidders valuations are distributed according to $F(v)$ in $[0,\ \ 1]$, and bidders have a utility function U that satisfies (1). Then,*

$$\frac{\partial b(1)}{\partial \alpha} > 0, \qquad 0 \le \alpha \le 1.$$

Proof. Let

$$V(v) = F^{n-1}(v)U(v - b(v)) + (1 - F^{n-1}(v)U(-\alpha b(v)) \tag{24}$$

be the expected utility of a bidder with value v. By Milgrom and Weber (1982),

$$V'^{n-1}(v)U'(v - b(v)). \tag{25}$$

In addition, differentiating (24) with respect to α gives

$$\begin{aligned} \frac{\partial V(v)}{\partial \alpha} &= -\frac{\partial b}{\partial \alpha}\left(F^{n-1}(v)U'(v - b(v)) + (1 - F^{n-1}(v))U'(-\alpha b(v))\right) \\ &\quad -b(v)(1 - F^{n-1}(v))U'(-\alpha b(v)). \end{aligned} \tag{26}$$

We now prove that $\frac{\partial V(v)}{\partial \alpha} < 0$ for all $0 < \alpha, v \leq 1$. By negation, assume that $\frac{\partial V(v)}{\partial \alpha} \geq 0$ for some $0 < v_1, \alpha_1 \leq 1$. Then, from equation (26) it follows that $\frac{\partial b}{\partial \alpha}|_{v_1,\alpha_1} < 0$. Hence, by risk aversion and (25),

$$\left.\frac{\partial}{\partial \alpha} V'(v)\right|_{v_1,\alpha_1} = \left.-\frac{\partial b}{\partial \alpha} F^{n-1}(v) U''(v - b(v))\right|_{v_1,\alpha_1} < 0. \tag{27}$$

Denote $y(v) = V_{\alpha_1+\Delta\alpha}(v) - V_{\alpha_1}(v)$, where $0 < \Delta\alpha$. By the negation assumption, if $\Delta\alpha$ is sufficiently small, then $y(v_1) \geq 0$. Hence, by (27), $y'(v_1) = V'_{\alpha+\Delta}(v_1) - V'_{\alpha}(v_1) < 0$. Thus, $y(t) = V_{\alpha+\Delta}(t) - V_{\alpha}(t) > 0$ for t slightly below v_1, and therefore by a continuation argument for every $0 \leq t < v_1$. This contradicts the fact that $y(0) = V_{\alpha+\Delta}(0) - V_{\alpha}(0) = 0$, since $V(0) = 0$ for every α.

We have thus proved that

$$0 > \frac{\partial V(1)}{\partial \alpha} = -\frac{\partial b(1)}{\partial \alpha} U'(1 - b(1)).$$

Therefore, the result follows. □

Therefore, the maximal bid approaches the maximal value:

Lemma 4 *Under the conditions of Lemma 3,*

$$\lim_{n \to \infty} b(1) = 1, \qquad \text{for} \;\; 0 \leq \alpha \leq 1.$$

Proof. From Lemma 3 we have that $b(1)$ is monotonically increasing in α. Therefore,

$$b(1; \alpha = 0) < b(1; \alpha) \leq 1.$$

Since for $\alpha = 0$ we have a first price auction, from equation (3) it follows that $\lim_{n\to\infty} b(1; \alpha = 0) = 1$. Therefore, the result follows. □

We now turn to the proof of Proposition 5. Let $V(v)$, defined by (24), be the expected utility of a bidder with value v in equilibrium. Then,

$$\begin{aligned} 0 &\leq n\int_0^1 V(v)f(v)\,dv = nU'(0)\int_0^1 [vF^{n-1}(v) - \alpha b - (1-\alpha)F^{n-1}(v)b]f(v)\,dv - C_n \\ &= U'(0)A_n - U'(0)R - C_n, \end{aligned}$$

where

$$\begin{aligned} C_n &= nU'(0)\int_0^1 [vF^{n-1}(v) - \alpha b - (1-\alpha)F^{n-1}(v)b]f(v)\,dv - n\int_0^1 V(v)f(v)\,dv, \\ A_n &= n\int_0^1 vF^{n-1}(v)f(v)\,dv, \\ R &= n\int_0^1 [bF^{n-1} + \alpha b(1-F^{n-1}]f\,dv. \end{aligned}$$

Therefore,

$$R \leq A_n - \frac{C_n}{U'(0)}.$$

Since

$$A_n = \int_0^1 v(F^n)' = 1 - \int_0^1 F^n = 1 + O(1/n),$$

see equation (17), then $\lim_{n\to\infty} A_n = 1$. Therefore, to finish the proof, we only need to show that

$$\lim_{n\to\infty} C_n > 0.$$

Now,

$$\begin{aligned} C_n &= -n\int_0^1 \left[F^{n-1}(v)\left(U(v-b) - (v-b)U'(0)\right) + (1-F^{n-1}(v))\left(U(-\alpha b) + \alpha b U'(0)\right)\right] f(v)\,dv \\ &= -n\int_0^1 \left[F^{n-1}(v)\frac{(v-b)^2}{2}U''(\theta_1(v)) + (1-F^{n-1}(v))\frac{\alpha^2 b^2}{2}U''(\theta_2(v))\right] f(v)\,dv, \end{aligned}$$

where $0 < \theta_1(v) < v - b(v)$ and $-b(v) < \theta_2(v) < 0$. Since $-U'' \geq M > 0$, we have that

$$\begin{aligned} C_n &\geq Mn \int_0^1 \left[F^{n-1}(v) \frac{(v-b)^2}{2} + (1 - F^{n-1}(v)) \frac{\alpha^2 b^2}{2} \right] f(v)\, dv \\ &\geq Mn \int_0^1 F^{n-1}(v) \frac{(v-b)^2}{2} f(v)\, dv. \end{aligned}$$

We now show that the limit of the right-hand-side is strictly positive. Indeed,

$$\begin{aligned} \int_0^1 nF^{n-1}(v) f(v)(v-b)^2\, dv &= \int_0^1 (F^n(v))' (v-b)^2\, dv \\ &= F^n(v)(v-b)^2 \Big|_0^1 - 2\int_0^1 F^n(v)(v-b)(1-b')\, dv \qquad (28) \\ &= (1-b(1))^2 - 2\int_0^1 F^n(v)(v-b)\, dv + 2\int_0^1 F^n(v)(v-b)b'\, dv. \end{aligned}$$

We claim that the first two terms go to zero, but the third term goes to a positive constant. Indeed, since $\lim_{n\to\infty} b(1) = 1$, see Lemma 4, the first term in (28) approaches zero. Since $(v - b)$ is bounded, the second term also goes to zero, see Lemma 2. As for the third term,

$$\int_0^1 F^n(v)(v-b)b' dv \geq \int_{1-1/n}^1 F^n(v)(v-b)b'\, dv.$$

Now, $F^n(1 - 1/n) \geq C_1 > 0$. Indeed,

$$F\left(1 - \frac{1}{n}\right) = 1 - \frac{1}{n} f(\theta), \qquad 1 - \frac{1}{n} < \theta < 1.$$

Therefore,

$$F^n\left(1 - \frac{1}{n}\right) \geq \left(1 - \frac{\max f}{n}\right)^n \longrightarrow e^{-\max f}.$$

Therefore,

$$\int_{1-1/n}^1 F^n(v)(v-b)b' dv \geq C_1 \int_{1-1/n}^1 (v-b)b' dv.$$

In addition,

$$\int_{1-1/n}^{1} (v-b)b'\,dv = vb\Big|_{1-1/n}^{1} - \int_{1-1/n}^{1} b\,dv - \frac{b^2}{2}\Big|_{1-1/n}^{1}$$
$$= b(1) - (1-1/n)b(1-1/n) - \int_{1-1/n}^{1} b\,dv - \frac{b^2(1)}{2} + \frac{b^2(1-1/n)}{2}.$$

As n goes to infinity, $b(1) \to 1$ and $\int_{1-1/n}^{1} b\,dv \to 0$. Hence,

$$\lim_{n\to\infty} \int_{1-1/n}^{1} (v-b)b'\,dv = \frac{1}{2}(1-X_\infty)^2,$$

where

$$X_\infty = \lim_{n\to\infty} X_n, \qquad X_n = b(1-1/n).$$

We now show that

$$X_\infty < 1, \tag{29}$$

and this will complete the proof. By Taylor expansion,

$$1 - X_n = 1 - b(1) + \frac{1}{n} b'(\theta), \qquad 1 - 1/n < \theta < 1. \tag{30}$$

Recall that

$$b'(v) = (n-1)F^{n-2}(v)f(v)\frac{U(v-b(v)) - U(-\alpha b(v))}{F^{n-1}(v)U'(v-b(v)) + \alpha(1-F^{n-1}(v))U'(-\alpha b(v))}.$$

Now, for $v \in (1-1/n, 1)$, as $n \longrightarrow \infty$,

$$F^{n-2}(v) \geq F^n(v) \geq C_1, \qquad f(v) \geq \min f(v),$$

$$U(v-b(v)) - U(-\alpha b(v)) = (v-(1-\alpha)b(v))U'(\theta_2) \geq (v-(1-\alpha)v)U'(\theta_2) \geq \alpha(1-1/n)U'(1),$$

and

$$F^{n-1}(v)U'(v-b(v)) + \alpha(1-F^{n-1}(v))U'(-\alpha b(v)) \leq U'(-1).$$

Therefore, there exists $C_2 > 0$ such that

$$b'(v) \geq (n-1)C_2, \qquad 1 - 1/n < v < 1.$$

Thus, since $1 - b(1) \to 0$, eq. (30) implies that $\lim_{n\to\infty}(1 - X_n) \geq C_2 > 0$.

F Last-price auctions

The equilibrium strategy in last-price auctions with $F(x) = x$ and a CARA utility $U(x) = 1 - e^{-\lambda x}$ is the solution of, see equation (21),

$$\int_0^v [1 - e^{-\lambda(v-b(t))}](v-t)^{n-3}\, dt = 0.$$

Therefore,

$$\int_0^v e^{\lambda b(t)}(v-t)^{n-3}\, dt = e^{\lambda v}\frac{v^{n-2}}{n-2}.$$

Differentiating $n-3$ times with respect to v gives

$$(n-3)! \int_0^v e^{\lambda b(t)}\, dt = \frac{d^{n-3}}{dv^{n-3}}\left(e^{\lambda v}\frac{v^{n-2}}{n-2}\right).$$

One more differentiation gives

$$(n-2)! e^{\lambda b(v)} = \frac{d^{n-2}}{dv^{n-2}}\left(e^{\lambda v} v^{n-2}\right).$$

Therefore,

$$\begin{aligned}
b(v) &= \frac{1}{\lambda}\ln\left[\frac{1}{(n-2)!}\frac{d^{n-2}}{dv^{n-2}}\left(e^{\lambda v}v^{n-2}\right)\right] = \frac{1}{\lambda}\ln\left[e^{\lambda v}\sum_{k=0}^{n-2}\binom{n-2}{k}\frac{1}{k!}\lambda^k v^k\right] \\
&= v + \frac{1}{\lambda}\ln\left[\sum_{k=0}^{n-2}\binom{n-2}{k}\frac{1}{k!}\lambda^k v^k\right] \leq v + \frac{1}{\lambda}\ln\left[\sum_{k=0}^{n-2}\binom{n-2}{k}\lambda^k v^k\right] \\
&= v + \frac{1}{\lambda}\ln\left[(1+\lambda v)^{n-2}\right] \leq v + \frac{1}{\lambda}\ln\left[(e^{\lambda v})^{n-2}\right] = (n-1)v = b_{rn}(v),
\end{aligned}$$

where $b_{rn}(v)$ is the equilibrium strategy in the risk-neutral case, see equation (9). Hence,

$$\begin{aligned}
\lambda(b_{rn}-b) &\geq \ln\left[\sum_{k=0}^{n-2}\binom{n-2}{k}\lambda^k v^k\right] - \ln\left[\sum_{k=0}^{n-2}\binom{n-2}{k}\frac{1}{k!}\lambda^k v^k\right] \\
&\geq \ln\left[\sum_{k=0}^{n-2}\binom{n-2}{k}\lambda^k v^k\right] - \ln\left[-\frac{1}{2}\binom{n-2}{2}\lambda^2 v^2 + \sum_{k=0}^{n-2}\binom{n-2}{k}\lambda^k v^k\right] \\
&= \ln\left[(1+\lambda v)^{n-2}\right] - \ln\left[-\frac{1}{2}\binom{n-2}{2}\lambda^2 v^2 + (1+\lambda v)^{n-2}\right].
\end{aligned}$$

Since $\ln b - \ln a \geq (b-a)/b$ for $0 < a < b$, we get that

$$\lambda(b_{rn}-b) \geq \frac{1}{2}\binom{n-2}{2}\lambda^2 v^2 \frac{1}{(1+\lambda v)^{n-2}}.$$

The distribution function of the lowest value is $F_{(n)} = 1-(1-v)^n$, hence the expected revenue is given by $R = \int_0^1 b(v)\,dF_{(n)} = n\int_0^1 b(v)(1-v)^{n-1}\,dv$. Therefore,

$$\begin{aligned}
R_{rn} - R &= n\int_0^1 [b_{rn}(v) - b(v)](1-v)^{n-1}\,dv \\
&\geq \frac{\lambda}{4}n(n-2)(n-3)\int_0^{1/n} v^2 \frac{1}{(1+\lambda v)^{n-2}}(1-v)^{n-1}\,dv \\
&\geq \frac{\lambda}{4}n(n-2)(n-3)\int_0^{1/n} v^2 \frac{1}{(1+\lambda/n)^{n-2}}(1-1/n)^{n-1}\,dv \\
&= \frac{\lambda}{4}n(n-2)(n-3)\frac{1}{3n^3}\frac{1}{(1+\lambda/n)^{n-2}}(1-1/n)^{n-1}.
\end{aligned}$$

Taking the limit, we have that

$$\lim_{n\to\infty}(R_{rn} - R) \geq \frac{\lambda}{12}e^{-\lambda-1} > 0.$$

G Proof of Theorem 1

Let $P(v) = F^{n-1}(v)$ be the probability of winning of a bidder with value v. Since $n\int_0^1 P(v)f(v)\,dv = 1$, from Condition (10) it follows that

$$n\int_0^1 E_{\mathbf{v}_{-i}}[\beta^{win}(v_i, \mathbf{v}_{-i}) - v_i]P(v_i)f(v_i)\,dv_i = O(\epsilon_n). \tag{31}$$

Let

$$S_i(v_i) = E_{\mathbf{v}_{-i}}\left[U(v_i - \beta(v_i, \mathbf{v}_{-i})) \,\middle|\, i \; wins\right] P(v_i) + E_{\mathbf{v}_{-i}}\left[U(-\beta(v_i, \mathbf{v}_{-i})) \,\middle|\, i \; loses\right](1 - P(v_i)),$$

be the expected surplus of a risk-averse bidder i when his type is v_i. From now on, we suppress the subindex i and the dependence on $\mathbf{v}_{-i}$, and introduce the notations β^{win} and β^{lose} for the equilibrium payment when bidder i wins or loses, respectively. Therefore, the last relation can be rewritten as

$$S(v) = E\left[U(v - \beta^{win}(v))\right] P(v) + E\left[U(-\beta^{lose}(v))\right](1 - P(v)). \tag{32}$$

Similarly, the expected revenue can be written as $R = R^{win} + R^{lose}$, where

$$R^{win} = n\int_0^1 E[\beta^{win}(v)]P(v)f(v)\,dv, \qquad R^{lose} = n\int_0^1 E[\beta^{lose}(v))](1 - P(v))f(v)\,dv.$$

From equation (31) it follows that

$$R^{win} = n\int_0^1 vP(v)f(v)\,dv + O(\epsilon_n) = 1 + O(1/n) + O(\epsilon_n) = R_{rn} + O(1/n) + O(\epsilon_n). \tag{33}$$

We now show that relation (31) implies that

$$R^{lose} = O(\epsilon_n). \tag{34}$$

Indeed, from equation (32) and the fact that $S_i \geq 0$, we have that

$$-E\left[U(-\beta^{lose}(v))\right](1-P(v)) \leq E\left[U(v-\beta^{win}(v))\right]P(v). \tag{35}$$

Since $U(-x) = -xU'(0) + x^2/2U''(\theta(-x)) < -xU'(0)$, it follows that $xU'(0) \leq -U(-x)$. Therefore, by (35) and the fact that the payments are positive,

$$0 \leq U'(0)E\left[\beta^{lose}(v)\right](1-P(v)) \leq E\left[U(v-\beta^{win}(v))\right]P(v).$$

Hence,

$$\begin{aligned} 0 &\leq U'(0)n\int_0^1 E\left[\beta^{lose}(v))\right](1-P(v))f(v)\,dv \\ &\leq n\int_0^1 E\left[U(v-\beta^{win}(v))\right]P(v)f(v)\,dv = O(\epsilon_n), \end{aligned}$$

where is the last stage we used (31). Therefore, we proved (34).

Combining (33,34) we get that $R = R_{rn} + O(1/n) + O(\epsilon_n)$. Therefore, we proved equation (11).

H Proof of Corollary 1

In the proof of Theorem 1 we used Condition (10) to conclude that

$$\lim_{n\to\infty} n\int_0^1 E[\beta^{win}(v) - v]P(v)f(v)\,dv = 0.$$

Therefore, we need to show that this limit does not change even if (10) holds "only" for $1 - C/n \leq v \leq 1$. To see that, we note that

$$n\int_0^1 E[\beta^{win}(v) - v]P(v)f(v)\,dv = I_1 + I_2,$$

where

$$I_1 = n\int_0^{1-C/n} E[\beta^{win}(v)-v]P(v)f(v)\,dv, \quad I_2 = n\int_{1-C/n}^{1} E[\beta^{win}(v)-v]P(v)f(v)\,dv.$$

Since $0 \le E[\beta^{win}(v)] \le v \le 1$,

$$I_1 \le n\int_0^{1-C/n} P(v)f(v)\,dv = F^n(1-C/n).$$

Now,

$$F\,(1-C/n) = 1 - C/nf(\theta_n), \qquad 1-C/n < \theta_n < 1.$$

Therefore, as $n \longrightarrow \infty$,

$$F^n\,(1-C/n) = \left(1 - \frac{Cf(\theta_n)}{n}\right)^n \longrightarrow e^{-Cf(1)}. \tag{36}$$

Therefore, we can choose C sufficiently large so that $|I_1| \le \epsilon/2$. In addition,

$$|I_2| \le \epsilon_n n\int_{1-C/n}^{1} P(v)f(v)\,dv \le \epsilon_n n\int_0^1 P(v)f(v)\,dv = \epsilon_n.$$

Therefore, we can choose n sufficiently large so that $|I_2| \le \epsilon/2$. Therefore, the result follows.

I Condition (10) in the risk-neutral case

- The risk-neutral equilibrium bids, hence payments, in the first-price and all-pay auctions are given by

$$\beta_{rn}^{1st}(v) = b_{rn}^{1st}(v) = v - \frac{1}{F^{n-1}(v)}\int_0^v F^{n-1}(s)\,ds, \qquad \beta_{rn}^{all}(v) = b_{rn}^{all}(v) = F^{n-1}(v)\beta_{rn}^{1st}(v).$$

Hence, by Lemma 2, as $n \longrightarrow \infty$,

$$\beta_{rn}^{1st}(v) \sim v - \frac{1}{n}\frac{F(v)}{f(v)}, \qquad \beta_{rn}^{all}(v) \sim F^{n-1}(v)[v - \frac{1}{n}\frac{F(v)}{f(v)}].$$

Therefore, Condition (10) is satisfied for (risk-neutral) first-price auction, but not for the all-pay auction.

- In the case of generalized all-pay auctions, the equilibrium bid function is the solution of

$$b'(v) = (n-1)F^{n-2}(v)f(v)\frac{U(v-b(v)) - U(-\alpha b(v))}{F^{n-1}(v)U'(v-b(v)) + \alpha(1-F^{n-1}(v))U'(-\alpha b(v))}, \qquad b(0) = 0.$$

This equation can be explicitly solved in the risk-neutral case, yielding

$$\begin{aligned} (\beta_{rn}^{gen-all})^{win}(v) &= b_{rn}^{gen-all}(v) = \frac{vF^{n-1}(v) - \int_0^v F^{n-1}(s)\,ds}{\alpha + (1-\alpha)F^{n-1}(v)} \\ &= \frac{F^{n-1}(v)}{\alpha + (1-\alpha)F^{n-1}(v)} b_{rn}^{1st}(v). \end{aligned} \tag{37}$$

Hence, $(\beta_{rn}^{gen-all})^{win}(v) \longrightarrow v$ provided that $\frac{F^{n-1}(v)}{\alpha+(1-\alpha)F^{n-1}(v)} \longrightarrow 1$. If α is held constant, then $\frac{F^{n-1}(v)}{\alpha+(1-\alpha)F^{n-1}(v)} \longrightarrow 1$ when $F^{n-1}(v) \longrightarrow 1$, i.e., for $1-v \ll 1/n$ but not for $1-v = O(1/n)$. Therefore, Condition (10) is not satisfied. If, however, $\alpha = \alpha(n)$ and $\lim_{n\to\infty} \alpha = 0$, then by (36) $\frac{F^{n-1}(v)}{\alpha+(1-\alpha)F^{n-1}(v)} \longrightarrow 1$ for $1-v = O(1/n)$, but not for all $0 \le v \le 1$. Therefore, Condition (10) is not satisfied, but its weaker form (see Corollary 1) is satisfied.

- In the case of k-price auctions, $\lim_{n\to\infty} b_{rn}^{k-price}(v) = v$, see equation (9). In addition, as $n \longrightarrow \infty$, the kth value approach the value of the winning bidder. Therefore, $\lim_{n\to\infty} \beta_{rn}^{k-price}(v) = v$, so that Condition (10) is satisfied.

- Finally, in the case of last-price auctions,

$$(\beta_{rn}^{last})^{win}(v_i, \mathbf{v}_{-i}) = b_{rn}^{last}(v_{\min}) = v_{\min} + (n-2)\frac{F(v_{\min})}{f(v_{\min})}, \qquad v_{\min} = \min_{j \neq i} v_j,$$

see equation (9). In addition, since $b_{rn}^{last}(v_{\min})$ is independent of v, then it is not converging to v for all v as $n \longrightarrow \infty$. Therefore, Condition (10) is not satisfied.

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