A note on approximating Max-Bisection on regular graphs

Uriel Feige Marek Karpinski Michael Langberg

November 12, 2000

Abstract

We design a 0.795 approximation algorithm for the Max-Bisection problem restricted to regular graphs. In the case of three regular graphs our results imply an approximation ratio of 0.834.

1 Introduction

Let $G = (V, E)$ be an undirected graph. The Maximum Cut of G is a partition of the vertex set V into two arbitrarily sized sets (X, Y) such that the number of edges with one end point in X and the other in Y is maximal. The Maximum Bisection of $G = (V, E)$ is a partition of V into two equally sized sets (X, Y) (*i.e.* a *bisection* of V) that maximizes the number of edges between X and Y (here we assume that $|V|$ is even). Given a graph G, we say that an algorithm A approximates the Max-Cut (Max-Bisection) of G within an approximation ratio of r, if by running A on the graph G we obtain a partition (bisection) (X, Y) such that the number of edges between X and Y is at least r times the number of edges between the two sides of an optimal partition (bisection). Note that $r \leq 1$.

A graph G is said to be *regular* if all its vertices have equal degree. In the following work we analyze the ratio between the Maximum Bisection of any given regular graph G, and its Maximum Cut. For general graphs it is not hard to see that this ratio can be arbitrarily close to 1/2. For regular graphs we show that this ratio is at least approximately 0.9027, and that there are infinitely many regular graphs which obtain a ratio arbitrarily close to 0.9027. We then use this property to present a 0.795 approximation algorithm for the Max-Bisection problem restricted to regular graphs. In the case of three regular graphs our results imply an approximation ratio of 0.834.

The best known approximation ratio for Max-Cut on regular graphs is 0.87856, presented in the work of Goemans and Williamson [GW95]. Extending their work, [FJ97] obtain a 0.6511 approximation algorithm for the general Max-Bisection problem. A further line of extensions by [Ye99, HZ00] improve this ratio to 0.701. Observe that $0.87856 \cdot 0.9027 \simeq 0.793$, and our approximation ratio for Max-Bisection on regular graphs slightly improves over this. On the negative side, it has been shown by [Hås97] that approximating the Max-Cut and Max-Bisection problems on general graphs beyond the ratio of $\frac{16}{17}$ is NP-hard. Furthermore, [BK98] show that approximating Max-Cut on 3-regular graphs beyond some explicit constant factor r strictly less than one is also NP-hard.

The structure of our paper is as follows. In Section 2 we analyze the ratio between the Maximum Bisection of any given regular graph G, and its Maximum Cut. In Section 3 we present and analyze an 0.795 approximation algorithm for the Max-Bisection problem restricted to regular graphs. Finally, in Section 4 we show that the ratio obtained in Section 2 is tight.

2 Max Cut vs. Max-Bisection

Let $G = (V, E)$ be a Δ regular graph, where $V = \{v_1 \dots v_n\}$ and n is even. For every $X \subseteq V$ and $Y = V \setminus X$, let $w(X)$ be the number of edges cut by the partition (X, Y). We call $w(X)$ the *value* of the partition (X, Y). Let $Cut(G)$ be the value of the Max-Cut of G, and $Bis(G)$ be the value of the Max-Bisection of G. Given a partition (X, Y), the *out-degree* of any $v \in V$ is the number of neighbors that v has on the opposite side of the partition, and the *in-degree* is the number of neighbors that v has on its own side of the partition. We say that a partition (X, Y) is *locally optimal* if there is no vertex v with out-degree smaller than in-degree. Clearly if (X, Y) is not locally optimal one may obtain a partition of value strictly greater than $w(X)$ by moving a vertex with out-degree smaller than its in-degree from one side of the partition to the other.

Theorem 2.1 *Given a regular graph G and any partition (X, Y) of G with value $w(X)$, one can efficiently find a bisection $(\hat{X}, \hat{Y})$ of G of value $\theta w(X)$, where $\theta \simeq 0.9027$. Specifically we have that $Bis(G) \geq \theta Cut(G)$.*

Proof : Let $G = (V, E)$ be a Δ regular graph and (X, Y) be any cut in G of value $w(X)$. As a preliminary step we would like to turn the partition (X, Y) into a locally optimal one. This is done by iteratively moving vertices with out-degree strictly smaller than their in-degree from one side of the partition to the other. Note that the new partition is of value greater than or equal to the original partition. We keep our original notation and denote this improved partition as (X, Y). Let cn be the size of X where $c \geq 1/2$, and $x|E|$ be the value of the partition (X, Y), *i.e.* $w(X) = x|E|$.

From the definition of c, we conclude that $cn\Delta/2 \leq w(X) \leq \Delta(1-c)n$ as there are only $(1-c)n$ vertices in Y, and the partition (X, Y) is locally optimal. Hence $c \leq x \leq 2-2c$ (and $c \leq 2/3$). We would now like to move vertices from X to Y in order to obtain a bisection of G, *i.e.* a partition $(\hat{X}, \hat{Y})$ for which $|\hat{X}| = |\hat{Y}| = n/2$.

Let $\hat{X}$ be a random subset of X of size $n/2$, and $\hat{Y}$ be $V \setminus \hat{X}$. Denote the value of the bisection $(\hat{X}, \hat{Y})$ by $w(\hat{X})$. In the following we compute the expected value of $w(\hat{X})$. Afterwards, using a greedy derandomization scheme, we show that a bisection of G with this expected value can be efficiently obtained from (X, Y).

Define the function $\theta(x)$ to be the expected ratio between the values $w(\hat{X})$ and $w(X)$ given $w(X) = x|E|$, and θ to be a lower bound of this ratio.

Lemma 2.2 *Let X and $\hat{X}$ be as above (in particular $w(X) = x|E|$). The expected ratio between $w(\hat{X})$ and $w(X)$ is at least $\theta(x) = \frac{1-x}{(2-x)x} + \frac{1}{(2-x)^2} \geq 0.9027$.*

Proof : Let $p = \frac{1}{2c}$. Recall that $\hat{X}$ is a random subset of X of size exactly $n/2$. The expected number of edges that were cut by the partition (X, Y) that are still cut in $(\hat{X}, \hat{Y})$ is $w(X)\binom{cn-1}{n/2-1}/\binom{cn}{n/2} = w(X)p = xp|E|$. As each vertex in G is of degree Δ, we have that the number of edges in the subgraph induced by X is $(\Delta cn - x|E|)/2 = (2c-x)|E|/2$. We conclude that the expected number of these edges cut by the partition $(\hat{X}, \hat{Y})$ is $(2c - x)|E|\binom{cn-2}{n/2-1}/\binom{cn}{n/2} \geq (2c-x)|E|p(1-p)$.

Thus, given any partition (X, Y) of value $w(X)$ we may obtain a partition $(\hat{X}, \hat{Y})$ such that

$$E\left[\frac{w(\hat{X})}{w(X)}\right] \geq \frac{xp|E| + (2c-x)|E|p(1-p)}{x|E|} = \frac{x + (2c-x)\left(1 - \frac{1}{2c}\right)}{2cx} = \frac{2c-1}{2cx} + \frac{1}{4c^2}.$$

The expression above, as a function of x is decreasing, and is thus minimal when $x = 2 - 2c$. Replacing c in the above expression with $1 - x/2$ we conclude that

$$E\left[\frac{w(\hat{X})}{w(X)}\right] \geq \frac{1-x}{(2-x)x} + \frac{1}{(2-x)^2}.$$

Using basic computations that are presented in the appendix, it can be seen that the above expression obtains a minimal value of $\theta \simeq 0.9027$ when $x^* \simeq 0.7932$ ($c^* = 1 - x^*/2 \simeq 0.6034 \leq x^*$). $\square$

It is left to show that given a partition (X, Y) of value $w(X)$ a bisection $(\hat{X}, \hat{Y})$ with value at least the expected can be obtained efficiently. Consider the random process analyzed above, that fixes a bisection by setting $\hat{X}$ to be a random subset of X of size $n/2$. Note that choosing a random subset $\hat{X} \subseteq X$ of size $n/2$ is equivalent to removing a random subset of size $|X| - n/2$ from X and setting $\hat{X}$ to be the remaining vertices of X. Furthermore, the latter is equivalent to the random process which iteratively removes one vertex in X at a time, uniformly at random, until X is of size $n/2$. For $i \in [0, |X| - n/2]$ let (X_i, Y_i) be the partition obtained by this last random process at step i (the size of X_i is $|X| - i$), and set $(\hat{X}, \hat{Y})$ to be the bisection $(X_{|X|-n/2}, Y_{|X|-n/2})$. Let $E[w(X_i)]$ be the expected value of the partition (X_i, Y_i). Given the value $w(X_i)$, the value of $E[w(X_{i+1})]$ can be explicitly computed. A random vertex in X_i has expected out-degree of $d = w(X_i)/|X_i|$ and expected in-degree $\Delta - d$ thus

$$E[w(X_{i+1}) \mid w(X_i)] = w(X_i)\left(1 - \frac{2}{|X_i|}\right) + \Delta.$$

We conclude that

$$E[w(X_{i+1})] = E[E[w(X_{i+1}) \mid w(X_i)]] = E[w(X_i)]\left(1 - \frac{2}{|X_i|}\right) + \Delta.$$

In the following we prove that the greedy process which at each step removes the vertex in X with lowest out-degree until the set X is of size exactly $n/2$, will obtain a bisection of value at least $E[w(X_i)]$ for each $i \in [0, |X| - n/2]$. As $\hat{X}$ is set to be X_i for $i = |X| - n/2$, this completes our proof.

Let (A_i, B_i) be the partition obtained by the above greedy process at step i, and let $w(A_i)$ be the value of the partition (A_i, B_i). Clearly $w(A_0) = E[w(X_0)] = w(X)$. Furthermore, using the fact that for each i there is a vertex in A_i with out-degree at most $w(A_i)/|A_i|$ we have that

$$w(A_{i+1}) \geq w(A_i)\left(1 - \frac{2}{|A_i|}\right) + \Delta.$$

We now conclude our proof using induction on i :

$$w(A_{i+1}) \geq w(A_i)\left(1 - \frac{2}{|A_i|}\right) + \Delta \geq E[w(X_i)]\left(1 - \frac{2}{|X_i|}\right) + \Delta = E[w(X_{i+1})].$$

□

In Section 4 we show that our result is tight, namely :

Proposition 2.3 *There are infinitely many regular graphs G for which the ratio between $Bis(G)$ and $Cut(G)$ is arbitrarily close to $\theta \simeq 0.9027$.*

3 Approximating Max-Bisection on regular graphs

In a recent breakthrough, Goemans and Williamson [GW95] present a 0.87856 approximation algorithm based on semidefinite programming for the general Max-Cut problem. Their work is extended in [FJ97, Ye99, HZ00] to obtain a 0.701 approximation algorithm for the general Max-Bisection problem. On the negative side, it has been shown by [Hås97] that approximating the Max-Cut and Max-Bisection problems on general graphs beyond the ratio of $\frac{16}{17}$ is NP-hard. Furthermore, [BK98] show that approximating Max-Cut on 3-regular graphs beyond some explicit constant factor r strictly less than one is also NP-hard.

In the following we extend the result of [BK98] to the Max-Bisection problem restricted to regular graphs, and use the work of [GW95] with the results of the previous section to achieve an approximation ratio of 0.795 on this restriction of Max-Bisection.

Proposition 3.1 *There exists some explicit constant $r < 1$ for which it is NP-hard to approximate the Max-Bisection of 3-regular graphs.*

Proof : Let $G = (V, E)$ be a 3-regular graph on n vertices. Consider the graph $\hat{G}$ consisting of two disjoint copies of G. Clearly $2Cut(G) = Bis(\hat{G})$. Thus approximating the Max-Bisection of $\hat{G}$ within an approximation ratio of r yields an approximation of the Max-Cut of G within the same ratio. Combining this with the result of [BK98] stated above, our proof is complete. □

Theorem 3.2 *The Max-Bisection problem on regular graphs can be approximated within a ratio of* 0.795*.*

Proof : Consider the well known Max-Cut algorithm based on semidefinite programming presented in [GW95]. In this algorithm, given a graph $G = (V, E)$, a semidefinite relaxation of the Max-Cut problem on G is solved yielding an embedding of G on the n dimensional unit sphere. This embedding is then rounded using the random hyperplane rounding technique, into a partition of G. In general, it is shown in [GW95] that the expected value of this partition is at least $\alpha = 0.87856$ times the value of the optimal cut in G.

Given a Δ regular graph $G = (V, E)$, using the algorithm of [GW95] one may obtain a partition (X, Y) of G of value $w(X) \geq \alpha Cut(G)$. Applying Theorem 2.1 on this partition naively, a bisection $(\hat{X}, \hat{Y})$ of value at least $\theta w(X) \geq \alpha\theta Cut(G) \geq \alpha\theta Bis(G) \simeq 0.793 Bis(G)$ may be obtained. We conclude that a 0.793 approximation algorithm for Max-Bisection on regular graphs is achieved by combining the algorithm of [GW95] and Theorem 2.1. A slight improvement in this ratio may be achieved by noticing that the *worst case* value of θ is obtained when $w(X)$ is of value $x^*|E| = 0.7932|E|$, while the *worst case* approximation ratio α of the [GW95] algorithm is obtained when $w(X)$ is of value $0.742|E|$. Details follow.

Denote the value of the semidefinite relaxation of G as $\delta|E|$. It is shown in [GW95] that the value $w(X)$ of the partition (X, Y) is at least $x(\delta)|E|$ where

$$x(\delta) = \begin{cases} \frac{acos(1-2\delta)}{\pi} & \delta \geq 0.8445 \\ \delta \cdot 0.87856 & \delta < 0.8445 \end{cases}$$

Assume that $w(X)$ is exactly of value $x(\delta)|E|$. Recall, using Lemma 2.2, that we may obtain a bisection $(\hat{X}, \hat{Y})$ of value at least $\theta(x(\delta))w(X)$ where

$$\theta(x(\delta)) = \frac{1 - x(\delta)}{(2 - x(\delta))x(\delta)} + \frac{1}{(2 - x(\delta))^2}.$$

We conclude that in such a case, the value of the bisection $(\hat{X}, \hat{Y})$ is at least $x(\delta)\theta(x(\delta))|E| \geq \frac{x(\delta)\theta(x(\delta))}{\delta} Bis(G)$. Using basic calculations which are described in the appendix it can be seen that the above is minimal when $\delta \simeq 0.8748$, yielding an approximation ratio of 0.7953.

It is left to show that if the partition (X, Y) is of value greater than that promised by the analysis of [GW95], we obtain a strictly higher approximation ratio. Assume that $w(X)$ is of value $y|E|$ for some y greater than $x(\delta)$. In such a case we may obtain a bisection of G of value $\frac{y\theta(y)}{\delta} Bis(G)$. We conclude that in order to prove our claim it is enough to show that the function $x\theta(x)$ is increasing. Using basic calculations, which are described in the appendix, it can be seen that this is true. $\square$

Two remarks regarding the result and proof of Theorem 3.2 are in place. The result above holds for regular graphs of arbitrary degree. Using the work of [FKL00], further improved approximation ratios for the Max-Bisection problem can be achieved when we assume the degree is constant. For instance, [FKL00] show that the Max-Cut problem on 3-regular graphs can be approximated within an approximation ratio of 0.924. Thus, combining this result with the result of Theorem 2.1, we conclude an 0.834 approximation ratio on the Max-Bisection problem restricted to 3-regular graphs.

Regarding the proof of Theorem 3.2, we use the results of [GW95] which are based on a semidefinite relaxation for the Max-Cut problem. As we are interested in approximating the maximum bisection, one may add additional constraints to this semidefinite relaxation as is done in [FJ97]. It would be interesting to see if such an addition can improve our results.

4 Upper bound

Proposition 2.3 *There are infinitely many regular graphs G for which the ratio between $Bis(G)$ and $Cut(G)$ is arbitrarily close to $\theta \simeq 0.9027$.*

Proof : We construct a constant degree regular graph $G = (V, E) = (X, Y; E)$ where X and Y are a partition of V, X is of size cn, and the ratio between $Bis(G)$ and $Cut(G)$ is arbitrarily close to $\theta \simeq 0.9027$. In general, our construction is random. We start by constructing a random regular graph H_x on the vertex set X, and a random regular bipartite graph H_{xy} on the vertex sets X and Y. Afterwards we show that for their union H the ratio between $Bis(H)$ and $Cut(H)$ is close to θ. Using the notation of Theorem 2.1 let $x \simeq 0.7932$ and $c = 1 - x/2$.

In the following we randomly construct a Δ_1 regular multi-graph H_x on the vertices of X. Consider the graph $\hat{H}_x$ consisting of $|X| = cn$ disjoint sets $\{S_1 \dots S_{|X|}\}$ of Δ_1 vertices each, *i.e.* a set of Δ_1 vertices corresponding to each vertex of H_x. Define the edge set of $\hat{H}_x$ to be a random perfect matching on its vertices. Note that $\hat{H}_x$ has exactly $\Delta_1|X|/2$ edges. Define H_x to be the multi-graph obtained by *shrinking* each set S_i of vertices in $\hat{H}_x$ into a single vertex i of H_x. That is H_x is the graph with a single vertex i corresponding to each set S_i in which each edge connecting S_i and S_j in $\hat{H}_x$ is expressed as an edge (i, j) in H_x. Following we analyze some properties of H_x.

Lemma 4.1 *For every constant $\varepsilon > 0$ there exists a constant Δ_1 such that with constant probability the following holds.* (a) *H_x will not include self loops or multiple edges, and* (b) *the number of edges in every partition $(A, X \setminus A)$ of H_x is at most $\frac{\Delta_1|A|(cn-|A|)}{cn} + \varepsilon\Delta_1 cn$.*

Proof : Part *(a)* of Lemma 4.1 is proven in [BMK86] where it is shown that with some constant probability (depending on Δ_1) H_x will not include self loops or multiple edges. For part *(b)*, let A be some subset of X of size at most $|X|/2$ and define B to be $X \setminus A$. The probability that the cut $(A, X \setminus A)$ has value k is exactly

$$\Pr(w(A) = k) \;=\; \frac{k! \begin{pmatrix} \Delta_1|A| \\ k \end{pmatrix} \begin{pmatrix} \Delta_1|B| \\ k \end{pmatrix} M(\Delta_1|A| - k) M(\Delta_1|B| - k)}{M(\Delta_1|X|)}$$

Where $M(i) = \frac{i!}{(\frac{i}{2})!\, 2^{\frac{i}{2}}}$ is the number of perfect matchings in a graph with i vertices. Using basic calculations which are described in the appendix it can be seen that $\Pr(w(A) = k)$ is at most $\delta^{\Delta_1 cn}$ for some constant $\delta < 1$ (dependent on ε) for any $\Delta_1|A| \geq k \geq \frac{\Delta_1|A|(cn-|A|)}{cn} + \varepsilon\Delta_1 cn$. As there are at most 2^{cn} subsets A of X and the range of k is polynomial in n, we conclude by choosing Δ_1 large enough that with overwhelming probability $(1 - \delta^n)$ part *(b)* of our lemma holds. Hence both properties *(a)* and *(b)* hold for the random graph H_x with some constant probability. □

We now construct the multi-graph H_{xy}, a bipartite graph on the vertex sets X and Y in which the degree of each vertex in Y is Δ and the degree of each vertex in X is $\Delta(1/c - 1)$. The construction is similar to the construction of H_x presented above. Consider the graph $\hat{H}_{xy}$ consisting of $|X| = cn$ disjoint sets $\{S_1 \dots S_{|X|}\}$ each of $\Delta(1/c - 1)$ vertices, and $|Y| = (1 - c)n$ disjoint sets $\{R_1 \dots R_{|Y|}\}$ each of Δ vertices. Define the edge set of $\hat{H}_{xy}$ to be a random bipartite perfect matching between the vertices in $\{S_1 \dots S_{|X|}\}$ and $\{R_1 \dots R_{|Y|}\}$. Define H_{xy} to be the multi-graph obtained by shrinking each vertex set S_i into a single vertex $i \in X$ and each vertex set R_i into a single vertex $i \in Y$.

Lemma 4.2 *For every constant $\varepsilon > 0$ there exists a constant Δ such that with constant probability the following holds.* (a) *H_{xy} will not include multiple edges, and* (b) *the number of edges in H_{xy} between every two subsets $A \subseteq X$ and $B \subseteq Y$ is at most $\frac{\Delta|A||B|}{cn} + \varepsilon\Delta(1 - c)n$.*

Proof : As in Lemma 4.1, part *(a)* of the lemma is stated in [BMK86] (with constant probability). For part *(b)*, let $N = (1 - c)\Delta n = \Delta|Y|$. Let A be some subset of X and B be some subset of Y. Let $\Delta(1/c - 1)|A| = \alpha N$, and $\Delta|B| = \beta N$. The probability that there are k edges between A and B in H_{xy} is exactly

$$\Pr(w(A, B) = k) = \frac{\binom{\alpha N}{k}\binom{\beta N}{k} k! \binom{N - \alpha N}{\beta N - k} (\beta N - k)!(N - \beta N)!}{N!}$$

Similarly to Lemma 4.1, basic calculations which are described in the appendix yield that $\Pr(w(A, B) = k)$ is at most δ^N for some constant $\delta < 1$ (dependent on ε) for any $k \geq \frac{\Delta|A||B|}{cn} + \varepsilon\Delta(1 - c)n$. As there are at most 2^{2n} subsets A,B of X,Y respectively and the range of k is polynomial in n, we conclude that part *(b)* of our lemma holds with overwhelming probability. Hence both properties *(a)* and *(b)* hold for the random graph H_{xy} with constant probability. □

Set Δ_1 to be $(2-1/c)\Delta$ and define H to be the union of the two graphs H_x and H_{xy}. Assume that H_x and H_{xy} have the properties stated in Lemma 4.1 and 4.2 (this happens with constant probability). It is not hard to verify that H is a Δ regular graph, and that the value of the partition (X, Y) in H is $\Delta(1-c)n$. Thus $Cut(H)$ is at least this value. Following, we show that the maximum bisection of H is at most $(\theta + 8\varepsilon)Cut(H)$, thus completing our proof.

Let (U, V) be some bisection of H, where $U = X_1 \cup Y_1$, $V = X_2 \cup Y_2$, the sets X_i are some partition of X, and the sets Y_i are some partition of Y. Denote the size of the set X_1 as γn, the size of X_2 as $(c - \gamma)n$, the size of Y_1 as $(1/2 - \gamma)n$, and the size of Y_2 as $(1/2 - c + \gamma)n$ for

$\gamma \in [c/2, 1/2]$. Let p_x be $\frac{\Delta(2-1/c)}{cn}$ and p_{xy} be $\frac{\Delta}{cn}$. We have that the value of the bisection (U, V) is at most

$$n^2\left(\gamma(c-\gamma)(p_x - 2p_{xy}) + \frac{c \cdot p_{xy}}{2}\right) + 3\varepsilon n\Delta.$$

Which is maximal when $\gamma = 1/2$. Thus we conclude using the fact that $c = 1 - x/2$ that

$$\frac{Bis(H)}{Cut(H)} \leq \left(\frac{1 - \frac{2c-1}{2c^2}}{x}\right) + 8\varepsilon = \left(\frac{1-x}{(2-x)x} + \frac{1}{(2-x)^2}\right) + 8\varepsilon \simeq \theta + 8\varepsilon.$$

□

References

[BK98] P. Berman and M. Karpinski. On some tighter inapproximability results, further improvements. *ECCC*, TR98-065, 1998. Extended abstract appears in *ICALP* 1999, pages 200-209.

[BMK86] B. Bollobás and B.D. McKay. The number of matchings in random graphs and bipartite graphs. *J. of Combinatorial Theory*, Ser. B 41:80–91, 1986.

[FJ97] A. Frieze and M. Jerrum. Improved approximation algorithms for Max-k-Cut and Max-Bisection. *Algorithmica*, 18:67–81, 1997.

[FKL00] U. Feige, M. Karpinski, and M. Langberg. Improved approximation of max-cut on graphs of bounded degree. *ECCC*, TR00-021, 2000.

[GW95] M.X. Goemans and D.P. Williamson. Improved approximation algorithms for maximum cut and satisfiability problems using semidefinite programming. *Journal of ACM*, 42:1115–1145, 1995.

[Hås97] J. Håstad. Some optimal inapproximability results. *In Proceedings of the 28th Annual ACM Symposium on Theory of Computing*, pages 1–10, 1997.

[HZ00] E. Halperin and U. Zwick. Improved approximation algorithms for maximum graph bisection problems. *Manuscript*, 2000.

[Hoe63] W. Hoeffding. Probability inequalities for sums of bounded random variables. *American Statistical Association Journal*, 58(301):13–30, 1963.

[Ye99] Y. Ye. A 0.699-approximation algorithm for Max-Bisection. *Submitted to Math Programming, available at URL http://dollar.biz.uiowa.edu/col/ye/*, 1999.

Appendix

Recall that

$$\begin{aligned} x(\delta) &= \begin{cases} \frac{acos(1-2\delta)}{\pi} & \delta \geq 0.8445 \\ \delta \cdot 0.87856 & \delta < 0.8445 \end{cases} \\ \theta(x) &= \frac{1-x}{(2-x)x} + \frac{1}{(2-x)^2}. \end{aligned}$$

Bounding $\theta(x)$ (Lemma 2.2) :

$$\theta'(x) = -\frac{x^3 - 2x^2 + 6x - 4}{x^2(x-2)^3}.$$

Using computer assisted analysis, it can be seen that $\theta'(x)$ is zero only when $x^* \simeq 0.7932$, yielding a lower bound of approximately 0.9027. Note that $\theta(x)$ is decreasing when $x \leq 0.7932$ and increasing otherwise.

Monotonicity of $x\theta(x)$ (Theorem 3.2) :

$$(x\theta(x))' = \frac{2x}{(2-x)^3}.$$

It can be easily seen that $(x\theta(x))'$ is positive for every $x \in [0.5, 1]$ (the range of our interest).

Bounding $\frac{x(\delta)\theta(x(\delta))}{\delta}$ (Theorem 3.2) :

We consider three cases, the first in which $\delta \in [0.5, 0.8445]$, the second in which δ is in the range $[0.8445, 0.8981]$, and the last in which $\delta \in [0.8981, 1]$. In the first case we have that $\frac{x(\delta)\theta(x(\delta))}{\delta}$ is equal to $0.87856 \cdot \theta(\delta \cdot 0.87856)$. As $\delta \cdot 0.87856 \in [0.4392, 0.7419]$, and $\theta(x)$ is monotone decreasing when $x \leq 0.7932$, we conclude that $\frac{x(\delta)\theta(x(\delta))}{\delta}$ is decreasing in this range and

$$\frac{x(\delta)\theta(x(\delta))}{\delta} \geq 0.87856 \cdot \theta(0.7419) \geq 0.7983.$$

In the third case, in which $\delta \in [0.8981, 1]$, we have that $x(\delta) \in [0.7932, 1]$. Using the fact that $x'(\delta) \geq 0$ we have

$$\left(\frac{x(\delta)}{\delta}\right)' = \left(\frac{2x(\delta)}{1 - \cos(\pi x(\delta))}\right)' = \frac{2(1 - \cos(\pi x(\delta)) - \pi x(\delta)\sin(\pi x(\delta)))}{(1 - \cos(\pi x(\delta)))^2} x'(\delta) \geq 0.$$

As $\theta(x)$ is monotone increasing when $x \geq 0.7932$, we conclude that $\frac{x(\delta)\theta(x(\delta))}{\delta}$ is increasing in the range $[0.8981, 1]$ and is of a minimal value of 0.7972 when $\delta = 0.8981$.

The final case in which $\delta \in [0.8445, 0.8981]$ is proven using computer assisted analysis. A plot of $\frac{x(\delta)\theta(x(\delta))}{\delta}$ in the above range is displayed in Figure 1. It can be seen that the function obtains a minimal value of approximately 0.7953 when $\delta \simeq 0.8748$. We thus conclude that $\frac{x(\delta)\theta(x(\delta))}{\delta}$ is bounded by 0.7953 in the range $\delta \in [0.5, 1]$.

Lemma 4.1 :

Recall that $|X|$ is of size cn, $A \subseteq X$, $k \geq \frac{\Delta_1|A|(cn-|A|)}{cn} + \varepsilon\Delta_1 cn$, and

$$\Pr(w(A) = k) \; = \; \frac{k! \binom{\Delta_1|A|}{k} \binom{\Delta_1|B|}{k} M(\Delta_1|A| - k) M(\Delta_1|B| - k)}{M(\Delta_1|X|)}$$

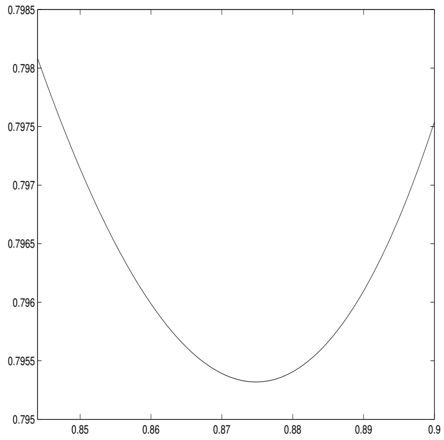

Figure 1: The function $\frac{x(\delta)\theta(x(\delta))}{\delta}$ in the range $\delta \in [0.8445, 0.8981]$.

Where $M(i) = \frac{i!}{(\frac{i}{2})!\ 2^{\frac{i}{2}}}$. Let $N = \Delta_1 cn$, $\Delta_1|A| = \alpha N$ and $k = \beta N$. The condition above on k implies that $\beta \geq \alpha(1-\alpha) + \varepsilon$. Ignoring factors which are polynomial in n we conclude using *Stirling's formula* that

$$\Pr(w(A) = k) \simeq \left(\frac{\alpha^\alpha (1-\alpha)^{(1-\alpha)}}{\beta^\beta (\alpha - \beta)^{\frac{\alpha-\beta}{2}} (1-\alpha-\beta)^{\frac{1-\alpha-\beta}{2}}} \right)^N$$

The above formula is decreasing in β (as long as $\beta \geq \alpha(\alpha - 1)$). Furthermore, using computer assisted analysis, it can be seen that by setting β to be $\alpha(1-\alpha) + \varepsilon$ the resulting formula is increasing in α (as long as $\alpha \leq 1/2$). We conclude that

$$\Pr(w(A) = k) \leq \left(2 \left(\frac{1}{4} + \varepsilon \right)^{\frac{1}{4}+\varepsilon} \left(\frac{1}{4} - \varepsilon \right)^{\frac{1}{4}-\varepsilon} \right)^{-N} \leq e^{-4\varepsilon^2 N}$$

where the second inequality is proven in [Hoe63].

Lemma 4.2 :

Recall that X is of size cn, Y is of size $(1-c)n$, $A \subseteq X$, $B \subseteq Y$, and $k \geq \frac{\Delta|A||B|}{cn} + \varepsilon\Delta(1-c)n$. Let $N = (1-c)\Delta n$, $\Delta(1/c - 1)|A| = \alpha N$, $\Delta|B| = \beta N$, and $k = \gamma N$. The condition above on k implies that $\gamma \geq \alpha\beta + \varepsilon$. Ignoring factors which are polynomial in n we conclude using *Stirling's formula* that

$$\Pr(w(A, B) = k) \simeq \frac{\alpha^\alpha \beta^\beta (1-\beta)^{(1-\beta)} (1-\alpha)^{(1-\alpha)}}{\gamma^\gamma (\alpha-\gamma)^{(\alpha-\gamma)} (\beta-\gamma)^{(\beta-\gamma)} (1-\alpha-\beta+\gamma)^{(1-\alpha-\beta+\gamma)}}$$

Using computer assisted analysis it can be seen that the above formula is maximal when $\alpha = \beta = 1/2$ and $\gamma = \alpha\beta + \varepsilon$. In such a case we obtain

$$\Pr(w(A,B) = k) \leq \left(2 \left(\frac{1}{4} + \varepsilon \right)^{\frac{1}{4}+\varepsilon} \left(\frac{1}{4} - \varepsilon \right)^{\frac{1}{4}-\varepsilon} \right)^{-2N} \leq e^{-8\varepsilon^2 N}.$$