On Minimum Power Connectivity Problems*

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Abstract

Given a (directed or undirected) graph with edge costs, the power of a node is the maximum cost of an edge leaving it, and the power of the graph is the sum of the powers of its nodes. Motivated by applications for wireless networks, we present polynomial and improved approximation algorithms, as well as inapproximability results, for some classic network design problems under the power minimization criteria. In particular, for the problem of finding a min-power subgraph that contains k internally-disjoint vs-paths from every node v to a given node s, we give a polynomial algorithm for directed graphs and a logarithmic approximation algorithm for undirected graphs. In contrast, we will show that the corresponding edge-connectivity problems are unlikely to admit a polylogarithmic approximation.

1 Introduction

1.1 Preliminaries

A large research effort focused on designing "cheap" networks that satisfy prescribed requirements. In wired networks, the goal is to find a subgraph of the minimum cost. In wireless networks, a range (power) of the transmitters determines the resulting communication network. We consider finding a power assignment to the nodes of a network such that the resulting communication network satisfies prescribed connectivity properties and the total power is minimized. Node-connectivity is more central here than edge-connectivity, as it models stations crashes. For motivation and applications to wireless networks (which is the same as of their min-cost variant for wired networks), see, e.g., [15, 1, 3, 16, 19].

Let $G = (V, E)$ be a (possibly directed) graph with edge costs $\{c(e) : e \in E\}$. For $v \in V$, the *power* $p(v) = p_c(v)$ *of* v in G (w.r.t. c) is the maximum cost of an edge leaving v in G (or zero, if no such edge exists). The power $p(G) = \sum_{v \in V} p(v)$ of G is the sum of powers of its nodes. Note

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that $p(G)$ differs from the ordinary cost $c(G) = \sum_{e \in E} c(e)$ of G even for unit costs; for unit costs, if G is undirected then $c(G) = |E|$ and $p(G) = |V|$. For example, if E is a perfect matching on V then $p(G) = 2c(G)$. If G is a clique then $p(G)$ is roughly $c(G)/\sqrt{|E|/2}$. For directed graphs, the ratio of cost over the power can be equal to the maximum outdegree of a node in G, e.g., for stars with unit costs. The following statement shows that these are the extremal cases for general costs.

Proposition 1.1 ([16]) *$c(G)/\sqrt{|E|/2} \leq p(G) \leq 2c(G)$ for any undirected graph $G = (V, E)$, and $c(G) \leq p(G) \leq 2c(G)$ if G is a forest. If G is directed then $c(G)/d_{\max} \leq p(G) \leq c(G)$, $d_{\max}$ is the maximum outdegree of a node in G.*

Simple connectivity requirements are: "st-path for a given node pair s, t", and "path from s to any other node". Min-cost variants are the (directed/undirected) Shortest Path problem and the Min-Cost Spanning Tree problem. In the min-power case, the directed/undirected Min-Power st-Path problem is solvable in polynomial time by a simple reduction to the min-cost case. The undirected Min-Power Spanning Tree problem is APX-hard and admits a $(5/3 + \varepsilon)$-approximation algorithm [1]. The directed case is at least as hard as the Set-Cover problem, and thus has an $\Omega(\log n)$-approximation threshold; the problem also admits an $O(\ln n)$-approximation algorithm [3, 4]. However, the "reverse" directed min-power spanning tree problem, when we require a path from every node to s, is equivalent to the min-cost case, and thus is solvable in polynomial time.

We note that for min-*cost* connectivity problems, a ρ-approximation algorithm for directed graphs usually implies a 2ρ-approximation for undirected graphs, c.f., [22]. For min-cost problems a standard reduction to reduce the undirected variant to the directed one is: replace every undirected edge uv by two opposite directed edges uv, vu of the same cost as e, find a solution G to the directed variant and take the underlying graph of G. This reduction does not work for min-power problems, since the power of the underlying graph of G can be much larger than that of G, e.g., if G is a star. The approximation algorithm for the directed case might select only one of the two opposite edges, and this does not correspond to a solution for the undirected case.

1.2 Problems considered

An important network property is fault-tolerance. A graph G is *k-outconnected from s* if it has k (pairwise) internally disjoint sv-paths for any $v \in V$; G is *k-inconnected to s* if its reverse graph is k-outconnected from s (for undirected graphs these two concepts are the same); G is *k-connected* if it has k internally disjoint uv-paths for all $u, v \in V$. When the paths are required only to be edge-disjoint, the graph is *k-edge outconnected from s*, *k-edge inconnected to s*, and *k-edge-connected*, respectively (for undirected graphs these three concepts are the same).

We consider the following classic problems in the power model, some of them generalizations of the problems from [1, 3, 4], that were already studied, c.f., [16, 19, 24]. These problems are defined for both directed and undirected graphs.

Min-Power k Disjoint Paths (MPk-DP)
Instance: A graph $\mathcal{G} = (V, \mathcal{E})$, edge-costs $\{c(e) : e \in \mathcal{E}\}$, $s, t \in V$, and an integer k.
Objective: Find a min-power subgraph G of $\mathcal{G}$ with k internally-disjoint st-paths.

Min-Power k-Inconnected Subgraph (MPk-IS)
Instance: A graph $\mathcal{G} = (V, \mathcal{E})$, edge-costs $\{c(e) : e \in \mathcal{E}\}$, $s \in V$, and an integer k.
Objective: Find a min-power k-inconnected to s spanning subgraph G of $\mathcal{G}$.

Min-Power k-Outconnected Subgraph (MPk-OS)
Instance: A graph $\mathcal{G} = (V, \mathcal{E})$, edge-costs $\{c(e) : e \in \mathcal{E}\}$, $s \in V$, and an integer k.
Objective: Find a min-power k-outconnected from s spanning subgraph G of $\mathcal{G}$.

Min-Power k-Connected Subgraph (MPk-CS)
Instance: A graph $\mathcal{G} = (V, \mathcal{E})$, edge-costs $\{c(e) : e \in \mathcal{E}\}$, and an integer k.
Objective: Find a min-power k-connected spanning subgraph G of $\mathcal{G}$.

When the paths are required only to be *edge-disjoint* we get the problems:
Min-Power k Edge-Disjoint Paths (MPk-EDP) (instead of MPk-DP);
Min-Power k-Edge-Inconnected Subgraph (MPk-EIS) (instead of MPk-IS);
Min-Power k-Edge-Outconnected Subgraph (MPk-EOS) (instead of MPk-OS);
Min-Power k-Edge-Connected Subgraph (MPk-ECS) (instead of MPk-CS).

Note that for undirected graphs, the three edge-connectivity problems MPk-EIS, MPk-EOS, and MPk-ECS, are equivalent, but none of them is known to be equivalent to the other for directed graphs. For node connectivity and undirected graphs, only MPk-IS and MPk-OS are equivalent.

We also consider the following related problem. An edge set E on V is a *k-(edge-)cover* (of V) if the degree (the indegree, in the case of directed graphs) of every $v \in V$ w.r.t. E is at least k.

Min-Power k-Edge-Multi-Cover (MP-k-EMC)
Instance: A graph $\mathcal{G} = (V, \mathcal{E})$, edge-costs $\{c(e) : e \in \mathcal{E}\}$, and an integer k.
Objective: Find a min-power k-edge-cover $E \subseteq \mathcal{E}$.

1.3 Previous work

Min-cost versions of the above problems were studied extensively, see, e.g., [9, 13, 14, 12], and surveys in [7, 11, 17, 22].

Previous results on MP-k-EMC: The min-cost variant of MP-k-EMC is essentially the classic b-matching problem, which is solvable in polynomial time, c.f., [7]. The min-power variant of undirected MP-k-EMC was shown recently in [19] to admit a $\min\{k+1, O(\ln n)\}$-approximation algorithm, improving the $\min\{k+1, O(\ln^4 n)\}$-approximation achieved in [16]. For directed graphs, MP-k-EMC admits an $O(\ln n)$-approximation algorithm, and this is tight, see [19].

Previous results on MPk-DP: The min cost variant of directed/undirected MPk-EDP/MPk-DP is polynomially solvable, as this is the classic (incapacitated) Min-Cost k-Flow problem, c.f., [7]. In the min-power case, the edge-connectivity variant MPk-EDP is substantially harder than the node-connectivity one MPk-DP. Directed MPk-DP is solvable in polynomial time by a simple reduction to the min-cost case, c.f., [16]; this implies a 2-approximation algorithm for undirected MPk-DP, which is however not known to be in P nor NP-hard. In the edge-connectivity case, directed MPk-EDP cannot be approximated within $O(2^{\log^{1-\varepsilon} n})$ for any fixed $\varepsilon > 0$, unless NP $\subseteq$ DTIME$(n^{\mathrm{polylog}(n)})$ [16]. The best known approximation ratio for directed MPk-EDP is k [24].

Previous results on MPk-IS and MPk-OS: For directed graphs the min-cost versions of MPk-EOS and MPk-OS are polynomially solvable, see [9] and [13, 12], respectively. This implies a 2-approximation algorithm for undirected graphs. In the min-power case, the best known approximation ratio for directed MPk-IS/MPk-EIS is k, and the best known ratio for directed MPk-OS/MPk-EOS is $O(k \ln n)$, see [24]. For undirected graphs, the previously best ratio was $2k - 1/3$ for both edge and node-connectivity versions of MPk-IS and MPk-OS, see [19].

Previous results on MPk-CS: Min-cost versions of MPk-CS/MPk-ECS were extensively studied, see surveys in [17] and [22]. The best known approximation ratios for the min-cost variant of MPk-CS are $O(\ln^2 k \cdot \min\{\frac{n}{n-k}, \frac{\sqrt{k}}{\ln k}\})$ for both directed and undirected graphs [21], and $O(\ln k)$ for undirected graphs with $n \geq 2k^2$ [6]. For the edge connectivity variant, there is a simple 2-approximation for both directed and undirected graphs [18]. For the min-power case, the best known ratio for undirected MPk-CS is $O(\alpha + \ln n)$ [19], where α is the best known ratio for the min-cost case. This result relies on the $O(\ln n)$-approximation for undirected MP-k-EMC of [19], and the observation from [16] that an α-approximation algorithm for the min-cost variant of MPk-CS and a β-approximation algorithm for MP-k-EMC implies a $(2\alpha + \beta)$-approximation algorithm for MPk-CS. For the edge-connectivity variant, the best known ratios are: $2k - 1/3$ for undirected graphs [19], and $O(k \ln k)$ for directed graphs [24].

1.4 Results in this paper

Our first two results, Theorems 1.2 (directed graphs) and 1.3 (undirected graphs), are for node-connectivity problems.

Theorem 1.2 *Directed* MPk-IS *admits a polynomial time algorithm.*

For undirected graphs, we show that MPk-CS/MPk-OS cannot achieve a better approximation ratio than MP-$(k-1)$-EMC. We also show that up to constants, MPk-OS and MP-k-EMC are equivalent w.r.t. approximation. We use this to get the first polylogarithmic approximation for MPk-OS, improving the ratio $2k - 1/3$ by [19]. Formally:

Theorem 1.3

(i) *If there exists a ρ-approximation algorithm for undirected* MPk-OS/MPk-CS *then there exist a ρ-approximation algorithm for* MP-$(k-1)$-EMC.

(ii) *If there exist a β-approximation algorithm for* MP-$(k-1)$-EMC*, then there exists a $(\beta+4)$-approximation algorithm for undirected* MPk-OS.

(iii) *Undirected* MPk-OS *admits a* $\min\{k+4, O(\log n)\}$*-approximation algorithm.*

Our next two results, Theorems 1.4 (directed graphs) and 1.5 (undirected graphs), are for edge-connectivity problems. In [19] is given a k-approximation algorithm for directed MPk-EDP and MPk-EIS, and in [24] is given an $O(k \ln n)$-approximation algorithm for MPk-EOS and MPk-ECS; these ratios are tight up to constant factor if k is "small", but may seem weak if k is large. We prove that for each one of these four problems a polylogarithmic approximation ratio is unlikely to exist for large values of k, even when the costs are symmetric.

Theorem 1.4 *Directed* MPk-EDP/MPk-EIS/MPk-EOS/MPk-ECS *cannot be approximated within* $O(2^{\log^{1-\varepsilon} n})$ *for any fixed $\varepsilon > 0$ even for symmetric costs, unless* $\mathrm{NP} \subseteq \mathrm{DTIME}(n^{\mathrm{polylog}(n)})$.

For undirected MPk-EDP and MPk-ECS we prove the following. As was mentioned, for $k=1$ directed/undirected MPk-EDP is easily reduced to the min-cost case. We give a strong evidence that for large values of k, a polylogarithmic approximation algorithm for undirected MPk-EDP/MPk-ECS may not exist even for highly restricted instances. For that, we show a reduction from the following extensively studied problem to the undirected MPk-EDP/MPk-ECS. For a graph $\mathcal{J} = (V, \mathcal{I})$ and $X \subseteq V$ let $\mathcal{I}(X)$ denote the edges in $\mathcal{I}$ with both ends in X.

Densest ℓ-Subgraph (Dℓ-S)
Instance: A graph $\mathcal{J} = (V, \mathcal{I})$ and an integer ℓ.
Objective: Find $X \subseteq V$ with $|X| \leq \ell$ and $|\mathcal{I}(X)|$ maximum.

The best known approximation ratio for Dℓ-S is roughly $|V|^{-1/3}$ [10] even for the case of bipartite graphs (which up to a factor of 2 is as hard to approximate as the general case), and this ratio holds for mare than 10 years.

We also consider the following "augmentation" version of undirected MPk-EDP (the directed case is easy, c.f., [24]), which already generalizes the case $k=1$ considered in [1].

Min-Power *k* Edge-Disjoint Paths Augmentation (MPk-EDPA)
Instance: A graph $\mathcal{G} = (V, \mathcal{E})$, edge-costs $\{c(e) : e \in \mathcal{E}\}$, $s, t \in V$, an integer k, and a subgraph $G_0 = (V, E_0)$ of $\mathcal{G}$ that contains $k-1$ pairwise edge-disjoint st-paths.
Objective: Find $F \subseteq \mathcal{E} - E_0$ so that $G_0 + F$ contains k pairwise edge-disjoint st-paths and with $p(G_0 + F) - p(G_0)$ minimum.

Theorem 1.5

(i) *Undirected* MPk-EDP/MPk-ECS *admit no* $C \ln n$*-approximation algorithm for some universal constant* $C > 0$*, unless* P=NP.

(ii) *If there exists a* ρ*-approximation algorithm for undirected* MPk-EDP *or to undirected* MPk-ECS*, then there exists a* $1/(2\rho^2)$*-approximation algorithm for* Dℓ-S *on bipartite graphs.*

(iii) *Undirected* MPk-EDPA *is in* P*; thus undirected* MPk-EDP *admits a* k*-approximation algorithm.*

Table 1 summarizes the currently best known approximation ratios and thresholds for the connectivity problems considered. Our results show that each one of the directed/undirected edge-connectivity problems MPk-EDP, MPk-EOS, MPk-EIS, MPk-ECS, is unlikely to admit a polylogarithmic approximation. For node connectivity problems, note again that *directed* MPk-OS and MPk-IS are *not* equivalent.

Problem	**Node-Connectivity**		**Edge-Connectivity**	
	Directed	*Undirected*	*Directed*	*Undirected*
MPk-DP	in P [16]	2 [16] $--$	k [24] $\Omega(2^{\log^{1-\varepsilon} n})$ [16]	k $\Omega(\max\{1/\sqrt{\sigma}, \ln n\})$
MPk-IS	k [24] in P	$\min\{k+4, O(\ln n)\}$ $\Omega(\beta)$	k [24] $\Omega(2^{\log^{1-\varepsilon} n})$	$2k - 1/3$ [19] $\Omega(\max\{1/\sqrt{\sigma}, \ln n\})$
MPk-OS	$O(k \ln n)$ [24] $\Omega(\ln n)$ for $k = 1$ [3]	$\min\{k+4, O(\ln n)\}$ $\Omega(\beta)$	$O(k \ln n)$ [24] $\Omega(2^{\log^{1-\varepsilon} n})$	$2k - 1/3$ [19] $\Omega(\max\{1/\sqrt{\sigma}, \ln n\})$
MPk-CS	$O(k \ln n)$ [24] $\Omega(\ln n)$ for $k = 1$ [3]	$O(\alpha + \ln n)$ [19] $\Omega(\alpha)$ [16], $\Omega(\beta)$	$O(k \ln n)$ [24] $\Omega(2^{\log^{1-\varepsilon} n})$	$2k - 1/3$ [19] $\Omega(\max\{1/\sqrt{\sigma}, \ln n\})$

Table 1: Currently best known approximation ratios and thresholds for min-power connectivity problems. Results without references are proved in this paper. β is the best ratio for MP-$(k-1)$-EMC; currently $\beta = \min\{k, O(\log n)\}$ [19]. σ is the best ratio for Dℓ-S; currently σ is roughly $O(n^{-1/3})$ [10]. α is the best ratio for the Min-Cost k-Connected Subgraph problem; currently, $\alpha = \lceil (k+1)/2 \rceil$ for $2 \le k \le 7$ (see [2] for $k = 2, 3$, [8] for $k = 4, 5$, and [20] for $k = 6, 7$); $\alpha = k$ for $k = O(\ln n)$ [20], $\alpha = 6H(k)$ for $n \ge k(2k-1)$ [6], and $\alpha = O(\ln k \cdot \min\{\sqrt{k}, \frac{n}{n-k} \ln k\})$ for $n < k(2k-1)$ [21].

This paper is organized as follows. In the rest of this section we introduce some notation used in the paper. Theorems 1.2, 1.3, 1.4, and 1.5, are proved in Sections 2, 3, 4, and 5, respectively.

A preliminary version of this paper is [23].

1.5 Notation

Here is some notation used in the paper. Let $G = (V, E)$ be a (possibly directed) graph. Let $\deg_E(v) = \deg_G(v)$ denote the degree of v in G. Given edge costs $\{c(e) : e \in E\}$, the power of v in G is $p_c(v) = p(v) = \max_{vu \in E} c(e)$, and the power of G is $p(G) = p(V) = \sum_{v \in V} p(v)$. Throughout the paper, $\mathcal{G} = (V, \mathcal{E})$ denotes the input graph with nonnegative costs on the edges. Let $n = |V|$ and $m = |\mathcal{E}|$. Given $\mathcal{G}$, our goal is to find a minimum power spanning subgraph $G = (V, E)$ of $\mathcal{G}$ that satisfies some prescribed property. We assume that a feasible solution exists; let opt denote the optimal solution value of an instance at hand.

2 Proof of Theorem 1.2

Theorem 1.2 easily follows by combining Proposition 1.1 with the following statement:

Lemma 2.1 *Let G_0 be a directed graph with $\deg_{G_0}(v) \geq k$ for all $v \in V$, and let F be an inclusion minimal augmenting edge set on V so that $G = G_0 + F$ is k-inconnected to s. Then $\deg_F(v) \leq 1$ for all $v \in V$.*

The algorithm is as follows:

1. Find a min-power subgraph G_0 of $\mathcal{G}$ with $\deg_{G_0}(v) = k$ for all $v \in V$.
2. Set $c'(uv) \leftarrow \max\{c(uv) - p_{G_0}(u), 0\}$ for all $uv \in \mathcal{E}$, and with the cost function c', find a minimum cost k-inconnected to s spanning subgraph G of $\mathcal{G}$.

Clearly, the computed solution is feasible, and we explain why it is optimal. Let H be an optimal solution. Since $\deg_H(u) \geq k$ for any $u \in V$, we must have $p_H(u) \geq p_{G_0}(u)$ for every $u \in V$. In Step 2 the algorithm assigns to every node $u \in V$ the power $p_{G_0}(u)$ and zero costs to edges in G_0, and subtracts the assigned power $p_{G_0}(u)$ from every edge $uv \in \mathcal{E} - E_0$ leaving u. Let F be any inclusion minimal augmenting edge set F so that $G_0 + F$ is k-inconnected to s. By Proposition 1.1 and Lemma 2.1, the cost and the power of F w.r.t. the modified cost function c' coincide. As the algorithm finds a minimum c'-cost augmenting edge set F, F is also a minimum c'-power augmenting edge set.

In the rest of this section we prove Lemma 2.1.

Definition 2.1 *An edge e of a k-inconnected to s graph G is* critical *if $G - e$ is not k-inconnected to s.*

We prove the following general statement that implies Lemma 2.1, and which is of independent interest.

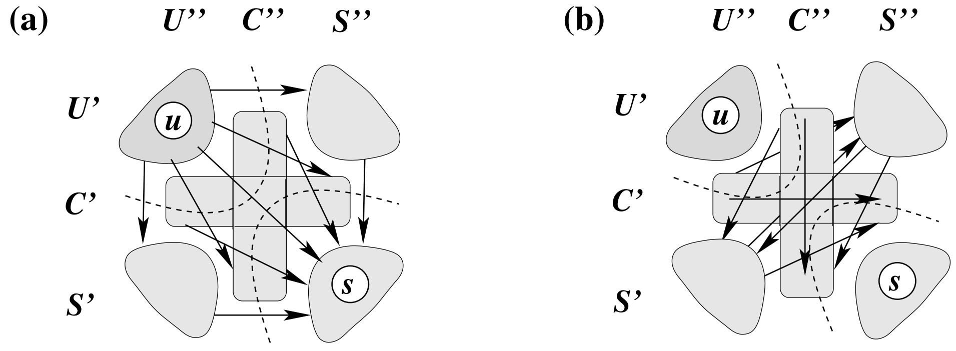

Figure 1: Illustration to the proof of Proposition 2.4.

Theorem 2.2 *Let uv' and uv'' where $v' \neq v''$ be two distinct critical edges of a k-inconnected to s graph G. Then* $\deg_G(u) = k$.

Definition 2.2 *An ordered pair (U, S) of disjoint subsets of V is called a* setpair*; (U, S) is a* us-setpair *if $u \in U$ and $s \in S$.*

Let $\delta(U, S)$ denote the set of edges in G from U to S, and let $d(U, S) = |\delta(U, S)|$. Let $\hat{d}(U, S) = d(U, S) + |V - (U + S)|$. The following statement stems from Menger's Theorem.

Proposition 2.3 *In a directed graph $G = (V, E)$ there are k internally disjoint us-paths if, and only if, $\hat{d}(U, S) \geq k$ for any us-setpair (U, S).*

Proposition 2.4 *Let k be the maximum number of internally disjoint us-paths in a directed graph $G = (V, E)$, and let (U', S') and (U'', S'') be two us-setpairs with $\hat{d}(U', S') = \hat{d}(U'', S'') = k$. Then $\hat{d}(U' \cap U'', S' \cup S'') = \hat{d}(U' \cup U'', S' \cap S'') = k$.*

Proof: Denote $C' = V - (U' + S')$, $C'' = V - (U'' + S'')$ and (see the dashed arcs in Fig. 1)

$$C_\cap = V - [(U' \cap U'') \cup (S' \cup S'')] \quad C_\cup = V - [(U' \cup U'') \cup (S' \cap S'')] \ .$$

It is easy to see that $C_\cap, C_\cup \subseteq C' \cup C''$ and that $|C_\cap| + |C_\cup| = |C'| + |C''|$. We claim that:

$$|C_\cap| + d(U' \cap U'', S' \cup S'') = |C_\cup| + d(U' \cup U'', S' \cap S'') = k \ .$$

We have $|C_\cap| + d(U' \cap U'', S' \cup S'') \geq k$ and $|C_\cup| + d(U' \cup U'', S' \cap S'') \geq k$, by Proposition 2.3. Also, $d(U', S') + |C'| = d(U'', S'') + |C''| = k$. On the other hand:

$$d(U', S') + d(U'', S'') \geq d(U' \cap U'', S' \cup S'') + d(U' \cup U'', S' \cap S'') \ .$$

The later inequality is easily verified by counting the contribution of every edge to each side of the inequality (see Fig. 1). Edges in Fig. 1(a) have the same contribution for both sides: every edge in $\delta(U' \cap U'', S' \cap S'')$ contributes 2 to both sides, while any other edge in Fig. 1(a) contributes 1 to both sides. Edges in Fig. 1(b) contribute only to the left hand side. Other edges (that are not shown in Fig. 1(a,b)) have no contribution. Thus we have:

$$\begin{aligned} k+k &= (|C'|+d(U',S'))+(|C''|+d(U'',S'')) \\ &\geq (|C_\cap|+d(U'\cap U'',S'\cup S''))+(|C_\cup|+d(U'\cup U'',S'\cap S'')) \\ &\geq k+k \ . \end{aligned}$$

Consequently, equality holds everywhere, and the statement follows. □

Let us now get back to the proof of Theorem 2.2. By Proposition 2.3 we have:

Fact 2.5 *An edge $e=uv$ of a k-inconnected to s graph $G=(V,E)$ is* critical *if, and only if, there exists a us-setpair (U,S) with $v\in S$ and $\hat{d}(U,S)=k$.*

Let now G be as in Theorem 2.2. Suppose to the contrary that there are two distinct critical edges leaving the same node u, say $e'=uv'$ and $e''=uv''$, $v'\neq v''$. By Fact 2.5 there exist us-setpairs (U',S') and (U'',S'') so that $v'\in S'$ and $v''\in S''$, and so that $\hat{d}(U',S')=\hat{d}(U'',S'')=k$. Let $U=U'\cap U''$ and $S=S'\cup S''$. Then (U,S) is a us-setpair with $v',v''\in S$, and $\hat{d}(U'\cap U'',S'\cup S'')=k$, by Proposition 2.4. As u has at least $k+1$ neighbors, $U-\{u\}\neq\emptyset$. Consider the setpair $(U-\{u\},S)$. We have:

$$\begin{aligned} \hat{d}(U-\{u\},S) &= d(U-\{u\},S)+|V-(U-\{u\}+S)| \\ &\leq (d(U,S)-2)+(|V-(U+S)|+1)=\hat{d}(U,S)-1=k-1 \ . \end{aligned}$$

Since $U-\{u\}\neq\emptyset$, Proposition 2.3 implies that G is not k-inconnected to s, which is a contradiction.

The proof of Theorem 2.2, and thus also of Lemma 2.1 and Theorem 1.2 is complete.

3 Proof of Theorem 1.3

In this section we consider undirected graphs only and prove Theorem 1.3. Part (iii) of Theorem 1.3 follows from Part (ii) and the fact that MP-$(k-1)$-EMC admits a $\min\{k,O(\log n)\}$-approximation algorithm [19]. In the rest of this section we prove Parts (i) and (ii) of Theorem 1.3.

We start by proving Part (i). The reduction for MPk-CS is a s follows. Let $\mathcal{G}=(V,\mathcal{E}),c$ be an instance of MP-$(k-1)$-EMC with $|V|\geq k$. Construct an instance $\mathcal{G}',c'$ for MPk-CS as follows. Add a copy V' of V and the set of edges $\{vv':v\in V\}$ of cost 0 ($v'\in V'$ is the copy of $v\in V$), and then add a clique of cost 0 on V'. Let $\mathcal{E}'$ be the edges of $\mathcal{G}'-\mathcal{E}$. We claim that $E\subseteq\mathcal{E}$ is a $(k-1)$-edge cover if, and only if, $G'=(V+V',E+\mathcal{E}')$ is k-connected.

Suppose that G' is k-connected. Then $\deg_{E+\mathcal{E}'}(v)\geq k$ and $\deg_{E'}(v)=1$ for all $v\in V$. Hence $\deg_{\mathcal{E}}(v)\geq k-1$ for all $v\in V$, and thus E is a $(k-1)$-edge-cover.

Suppose that $E \subseteq \mathcal{E}$ is a $(k-1)$-edge cover. We will show that G' has k internally disjoint vu-paths for any $u, v \in V + V'$. It is clear that $G' - E$, and thus also G', has k internally disjoint vu-paths for any $u, v \in V'$. Let $v \in V$. Consider two cases: $u \in V'$ and $u \in V$. Assume that $u \in V'$. Every neighbor v_i of v in (V, E) defines the vu path (v, v_i, v'_i, u) (possibly $v'_i = u$), which gives $\deg_E(v) \geq k-1$ internally disjoint vu-paths. An additional path is (v, v', u). Now assume that $u \in V$. Every common neighbor a of u and v defines the vu-path (v, a, u), and suppose that there are q such common neighbors. Each of v and u has at least $k-1-q$ more neighbors in G, say $\{v_1, \ldots, v_{k-1-q}\}$ and $u_1, \ldots, u_{k-1-q}$, respectively. This gives $k-1-q$ internally disjoint vu-paths (v, v_i, v'_i, u'_i, u), $i = 1, \ldots, k-1-q$. An additional path is (v, v', u', u). It is easy to see that these k vu-paths are internally disjoint. The proof for MPk-CS is complete.

The reduction for MPk-OS is the same, except that in the construction of $\mathcal{G}'$ we also add a node s and edges $\{sv' : v' \in V'\}$ of cost 0.

We now prove Part (ii); the proof is similar to the proof of Theorem 3 from [16]. Given a graph G which is k-outconnected from s, let us say that an edge e of G is *critical* if $G - e$ is not k-outconnected from s. We need the following fundamental statement:

Theorem 3.1 ([5]) *In a k-outconnected from s undirected graph G, any cycle of critical edges contains a node $v \neq s$ whose degree in G is exactly k.*

The following corollary (e.g., see [5]) is used to get a relation between $(k-1)$-edge covers and k-outconnected subgraphs.

Corollary 3.2 *If $\deg_J(v) \geq k-1$ for every node v of an undirected graph J, and if F is an inclusion minimal edge set such that $J \cup F$ is k-outconnected from s, then F is a forest.*

Proof: If not, then F contains a critical cycle C, but every node of C is incident to 2 edges of C and to at least $k-1$ edges of J, contradicting Theorem 3.1. □

We now finish the proof of Part (ii). By the assumption, we can find a subgraph J with $\deg_J(v) \geq k-1$ of power at most $p(J) \leq \rho\mathsf{opt}$. We reset the costs of edges in J to zero, and apply a 2-approximation algorithm for the Min-Cost k-Outconnected Subgraph problem (c.f., [13]) to compute an (inclusion) minimal edge set F so that $J + F$ is k-outconnected from s. By Corollary 3.2, F is a forest. Thus $p(F) \leq 2c(F) \leq 4\mathsf{opt}$, by Proposition 1.1. Combining, we get Part (ii).

4 Proof of Theorem 1.4

4.1 Arbitrary costs

We first prove Theorem 1.4 for arbitrary costs, not necessarily symmetric. For that, we use the hardness result for MPk-EDP of [16], to show a similar hardness for the other three problems MPk-

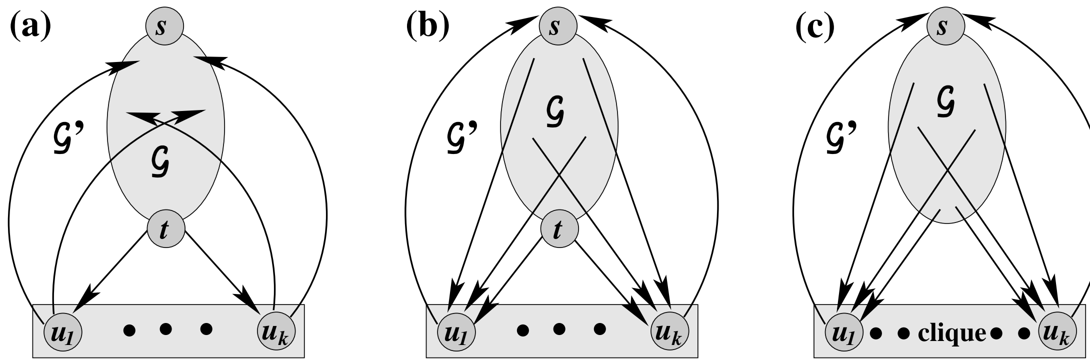

Figure 2: Reductions for asymmetric costs: (a) directed MPk-EDP to directed MPk-EOS; (b) directed MPk-EDP to directed MPk-EIS; (c) directed MPk-EOS to directed MPk-ECS.

EOS, MPk-EIS, and MPk-ECS. Loosely speaking, we show that each of directed MPk-EOS/MPk-EIS is at least as hard as MPk-EDP, and that MPk-ECS is at least as hard as MPk-EOS.

We start by describing how to reduce directed MPk-EDP to directed MPk-EOS. Given an instance $\mathcal{G} = (V, \mathcal{E}), c, (s,t), k$ of MPk-EDP construct an instance of $\mathcal{G}' = (V', \mathcal{E}'), c', s, k$ of directed MPk-EOS as follows. Add to $\mathcal{G}$ a set $U = \{u_1, \ldots, u_k\}$ of k new nodes, and then add an edge set E_0 of cost zero: from t to every node in U, and from every node in U to every node $v \in V - \{s,t\}$. That is

$$V' = V + U = V + \{u_1, \ldots, u_k\},$$
$$\mathcal{E}' = \mathcal{E} + E_0 = \mathcal{E} + \{tu : u \in U\} + \{uv : u \in U, v \in V' - \{s,t\}\},$$
$$c'(e) = c(e) \text{ if } e \in \mathcal{E} \text{ and } c'(e) = 0 \text{ otherwise .}$$

Since in the construction $|V'| = |V| + k \leq |V| + |V|^2 \leq 2|V|^2$, Theorem 1.5 together with the following claim implies Theorem 1.4 for asymmetric MPk-EOS.

Claim 4.1 *$G = (V, E)$ is a solution to the* MPk-EDP *instance if, and only if, $G' = (V', E' = E + E_0)$ is a solution to the constructed* MPk-EOS *instance.*

Proof: Let E be a solution to the MPk-EDP instance and let $\Pi = \{P_1, \ldots, P_k\}$ be a set of k pairwise edge-disjoint st-paths in E. Then in $G' = (V', E + E_0)$ for every $v \in V' - s$ there is a set $\Pi' = \{P'_1, \ldots, P'_k\}$ of k pairwise edge-disjoint sv-paths: if $v = t$ then $\Pi' = \Pi$; if $v \neq s$ then $P'_j = P_j + tu_j + u_j v$, $j = 1, \ldots, k$.

Now let $E' = E + E_0$ be a solution to constructed MPk-EOS instance. In particular, (V', E') contains a set Π of k-edge disjoint st-paths, none of which has t as an internal node. Consequently, no path in Π passes through U, as t is the tail of every edge entering U. Thus Π is a set k-edge disjoint st-paths in G, namely, $G = (V, E)$ is a solution to the original MPk-EDP instance. □

Asymmetric MPk-EIS*:* The reduction of asymmetric MPk-EIS to MPk-EDP is similar to the one described above, except that here set $E_0 = \{us : u \in U\} + \{vu : v \in V - \{s,t\}, u \in U\}$; namely, connect every $u \in U$ to s, and every $v \in V - \{s,t\}$ to every $u \in U$. Then in the obtained MPk-EIS instance, require k internally edge-disjoint vt-paths for every $v \in V$, namely, we seek a graph that is k-edge-inconnected to t. The other parts of the proof for MPk-EIS are identical to those for MPk-EOS described above.

Asymmetric MPk-ECS: Reduce the directed MPk-EOS to the directed MPk-ECS as follows. Let $\mathcal{G} = (V, \mathcal{E}), c, s, k$ be an instance of MPk-EOS. Construct an instance of $\mathcal{G}' = (V', \mathcal{E}'), c', s, k$ of MPk-EOS as follows. Add to $\mathcal{G}$ a set $U = \{u_1, \ldots, u_k\}$ of k new nodes, and then add an edge set $E_0 = \{uu' : u, u' \in U\} + \{vu : v \in V - s, u \in U\} + \{us : u \in U\}$ of cost 0; namely, E_0 is obtained by taking a complete graph on U and adding all edges from $V - s$ to U and all edges from U to s. It is not hard to verify that if $E \subseteq \mathcal{E}$, then $G = (V, E)$ is k-edge-outconnected from s if, and only if, $G' = (V', E_0 + E)$ is k-edge-connected.

4.2 Symmetric costs

We show that the directed problems MPk-EDP, MPk-EOS, MPk-EIS, MPk-ECS are hard to approximate even for symmetric costs. We start with directed symmetric MPk-EDP. We use a refinement of a result from [16] which states that directed MPk-EDP cannot be approximated within $O(2^{\log^{1-\varepsilon} n})$ for any fixed $\varepsilon > 0$, unless NP $\subseteq$ DTIME$(n^{\text{polylog}(n)})$. In [16] it is shown that this hardness result holds for simple graphs with costs in $\{0, n^3\}$, where $n = |V|$. If we change the cost of every edge of cost 0 to 1, it will add no more then n^2/n^3 to the total cost of any solution that uses at least one edge of cost n^3. Thus we have:

Corollary 4.2 ([16]) *Directed* MPk-EDP *with costs in* $\{1, n^3\}$ *cannot be approximated within* $O(2^{\log^{1-\varepsilon} n})$ *for any fixed* $\varepsilon > 0$*, unless* NP $\subseteq$ DTIME$(n^{\text{polylog}(n)})$.

We show that a ρ-approximation algorithm for directed symmetric MPk-EDP implies a ρ-approximation algorithm for directed MPk-EDP with costs in $\{1, n^3\}$, for any $\rho < n^{1/7}$. Let $\mathcal{G} = (V, \mathcal{E}), c, (s,t), k$ be an instance of MPk-EDP with costs in $\{1, n^3\}$. Let opt be an optimal solution value for this instance. Note that opt $\leq n^4$. Let $N = n^5$. Define an instance $\mathcal{G}' = (V', \mathcal{E}'), c', (s,t), k' = kN$ for directed symmetric MPk-EDP as follows. First, obtain $\mathcal{G}^+ = (V', \mathcal{E}^+), c^+$ by replacing every edge $e = uv \in E$ by N internally-disjoint uv-paths of the length 2 each, where the cost of the first edge in each paths is $c(e)$ and the cost of the second edge is 0. Second, to obtain a symmetric instance $\mathcal{G}', c'$, for every edge $ab \in \mathcal{E}^+$ add the opposite edge ba of the same cost as ab.

For a path P^+ in $\mathcal{E}^+$, let $\psi(P^+)$ denote the unique path in $\mathcal{E}$ corresponding to P^+. For any path P in $\mathcal{E}$, the paths in the set $\psi^{-1}(P)$ of the paths in $\mathcal{E}^+$ that corresponds to P are edge-disjoint. Hence, any set Π of paths in $\mathcal{E}$ is mapped by ψ^{-1} to a set $\Pi^+ = \psi^{-1}(\Pi)$ of exactly $N|\Pi|$ edge-

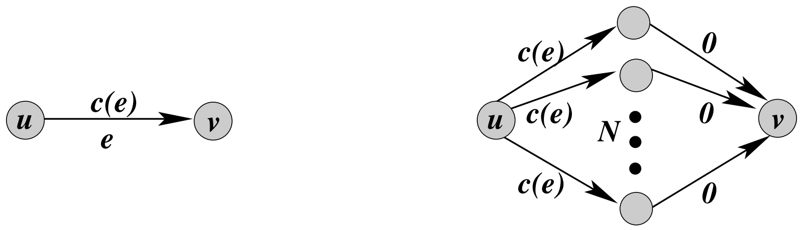

Figure 3: Reductions for symmetric costs: transforming an edge into N paths.

disjoint paths in $\mathcal{E}^+$ of the same power, namely $|\Pi^+| = N|\Pi|$ and $p_c(\Pi) = p_{c^+}(\Pi^+)$. Conversely, any set Π^+ of paths in $\mathcal{E}^+$ is mapped by ψ to a set $\Pi = \psi(\Pi^+)$ of at least $\lceil |\Pi^+|/N \rceil$ edge-disjoint paths in $\mathcal{E}$ of the same power, namely, $|\Pi| = \lceil |\Pi|/N \rceil$ and $p_c(\Pi) = p_{c^+}(\Pi^+)$. In particular:

Corollary 4.3 $\mathsf{opt}' \leq \mathsf{opt} \leq n^4$, *where* opt' *is an optimal solution value for* $\mathcal{G}'$.

Note that $|V'| = n' \leq n^7$, hence to prove Theorem 1.4 for directed symmetric MPk-EDP it is sufficient to prove that a $\rho(n')$-approximation algorithm for $\mathcal{G}', c', (s,t), k'$ with $\rho(n') < n'^{1/7}$ implies a $\rho(n)$-approximation algorithm for the original instance. Suppose that we have a $\rho(n')$-approximation algorithm that computes an edge set $E' \subseteq \mathcal{E}'$ that contains a set Π' of kN edge-disjoint paths in $\mathcal{G}'$ of power $p_{c'}(E') \leq \rho \cdot \mathsf{opt}'$, where $\rho = \rho(n') < n'^{1/7} \leq n$. Then $|E' - \mathcal{E}^+| \leq \rho \cdot \mathsf{opt}' \leq \rho \cdot n^4$, since every edge in $|E' - \mathcal{E}^+|$ adds at least one to $p_{c'}(E')$. Consequently, there is a set $\Pi^+ \subseteq \Pi'$ of at least $kN - \rho \cdot n^4$ paths in Π that are contained in $E^+ = E' \cap \mathcal{E}^+$. Hence, since $\rho = \rho(n') > n'^{1/7} \geq n$, the number of paths in $\Pi = \psi(\Pi^+)$ is at least

$$|\Pi| \geq \left\lceil \frac{kN - \rho \cdot n^4}{N} \right\rceil \geq \left\lceil k - \frac{\rho \cdot n^4}{N} \right\rceil = \left\lceil k - \frac{\rho}{n} \right\rceil \geq k \ .$$

Consequently, the set E of edges of Π is a feasible solution for $\mathcal{G}, c, (s,t), k$ of power at most $p_c(E) \leq p_{c'}(E') \leq \rho\mathsf{opt}' \leq \rho\mathsf{opt}$. Since in the construction $|V'| \leq |V|^7$, Corollary 4.2 implies Theorem 1.4 for directed symmetric MPk-EDP.

The proof for the other problems MPk-EOS, MPk-EIS, and MPk-ECS, is similar, with the help of reductions described for the asymmetric case.

5 Proof of Theorem 1.5

It is easy to see that MPk-EDP is "Set-Cover hard". Indeed, the Set-Cover problem can be formulated as follows. Given a bipartite graph $J = (A + B, E)$, find a minimum size subset $S \subseteq A$ such that every node in B has a neighbor in S. Construct an instance of MPk-EDP by adding two new nodes s and t, edges $\{sa : a \in A\} \cup \{bt : b \in B\}$, and setting $c(e) = 1$ if e is incident to s and $c(e) = 0$ otherwise. Then replace every edge not incident to t by $|B|$ parallel edges. For $k = |B|$, it is easy to see that S is a feasible solution to the Set-Cover instance if, and only if, the subgraph induced by $S \cup B \cup \{s,t\}$ is a feasible solution of power $|S|+1$ to the obtained MPk-EDP instance; each node in $S \cup \{t\}$ contributes 1 to the total power, while the contribution of any other node is 0. Combining this with the hardness results of [26] for Set-Cover, we get Part (i).

The proof of Part (ii) is similar to the the proof of Theorem 1.2 from [25], where the related Node-Weighted Steiner Network was considered; we provide the details for completeness of exposition. We need the following known statement (c.f., [10, 25]).

Lemma 5.1 *There exists a polynomial time algorithm that given a graph $G=(V,E)$ and an integer $1 \leq \ell \leq n = |V|$ finds a subgraph $G'=(V',E')$ of G with $|V'|=\ell$ and $|E'| \geq |E| \cdot \frac{\ell(\ell-1)}{n(n-1)}$.*

Given an instance $\mathcal{J}=(A+B,\mathcal{I})$ and ℓ of bipartite Dℓ-S, define an instance of (undirected) unit-cost MPk-EDP/MPk-ECS by adding new nodes $\{s,t\}$, a set of edges $E_0=\{aa' : a,a' \in A+s\} \cup \{bb' : b,b' \in B+t\}$ of multiplicity $|A|+|B|$ and cost 0 each, and setting $c(e)=1$ for all $e \in \mathcal{I}$. It is easy to see that any $E \subseteq \mathcal{I}$ determines $|E|$ edge-disjoint st-paths, and that $(A+B+\{s,t\}, E_0+|E|)$ is k-connected if, and only if, $|E| \geq k$. Thus for any integer $k \in \{1,\ldots,|\mathcal{I}|\}$, if we have a ρ-approximation algorithm for undirected MPk-EDP/MPk-ECS, then we have a ρ-approximation algorithm for

$$\min\{|X| : X \subseteq A+B, |\mathcal{I}(X)| \geq k\} \ .$$

We show that this implies a $1/(2\rho^2)$-approximation algorithm for the original instance of bipartite Dℓ-S, which is

$$\max\{|\mathcal{I}(X)| : X \subseteq A+B, |X| \leq \ell\} \ .$$

For every $k=1,\ldots,|\mathcal{I}|$, use the ρ-approximation algorithm for undirected MPk-EDP/MPk-ECS to compute a subset $X_k \subseteq A+B$ so that $|\mathcal{I}(X_k)| \geq k$, or to determine that no such X_k exists. Set $X=X_k$ where k is the largest integer so that $|X_k| \leq \min\{\lfloor \rho \cdot \ell \rfloor, |A|+|B|\}$ and $|\mathcal{I}(X_k)| \geq k$. Let X^* be an optimal solution for Dℓ-S. Note that $|\mathcal{I}(X)| \geq |\mathcal{I}(X^*)|$ and that $\frac{\ell(\ell-1)}{|X|(|X|-1)} \geq 1/(2\rho^2)$. By Lemma 5.1 we can find in polynomial time $X' \subseteq X$ so that $|X'|=\ell$ and $|\mathcal{I}(X')| \geq |\mathcal{I}(X)| \cdot \frac{\ell(\ell-1)}{|X|(|X|-1)} \geq |\mathcal{I}(X^*)| \cdot 1/(2\rho^2)$. Thus X' is a $1/(2\rho^2)$-approximation for the original bipartite Dℓ-S instance.

We now prove part (iii) of Theorem 1.5. It would be convenient to describe the algorithm using "mixed" graphs that contain both directed and undirected edges. Given such mixed graph with weights on the nodes, a minimum weight path between two given nodes can be found in polynomial time using Dikjstra's algorithm and elementary constructions. The algorithm for undirected MPk-EDPA is as follows (see Fig. 4).

1. Construct a graph $\mathcal{G}'$ from $\mathcal{G}$ as follows (see Fig. 4(a,b)). Let $p_0(v)$ be the power of v in G_0. For every $v \in V$ do the following. Let $p_0(v) \leq c_1 < c_2 < \cdots$ be the costs of the edges in $\mathcal{E}$ leaving v of cost at least $p_0(v)$ sorted in increasing order. For every c_j add a node v_j of the weight $w(v_j)=c_j-p_0(v)$. Then for every $u_{j'}, v_{j''}$ add an edge $u_{j'}v_{j''}$ if $w(u_{j'}), w(v_{j''}) \geq c(uv)$. Finally, add two nodes s,t and an edge from s to every s_j and from every t_j to t.

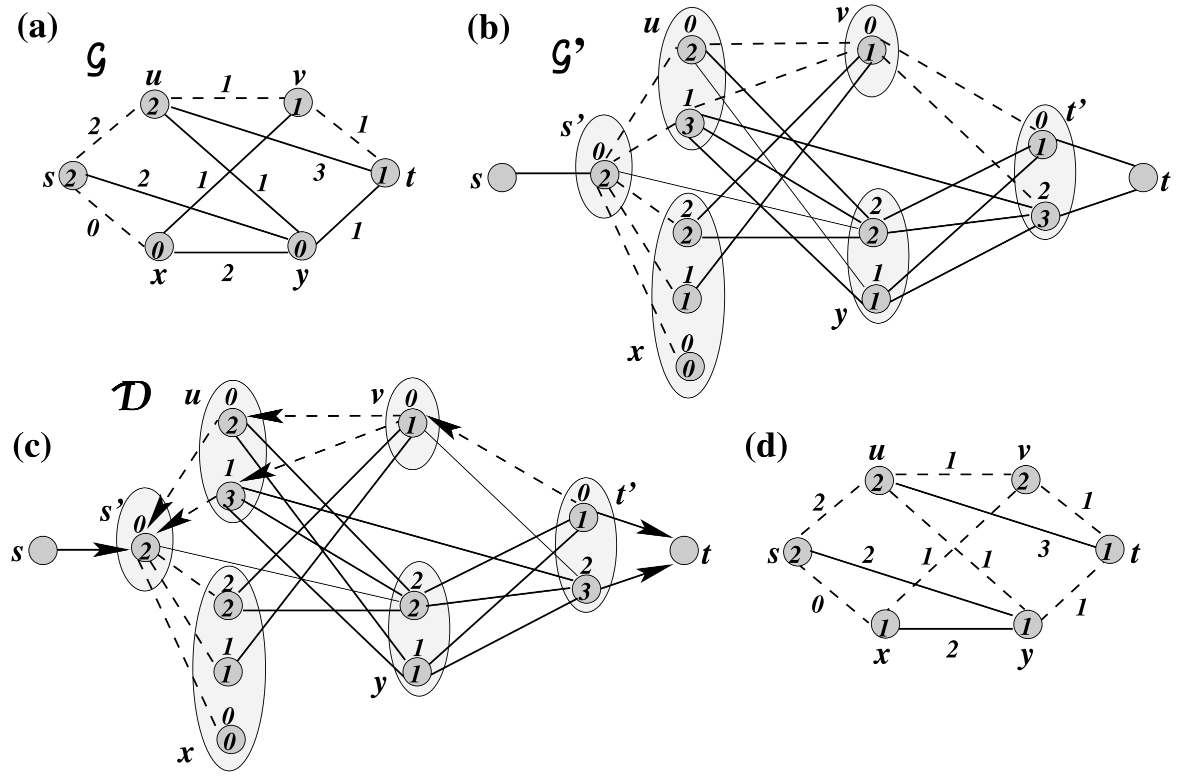

Figure 4: (a) The input graph $\mathcal{G}$ and its subgraph G_0; $k = 2$, edges of G_0 are shown by dashed lines, the number in each node v is $p_0(v)$. (b) The graph $\mathcal{G}'$; the number above each node is its weight $w(v)$. (c) The graph $\mathcal{D}$; the optimal augmenting path is $P = (s, s', x, v, u, y, t', t)$ has weight 2, the power of each one of x, y is increased by 1. (d) An optimal solution $E_0 + F$, $F = \{xv, yu\}$, is shown by dashed lines; the two edge-disjoint paths are $(s, x, v, t), (s, u, y, t)$.

2. Construct a mixed graph $\mathcal{D}$ from $\mathcal{G}'$ as follows (see Fig. 4(c)). Let I be an inclusion minimal edge set in G_0 that contains $k - 1$ pairwise edge-disjoint st-paths. Direct those paths from t to s, and direct accordingly every edge of $\mathcal{G}'$ that corresponds to an edge in I.

3. In $\mathcal{D}$, compute a minimum weight st-path P (see Fig. 4(c,d)). Return the set of edges of $\mathcal{G}$ that correspond to P that are not in E_0.

We now explain why the algorithm is correct. It is known that the following "augmenting path" algorithm solves the **Min-Cost k Edge-Disjoint Paths Augmentation** problem (the min-cost version of MPk-EDPA, where the edges in G_0 have cost 0) in undirected graphs (c.f., [7]).

1. Let I be an inclusion minimal edge set in G_0 that contains $k-1$ pairwise edge-disjoint st-paths. Construct a mixed graph $\mathcal{D}$ from $\mathcal{G}$ by directing these paths from t to s.

2. Find a min-cost path P in $\mathcal{D}$. Return $P - E_0$.

Our algorithm for MPk-EDPA does the same but on the graph $\mathcal{G}'$. The feasibility of the solution follows from standard network flow arguments. The key point in proving optimality is that in $\mathcal{G}'$ the weight of a node is the increase of its power caused by taking an edge incident to this node. It can be shown that for any feasible solution F corresponds a unique path P in $\mathcal{D}$ so that $p(G_0 + F) - p(G_0) = w(P)$, and vice versa. As we choose the minimum weight path in $\mathcal{D}$, the returned solution is optimal.

6 Conclusions

In this paper we showed that MPk-OS and MP-$(k-1)$-EMC are equivalent w.r.t. approximation, and used this to derive a logarithmic approximation for MPk-OS. We also showed that the edge-connectivity problems MPk-EDP, MPk-EOS/MPk-EIS, and MPk-ECS, are unlikely to admit a polylogarithmic approximation ratio for both directed and undirected graphs, and for directed graphs this is so even if the costs are symmetric. In contrast, we showed that the augmentation version MPk-EDPA of MPk-EDP, can be reduced to the shortest path problem.

We now list some open problems, that follow from Table 1. Most of them concern node-connectivity problems. One open problem is to determine whether the *undirected* MPk-DP is in P or is NP-hard (as was mentioned, the directed MPk-DP is in P, c.f., [16]). In fact, we do not even know whether the augmentation version MPk-DPA of undirected MPk-DP is in P. A polynomial algorithm for undirected MPk-DPA can be used to slightly improve the known ratios for the undirected MPk-CS: from 9 (follows from [20]) to $8\frac{2}{3}$ for $k = 4$, and from 11 (follows from [8]) to 10 for $k = 5$. This is achieved as follows. In [8] it is shown that if G is k-outconnected from r and $\deg_G(r) = k$ then G is $(\lceil k/2 \rceil + 1)$-connected; furthermore, for $k = 4, 5$, G contains two nodes s, t so that increasing the connectivity between them by one results in a k-connected graph. Hence for $k = 4, 5$, we can get approximation ratio $\gamma + \delta$, where γ is the ratio for undirected MPk-OS, and δ is the ratio for undirected MPk-DPA. As MPk-OS can be approximated within $\min\{2k - 1/3, k + 4\}$, then if MPk-DPA is in P, for MPk-CS we can get approximation ratios $8\frac{2}{3}$ for $k = 4$ and 10 for $k = 5$.

Except directed MPk-DP and MPk-IS that are in P, there is still a large gap between upper and lower bounds of approximation for all the other min-power node connectivity problems, for both directed and undirected graphs.

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