The Open University of Israel
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Approximating survivable networks with β-quasi-metric costs

Thesis submitted as partial fulfillment of the requirements
towards an M.Sc. degree in Computer Science
The Open University of Israel
Computer Science Division

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Prepared under the supervision of Prof. Zeev Nutov

June 2009

Abstract

The Survivable Network Design (SND) problem seeks a minimum-cost subgraph that satisfies prescribed node-connectivity requirement. We consider SND on both directed and undirected complete graphs with β-*quasi-metric costs* when $c(xz) \leq \beta[c(xy) + c(yz)]$ for all $x, y, z \in V$, which varies from uniform costs ($\beta = 1/2$) to metric costs ($\beta = 1$).

For the k-Connected Subgraph (k-CS) problem our ratios are: $1 + \frac{2\beta}{k(1-\beta)} - \frac{1}{2k-1}$ for undirected graphs, and $1 + \frac{4\beta^3}{k(1-3\beta^2)} - \frac{1}{2k-1}$ for directed graphs and $\frac{1}{2} \leq \beta < \frac{1}{\sqrt{3}}$. For undirected graphs this improves the ratios $\frac{\beta}{1-\beta}$ of [2] and $2 + \beta\frac{k}{n}$ of [10] for all $k \geq 4$ and $\frac{1}{2} + \frac{1}{2k} \leq \beta \leq \frac{k^2}{(k+1)^2-2}$. We also show that SND admits the approximation ratios $\frac{2\beta}{1-\beta}$ for undirected graphs, and $\frac{4\beta^3}{1-3\beta^2}$ for directed graphs and $1/2 \leq \beta \leq 1/\sqrt{3}$. For two important particular cases of SND, so called Rooted SND and Subset k-CS, our ratios are $\frac{2\beta^3}{1-3\beta^2}$ for directed graphs and $\frac{\beta}{1-\beta}$ for Subset k-CS on undirected graphs.

Acknowledgments

I wish to thank my thesis advisor, Prof. Zeev Nutov, for excellent guidness, and for providing numerous ideas, valuable suggestions, help in making usefull discussions, and his encouragement during this study. Your guidness has significantly improved the way I approach optimization problems.

Contents

List of Figures

1 Introduction

1.1 Problems considered

For a graph H let $\kappa_H(u,v)$ denote the uv-connectivity of H, that is the maximum number of internally node disjoint uv-paths in H. We consider variants the following fundamental problem:

Survivable Network Design (SND)
Instance: A directed/undirected graph $G=(V,E)$ with edge-cost $\{c(e) : e \in E\}$, and connectivity requirements $\{r(u,v) : u,v \in V\}$.
Objective: Find a min-cost subgraph H of G satisfying $\kappa_H(u,v) \geq r(u,v)$ for all $u,v \in V$.

Let $k = \max_{u,v \in V} r(u,v)$ denote the maximum requirement of an SND instance. Important particular cases of SND are:

- k-Connected Subgraph (k-CS), when $r(u,v)=k$ for all $u,v \in V$.
- Subset k-CS, when $r(u,v)=k$ for all $u,v \in T \subseteq V$ and $r(u,v)=0$ otherwise.
- Rooted SND, when there is a node $s \in V$ so that $r(u,v)>0$ implies $u=s$.

We consider instances of SND with *β-quasi-metric costs*, namely, when the input graph is complete and the costs satisfy the *β-triangle inequality* $c(xz) \leq \beta(c(xy)+c(yz))$ for all $x,y,z \in V$. When $\beta=\frac{1}{2}$ the costs are uniform, and we have the "cardinality version" of the problem (in a complete graph). When $\beta=1$ the costs satisfy the triangle inequality and we have the metric version of the problem. Many practical instances of the problem may have costs which are between metric and uniform.

1.2 Previous work on general and metric costs

k-CS with β-quasi-metric costs is APX-hard [1] for $k=2$ and any $\beta>1/2$, and thus also Subset k-CS, and SND with with β-quasi-metric costs, metric costs, or general costs. No previous result for general directed/undirected SND with β-quasi-metric cost. For undirected SND with metric costs an $O(\log k)$-approximation algorithm was given by Cheriyan and Vetta [5]. This algorithm applies also for β-quasi-metric costs. On the other hand, the directed SND with metric costs is unlikely to admit a polylogarithmic approximation ratio even for $k=1$ [8].

Costs	Requirements	Approximability	
		Undirected	*Directed*
general	general	$O(\min\{k^3 \log n, n^2\}$ [7], $\Omega(k^\varepsilon)$ [6]	$\Omega(2^{\log^{1-\varepsilon} n})$ for $k = 1$ [8]
general	k-CS	$O(\log \frac{n}{n-k} \log k)$ [13]	$O(\log \frac{n}{n-k} \log k)$ [13]
general	Subset k-CS	$O(\min\{k^2 \log k, n^2\})$ [14], $\Omega(k^\varepsilon)$ [6]	$O(n^2)$
general	Rooted SND	$O(\min\{k^2, n\})$ [14]	$O(n)$
metric	general	$O(\log k)$ [5]	$\Omega(2^{\log^{1-\varepsilon} n})$ for $k = 1$ [8]
metric	k-CS	$2 + (k-1)/n$ [10]	$2 + k/n$ [10]
β-quasi-metric	general	–	–
β-quasi-metric	k-CS	$2 + \beta\frac{k}{n}$ [10], $\frac{\beta}{1-\beta}$ [2], APX-hard [1]	–

Table 1: Approximation ratios and hardness of approximation results for SND and k-CS.

We now survey some recent work on SND problems with general costs from [7, 13, 14, 12]. For general costs, the currently best known ratio for undirected SND problems are: $O(\min\{k^3 \log n, n^2\})$ due to chuzhoy and Khanna [7], $O(\min\{k^2, n\})$ for Rooted SND and $O(\min\{k^2 \log k, n^2\})$ for Subset k-Connected Subgraph due to Nutov [14]; the latter problem has an $\Omega(k^\varepsilon)$-approximation threshold [6]. For k-CS with general costs the currently best known ratio is $O(\log \frac{n}{n-k} \log k)$ for both directed and undirected graphs due to Nutov [13]. In [12] it is proved that for $k = n/2 + k'$ the approximability of undirected SND is the same as that of directed SND with maximum requirement k'. This is so also for k-CS. However, the reduction in [12] does not preserve metric costs.

For further approximation ratios and hardness of approximation results for SND and k-CS are summarized in Table 1. For a survey on various min-cost connectivity problems see [11].

We note that in [2] is also given a $(1 + \frac{5(2\beta-1)}{9(1-\beta)})$-approximation algorithm for undirected 3-CS with β-quasi-metric costs, however this approximation is for $\beta < \frac{2}{3}$, and our approximation improvment range is disjoint to this improvment range.

1.3 Our results

We analyze the algorithm of Cheriyan & Thurimella [4] originally suggested for k-CS with $1, \infty$-costs, and show that for β-quasi-metric costs it achieves the following ratios:

Theorem 1.1 k-CS *with β-quasi-metric costs admits the following approximation ratios:*

Graph	Requirements	Approximability	Improvement Range
undirected	general	$\frac{2\beta}{1-\beta}$	$1/2 \leq \beta < 1$
”	subset k-CS	$\frac{\beta}{1-\beta}$	”
”	k-CS	$1+\frac{2\beta}{k(1-\beta)}-\frac{1}{2k-1}$	$k \geq 3$, $\frac{1}{2}+\frac{3k-2}{2(4k^2-7k+2)} < \beta < \frac{k^2}{(k+1)^2-2}$
directed	general	$\frac{4\beta^3}{1-3\beta^2}$	$\frac{1}{2} \leq \beta < \frac{1}{\sqrt{3}}$
”	subset k-CS	$\frac{2\beta^3}{1-3\beta^2}$	”
”	rooted	$\frac{2\beta^3}{1-3\beta^2}$	”
”	k-CS	$1+\frac{4\beta^3}{k(1-3\beta^2)}-\frac{1}{2k-1}$	”

Table 2: Improvement ranges of our results.

- $1+\frac{2\beta}{k(1-\beta)}-\frac{1}{2k-1} \leq 1+\frac{1}{k}\left(\frac{2\beta}{1-\beta}-\frac{1}{2}\right)$ *for undirected graphs.*
- $1+\frac{4\beta^3}{k(1-3\beta^2)}-\frac{1}{2k-1} \leq 1+\frac{1}{k}\left(\frac{4\beta^3}{1-3\beta^2}-\frac{1}{2}\right)$ *for directed graphs and* $1/2 \leq \beta \leq 1/\sqrt{3}$.

For directed k-CS we offered an approximation ratio of $1+\frac{4\beta^3}{k(1-3\beta^2)}-\frac{1}{2k-1}$ in the range $1/2 \leq \beta \leq 1/\sqrt{3}$, and improved previous approximation ration of $O(\log\frac{n}{n-k}\log k)$ [13] in this entire range. For undirected k-CS we offered an approximation ratio of $1+\frac{2\beta}{k(1-\beta)}-\frac{1}{2k-1}$, this result improves the approximation ration of $\frac{\beta}{1-\beta}$ [2] for all $k \geq 3$ and $\frac{1}{2}+\frac{3k-2}{2(4k^2-7k+2)} < \beta$, and improves the approximation ration of $2+\beta\frac{k}{n}$ [10] for any[1] $\beta < \frac{k^2}{(k+1)^2-2}$. We note that [2] also introducted an improvement for $\frac{\beta}{1-\beta}$ when $k=3$, and $\beta \leq \frac{2}{3}$, and thus their improvement range is dosjoint to ours in this case.

For other versions of the problem our results are as follows.

Theorem 1.2 SND *with β-quasi-metric costs admits approximation ratios $\frac{2\beta}{1-\beta}$ for undirected graphs and $\frac{4\beta^3}{1-3\beta^2}$ for directed graphs and $1/2 \leq \beta \leq 1/\sqrt{3}$. For* Subset k-CS *the ratios are $\frac{\beta}{1-\beta}$ for undirected graphs and $\frac{2\beta^3}{1-3\beta^2}$ for directed graphs; for directed* Rooted SND *the ratio is $\frac{2\beta^3}{1-3\beta^2}$.*

For directed SND no previous results existed, we introduce an approximation ratio of $\frac{4\beta^3}{1-3\beta^2}$ for any $\frac{1}{2} \leq \beta < \frac{1}{\sqrt{3}}$. For both directed Subset k-CS and directed Rooted SND there exists previous results of $O(n^2)$ and $O(n)$ respectivly, we provide an approximation ratio of $\frac{2\beta^3}{1-3\beta^2}$ and improve both results for any $\frac{1}{2} \leq \beta < \frac{1}{\sqrt{3}}$.

The improvement ranges of our results a summarized in Table 2.

[1] This calculation assumed k might be relevantly small and compared out results to a 2-approximation

2 Harray graphs and β-quasi-metric costs

Our work relies on three important results. First is Cheriyan & Thurimella [4] approximation for Subset k-CS. Second is the costs relations in β-quasi-metric as decribes in the following statment:

Lemma 2.1 ([1, 3]) *Let e, e' be a pair of edges in a complete graph G with β-quasi-metric costs.*

(i) *If G is undirected, and if e, e' are adjacent then $c(e) \leq \frac{\beta}{1-\beta} c(e')$.*

(ii) *If G is directed, and if $\frac{1}{2} \leq \beta \leq \frac{1}{\sqrt{3}}$, then $c(e) \leq \frac{2\beta^3}{1-3\beta^2} c(e')$.*

And third Harrary Graphs. These graphs are explicit constructions for undirected k-CS with uniform costs, that is a graph with minimum number of edges such that it is k-connected. We extend this construction for the general case of SND. In this section we will introduce the construction of Harrary, and summerize the results achived in [[1, 3]] on the relations between costs in β-quasi-metric.

2.1 Harrary Graphs

In the sixties Harary [9] showed that for any k, n, and $k < |V|$ there exist a n-nodes, k-connected undirected graph $H_{n,k} = (V, E)$ such that $|E| = \lceil \frac{|V| \cdot k}{2} \rceil$, and that every n-nodes, k-connected undirected graph have at least $\lceil \frac{|V| \cdot k}{2} \rceil$ edges. Given n nodes $V = \{v_0, \cdots, v_{n-1}\}$, the construction of $H_{n,k}$ is divided to three cases.

(i) If both n, k are even, then an edge (v_i, v_j) is in $H_{n,k}$ if $j - i \leq \frac{k}{2} \text{mod } n$.

(ii) If both n is even, and k is odd, then an edge (v_i, v_j) is in $H_{n,k}$ if $j - i \leq \frac{k}{2} \text{mod } n$ or $j - i \equiv \frac{n}{2} \text{mod } n$.

(iii) If both n is even, and k is odd, then an edge (v_i, v_j) is in $H_{n,k}$ if $j - i \leq \frac{k}{2} \text{mod } n$ or $i \leq \frac{n+1}{2}, j - i \equiv \frac{n}{2} \text{mod } n$.

the follwing figure demonstrates the construction on small n, k.

For directed graph a similar construction exists, it is a well known simple construction even tough we cannot find reference to it in the litreture.

Given n nodes $V = \{v_0, \cdots, v_{n-1}\}$, we include all edges (v_i, v_j) such that $j - i \leq k \text{mod } n$.

the follwing figure demonstrates the construction on small n, k.

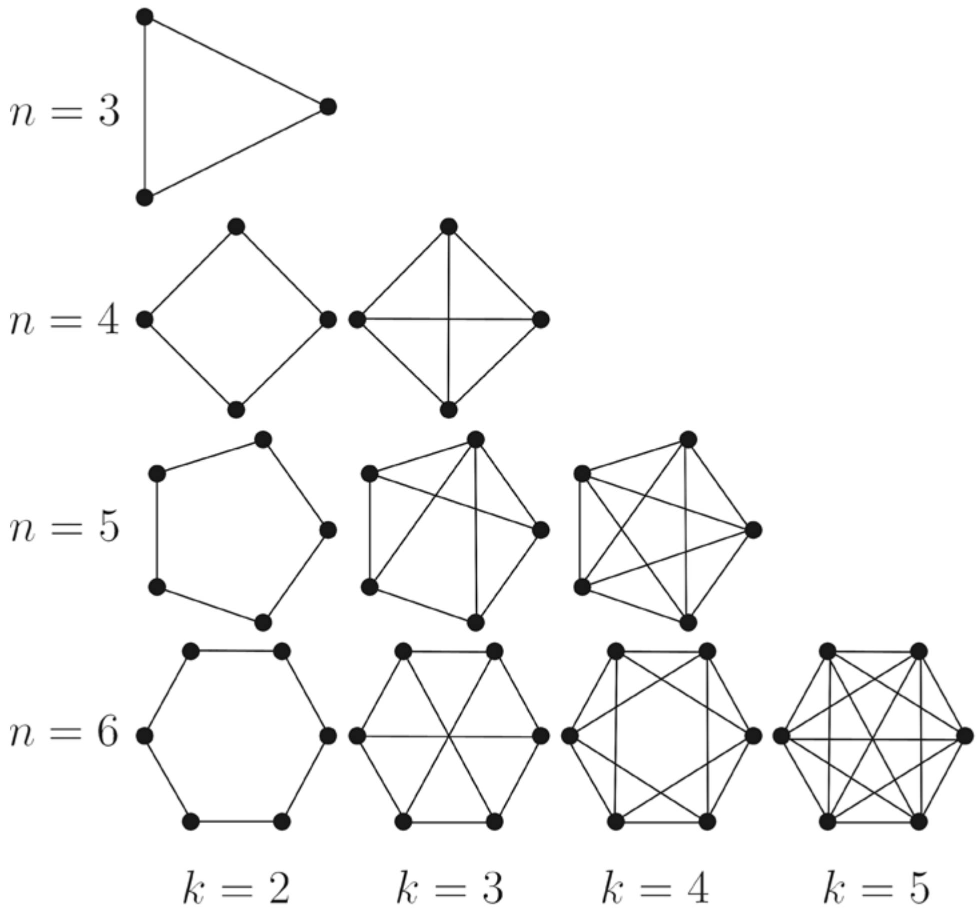

Figure 1: Harary Graphs

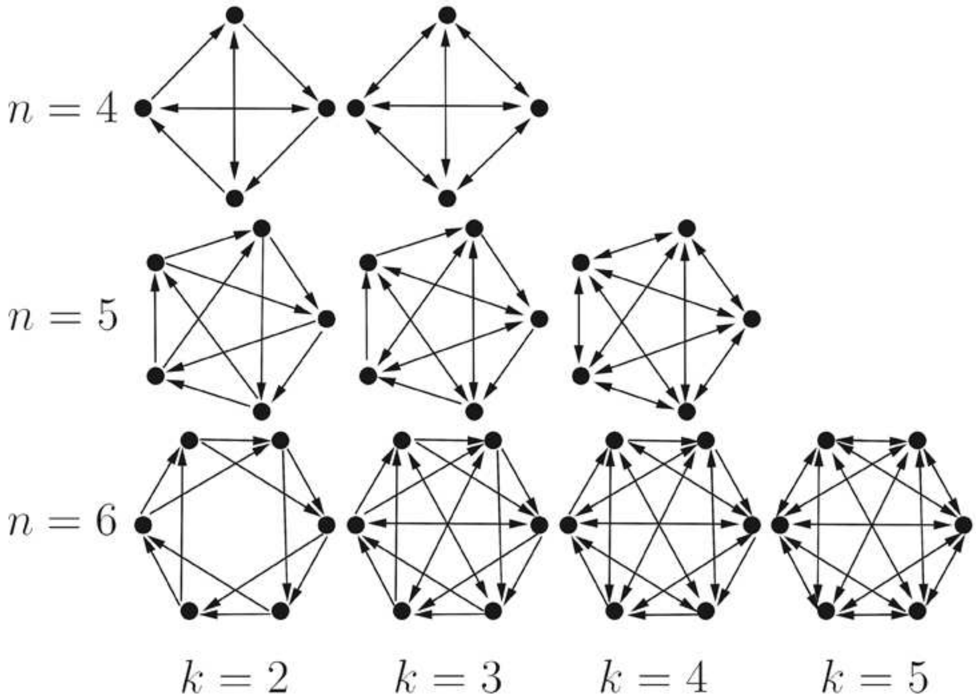

Figure 2: Directed "Harrary Graphs"

2.2 Proof of lemma 2.1

Part *(i)* is simple, given an undirected graph $G=(V,E)$ with costs satisfying the β-triangle, and two edges uv, uw. $c(u,v) \leq \beta[c(u,w)+c(w,v)] \leq \beta c(u,w)+\beta^2[c(u,w)+c(u,v)]$, rearranging we have $c(u,v) \leq \frac{\beta+\beta^2}{1-\beta^2}c(u,w) = \frac{\beta}{1-\beta}c(u,w)$.

In the rest of this section assumes that $\frac{1}{2} \leq \beta < \frac{1}{\sqrt{3}}$, and w.l.o.g. that the edge of minimum cost $c_{min}=1$. We will first introduce a proof for the following lemma, and then use it to proof part *(ii)*.

Lemma 2.2 *Given a directed graph with β-quasi-metric. Assume w.l.o.g. that uv is an edge of minimum cost and $c(u,v)=1$, let $w \in V$, and let c_{max} be the maximum cost of an edge in the subgraph induces by $\{u,v,w\}$. Then the following statement holds:*

$$c(u,w) \leq \frac{\beta+\beta^2 c_{max}}{1-\beta^2}, \qquad c(v,w) \leq \frac{\beta^2+\beta c_{max}}{1-\beta^2} \tag{1}$$

$$c(w,v) \leq \frac{\beta+\beta^2 c_{max}}{1-\beta^2}, \qquad c(w,u) \leq \frac{\beta^2+\beta c_{max}}{1-\beta^2} \tag{2}$$

Proof: Let define a sequence: $x_0 = c_{max}$, $x_i = \beta+\beta^2 c_m ax+\beta^2 x_{i-1}$ and show that $c(u,w) \leq x_i$ for all i, obviously $c(u,w) \leq x_0 = c_m ax$, now assume $c(u,w) \leq x_{i-1}$, then $c(u,w) \leq \beta(1 + c(v,w)) \leq \beta + beta^2 c_{max} + \beta^2 x_{i-1} = x_i$. Now x_i is monotone and $\lim_{i\to\infty} x_i = \frac{\beta+\beta^2 c_{max}}{1-\beta^2}$. Thus $C(u,w) \leq \frac{\beta+\beta^2 c_{max}}{1-\beta^2}$, and $c(v,w) \leq \beta[c_{max} + c(u,w)] = \frac{\beta^2+\beta c_{max}}{1-\beta^2}$. And the same arrguments can be made to prove (2). $\square$

Now given a directed graph $G = (V,E)$ with costs satisfying the β-triangle inequality. If $\frac{c_{max}}{c_{min}} \leq \frac{\beta^2}{1-\beta-\beta^2} \leq \frac{2\beta^3}{1-3\beta^2}$ the lemma follows immediatly, otherwise $\frac{c_{max}}{c_{min}} > \frac{\beta^2}{1-\beta-\beta^2}$.

In this case, let assume uv is a minimum cost edge, and $c(u,v) = 1$. For any edge $e = xy$ such that e is not incident to either u or v, $c(x,y) \leq \beta[c(x,u) + c(u,y)]$ and by lemma 2.2 $\leq \frac{\beta^2}{1-\beta}(1 + c_{max})$, from $c_{max} > \frac{\beta^2}{1-\beta-\beta^2}$ we can derive that $c_{max} > \frac{\beta^2}{1-\beta}(1 + c_{max})$, and thus $c(x,y) < c_{max}$. Now if $c_{max} > c_{min}$ then also $c_{max} > \frac{\beta^2+\beta c_{max}}{1-\beta^2} \geq \frac{\beta+\beta^2 c_{max}}{1-\beta^2}$, thus by lemma 2.2 for any node $w \neq u, v$ all edges uw, wu, wv, vw are strictly lower then c_{max}. Thus all edges in G except vu are strictly lower then c_{max}, we means that vu is the unique edge of maximum cost, and by lemma 2.2 (for some $w \in V$, $c_{max} = c(v,u) = \beta[c(v,w) + c(w,u)] \leq \frac{2\beta(\beta c_{max}+\beta^2)}{1-\beta^2}$ rearranging, the lemma follows.

3 Proof of Theorem 1.1

For an edge set F and a node v, let $\deg_F(v)$ denote the degree of v in F. For directed graphs, let $\deg_F^{in}(v)$ and $\deg_F^{out}(v)$ denote the indegree and the outdegree of v in F.

Definition 3.1 *An edge set F on node set V is a k-*cover *if for all $v \in V$:*

(i) $\deg_F(v) \geq k$ *if F is undirected.*

(ii) $\deg_F^{in}(v) \geq k$ *and* $\deg_F^{out}(v) \geq k$ *if F is directed.*

Lemma 3.1 *For both directed and undirected graphs, any k-cover J contains a $(k-1)$-cover F of cost $c(F) \leq \left(1 - \frac{1}{2k-1}\right) c(J)$.*

Proof: The following procedure finds $M \subseteq J$ such that $F = J - M$ is a $(k-1)$-cover and $c(M) \geq c(J)/(2k-1)$. Start with $M = \emptyset$, $F = J$, and all edges in F unmarked, and iteratively do the following, until all edges that remain in F are marked. Among all unmarked edges in F, let $e = uv$ be one of the maximum cost. Remove e from F and add it to M. In the case of undirected graphs, if the degree in F of an endnode of e is exactly $k-1$, mark all edges incident to this endnode. In the case of directed graphs, if $\deg_F^{out}(u) = k-1$ mark all edges leaving u, and if $\deg_F^{in}(v) = k-1$ mark all edges entering v. It is easy to see that at the end $F = J - M$ is a $(k-1)$-cover. At every iteration, at most $2k-1$ edges in F are removed or marked, and each of them is cheaper than the edge e added to M. Hence $c(M) \geq c(J)/(2k-1)$. $\square$

Let $F \subseteq E$ be a minimum cost $(k-1)$-edge-cover. Such F of minimum costs can be computed in polynomial time, for both directed and undirected graphs, c.f. [15]. As any feasible solution to k-CS is a k-edge-cover, $c(F) \leq \left(1 - \frac{1}{2k-1}\right) \mathsf{opt}$, by Lemma 3.1. Now let $I \subseteq E - F$ be an inclusion minimal augmenting edge set so that $H = (V, F + I)$ is k-connected. It is known that I is a forest in the case of undirected graphs, and $|I| \leq 2n-1$ in the case of directed graphs.

In the case of undirected graphs, since I is a forest, there exists an orientation D of I so that the outdegree of every node w.r.t. D is at most 1. Let D_i be the set of edges in D leaving v_i, so either $D_i = \emptyset$ or $|D_i| = 1$ for all i. As $J_i \geq k$, we have $c(D_i) \leq c(J_i)\frac{\beta}{k(1-\beta)}$, by Lemma 2.1. Hence

$$c(I) = \sum_{i=1}^{n} c(D_i) \leq \frac{\beta}{k(1-\beta)} \sum_{i=1}^{n} c(J_i) \leq \frac{2\beta}{k(1-\beta)} c(J) = \frac{2\beta}{k(1-\beta)} \cdot \mathsf{opt} \ .$$

Consequently,

$$c(H) = c(F) + c(I) \le \left(1 - \frac{1}{2k-1}\right) \cdot \mathsf{opt} + \frac{2\beta}{k(1-\beta)} \cdot \mathsf{opt} = \left(1 + \frac{2\beta}{k(1-\beta)} - \frac{1}{2k-1}\right) \cdot \mathsf{opt}\ .$$

In the case of directed graphs, $|I| \le 2n-1$. As any feasible solution has at least kn edges, we have

$$c(I) \le \frac{2n-1}{kn} \cdot \frac{2\beta^3}{1-3\beta^2} \cdot \mathsf{opt} \le \frac{4\beta^3}{k(1-3\beta^2)} \cdot \mathsf{opt}\ .$$

Consequently,

$$c(H) = c(F) + c(I) \le \left(1 - \frac{1}{2k-1}\right) \cdot \mathsf{opt} + \frac{4\beta^3}{k(1-3\beta^2)} \cdot \mathsf{opt} = \left(1 + \frac{4\beta^3}{k(1-3\beta^2)} - \frac{1}{2k-1}\right) \cdot \mathsf{opt}\ .$$

4 Proof of Theorem 1.2

Our strategy to prove Theorem 1.2 is to give an explicit construction of a graph H so that the following holds. In the case of directed graphs, the number of edges in H is at least α times a lower bound on the number of edges in any feasible solution. Using part (ii) from Lemma 2.1, this immediately implies the ratio $\alpha \cdot \frac{2\beta^3}{1-3\beta^2}$. In the case of undirected graphs, we will show that we can orient the edges of H so that the number of the edges leaving every node v is at most α times the number of edges incident to v in any feasible solution. Using part (i) from Lemma 2.1, this immediately implies the ratio $\alpha \cdot \frac{\beta}{1-\beta}$. For both directed and undirected graphs, we will have $\alpha = 2$ for SND and $\alpha = 1$ for Subset k-CS. For directed $\mathsf{Rooted\ SND}$ we will also have $\alpha = 1$.

Given an instance $G = (V, E), c, r$ of SND we use the following notation. Let $V = \{v_1, \ldots, v_n\}$. Let us fix some optimal solution J. For undirected graphs, the *requirement* r_i *of* v_i is the maximum requirement of a pair containing v_i, and J_i is the set of edges in J incident to v_i. For directed graphs $r_i^{out} = \max_{v_j \in V} r(v_i, v_j)$ is the *out-requirement of* v_i, and $r_i^{in} = \max_{v_j \in V} r(v_j, v_i)$ is the *in-requirement of* v_i; J_i^{out} and J_i^{in} be the set of edges in J leaving and entering v_i, respectively.

4.1 General SND

For general SND we use the following simple construction.

Lemma 4.1 *Let $V = \{v_1, \ldots, v_n\}$ be a node set, and for $i = 1, \ldots, n$ let $r_i^{out}, r_i^{in} \leq n-1$ be non-negative integers. Let A_i^{out} be the set of edges from v_i to the first $r^{out}(v_i)$ nodes in $V - \{v_i\}$, and A_i^{in} be the set of edges from the first $r^{out}(v_i)$ nodes in $V - \{v_i\}$ to v_i. Namely:*

$$A_i^{out} = \begin{cases} \{v_i v_j : 1 \leq j \leq r(v_i)\} & \textit{if } \ r^{out}(v_i) < i \\ \{v_i v_j : 1 \leq j \leq r(v_i) + 1, j \neq i\} & \textit{otherwise} \end{cases}$$

$$A_i^{in} = \begin{cases} \{v_j v_i : 1 \leq j \leq r(v_i)\} & \textit{if } \ r^{in}(v_i) < i \\ \{v_j v_i : 1 \leq j \leq r(v_i) + 1, j \neq i\} & \textit{otherwise} \end{cases}$$

Then for any $i \neq j$, the graph $H_{ij} = (V, A_i^{out} \cup A_j^{in})$ contains at least $\min\{r_i^{out}, r_j^{in}\}$ internally disjoint $v_i v_j$-paths.

Proof: Note that there is a set C of $\min\{r(v_i), r(v_j)\} - 1$ nodes so that in H_{ij} there is an edge from v_i to every node in C and from every node in C to v_j; furthermore, either $v_i v_j \in H_{ij}$ or $v_i v_j$ there is one more node that can be added to C. The statement follows. $\square$

The algorithm is as follows. In the case of directed graphs, we compute the edge sets A_i^{out} and A_i^{in} as in Lemma 4.1, and output their union graph H. In the case of undirected graphs, we consider the directed problem on the bi-direction of G with the requirements $r^{in}(v_i) = 0$ for all i, $r^{out}(v_i, v_j) = \max\{r(v_i, v_j), r(v_j, v_i)\}$ for $i > j$ and $r^{out}(v_i, v_j) = 0$ otherwise. Hence we will have $A_i^{in} = \emptyset$ for all i. The graph H is the underlying graph of the union of the sets A_i^{out}. For both directed and undirected graphs we have $\kappa_H(v_i, v_j) \geq \min\{r(v_i), r(v_j)\} \geq r(v_i, v_j)$, hence H is a feasible solution.

To establish the approximation ratio, we will use Lemma 2.1. In the case of directed graphs, note that $|A_i^{out}| = r_i^{out}$ and $|A_i^{in}| = r_i^{in}$ while $|J_i^{out}| \geq r_i^{out}$ and $|J_i^{in}| \geq r_i^{in}$. Hence the number of edges in the constructed solution is $\sum_{i=1}^{n}(r_i^{out}+r_i^{in})$, while any feasible solution has at least half this number of edges. Combined with part (ii) of Lemma 2.1, this immediately implies the ratio $\frac{4\beta^3}{1-3\beta^2}$.

In the case of undirected graphs, let A_i be the set of undirected edges that corresponding to A_i^{out} in the bi-direction of G. Note that $|A_i| = r(v_i)$ and that $|J_i| \geq r_i$ for all i. Hence $c(A_i) \leq \frac{\beta}{1-\beta}c(J_i)$, by part (i) of Lemma 2.1. Thus

$$c(H) \leq \sum_{i=1}^{n} c(A_i) \leq \frac{\beta}{1-\beta}\sum_{i=1}^{n} c(J_i) \leq \frac{2\beta}{1-\beta}c(J) = \frac{2\beta}{1-\beta} \cdot \mathsf{opt}\ .$$

4.2 Subset k-CS

Recall that Subset k-CS is the case of SND when for some $T \subseteq V$ we have $r(u, v) = k$ for all $u, v \in T$. Let $t = |T|$. For the case $t > k$ we can apply our algorithm for k-CS while ignoring the nodes in $V - T$, thus obtaining ratios as in Theorem 1.1. We can also obtain the ratios as in Theorem 1.2. Such an algorithm is described in [2] for undirected graphs, and we extend it to directed graphs.

Let $\ell \leq t - 1$ be an integer. Let $H(\ell)$ be an ℓ-connected graph on T with the following property. In the case of undirected graphs, we require that $H(\ell)$ has an orientation so that the outdegree of every node is exactly k. In the case of directed graphs, we require that $H(\ell)$ has ℓn edges. Such graphs are known to exist. If $t \geq k + 1$ then our algorithm for k-CS returns any graph $H(k)$ as above. The approximation ratio is shown as follows. In the case of undirected graphs, let A_i be the set of edges corresponding to the edges leaving v_i in the above orientation of $H(k)$. For any feasible solution, the degree of every node in T is at least k. The ratio of $\frac{\beta}{1-\beta}$ now immediately follows from part (i) of Lemma 2.1. In the case of directed graphs, any feasible solution has at least kt edges. The ratio of $\frac{2\beta^3}{1-3\beta^2}$ now immediately follows from part (ii) of Lemma 2.1.

Our construction for the case $t \leq k$ is a slight extension of this construction. Note that $|V| \geq k+1$, as otherwise the problem has no feasible solution. We choose a set $U \subseteq V-T$ of arbitrary $k-t+1$ nodes, and obtain a graph H by adding all possible edges between $H(t-1)$ and U. It is easy to see that H is a feasible solution. For the analysis of the approximation ratio, we use the following simple observation.

Lemma 4.2 *Let J be a feasible solution to a* Subset k-CS *instance. Then:*

(i) *For undirected graphs, every node in T has in J at least $k-t+1$ neighbors in $V-T$.*

(ii) *For directed graphs, J has at least $t(t-1)+2t(k-t+1)$ edges.*

Proof: In undirected J, every node in T has at least k neighbors. At most $t-1$ of these neighbors can lie in T, hence all the other at least $k-t+1$ neighbors are in $V-T$. In directed J, every node has outdegree and indegree at least k. At most $t-1$ edges can enter a node from nodes in T, or leave a node to a node in T. Hence for every $v \in T$, at least $k-t+1$ edges go from v to $V-T$, and at least $k-t+1$ edges go from $V-T$ to v. Thus the number of edges in J is at least $t(t-1)+2t(k-t+1)$, as claimed. $\square$

For undirected graphs, we orient the edges of our solution H as follows. We can orient the edges of $H(t-1)$ so that the outdegree of every node is k, and we orient the edges between $H(t-1)$ and U from T to U. In this orientation, the outdegree of every node is exactly $k-t+1$. For directed graphs our solution H has exactly $t(t-1)+2t(k-t+1)$ edges, Thus the ratios $\frac{\beta}{1-\beta}$ for undirected graphs and $\frac{2\beta^3}{1-\beta^2}$ for directed graphs follow from Lemmas 4.2 and 2.1.

4.3 Directed Rooted SND

Let $G=(V,E),c,r,s$ be an instance of directed Rooted SND where s is the root. Let $T=\{u \in V : r(s,u)>0\}$ be the set of nodes with positive in-requirements.

Lemma 4.3 *Any feasible solution J for directed* Rooted SND *has at least $\sum_{v \in T} r^{in}(v)$ edges, and at least $\sum_{v \in T} r^{in}(v)+k-|T|$ edges if $k>|T|$.*

Proof: Clearly, $\deg_J^{in}(v) \geq r^{in}(v)$ and $\deg_J^{out}(r) \geq k$. Now consider the edges in J leaving r. At most $|T|$ of these edges can go to nodes in T, hence if $k>|T|$ then there are at least $k-|T|$ edges that go to nodes in $V-T$. The statement follows. $\square$

Now we show that the lower bound in Lemma 4.3 is achievable. Construct a graph H as follows. Let $U \subseteq V \setminus \{s\}$ be an arbitrary set of $\max\{k,|T|\}$ nodes containing T, so $U=T$ if

$|T| \geq k$. Take an edge from s to every node in U, and for every $v \in T$ take arbitrary $r^{in}(v)-1$ edges entering v from any $r^{in}(v)-1$ nodes in $U \setminus \{v\}$. It is easy to see that H is a feasible solution for the directed Rooted SND instance, and the number of edges in H coincides with the lower bound in Lemma 4.3. Applying Lemma 2.1(ii) we obtain $c(H) \leq \frac{2\beta^3}{1-3\beta^2} \cdot \mathsf{opt}$.

The proof of Theorem 1.2 is complete.

5 Conclusions

We have analized and shown that the algorithm of Cheriyan & Thurimella [4] achives $1 + \frac{2\beta}{k(1-\beta)} - 12k - 1$, and $1 + \frac{4\beta^3}{k(1-3\beta^2)} - 12k - 1$ for undirected and directed k-CS with β-quasi-metric costs.

We used Harrary construction for undirected k-CS, and provided explicit construction for directed k-CS, undirected subset k-CS, directed subset k-CS. and directed rooted SND. All of which gives an optimal solution when the edge costs are uniform. For the general case of SND we provided an explicit consturction that achives 2-approximation for unifrom costs.

Using those construction, and properties of β-quasi-metric we provided approximation ratios for subset k-CS, SND, and an improvment for rooted SND with directed graphs.

Still some questions remains unanswered. Is there any explicit construction for SND with uniform costs that provide an optimal solution?, and if not is there a better approximation then 2?. Are there any better approximation for the problems of subset k-CS, k-CS, and SND?. And for the directed SND, is there any approximation for $\beta > \frac{1}{\sqrt{3}}$?, note that for $\beta = 1$ there is a lower bound of $\Omega(2^{\log^{1-\epsilon} n})$.

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