The ROBDD size of simple CNF formulas

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Abstract. Reduced Ordered Binary Decision diagrams (ROBDDs) are nowadays one of the most common dynamic data structures for Boolean functions. Among the many areas of application are verification, model checking, and computer aided design. In the last few years, SAT checkers, based on the CNF representation of Boolean functions are getting more and more attention as an alternative to the ROBDD based methods. We show the difference between the CNF representation and the ROBDD representation in one of the most degenerate cases — random monotone 2CNF formulas. We examine this model and give almost matching lower and upper bounds for the ROBDD size in different cases, and show that as soon as the formulas are non-trivial the ROBDD size becomes exponential, thus showing perhaps one of the most fundamental advantages of SAT solvers over ROBDDs.

1 Introduction

Automatic manipulation of formulas in propositional logic is of major importance in both theoretical and practical computer science. In the VLSI and process analysis communities Reduced Ordered Binary Decision Diagrams (ROBDDs) are popular. Their usage, initiated by Bryant [B86], has caused a considerable increase of the scale of systems that can be verified. In the last few years SAT checkers have appeared as a very competitive alternative to the ROBDD based techniques, Clarke et al. [BCCF99] probably being the initiator of this trend.

It is a common place saying that ROBDDs and SAT complement each other, *i.e.*, there are cases where the ROBDD technique will work better, and those where SAT will. Indeed, Groote and Zantema [GZ01] show that the ROBDD proof of the pigeon hole principal takes exponential size ROBDDs while the unit resolution proof is polynomial. In the other direction, they also give a family of formulas, where an ROBDD based proof is polynomial, while already the CNF representation is exponential. Ideally, for understanding the different faults and merits of both techniques, we would like to have a characterization of the size relation between the two representations of boolean formulas — in CNF form, and in ROBDD form. Hopefully, such an understanding will help in the construction of a new data structure which will combine the good qualities of both ROBDDs and SAT solvers.

There has been some previous work on the size of ROBDDs, Gropl et al. [GPS01] for example, investigates the largest possible size of an ROBDD over all functions over n variables. Bollig and Wegener [BW00] examine the worst case ROBDD size of a function with a given number of 1-inputs (among other questions). Woelfel [W01] gives very tight bounds on the ROBDD size of the integer multiplication function, which was one of the first examples of a function with a polynomially sized circuit but an exponential size ROBDD, proved originally by Bryant [B86].

In this paper we examine a very degenerate type of CNF formulas, monotone 2CNF formulas, consisting only of clauses with 2 variables, and no negation. We consider *random* monotone 2CNF formulas with n variables where each of the $\binom{n}{2}$ possible clauses is chosen with probability p. These formulas are clearly always satisfiable, and the (expected) number of satisfying assignments depends on p (this number decreases as p increases). Moreover, the simple syntactic structure of these formulas may lead one to believe that their ROBDD structure is succinct. We show that this is far from being true.

In this work, we present a full characterization of the ROBDD size of random monotone 2CNF formulas. Namely, for practically every value of p, we study the ROBDD size of such random formulas and present matching (up to low order terms) lower and upper bounds on this size. Our results show that except for very small p, where the formula is degenerate, or very large p, where the formula has only a polynomial number of satisfying assignments, the most probable ROBDD size (under *any* ordering of the variables in the formula) is highly exponential, very closely related to the number of satisfying assignments to the formula. Thus we show that the ROBDD reductions are of little use when handling these simple CNF formulas.

Let φ_p be a random monotone 2CNF formula with n variables, in which each of the $\binom{n}{2}$ possible clauses is chosen with probability p. Our results can be (roughly) summarized as follows:

1. Let $p < (1-\epsilon)\frac{1}{n}$, where $\epsilon > 0$ is constant. Notice that in this case a random formula φ_p is expected to have less than $n/2$ clauses (implying that each variable is expected to appear at most once in φ_p). Then w.h.p. the ROBDD size of φ_p is polynomial.
2. Let p satisfy (a) $(1+\epsilon)\frac{1}{n} < p$ for some constant $\epsilon > 0$, and (b) For every constant $\alpha > 0$, $p < 1 - \frac{1}{n^\alpha}$ (*i.e.* p is not *very* small or large). Then w.h.p. the ROBDD size of φ_p is super polynomial. Specifically, we show that for small values of p in the range defined above, the ROBDD size of φ_p is in the range $\left[2^{\frac{1}{p}\frac{1}{\mathrm{polylog} n}}, 2^{\frac{1}{p}\mathrm{polylog} n}\right]$; and for large values of p, the ROBDD size of φ_p is equal to $2^{\Theta\left(\frac{\log^2 n}{\log 1/(1-p)}\right)}$ (w.h.p.). For example for $p = 1/\sqrt{n}$ the ROBDD size of φ_p is roughly $2^{\sqrt{n}\mathrm{polylog} n}$, and for $p = 1/2$ this size is roughly $2^{\log^2 n} = n^{\log n}$. Notice the sharp jump in the ROBDD size, with respect to case 1 above, with a very small increase of p.
3. If there exists some constant $\alpha > 0$ such that $p > 1 - \frac{1}{n^\alpha}$, then w.h.p. the ROBDD size of φ_p is again polynomial.

An important point in these bounds, is that the upper bounds in items 2 and 3 above are derived by showing an upper bound to the number of satisfying assignments to the formula. The fact that these bounds practically match the lower bounds means that the ROBDD reductions are of very little use for these kinds of formulas — we might as well have written a list of all satisfying assignments as a description of the formula.

Along the way, we show that for small p, it is the *pathwidth* of the formula which determines the optimal ROBDD size. This parameter captures in a simple manner the concept of information flow that is caused by the variable ordering in the ROBDD method. In our restricted setting, this result can be seen as a matching lower bound to Berman's [B89] classic upper bound on ROBDD size, relating circuit structure and ROBDD size using a notion similar to our pathwidth. Also, this result formalizes the common sense intuition of ROBDD ordering, and thus shows one of the fundamental drawbacks of ROBDDs, if an ordering does not put related variables close to one another — the ROBDD size will be large.

The remainder of this paper is organized as follows. In Section 2 we present the main definitions and notation that will be used throughout this work. Specifically we show a natural characterization of random monotone 2CNF formulas φ_p on n variables by the distribution $\mathcal{G}_{n,p}$ on graphs with n vertices. In Section 3 we show a connection between the ROBDD size of monotone 2CNF formulas and certain combinatorial graph properties. We then define the *pathwidth* of a formula, a notion which plays a major role in our analysis. Finally, in Section 4 we state the upper and lower bounds sketched above rigorously and proceed in their proof. Due to space limitations, some of our results appear without detailed proof. A full version can be found at,

`http://www.wisdom.weizmann.ac.il/~verify/publications/2003/LPR03.html`

2 Preliminaries and notation

2.1 Graphs

For a graph G, denote its set of vertices by V, and its set of edges by E. Let n be the size of V, and m be the size of E. We denote by $d(G)$ the maximum degree of a vertex in G. For a set of vertices $U \subseteq V$ define its set of neighbors as $\Gamma_G(U) = \{v \in V \mid v \notin U, \exists u \in U, (u,v) \in E\}$. Denote the subgraph induced by a subset U of vertices as $G_{|_U}$, *i.e.*, $G_{|_U} = \langle U, E \cap (U \times U)\rangle$. We say $U \subseteq V$ is an independent set if the edge set of $G_{|_U}$ is empty. Let $\mathrm{ID}(G)$ denote the set of independent sets of the graph G. Denote the size of the largest independent set in G by $\mathrm{maxID}(G)$. The definitions above imply that,

Proposition 1. $|\mathrm{ID}(G)| \leq n^{\mathrm{maxID}(G)}$

Let $\mathcal{G}_V$ be the set of graphs on vertex set V. For short, we mark $\mathcal{G}_n = \mathcal{G}_{[1,n]}$.

2.2 Boolean formulas

Let Δ_V denote the set of Boolean assignments to the variable set V, $\Delta_V = \{\alpha \mid \alpha : V \to \{0,1\}\}$. Let $\Phi_V = \{\varphi \mid \varphi \subseteq \Delta_V\}$ denote the set of all Boolean formulas on the variable set V (φ is characterized by its set of satisfying assignments). For $\alpha \in \Delta_V$, $U \subseteq V$, denote by $\alpha_{|_U} \in \Delta_U$ the restriction of assignment α to the set U. We would also like to consider the restriction of the formula φ to a partial assignment. For $\varphi \in \Phi_V$, $U \subseteq V$, and some $\alpha \in \Delta_U$, let

$$\varphi_{|_\alpha} = \left\{ \beta \in \Delta_{V \setminus U} \;\middle|\; \exists \gamma \in \varphi, \gamma_{|_U} = \alpha \text{ and } \gamma_{|_{V\setminus U}} = \beta \right\}$$

Again we will mark $\Phi_n = \Phi_{[1,n]}$, and $\Delta_n = \Delta_{[1,n]}$.

2.3 Random Monotone 2CNF formulas

In 2.2 we considered only the semantics of boolean formulas by characterizing them using their satisfying set of assignments. We now proceed to consider the representation of a formula, its syntax. We consider a restricted class of CNF formulas, monotone 2CNF formulas. A monotone 2CNF formula over variable set V is the conjunction of a set of clauses of the form $(a \vee b)$ where a, b are in V. We can equivalently model such a formula by a graph $G \in \mathcal{G}_V$, where each edge (a, b) in the graph stands for the clause $(a \vee b)$. We then get that the formula corresponding to the graph G is

$$\varphi_G = \{\alpha \in \Delta_V \mid \forall (i,j) \in E(G), \alpha(i) = 1 \text{ or } \alpha(j) = 1\}$$

We will consider such random formulas, using the random model $\mathcal{G}_{n,p}$, where $G \in \mathcal{G}_{n,p}$ is a graph on vertices $[1,n]$, where each possible edge is in the graph with probability p, uniformly and independently. We will say an event in $\mathcal{G}_{n,p}$ happens with high probability if it happens with probability tending to 1 as n approaches infinity.

2.4 ROBDDs — Reduced Ordered Binary Decision Diagrams

Definition 1. *An OBDD on* $[1,n]$ *is a edge labeled directed graph, whose sinks are labeled by Boolean constants* FALSE *and* TRUE, *and whose non sink (or inner) nodes are labeled by elements of* $[1,n]$. *Each inner node has two outgoing edges, one labeled by 0 and the other by 1. An edge leading from an* i*-node must end in a sink or a* j*-node, where* $j > i$. *Each inner node* v *with label* k, *represents a Boolean formula* $\varphi_v \in \Phi_{[k,n]}$ *defined in the following way. In order to check if* $\alpha \in \varphi_v$, $\alpha \in \Delta_{[k,n]}$, *start at* v. *After reaching an* i*-node, choose the outgoing edge with label* $\alpha(i)$, *until a sink is reached. If the label of the sink is* TRUE *then* $\alpha \in \varphi_v$, *if it is* FALSE *then* $\alpha \notin \varphi_v$. *The size of the OBDD is defined to be its number of nodes.*

Bryant [B86] has already shown that the minimal size OBDD for a formula $\varphi \in \Phi_n$ is unique (up to isomorphism), and is called the ROBDD of φ. If we add an additional requirement, that every edge leaving an i-node, reaches a sink or an $(i+1)$-node, then we get a slightly different version of ROBDDs, called Quasi-reduced OBDDs (QOBDDs). In this paper we will actually consider this latter type, because of the following two lemmas (see [BW00] for example):

Lemma 1. *The number of i-nodes, $1 < i \leq n$, of the QOBDD of $\varphi \in \Phi_n$ is* $\left|\left\{ \varphi_{|_\alpha} \mid \alpha \in \Delta_{i-1} \right\}\right|$.

Lemma 2. *If s_R is the size of the ROBDD of $\varphi \in \Phi_n$, and s_Q is the size of its QOBDD, then $\frac{1}{n} s_Q \leq s_R \leq s_Q$.*

The first Lemma allows us to deal with the size of QOBDD in a simple manner, and the second Lemma shows that the size of QOBDDs is practically the same as that of ROBDDs, especially since all size lower bounds we show will have an exponential nature. Therefore, for the remainder of the paper, we will examine only QOBDDs. For $\varphi \in \Phi_n$, we denote by $\mathrm{BDD}(\varphi)$, the size of φ's QOBDD. For simplicity, we will not count the root node and the two leaf nodes of the QOBDD when calculating $\mathrm{BDD}(\varphi)$, this changes the QOBDD size by at most 3, and so is immaterial. We get the following proposition,

Proposition 2. *For $\varphi \in \Phi_n$, $\mathrm{BDD}(\varphi) = \sum_{k=1}^{n-1} \left|\left\{ \varphi_{|_\alpha} \mid \alpha \in \Delta_k \right\}\right|$*

We note the following useful upper bound on QOBDD size.

Proposition 3. *For $\varphi \in \Phi_n$, $\mathrm{BDD}(\varphi) < n(|\varphi| + 1)$.*

Proof. By Proposition 2,

$$\mathrm{BDD}(\varphi) = \sum_{k=1}^{n-1} \left|\left\{ \varphi_{|_\alpha} \mid \alpha \in \Delta_k \right\}\right| \leq \sum_{k=1}^{n-1} \left(\left|\left\{ \alpha \in \Delta_k \mid \varphi_{|_\alpha} \neq \emptyset \right\}\right| + 1 \right)$$

For every $\alpha \in \Delta_k$, such that $\varphi_{|_\alpha} \neq \emptyset$, there is at least one $\beta \in \varphi$ s.t. $\beta_{|_{[1,k]}} = \alpha$. Choose one of these β and mark it by β_α. Clearly if $\alpha_1 \neq \alpha_2$ then $\beta_{\alpha_1} \neq \beta_{\alpha_2}$, and so $\left|\left\{ \alpha \in \Delta_k \mid \varphi_{|_\alpha} \neq \emptyset \right\}\right| \leq |\varphi|$ and we conclude, $\mathrm{BDD}(\varphi) \leq (n-1)(|\varphi| + 1) < n(|\varphi| + 1)$. □

As is well known, the QOBDD of a formula φ depends on the specific ordering of variables in φ. Denote by S_n the set of permutations on the set $[1, n]$. For a formula $\varphi \in \Phi_n$, and a permutation $\sigma \in S_n$, denote

$$\varphi^\sigma = \{ \alpha \mid \exists \beta \in \varphi, \forall v \in V, \alpha(\sigma(v)) = \beta(v) \}$$

φ^σ is the result of changing the names of the variables of φ. This change may result in a change of $\mathrm{BDD}(\varphi)$, and in fact there are known examples (see for example [CGP]), where $\mathrm{BDD}(\varphi)$ is polynomial, while for some σ, $\mathrm{BDD}(\varphi^\sigma)$ is exponential. We therefore denote,

$$\mathrm{mBDD}(\varphi) = \min_{\sigma \in S_n} \mathrm{BDD}(\varphi^\sigma)$$

Clearly, Proposition 3 applies also to $\mathrm{mBDD}(\varphi)$.

3 QOBDD size vs. combinatorial graph properties

Let G be a graph in $\mathcal{G}_n$. Let $\varphi = \varphi_G \in \Phi_n$ be the 2CNF formula corresponding to G. In this section we show various connections between combinatorial properties of G and the size of the QOBDD of φ. We will need the following definition. For $\alpha \in \Delta_n$ denote $Z_\alpha = \{v \in V \mid \alpha(v) = 0\}$.

Lemma 3. $\mathrm{ID}(G) = \{Z_\alpha \mid \alpha \in \varphi\}$

Proof. Let Z be an independent set in G. Consider the assignment α which assigns a value of 0 to every vertex in Z and a value of 1 to the remaining vertices in $V \setminus Z$. Clearly $Z = Z_\alpha$, furthermore as Z is independent we conclude that $\alpha \in \varphi$ implying that $Z \in \{Z_\alpha \mid \alpha \in \varphi_G\}$. For the other direction, consider an assignment $\alpha \in \varphi$. By the definitions above, Z_α must be an independent set in G. □

Corollary 1. *For* $\varphi \in \Phi_n$, $\mathrm{BDD}(\varphi) < n(|\mathrm{ID}(G)| + 1)$.

Theorem 1. *For* $G \in \mathcal{G}_n$*, Setting,*

$$\Lambda_G = \left\{ \Gamma \;\middle|\; \Gamma = \Gamma_G(I) \cap [k+1, n], \quad I \in \mathrm{ID}\left(G_{|[1,k]}\right) \right\}$$

The size of the $k+1$ *level in* φ*'s QOBDD (under natural ordering) is either* $|\Lambda_G|$ *or* $|\Lambda_G| + 1$

Proof. Consider the set

$$\Lambda_\varphi = \{\varphi_{|\alpha} \mid \alpha \in \Delta_{[1,k]}, \quad \varphi_{|\alpha} \neq \emptyset\}.$$

The size of the $k+1$ level in φ's QOBDD (under natural ordering) is exactly the size of Λ_φ, possibly plus 1, if there is some α s.t. $\varphi_{|\alpha} = \emptyset$. Hence, it suffices to present a one to one function from Λ_φ to Λ_G and vice versa. For the first direction consider the function which associates with every $\varphi_{|\alpha}$ the set $\Gamma_G(Z_\alpha) \cap [k+1, n]$ (where Z_α is as defined above). As $\varphi_{|\alpha} \neq \emptyset$ we have that Z_α in as independent set in $G_{|[1,k]}$. Now assume two formulas $\varphi_{|\alpha_1}$ and $\varphi_{|\alpha_2}$ that are not equal. Namely (w.l.o.g.) there exists some assignment $\beta \in \Delta_{[k+1,n]}$ such that $\beta \in \varphi_{|\alpha_1}$ but $\beta \notin \varphi_{|\alpha_2}$. For $i = 1, 2$ let $\gamma_i \in \Delta_{[1,n]}$ be the assignment obtained by concatenating α_i and β. By these definitions $\gamma_1 \in \varphi$ and $\gamma_2 \notin \varphi$. Hence, it must be the case that γ_2 violates some clause, say the clause including the i'th and j'th variables, where $i < j$ (that is $\gamma_2(i) = \gamma_2(j) = 0$).

Now (by contradiction) assume that $\Gamma_1 = \Gamma_G(Z_{\alpha_1}) \cap [k+1, n]$ is equal to $\Gamma_2 = \Gamma_G(Z_{\alpha_2}) \cap [k+1, n]$. Recall that φ is a monotone 2CNF formula, it is satisfied by $\gamma_1 = \alpha_1\beta$, and it is not satisfied by $\gamma_2 = \alpha_2\beta$. Moreover, $\varphi_{|\alpha_2}$ is not equal to $\emptyset$. By the fact that φ is satisfied by γ_1 we conclude that all variables in $\Gamma_1 = \Gamma_2$ have value 1 under the assignment β implying that they have value 1 both in the assignment γ_1 and γ_2. Hence, it cannot be the case that i or j belong to Γ_2. By the fact that $[1, k] \setminus Z_{\alpha_2}$ is set to 1 in γ_2 it cannot be the case that i

or j are in $[1,k] \setminus Z_{\alpha_2}$. By the fact that $\varphi_{|_{\alpha_2}} \neq \emptyset$ it cannot be the case that both i and j are in Z_{α_2}. We conclude that it must be the case that both i and j are in $[k+1,n] \setminus \Gamma_2$. But the value of such i and j are determined by β, and by the fact that $\gamma_1 = \alpha_1\beta \in \varphi$ we conclude that either the value of i or j is 1 in γ_2.

For the other direction, consider the function which associates with each $\Gamma \in \Lambda_G$ the assignment $\alpha \in \Delta_{[1,k]}$ which is defined as follows. Let Z be some independent set in $G_{|_{[1,k]}}$ such that $\Gamma_G(Z) \cap [k+1,n] = \Gamma$, define $\alpha(i)$ to be zero iff $i \in Z$. As Z in an independent set in $G_{|_{[1,k]}}$ it is the case that $\varphi_{|_\alpha} \neq \emptyset$ and thus in Λ_φ. Let $\Gamma_1 = \Gamma_G(Z_1) \cap [k+1,n]$ and $\Gamma_2 = \Gamma(Z_2) \cap [k+1,n]$ be two different subsets in Λ_G. We will show that for corresponding α_1 and α_2 as defined above the functions $\varphi_{|_{\alpha_1}}$ and $\varphi_{|_{\alpha_1}}$ differ. Let (w.l.o.g.) i be a vertex in $\Gamma_1 \setminus \Gamma_2$ (note that $i \in [k+1,n]$). Let $\beta \in \Delta_{[k+1,n]}$ be defined such that $\beta(i) = 0$ and $\beta(j) = 1$ for all $j \neq i$. The vertex i is connected by an edge to Z_1 implying that the assignment γ_1 which is the concatenation of α_1 and β does not satisfy φ. We conclude that $\beta \notin \varphi_{|_{\alpha_1}}$. On the other hand , the vertex i is not connected to any vertices in Z_2, implying (in a similar manner) that $\beta \in \varphi_{|_{\alpha_2}}$. □

In the following, we define the notion of the *pathwidth* of a graph (as introduced in [RS83]). Given an ordering of the vertices of a given graph G the pathwidth of G is defined as follows:

Definition 2. *For $G \in \mathcal{G}_n$, denote* $\mathrm{PW}(G) = \max_{k \in [1,n]} |\Gamma_G([1,k])|$.

Next we present upper and lower bounds on the QOBDD size of φ using the pathwidth notion. Afterwards we show that the pathwidth of a graph is monotone with respect to edge contractions and vertex and edge deletions. We will use this property later on in Section 4.

3.1 Upper bound

Lemma 4. $\mathrm{BDD}(\varphi) \leq n(2^{\mathrm{PW}(G)} + 1)$

Proof. Using Theorem 1 we need to show that for every k the size of the set

$$\left\{ \Gamma_G(I) \cap [k+1,n] \;\middle|\; I \in \mathrm{ID}(G_{|_{[1,k]}}) \right\}$$

is of size at most $2^{\mathrm{PW}(G)}$. However, since $I \subseteq [1,k]$, then $|\Gamma_G(I) \cap [k+1,n]| \leq |\Gamma_G([1,k])| \leq \mathrm{PW}(G)$, and therefore the number of possible sets of the form $\Gamma_G(I) \cap [k+1,n]$ is at most $2^{\mathrm{PW}(G)}$. □

3.2 Lower bound

We first state without proof the following lemma, which is proved using a simple greedy strategy.

Lemma 5. *For $G \in \mathcal{G}_n$,* $\mathrm{maxID}(G) \geq \frac{n}{d(G)+1}$

Lemma 6. $\mathrm{BDD}(\varphi) \geq 2^{\frac{\mathrm{PW}(G)}{(d(G)+1)^4}}$

Proof. Mark $h = \mathrm{PW}(G)$ and $d = d(G)+1$. Set k to be such that $|\Gamma_G([1,k])| = h$. Using Theorem 1 we want to show that

$$\left| \left\{ \Gamma \;\middle|\; \Gamma = \Gamma_G(I) \cap [k+1,n], \quad I \in \mathrm{ID}\left(G_{|_{[1,k]}}\right) \right\} \right| \geq 2^{\frac{h}{d^4}} \tag{1}$$

For every vertex $v \in [1,k]$ denote $A_v = \Gamma_G(\{v\}) \cap [k+1,n]$. We will find a specific independent set $\mathcal{I}$ of $G_{|_{[1,k]}}$ such that

1. For every $u \in \mathcal{I}$, $A_u \neq \emptyset$.
2. For every $u, v \in \mathcal{I}$, $A_v \cap A_u = \emptyset$
3. $|\mathcal{I}| \geq \frac{h}{d^4}$

Finding such an $\mathcal{I}$ will prove Equation (1), by letting I run over all subsets of $\mathcal{I}$.

Since $|\Gamma_G([1,k])| = h$, then $|\cup A_v| \geq h$. Therefore there are at least $\frac{h}{d}$ such sets $A_v \neq \phi$. Noticing that each vertex $w \in [k+1,n]$ can appear in at most d sets A_v, and since $|A_v| < d$, we have that each A_v intersects at most d^2 other such sets. By Lemma 5, there are at least $\frac{h}{d} \cdot \frac{1}{d^2} = \frac{h}{d^3}$ such sets that do not intersect each other. Denote by $H \subseteq [1,k]$ the set of v's corresponding to these A_v's. Again, using Lemma 5, and by the fact that $|H| \geq \frac{h}{d^3}$, we can find a subset $\mathcal{I}$ of H that is an independent set in G. This $\mathcal{I}$ satisfies all three properties above. $\square$

3.3 Optimal ordering

The previous results we have shown all consider the natural ordering of variables in φ. In the following we extend these results naturally to obtain the connections needed between the properties of G and the QOBDD size of an arbitrary ordering of φ. Let $\sigma \in S_n$ and $G \in \mathcal{G}_n$. The graph G obtained after a renaming of V according to σ is defined as

$$G^\sigma = (V, \{(\sigma(i), \sigma(j)) \mid (i,j) \in E(G)\}).$$

It is not hard to verify that $(\varphi_G)^\sigma = \varphi_{(G^\sigma)}$, implying that $\mathrm{mBDD}(\varphi_G) = \min_\sigma \mathrm{BDD}(\varphi_{(G^\sigma)})$. We now define the minimal pathwidth of a graph.

Definition 3. *The minimal pathwidth of G is* $\mathrm{mPW}(G) = \min_\sigma \mathrm{PW}(G^\sigma)$.

It is straightforward to verify that Lemma 6 and Lemma 4 now imply:

Theorem 2. $2^{\frac{\mathrm{mPW}(G)}{(d(G)+1)^4}} \leq \mathrm{mBDD}(\varphi_G) \leq n(2^{\mathrm{mPW}(G)} + 1)$.

We believe this result to be of independent interest, since it shows the close connection between the pathwidth of the graph and the QOBDD size of the formula. If all orderings of the vertices result in many clauses being separated — the QOBDD size will be large, exponential in the pathwidth.

3.4 Minors

For a graph $G \in \mathcal{G}_n$, and an edge $(i,j) \in E(G)$, the result of *contracting* the edge (i,j) in G is the graph $G_{|[1,n]\setminus\{i\}}$ with the addition of the edges $\{(j,x) \mid (i,x) \in E(G)\}$. We say H is a *minor* of G if it is the result of consecutive edge contractions of G, vertex deletions and edge deletions of G. In our application, H does not have any multiple edges (*i.e.* H is not a multi graph).

Lemma 7. *If H is a minor of G then* $\mathrm{mPW}(H) \leq \mathrm{mPW}(G)$.

Proof. For one vertex or edge deletion the result is trivial. We therefore prove it for one edge contraction and the Lemma follows by induction. Let $G \in \mathcal{G}_n$, and assume w.l.o.g. that $\mathrm{PW}(G) = \mathrm{mPW}(G)$. Assume an edge (i,j) is contracted in G to give H, where $i < j$. We claim that the following ordering of H's vertices gives a pathwidth of H which is at most $\mathrm{PW}(G)$: $1, 2, \ldots, i-1, i+1, \ldots, n$.

1. For all $k \leq i-1$, $\Gamma_H([1,k]) = \Gamma_G([1,k]) \setminus \{i\}$.
2. For all $k \geq j$, $\Gamma_H([1,k] \setminus \{i\}) = \Gamma_G([1,k])$.
3. For all $i < k < j$, $\Gamma_H([1,k] \setminus \{i\}) \subseteq \Gamma_G([1,k] \setminus \{i\}) \setminus \{i\} \cup \{j\} \subseteq \Gamma_G([1,k])$

And so, for all k: $|\Gamma_H([1,k] \setminus \{i\})| \leq |\Gamma_G([1,k])|$, to conclude. □

4 QOBDD size of random 2CNF

We now proceed to examine the most probable QOBDD size of a random formula in $\mathcal{G}_{n,p}$ for different values of p. Our analysis is divided into several cases, each examining a different range of values for pn. The value pn is (approximately) twice the expected ratio between the number of clauses and the number of variables in the formula, and is therefore a good indicator for the expected structure and complexity of the formula. We prove the following results (with high probability over the random formula φ).

1. For $pn < 1-\epsilon$, where $\epsilon > 0$ is constant, $\mathrm{mBDD}(\varphi) = O(n \log n)$. We will see that the probable formulas in this case are very degenerate, since the graph will most probably contain only very small connected components.
2. For $1+\epsilon < pn < o(n)$, where $\epsilon > 0$ is constant,

$$2^{\Omega(\frac{1}{p}\log^{-6} n)} < \mathrm{mBDD}(\varphi) < 2^{O(\frac{1}{p}\log^{2} n)}$$

   This implies that the QOBDD size is highly exponential[1] for small values of p, and slowly decreases as p approaches 1. For example, when $pn = \sqrt{n}$, the QOBDD size is $2^{\sqrt{n}\cdot\mathrm{polylog}n}$ (which is still highly exponential). Notice the sharp jump in the QOBDD size, with respect to the previous case, with a very small increase of pn.

[1] For $pn \geq 12$ we show an improved lower bound of $2^{\Omega(\frac{1}{p}\log^{-4} n)}$.

3. We improve the bounds above for large values of p. Let p satisfy (a) For every constant $\epsilon > 0$, $pn \geq n^{1-\epsilon}$, and (b) For every constant $\alpha < 1$, $pn \leq n - n^{\alpha}$. (*I.e.* pn is large but not *too* large). Then

$$\mathrm{mBDD}(\varphi) = 2^{\Theta(\frac{\log^2 n}{\log 1/(1-p)})}$$

In this case we get matching lower and upper bounds (up to constant factors in the exponent). Since $pn < n - n^{\alpha}$ for all $\alpha < 1$, this means that $\mathrm{mBDD}(\varphi)$ is super polynomial. For example, when $p = \frac{1}{2}$, $\mathrm{mBDD}(\varphi) = 2^{\Theta(\log^2 n)} = n^{\Theta(\log n)}$
4. If there exists a constant $0 < \alpha < 1$ s.t. $pn > n - n^{\alpha}$, then $\mathrm{mBDD}(\varphi) = n^{O(1)}$, *i.e.*, is polynomial.

An important point in these bounds, is that all upper bounds (except the one for $pn < 1 - \epsilon$) are derived using Corollary 1, by showing an upper bound to the number of satisfying assignments to the formula. The fact that these bounds practically match the lower bounds means that the QOBDD reductions are of very little use for these kinds of formulas — we might as well have written a list of all satisfying assignments as a description of the formula.

4.1 Case 1: $pn < 1 - \epsilon$

We start by stating the following theorem appearing in [JLR] which states that w.h.p. G's connected components are all of size at most $O(\log n)$ and are all *almost* trees

Theorem 3. ([JLR]): *If $G \in \mathcal{G}_{n,p}$, where $pn < 1 - \epsilon$ for some constant $\epsilon > 0$, then w.h.p. G's connected components are of size $O(\log n)$, and are either trees, or trees with one extra edge.*

We now show that the QOBDD size of a graph that is a tree is small. This is done by showing that the pathwidth of a tree is small. Combining these two facts we will conclude that w.h.p. $\mathrm{mBDD}(\varphi) < O(n \log n)$.

Lemma 8. *For $T \in \mathcal{G}_n$, where T is a tree,* $\mathrm{mPW}(T) \leq \log_2 n$

Proof. If $n = 1$ then clearly $\mathrm{mPW}(T) = 0 = \log_2(1)$. We order the vertices of the tree recursively. Number the s subtrees rooted at the children of the root vertex r according to their size, *i.e.*, T_1 is the largest, T_2 the second, and so on until T_s, the smallest subtree. Order each of the subtrees recursively, the vertices of T_1 are ordered $t_1^1, t_2^1, \ldots t_{k_1}^1$, the vertices of T_2 are ordered $t_1^2, t_2^2, \ldots t_{k_2}^2$ and so on. Now order all the vertices in the following way:

$$t_1^1, t_2^1, \ldots t_{k_1}^1, t_1^2, t_2^2, \ldots t_{k_2}^2, \ldots t_1^s, t_2^s, \ldots t_{k_s}^s, r$$

We claim that this ordering gives a pathwidth of at most $\log_2 n$.

1. For $k \in [1, k_1 - 1]$, $\Gamma_T(\{t_1^1, \ldots, t_k^1\}) = \Gamma_{T_1}(\{t_1^1, \ldots, t_k^1\})$. By the induction hypothesis this set is of size at most $\log_2 |T_1| \leq \log_2 n$.

2. For $k = k_1$, $\Gamma_T(\{t_1^1, \dots, t_k^1\}) = |\{r\}| = 1 \le \log_2 n$, since n is at least 2.
3. For $1 < i \le s$, for $k \in [1, k_i]$ $\Gamma_T(\{t_1^1, \dots, t_{k_1}^1, \dots, t_1^i, \dots t_k^i\}) = \Gamma_T(T_1) \cup \dots \cup \Gamma_T([t_1^i, t_k^i]) = \{r\} \cup \Gamma_{T_i}([t_1^i, t_k^i])$. By the induction hypothesis we get that this set is of size at most $\log_2 |T_i| + 1$. However, since $i > 1$, then T_i is not the largest subtree child of r, and therefore must satisfy $|T_i| < \frac{1}{2}|T|$. Which gives $\log_2 |T_i| + 1 \le \log_2 n$.

□

Theorem 4. *If $G \in \mathcal{G}_{n,p}$ where $pn < 1 - \epsilon$ for some constant $\epsilon > 0$, then w.h.p. $\mathrm{mBDD}(\varphi_G) = O(n \log n)$.*

Proof. According to Theorem 3, w.h.p. G's connected components $C_1, \dots C_k$ are all of size at most $O(\log n)$ and are each a tree with maybe an addition of one edge. Since an extra edge can increase the pathwidth of a graph by at most 1, then by Lemma 8 we have that for all i, $\mathrm{mPW}(G_{|_{C_i}}) \le \log_2 |C_i| + 1$. Therefore, by Theorem 2 we have $\mathrm{mBDD}(G_{|_{C_i}}) \le |C_i| \cdot (2^{\log_2 |C_i| + 1} + 1) < 3|C_i|^2$. It is not hard to verify that this implies

$$\mathrm{mBDD}(\varphi_G) \le n + \sum_{i=1}^{k} \mathrm{mBDD}(G_{|_{C_i}}) \le n + 3 \sum_i |C_i|^2$$

Denoting $M = \max_i |C_i|$, we have that $\mathrm{mBDD}(\varphi_G) \le n + 3\frac{n}{M} M^2$, and since for all i, $|C_i| = O(\log n)$, $\mathrm{mBDD}(\varphi_G) = O(n \log n)$. □

4.2 Lower bound of case 2: $1 + \epsilon < pn = o(n)$

We start by showing that for $pn > 12$ w.h.p. $\mathrm{mPW}(G) > \frac{1}{4}n$. We also show that for $pn = O(1)$, w.h.p. $d(G) = O(\log n)$, and now using Theorem 2 we get an exponential lower bound for $\mathrm{mBDD}(\varphi)$ in the case $12 < pn = O(1)$. From this we easily derive a lower bound for larger pn, while $pn = o(n)$.

The result for $1 + \epsilon < pn \le 12$ now follows by finding a minor H of G, that has a large pathwidth. We show that G contains a minor H which is actually an element of $\mathcal{G}_{l,p'}$, where $lp' > 12$, and since $\mathrm{mPW}(G) \ge \mathrm{mPW}(H)$, we get an exponential (in l) lower bound for $\mathrm{mBDD}(\varphi)$. Details follow.

Lemma 9. *For $G \in \mathcal{G}_{n,p}$, where $pn > 12$, w.h.p., $\mathrm{mPW}(G) > \frac{1}{4}n$.*

Proof. We show that if $pn > 12$, then w.h.p., for $G \in \mathcal{G}_{n,p}$, every set $V \subseteq V(G)$, where $|V| = \frac{1}{2}n$, satisfies $|\Gamma_G(V)| > \frac{1}{4}n$. This will prove the lemma.

For fixed $A, B \subseteq V$, where $|A| = \frac{1}{2}n$ and $|B| = \frac{1}{4}n$,

$$\Pr[\Gamma_G(A) \subseteq B] = (1-p)^{|A|(n-(|A|+|B|))} = (1-p)^{\frac{1}{2}n\frac{1}{4}n} < e^{-\frac{pn^2}{8}}$$

If we have that for all relevant A and B, $\Gamma_G(A) \nsubseteq B$ then the graph is as we want it. We bound the probability of this not happening using a simple union bound:

$$2^n \cdot 2^n \cdot e^{-\frac{pn^2}{8}} = e^{n(2\log 2 - \frac{1}{8}pn)}$$

This tends to zero if $pn > 12$. □

It is not hard to verify that w.h.p. the maximal degree $d(G)$ of a graph $G \in \mathcal{G}_{n,p}$ with $pn = O(1)$ is of size $O(\log n)$. We thus conclude, by Theorem 2 that

Corollary 2. *For $G \in \mathcal{G}_{n,p}$ where $12 < pn = O(1)$, w.h.p.,* $\mathrm{mBDD}(\varphi_G) > 2^{\Omega\left(\frac{n}{\log^4 n}\right)}$.

We now turn to study values of p that satisfy $12 < pn = o(n)$.

Theorem 5. *For $G \in \mathcal{G}_{n,p}$, where $12 < pn = o(n)$, w.h.p.,* $\mathrm{mBDD}(\varphi_G) > 2^{\Omega\left(\frac{1}{p}\log^{-4} n\right)}$.

Proof. Set $k = \frac{13}{p}$, and examine the random behavior of $G_{|_{[1,k]}}$, which is actually an element of $\mathcal{G}_{k,p}$. Since $pn = o(n)$, $p = o(1)$ and therefore k is unbounded, so by Corollary 2, w.h.p. $\mathrm{mBDD}(\varphi_{G_{|_{[1,k]}}}) = 2^{\Omega\left(\frac{k}{\log^4 k}\right)} = 2^{\Omega\left(\frac{1}{p\log^4 1/p}\right)}$. Since $\frac{1}{p} < n$, we get $\frac{1}{p}\log^{-4}\frac{1}{p} > \frac{1}{p}\log^{-4} n$.

A simple observation is that if $H = G_{|_U}$, then $\mathrm{mBDD}(\varphi_G) \geq \mathrm{mBDD}(\varphi_H)$, and this gives us the desired result. $\square$

It is left to show our bounds for $1 + \epsilon < pn \leq 12$. To do so we show that for $G \in \mathcal{G}_{n,p}$, $pn > 1 + \epsilon$, G contains a minor H that behaves as a random graph in $\mathcal{G}_{k,p'}$, where $p'k > 12$. This, combined with the analysis above will prove that H has large pathwidth.

Theorem 6. ([JLR]): *If $G \in \mathcal{G}_{n,p}$ and $pn > 1 + \epsilon$, for some constant $\epsilon > 0$, then there is some constant θ s.t. w.h.p. the biggest connected component of G is of size at least θn.*

Theorem 7. *For $G \in \mathcal{G}_{n,p}$, where $1+\epsilon < pn \leq 12$ and $\epsilon > 0$ is constant, w.h.p.* $\mathrm{mBDD}(\varphi_G) > 2^{\Omega\left(\frac{n}{\log^6 n}\right)}$.

Proof. For two reals $0 \leq p_1, p_2, \leq 1$, s.t., $p_1 + (1 - p_1)p_2 = p$, we can view G as the union of two graphs, G_1 and G_2, where $G_1 \in \mathcal{G}_{n,p_1}$, and $G_2 \in \mathcal{G}_{n,p_2}$. Setting $p_1 = \frac{1}{n}(1 + \frac{\epsilon}{2})$, we get that $\frac{\epsilon}{2} < np_2 \leq 12$.

In the following, we find a minor H_1 of G_1 which will contain no edges at all, and then consider how the edges of G_2 appear in H_1. This gives us a minor H of G which will have a large pathwidth.

By Theorem 6, we have that G_1 contains a tree of size θn. As before we may assume that the maximum degree in this tree is $d = O(\log n)$ (this will happen w.h.p.). It is not hard to verify that this implies that for any k, G_1 contains $l = \frac{\theta n}{kd}$ disjoint connected sets $V_1, \ldots V_l$, each of size k (such a partition can be obtained by traversing the tree mentioned above). Now set $k = \frac{24e}{\epsilon\theta}d = O(\log n)$, notice that l is unbounded. In the following we assume that both k and l are integers, otherwise we must use the $\lfloor\cdot\rfloor$ notation.

Define a minor H_1 of G_1, by contracting all of the edges internal to each V_i, and removing all vertices outside of $\cup_i V_i$, and all edges not internal to the V_i's

— in other words, H_1 contains l vertices, and no edges. Define a minor H of G, by considering the edges of G_2 as they appear in H_1. An edge of H corresponds to k^2 (possible) edges of G_2, and so will appear with probability p_3,

$$p_3 = p_2(1 + (1 - p_2) + \ldots + (1 - p_2))^{k^2-1} \geq p_2 k^2 (1 - p_2)^{k^2-1} \geq p_2 k^2 \frac{1}{e}$$

Now,

$$lp_3 \geq \frac{\theta n}{kd} p_2 k^2 \frac{1}{e} \geq \frac{\theta \epsilon k}{2ed} = 12.$$

According to Lemma 9, w.h.p. $\mathrm{mPW}(H) > \frac{1}{4}l = \Omega(\frac{n}{\log^2 n})$, and by Lemma 7, $\mathrm{mPW}(G) \geq \mathrm{mPW}(H) > \Omega(\frac{n}{\log^2 n})$. Lastly, w.h.p. $d(G) = O(\log n)$, and then by Theorem 2 we have that w.h.p. $\mathrm{mBDD}(\varphi_G) > 2^{\Omega\left(\frac{n}{\log^6 n}\right)}$, to conclude. □

4.3 Lower bound of case 3: $n^{1-\epsilon} < pn < n - n^{\alpha}$

Notice that the lower bound presented in the previous Section 4.2 is not super polynomial if p is taken to be very large (namely for values of p greater than $1/\log^6 n$). In the following section, we study large values of p and obtain super polynomial lower bounds. To show a lower bound in these cases, we will work directly with Theorem 1 and not with the pathwidth of the graph. To get a lower bound using this theorem we need to first estimate the number of independent sets in a random graph of $\mathcal{G}_{n,p}$.

For the reminder of this section, we will assume (a) For every constant $\epsilon > 0$, $pn > n^{1-\epsilon}$, and (b) For every constant $\alpha < 1$, $pn < n - n^{\alpha}$.

4.3.1 Independent sets in $\mathcal{G}_{n,p}$

Recall that Theorem 1 shows a connection between certain combinatorial properties of G and the QOBDD size of φ_G. In particular, a necessary condition for a large $\mathrm{mBDD}(\varphi_G)$ is the existence of many (super polynomial) number of independent sets in G. We start by showing this condition holds w.h.p. on random graphs in $\mathcal{G}_{n,p}$, and then use it for proving the lower bound of case 3.

Denote $q = 1 - p$. We will consider the number of independent sets of size $k = k_c$ in $\mathcal{G}_{n,p}$, where $k = c\frac{\log n}{\log 1/q}$, and therefore $q^k = n^{-c}$. Since $q > n^{\alpha-1}$ for every constant $\alpha < 1$, we get that k is unbounded, and we can therefore assume k is a natural number. We take c to be a small constant. Since $pn > n^{1-\epsilon}$ for every constant $\epsilon > 0$, we have $k = O(n^{\epsilon} \log n)$ for every constant $\epsilon > 0$. Let $\gamma > 0$ be an arbitrarily small constant, in the following we will use the fact that $k \leq n^{\gamma}$.

Denote the expected number of independent sets of size k_c by $E = E_c$. Clearly, $E = \binom{n}{k} q^{\binom{k}{2}}$. It is not hard to verify that $E = n^{\Omega(k)}$ given c is small enough. Furthermore, it can be seen (using standard techniques) that the variance V of the number of independent sets of size k is at most $\frac{1}{4}E^2$. Thus, by Chebyshev's inequality,

Corollary 3. *For small enough c, the number of independent sets of size k in $G \in \mathcal{G}_{n,p}$ is $n^{\Omega(k)}$ with probability greater than $\frac{1}{2}$.*

The constant $\frac{1}{2}$ bound on the probability obtained in Corollary 3 will not suffice for our purpose, and we will therefore amplify the probability of this result. Roughly speaking, this is done by applying Corollary 3 on a large class of *almost* disjoint subsets of vertices in G (namely subsets that share at most a single vertex) where each subset is of polynomial size. If one of these subsets has many independent sets, so does G. Due to space limitations, full proof is omitted.

Lemma 10. *For small enough c, the number of independent sets of size k in $G \in \mathcal{G}_{n,p}$ is $n^{\Omega(k)}$ with probability greater than $1 - 2^{-n^{1.5}}$.*

4.3.2 QOBDD size lower bound

We will now use Theorem 1 to prove the lower bound of case 3 on the QOBDD size of $G \in \mathcal{G}_{n,p}$. It is not hard to verify that it suffices to prove

Lemma 11. *Let $G \in \mathcal{G}_{n,p}$. Let $k = k_c$ be as defined in Section 4.3.1, For small enough c, w.h.p.* $\mathrm{mBDD}(\varphi_G) = n^{\Omega(k)}$.

Proof. By Theorem 1 it is enough to show that w.h.p., for every set $U \subseteq [1, n]$, $|U| = \sqrt{n}$,

$$\left|\left\{\Gamma_G(I) \cap ([1,n] \setminus U) \;\middle|\; I \in \mathrm{ID}\left(G_{|U}\right)\right\}\right| \geq n^{\Omega(k)}.$$

Since this will show, that for every ordering of the vertices of G, the size of the $\sqrt{n}+1$ row in φ_G's QOBDD is at least $n^{\Omega(k)}$. We will therefore show that for every such U this happens with probability greater than $1 - \frac{1}{n}\binom{n}{\sqrt{n}}^{-1}$, and so using the union bound, we get that it is true for all U w.h.p.

Let U_1 and U_2 be two independent sets of size k in $G_{|U}$. For $i = 1, 2$, let $\Gamma_i = \Gamma_G(U_i) \cap ([1,n] \setminus U)$. The probability that a specific vertex is in Γ_1 but not Γ_2 is greater than pq^k, and therefore the probability that there is no such vertex in $[1,n] \setminus U$, *i.e.*, $\Gamma_1 = \Gamma_2$, is at most,

$$(1 - pq^k)^{n-\sqrt{n}} < (1 - \frac{p}{n^c})^{\frac{n}{2}} < e^{-\frac{n}{2}\frac{p}{n^c}} < e^{-\frac{1}{2}n^{1-\gamma-c}} < e^{-n^{3/4}},$$

where $\gamma > 0$ is an arbitrarily small constant. Since the number of independent sets U_i in U is at most $|U|^k < e^{k \log n} < e^{n^\gamma \log n}$, then the probability that all the sets $\Gamma_G(U_i) \cap ([1,n] \setminus U)$ differ is at least

$$1 - e^{2n^\gamma \log n} e^{-n^{3/4}} > 1 - e^{-n^{2/3}}$$

For a specific U, by Lemma 10, with probability at least $1 - 2^{-n^{3/4}}$, the number of independent sets of size k in U, is $\sqrt{n}^{\Omega(k)} = n^{\Omega(k)}$. To conclude,

$$1 - (e^{-n^{2/3}} + 2^{-n^{3/4}}) > 1 - e^{-\sqrt{n}\log n} > 1 - \frac{1}{n}\binom{n}{\sqrt{n}}^{-1}$$

□

4.4 Upper bounds of cases 2,3 and 4

We now prove the upper bound of case 4. The upper bounds of cases 2 and 3 are proven similarly (their proof involves setting the parameter k in the proof below to $4\frac{\log n}{\log 1/q}$).

Theorem 8. *Let $G \in \mathcal{G}_{n,p}$, where $pn > n - n^{\alpha}$ for some constant $0 < \alpha < 1$. Then, w.h.p.* $\mathrm{mBDD}(\varphi_G) = n^{O(1)}$.

Proof. The expectation of the number of independent sets of size $k = \lceil \frac{3}{1-\alpha} \rceil + 1$ is at most,

$$\binom{n}{k}(1-p)^{\binom{k}{2}} = \binom{n}{k}(n^{\alpha-1})^{\binom{k}{2}} \leq n^k n^{(\alpha-1)\frac{k(k-1)}{2}} = n^{\frac{1}{2}k(2+(\alpha-1)(k-1))}.$$

Since $(\alpha-1)(k-1) = (\alpha-1)\lceil \frac{3}{1-\alpha} \rceil \leq -3$, the expectation is at most $n^{-\frac{1}{2}k} = o(1)$, and so by Markov's inequality w.h.p. $\mathrm{maxID}(G) \leq k$. By Proposition 1 and Corollary 1, $\mathrm{mBDD}(\varphi_G) \leq n \cdot n^k = n^{O(1)}$. □

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