On Rooted Node-Connectivity Problems

Joseph Cheriyan [*] Tibor Jordán [†] Zeev Nutov [‡]

March 10, 2000, Revised version

Abstract

Let G be a graph which is k-outconnected from a specified root node r, that is, G has k openly disjoint paths between r and v for every node v. We give necessary and sufficient conditions for the existence of a pair rv, rw of edges for which replacing these edges by a new edge vw gives a graph that is k-outconnected from r. This generalizes a theorem of Bienstock, Brickell and Monma on splitting off edges while preserving k-node-connectivity.

We also prove that if C is a cycle in G such that each edge in C is critical with respect to k-outconnectivity from r, then C has a node v, distinct from r, which has degree k. This result is the rooted counterpart of a theorem due to Mader.

We apply the above results to design approximation algorithms for the following problem: given a graph with nonnegative edge weights and node requirements c_u for each node u, find a minimum-weight subgraph that contains $\max\{c_u, c_v\}$ openly disjoint paths between every pair of nodes u, v. For metric weights, our approximation guarantee is 3. For uniform weights, our approximation guarantee is $\min\{2, \frac{k+2q-1}{k}\}$. Here k is the maximum node requirement, and q is the number of positive node requirements.

1 Introduction

A graph is said to be *k-outconnected from node r* if there exist k openly disjoint paths from r to every node v, $v \neq r$. Node r is called the *root.*

Splitting off two edges ru, rv means deleting ru and rv and adding a new edge uv. Splitting off is a basic operation in graph connectivity with a broad range of applications. There are a number of results asserting the existence of pairs of edges that can be split off preserving certain edge-connectivity conditions or directed node-connectivity conditions, see the survey by Frank [5].

For the node-connectivity of undirected graphs, only one general splitting-off result has been proved so far. This result is due to Bienstock, Brickell and Monma [1], see Theorem 4. (Related results are in [8] and [4].) We generalize this result from k-node-connected graphs to k-outconnected graphs by giving necessary and sufficient conditions for the existence of a pair of edges incident to the root r that can be split off while preserving k-outconnectivity from r, see Theorem 3.

Mader's theorem (Theorem 19 below) on "cycles of critical edges" in k-node-connected graphs has applications in extremal graph theory, connectivity augmentation, and approximation algorithms, see [2], [3], [8], [11]. We prove a related but independent result for k-outconnected graphs in Theorem 21.

[*]jcheriyan@math.uwaterloo.ca Department of Combinatorics and Optimization, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1. Supported in part by NSERC research grant no. OGP0138432.

[†]jordan@cs.elte.hu Department of Operations Research, Eőtvős University, 1088 Budapest, Hungary.

[‡]nutov@oumail.openu.ac.il Open University, Israel. Supported in part by NSERC research grant no. OGP0138432.

We use the above structural results to design and analyze approximation algorithms for network design problems, see Sections 5.2,5.3.

In the rest of the Introduction, we discuss some network design problems of interest to us, list some related results, and state our new approximation results. A basic problem in network design is to find a minimum-weight spanning subgraph of a given graph satisfying certain connectivity conditions. For the following general problem no algorithm is known that achieves a non-trivial approximation guarantee in polynomial time.

Problem A: Given a graph G with a nonnegative weight function w on the edges, and node connectivity requirements $c(u,v)$ for each pair of nodes u,v, find a minimum-weight spanning subgraph H such that for every node pair u,v there exist at least $c(u,v)$ openly disjoint paths between u and v in H.

The case when $c(u,v) \equiv k \geq 2$ for each pair u,v is called the minimum-weight k-node-connected subgraph problem. This problem is already NP-hard, even for metric weights (that is, when the weight function satisfies the triangle inequality) and uniform weights (that is, when the weight of every edge is the same), and there has been extensive recent research on approximation algorithms for this and related problems with uniform weights and with metric weights, see [1, 3, 7, 9, 10]. For metric weights, Khuller and Raghavachari [10] developed a $(2+2(k-1)/n)$-approximation algorithm for the minimum-weight k-node-connected subgraph problem. For uniform weights, Cheriyan and Thurimella [3] gave a $(1+\frac{1}{k})$-approximation algorithm.

We design approximation algorithms for the following problem that is sandwiched between the minimum weight k-node-connected subgraph problem and Problem A.

Problem B: Given a graph G with a nonnegative weight function w on the edges, and a node requirement c_u for each node u, find a minimum-weight spanning subgraph H such that for every node pair u,v there exist at least $\max\{c_u,c_v\}$ openly disjoint paths between u and v in H.

Some special cases of Problem B have been investigated previously by Nutov et al [13]. Motivated by an application in so-called mobile robot flow networks, they gave approximation algorithms for cases where $c_u \leq 3$ for each node u. Stoer [14] discusses another special case of Problem A where the local requirements are of the form $c(u,v) = \min\{c_u,c_v\}$, for given node requirements c_u. Problem B may find applications elsewhere, say, in network design problems where some distinguished nodes are required to have a large number of connections to all the other nodes of the network.

It is easy to see that a graph H is a feasible solution for Problem B if and only if H is c_u-outconnected from each node u. We call Problem B the *minimum-weight multi-root outconnected subgraph problem*, or the *multi-root problem*. Given an instance of the multi-root problem, q is defined to be the number of nodes u with positive node requirement c_u. Notice that Problem B is a special case of Problem A, and the minimum-weight k-node-connected subgraph problem is a special case of Problem B. Thus Problem B is NP-hard, even for metric weights and uniform weights. To the best of our knowledge, the previous best approximation guarantee known for the (general) q-root problem is $2q$, even for uniform weights and for metric weights; see Section 5.1 for a discussion of the $2q$-approximation algorithm.

For the metric weight multi-root problem (with q roots) we improve the approximation guarantee from $2q$ to 3, see Theorem 27. Our approximation algorithm is based on Theorem 17 which is a weaker version of Theorem 3 (our splitting-off result for k-outconnectivity). For the uniform weight multi-root problem we improve the approximation guarantee from $2q$ to $\min\{2, \frac{k+2q-1}{k}\}$, where k is the largest node requirement, see Theorem 29. This implies a $(1+\frac{1}{k})$ approximation algorithm for the uniform weight single-root problem. This approximation algorithm is based on

Theorem 21, our rooted counterpart of Mader's theorem.

More definitions and preliminaries are given in Section 2. The splitting-off theorem for k-outconnectivity is given in Section 3, and the rooted version of Mader's theorem is given in Section 4. The approximation algorithms are in Section 5.

2 Definitions and preliminary results

Graphs in this paper are undirected and have no multiple edges and no loops. A pair of sets A, B is called *properly intersecting* if each of the three sets $A-B$, $B-A$, $A\cap B$ is nonempty. A *separator* of a connected graph is a set of nodes whose deletion results in a disconnected graph. Given a connected graph G, we say that a node set S *separates* a pair u, v of nodes, or simply that S is an (u,v)*-separator* if the two nodes are in different components of $G-S$. Two distinct paths are called *openly disjoint* if every node common to both paths is an end node of both paths. For a graph G and a pair of nodes u, v, $\kappa_G(u,v)$ denotes the maximum number of pairwise openly disjoint paths between u and v. A connected graph is said to be k*-node-connected* if it has at least $k+1$ nodes and it has no separator of cardinality $k-1$. From now on, k*-connected* refers to k-node-connected. By Menger's theorem, $\kappa_G(u,v) \geq k$ for every pair of nodes u, v in a k-connected graph G. Let G be a graph that is k-outconnected from a node r (that is, $\kappa_G(v,r) \geq k$ for every node v, $v \neq r$); a pair of edges incident to r is called *admissible* if splitting off the pair preserves k-outconnectivity from r, otherwise the edge pair is called *illegal.* An edge uv of a k-connected graph H is called *critical* (with respect to k-connectivity) if $H-uv$ is not k-connected. Similarly, an edge uv of a graph H' that is k-outconnected from a node r is called *critical* (with respect to k-outconnectivity) if $H'-uv$ is not k-outconnected from r. We say that a graph is $(c_1,...,c_q)$*-outconnected* from roots $(r_1,\ldots,r_q)$ if it is simultaneously c_i-outconnected from each r_i, $i=1,...,q$. Given an instance of Problem B, the vector $\vec{c}=(c_1,...,c_q)$ of positive node requirements is called the *connectivity requirement vector* and the corresponding vector $\vec{R}=(r_1,\ldots,r_q)$ is called the *root vector.* We shall always assume, without loss of generality, that $c_1 \geq c_2 \geq ... \geq c_q$ holds for the connectivity requirement vector.

For a graph $G=(V,E)$ and a nonempty set $X \subseteq V$ of nodes, $\Gamma_G(X)$ or $\Gamma(X)$ denotes the set $\{y \in V-X : xy \in E \text{ for some } x \in X\}$ of *neighbours* of X. The following proposition is well-known (see [8, Lemma 1.2]) and is easy to verify by counting the contribution of each node to the two sides of the inequality.

Proposition 1 *In a graph $H=(V,E)$ every pair $X, Y \subseteq V$ satisfies*

$$\Gamma(X)+\Gamma(Y) \geq \Gamma(X\cap Y)+\Gamma(X\cup Y). \tag{1}$$

Moreover, if equality holds, then there are no edges from $X-Y$ to $Y-X-\Gamma(X\cap Y)$. $\square$

A ρ*-approximation algorithm* for a minimization problem runs in polynomial time and delivers a solution whose value is always within the factor ρ of the optimum value. The quantity ρ is called the *approximation guarantee* of the algorithm.

2.1 Irreducible node requirements

Some entries in a connectivity requirement vector may be redundant: no matter what is the underlying graph, we can reset them to zero (and get a shorter vector) without changing the problem. In this section we characterize when such a reduction is possible. This will enable us later to assume that the number q of roots is at most the value of the largest node requirement.

A connectivity requirement vector $\vec{c} = (c_1, \ldots, c_q)$ is called *n-reducible* if it has a proper subvector $\vec{c^*}$ that implies the connectivity requirements of $\vec{c}$. That is, for every graph G on n nodes and with an arbitrary choice of q roots we have that G is $\vec{c}$-outconnected from the roots if and only if it is $\vec{c^*}$-outconnected from the (reduced vector of) roots. Otherwise, $\vec{c}$ is called *n-irreducible*. For example, $\vec{c} = (2, 1)$ is n-reducible for every n since a 2-outconnected graph is 1-outconnected from every node. The next result characterizes reducible requirement vectors.

Proposition 2 *Let $\vec{c} = (c_1, \ldots, c_q)$ be a connectivity requirement vector with $c_1 \geq \ldots \geq c_q$. Then $\vec{c}$ is n-irreducible if and only if*

(1) $c_j \geq j$ for $1 \leq j \leq q$, and

(2) $c_1 \leq \dfrac{n + c_q - 3}{2}$.

Proof: First observe that a graph G that is $(c_1, \ldots, c_q)$-outconnected from a vector of q root nodes must be ℓ-connected, where $\ell = \min\{q, c_1, \ldots, c_q\}$.

Let G be a graph on n nodes which is $(c_1, \ldots, c_q)$-outconnected from $(r_1, \ldots, r_q)$. Suppose that condition (1) fails. Let ℓ be the lowest index such that for $j = \ell + 1$ we have $c_j < j$. Clearly, $\ell \geq 1$, since $c_1 \geq 1$. Note that $c_\ell \geq \ell$. Let $\vec{c^*} = (c_1, \ldots, c_\ell)$ and let $\vec{R^*} = (r_1, \ldots, r_\ell)$. Suppose that H is a graph that is $\vec{c^*}$-outconnected from $\vec{R^*}$. Each of the roots in $\vec{R^*}$ has connectivity requirement at least ℓ, and also the number of roots in $\vec{R^*}$ is ℓ. By our first observation H is ℓ-connected, and so $\kappa_H(v, r_i) \geq \ell$, for each node $v \in V(H) - \{r_i\}$ and each root r_i, $i = \ell + 1, \ldots, q$ (i.e., each root in $\vec{R}$ not in $\vec{R^*}$). Then H is $\vec{c}$-outconnected from $\vec{R}$. Hence, if condition (1) fails, then $\vec{c}$ is reducible.

Suppose that condition (2) fails. Clearly, we may assume $q \geq 2$. We claim that every subgraph satisfying the requirements $\vec{c^*} = (c_1, \ldots, c_{q-1})$ is c_q-outconnected from r_q. Suppose H is a counterexample to this claim. Then there exists a node w in H with $\kappa_H(r_q, w) \leq c_q - 1$. Thus there exists a separator S in H (or, if $wr_q \in E(H)$, then in $H - wr_q$) such that $|S| \leq c_q - 1$ and $r_q \notin S$. Now $r_1 \in S$ holds, since H is c_1-outconnected from r_1. Let D be a component of $H - S$ with $|V(D)| \leq \frac{(n - c_q + 1)}{2}$. Clearly, such a D exists. Focus on a node v in D. Since $(V(D) - \{v\}) \cup (S - \{r_1\})$ separates r_1 and v in H (provided we delete the edge $r_1 v$, if it exists), we have

$$c_1 \leq \kappa_H(r_1, v) \leq |V(D)| - 1 + |S| - 1 + 1 \leq \frac{(n - c_q + 1)}{2} - 1 + c_q - 1 \leq \frac{n + c_q - 3}{2},$$

and this contradicts our assumption on condition (2). This proves that an n-irreducible requirement vector must satisfy both conditions (1) and (2).

Conversely, we can show that a requirement vector $\vec{c}$ is n-irreducible if conditions (1) and (2) hold. To see this, we take the proper subsequence $(c_1, \ldots, c_{q-1})$, and show that it does not always imply the connectivity requirements of $\vec{c}$; a similar argument applies for any other proper subsequence. Let G be a graph on n nodes that contains a separator S such that $|S| = c_q - 1$, $\{r_1, \ldots, r_{q-1}\} \subseteq S$, and $r_q \notin S$, and let S induce a complete subgraph in G. Furthermore, let $G - S$ consist of two complete graphs C_1, C_2 with the same number of nodes, and let G contain all possible edges between S and $V(C_1) \cup V(C_2)$. Then G satisfies the requirement vector $(c_1, \ldots, c_{q-1})$ (since condition (2) holds), but it is not c_q-outconnected from r_q. $\square$

Note that condition (1) does not depend on n. This implies that an irreducible requirement vector must satisfy $c_1, \ldots, c_q \geq q$ in a graph of arbitrary order. For our multi-root outconnected subgraph problem it implies that as long as $c_q < q$ holds we can reset the smallest positive node requirement c_q to zero without changing the problem. Thus we can assume $c_q \geq q$ and, in particular, that the number q of roots is not more than the maximum node requirement.

3 Splitting off edges from the root in a k-outconnected graph

This section contains our splitting-off theorem. Given an integer $k \geq 1$, a graph H, and a specified node r of H, *Property (T)* is said to hold if

> H is k-connected, and there exists a node set T such that $|T| = k$, $r \in T$, and the number of components of $H - T$ equals $\deg_H(r)$.

For example, Property (T) holds for a complete bipartite graph $K_{k,p}$, $p \geq k \geq 1$, with r a node of degree p. If we take a graph H that satisfies property (T) and split off any pair of edges incident to r, then the resulting graph is not k-outconnected from r. (To see this, consider any edge pair rv, rw and let D_v, D_w be the node sets of the components containing v, w respectively in $H - T$. If we split off rv, rw, then r can be separated from $D_v \cup D_w$ (in the new graph) by deleting the $k-1$ nodes in $T - \{r\}$.)

Theorem 3 *Let $G = (V, E)$ be a graph with $|V| \geq 2k$ which is k-outconnected from a root node $r \in V$ and suppose that $\deg(r) \geq k + 2$ and every edge incident to r is critical with respect to k-outconnectivity from r. Then either*

(a) G satisfies Property (T), or

(b) there exists a pair of edges incident to r that can be split off preserving k-outconnectivity.

Note that the condition $|V| \geq 2k$ in the theorem is necessary. (To see this take the complete bipartite graph $K_{k-1,k-1}$ and an additional root node r which is adjacent to all the other nodes.)

Our theorem generalizes the following theorem of Bienstock et al [1].

Theorem 4 ([1]) *Let $G = (V, E)$ be a k-connected graph with $|V| \geq 2k$ and let $r \in V$ be a node such that* $\deg(r) \geq k+2$ *and every edge incident to r is critical with respect to k-connectivity. Then either*

(a) G satisfies Property (T), or

(b) there exists a pair of edges incident to r that can be split off preserving k-connectivity. $\square$

We remark that Theorem 4 here differs from the statement of [1, Theorem 3]. In [1] part (a) is replaced by part (a$'$): "for any edge pair ru, rv, there exists another edge pair sw, sz such that splitting off both edge pairs preserves k-connectivity." Part (a) implies part (a$'$) by a short proof, but part (a$'$) does not imply part (a). However, part (a) is implicitly proved in [8].

To see that Theorem 3 implies Theorem 4 we need two observations: (i) if we split off a pair of edges from a node r in a k-connected graph, and this preserves k-outconnectivity from r, then we preserve k-connectivity as well (otherwise, if the resulting graph has a separator S with $|S| < k$, then $r \in S$ by k-outconnectivity, but then S is a separator of the original graph); (ii) in a k-connected graph, an edge incident to r is critical with respect to k-connectivity if and only if the edge is critical with respect to k-outconnectivity from r.

A proof of Theorem 4 can be extracted from our proof of Theorem 3 by omitting Lemmas 9, 11, and Claims 12–15.

Proof: (of Theorem 3) Let $G = (V, E)$, the root r, and $k \geq 1$ be given and assume that G satisfies all the conditions in Theorem 3. Let R denote the set of neighbours of r.

For nonempty subsets X of $V-r$ let $N(X) := \Gamma_{G-r}(X)$ and let $g(X) := |N(X)| + |X \cap R|$. We fix $g(\emptyset) = 0$. Proposition 1 implies that $g(X) + g(Y) \geq g(X \cap Y) + g(X \cup Y)$ holds for every $X, Y \subseteq V - r$. Moreover, if equality holds, then there is no edge between $X - Y$ and $Y - X - N(X \cap Y)$. We shall refer to these properties as the *submodularity* of g.

Lemma 5 *$G = (V, E)$ is k-outconnected from r if and only if*

$$g(X) \geq k \quad \textit{for every} \quad \emptyset \neq X \subseteq V - r. \tag{2}$$

Proof: We prove one direction; the other one is easy. Suppose that G satisfies (2), but is not k-outconnected from r. Then there is a node $v \neq r$ with $\kappa_G(v, r) < k$. By Menger's theorem, either G or $G - vr$ has a (v, r)-separator C with $|C| \leq k - 1$ or $|C| \leq k - 2$, respectively. Let X be the node set of the component containing v in $G - C$ or in $G - vr - C$. Then $g(X) < k$, a contradiction to (2). $\square$

Recall that a pair of edges incident to r is called admissible if splitting off the pair preserves the k-outconnectivity, otherwise the edge pair is called illegal. By Lemma 5, a pair of edges ru, rv is admissible if and only if removing ru and rv and adding the new edge uv preserves (2). For a pair of edges rx, ry let G' be the graph obtained from G by splitting off rx, ry. If rx, ry is illegal, there must be a node set $X \subseteq V - r$ with $g'(X) < k$ (here g' denotes g on G'). But in G we have $g(X) \geq k$, hence it can be seen that a pair rx, ry is illegal if and only if there exists a set $X \subseteq V - r$ with one of the following properties:

(i) $x, y \in X$, $g(X) \leq k + 1$,

(ii) $x \in X$, $y \in N(X)$, $g(X) = k$, or

(iii) $y \in X$, $x \in N(X)$, $g(X) = k$.

We call a nonempty set $X \subseteq V - r$ *dangerous* if $g(X) \leq k + 1$. If $g(X) = k$ holds then we call X *critical*. Observe that for every neighbour x of the root, the edge rx is critical with respect to k-outconnectivity (thus, by Lemma 5, (2) fails if rx is removed from G), and hence there exists a critical set $X \subseteq V - r$ with $x \in X$. The next lemma establishes some properties of dangerous and critical sets.

Lemma 6 *(1) The intersection and union of two intersecting critical sets are both critical;*

(2) for every node $x \in R$, there exists a unique maximal critical set containing x, denoted S_x;

for $x, y \in R$ and the two sets S_x, S_y, either $S_x = S_y$ or $S_x \cap S_y = \emptyset$ holds;

(3) for two properly intersecting maximal dangerous sets X, Y, we have $g(X \cap Y) = k$ and $g(X \cup Y) = k + 2$;

(4) if X is a maximal dangerous set and Y is a critical set, then either $X \cap Y = \emptyset$ or $Y \subseteq X$;

(5) if D_1, D_2 are distinct maximal dangerous sets containing a node $x \in R$, then $D_1 \cap D_2 = S_x$.

Proof: (1) Let X and Y be two critical sets with $X \cap Y \neq \emptyset$. By criticality we have $g(X) = g(Y) = k$ and (2) (in Lemma 5) implies $g(X \cap Y), g(X \cup Y) \geq k$. Applying the submodularity of g this gives

$$k + k = g(X) + g(Y) \geq g(X \cap Y) + g(X \cup Y) \geq k + k.$$

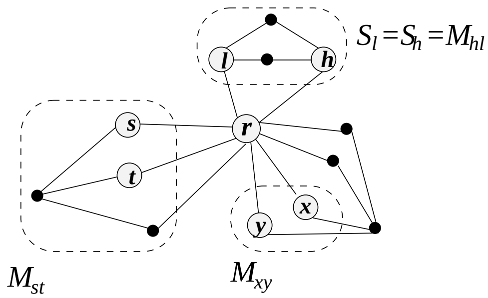

Figure 1: An example illustrating the notation in the proof of Theorem 3. Here $k = 2$. For a neighbour v of r, S_v is a maximal set containing v with $g(S_v) = k$. For a pair of neighbours u, v of r, M_{uv} (if it exists) is a maximal set containing u, v with $g(M_{uv}) \leq k+1$. In the example, $S_v = \{v\}$ for each neighbour v of r except h, ℓ.

Hence equality holds everywhere and $g(X \cap Y) = k$ and $g(X \cup Y) = k$ follow.
(2) Since the edge rx is critical, there is a critical set containing the node $x \in R$. The maximal critical set containing x is unique by part (1).

The second statement follows from part (1) and the first statement, since $S_x \cap S_y \neq \emptyset$ implies that $S_x \cup S_y$ is a critical set.
(3) We have $g(X) \leq k+1$, $g(Y) \leq k+1$, $X \cap Y \neq \emptyset$, and $g(X \cap Y) \geq k$ by (2) (in Lemma 5). Moreover, since $X - Y \neq \emptyset \neq Y - X$ and X, Y are maximal dangerous sets we get $g(X \cup Y) \geq k+2$. Applying the submodularity of g gives

$$k+1+k+1 \geq g(X) + g(Y) \geq g(X \cap Y) + g(X \cup Y) \geq k+k+2.$$

Hence equality holds everywhere and $g(X \cap Y) = k$ and $g(X \cup Y) = k+2$ follow.
(4) We have $g(X) \leq k+1$ and $g(Y) = k$. If $X \cap Y \neq \emptyset$ and $Y - X \neq \emptyset$ then the maximality of X implies that X and Y properly intersect and $g(X \cup Y) \geq k+2$. Applying the submodularity of g this leads to a contradiction:

$$k+k+1 \geq g(X) + g(Y) \geq g(X \cap Y) + g(X \cup Y) \geq k+k+2.$$

(5) By part (4) we have $S_x \subseteq D_i$ for $i = 1, 2$. Suppose $D_1 \cap D_2 \neq S_x$. Then, since D_1, D_2 properly intersect, part (3) implies $D_1 \cap D_2$ is a critical set that properly contains S_x, and this contradicts the maximality of S_x. This proves the lemma. □

Focus on a fixed node pair $i, j \in R$ (and note that $S_i = S_j$ may hold for different nodes $i, j \in R$). If there exists a dangerous set X containing both i and j, then let M_{ij} be defined as an (arbitrarily chosen) maximal dangerous set with $i, j \in M_{ij}$. In this case, we have $S_i \subseteq M_{ij}$, $S_j \subseteq M_{ij}$, by Lemma 6(4). To illustrate our notation, note that if Property (T) holds for some graph H, then the sets S_i are the connected components of $H - T$, $S_i \cap R = \{i\}$ for all $i \in R$, and for all $i \neq j \in R$ $N(S_i)$ is disjoint from S_j and $M_{ij} = S_i \cup S_j$ (so M_{ij} always exists). Also see Figure 1.

Two disjoint sets $A, B \subseteq V - r$ are said to be *adjacent* if there is an edge with one end in A and the other end in B, otherwise A and B are said to be *nonadjacent*. Note that A and B are adjacent if and only if $N(A) \cap B \neq \emptyset$.

The next lemma is the key for demonstrating Property (T).

Lemma 7 *Let i, j, ℓ be nodes in R such that the sets S_i, S_j, S_ℓ are distinct, and the sets $M_{ij}, M_{i\ell}, M_{j\ell}$ exist and are distinct. If S_i, S_j are nonadjacent, then*

$$N(S_i) = N(S_j) \quad \textit{and} \quad |N(S_i)| = k-1.$$

Proof: First observe that $M_{ij} \cap S_\ell = \emptyset$, otherwise $M_{ij} \supseteq S_\ell$ (by Lemma 6(4)), but then $M_{ij} \cap M_{i\ell} \supseteq S_i \cup S_\ell$, a contradiction to Lemma 6(5). Similarly, $M_{i\ell} \cap S_j = \emptyset = M_{j\ell} \cap S_i$. That is, every pair among the three sets $M_{ij}, M_{i\ell}, M_{j\ell}$ is properly intersecting. Now we apply the submodular inequality of g to $A = M_{ij}$ and $B = M_{i\ell} \cup M_{j\ell}$, noting that $A \cap B = S_i \cup S_j$ (by Lemma 6(5)), $g(A) \le k+1$ (since A is a dangerous set), $g(B) = k+2$ (by Lemma 6(3)), $g(A \cup B) \ge k+2$ (by maximality of the dangerous set A), $g(A \cap B) \ge (k+1)$ (by maximality of the critical set S_i).

$$k+1+k+2 \ge g(A) + g(B) \ge g(A \cap B) + g(A \cup B) = g(S_i \cup S_j) + g(A \cup B) \ge k+1+k+2.$$

This means equality holds everywhere, and so $g(S_i \cup S_j) = k+1$. Moreover, S_i and S_j are disjoint nonadjacent sets with $|N(S_i)|, |N(S_j)| \le k-1$, hence,

$$g(S_i \cup S_j) = g(S_i) + g(S_j) - |N(S_i) \cap N(S_j)| \ge 2k - (k-1).$$

Thus $g(S_i \cup S_j) = k+1$ implies $|N(S_i) \cap N(S_j)| = k-1$. Since both $N(S_i)$ and $N(S_j)$ have cardinality at most $k-1$, we have $N(S_i) = N(S_j)$ and $|N(S_i)| = k-1$, as required. □

Lemma 8 *Suppose that $|S_i \cap R| = 1$, for all $i \in R$, and suppose that every pair of edges incident to the root is illegal. Let $i, j \in R$ be such that S_i, S_j are nonadjacent. Then*

$$N(S_i) = N(S_j) \quad \textit{and} \quad |N(S_i)| = k-1.$$

Proof: We will show that the conditions of Lemma 7 hold for i, j and some $\ell \in R$. Note that for every node pair $u, v \in R$, the sets S_u, S_v are distinct by the assumption of the lemma.

The illegal pair ri, rj does not satisfy cases (ii) or (iii) (stated after Lemma 5), since S_i, S_j are nonadjacent. Hence, the maximal dangerous set M_{ij} of case (i) exists. We claim that there is a node $\ell \in R - M_{ij}$ such that both the maximal dangerous sets $M_{i\ell}, M_{j\ell}$ exist. Otherwise, for each $\ell \in R - M_{ij}$, either $M_{i\ell}$ or $M_{j\ell}$ does not exist. Suppose that $M_{j\ell}$ does not exist (the argument is similar if $M_{i\ell}$ does not exist). Then the illegality of $rj, r\ell$ shows that S_j and S_ℓ are adjacent, so $|N(S_j) \cap S_\ell| \ge 1$. Note that S_ℓ is disjoint from M_{ij} by Lemma 6(4), hence, $N(S_j) \cap S_\ell \subseteq N(M_{ij}) \cap S_\ell$, and so $|N(M_{ij}) \cap S_\ell| \ge 1 = |S_\ell \cap R|$. Thus each distinct set S_ℓ ($\ell \in R$) contributes at least one node to $(M_{ij} \cap R) \cup N(M_{ij})$, giving the contradiction

$$g(M_{ij}) = |M_{ij} \cap R| + |N(M_{ij})| \ge |R| \ge k+2.$$

Hence, there is an $\ell \in R - M_{ij}$ such that both $M_{i\ell}, M_{j\ell}$ exist. Clearly, $M_{i\ell}$ and $M_{j\ell}$ are distinct, otherwise this set contains $S_i \cup S_j$ (by Lemma 6(4)), and so $M_{i\ell} \cap M_{ij}$ contains $S_i \cup S_j$, which is impossible by Lemma 6(5) (with $D_1 = M_{i\ell}, D_2 = M_{ij}$). Since S_i, S_j, S_ℓ are distinct and $M_{ij}, M_{j\ell}, M_{i\ell}$ exist and are distinct, Lemma 7 can be applied and the statement follows. □

Lemma 9 *Let i, j, ℓ be nodes in R such that the sets S_i, S_j, S_ℓ are distinct, and the sets $M_{ij}, M_{i\ell}, M_{j\ell}$ exist and are distinct. Then*

$$N(S_j) \cap S_i = N(S_\ell) \cap S_i.$$

Proof: First observe that $M_{ij} \cap S_\ell = M_{i\ell} \cap S_j = M_{j\ell} \cap S_i = \emptyset$, by Lemma 6. That is, every pair among the three sets $M_{ij}, M_{i\ell}, M_{j\ell}$ is properly intersecting. Take M_{ji} and $M_{j\ell}$, and use the submodularity of g and (2):

$$k+1+k+1 \ge g(M_{ji}) + g(M_{j\ell}) \ge g(M_{ji} \cap M_{j\ell}) + g(M_{ji} \cup M_{j\ell}) \ge k + (k+2),$$

where $g(M_{ji} \cup M_{j\ell}) \geq k+2$ follows from the maximality of M_{ji}. Thus equality holds everywhere and, again by the submodular property of g, there is no edge between $M_{j\ell} - M_{ji}$ and $M_{ji} - M_{j\ell} - N(M_{ji} \cap M_{j\ell}) = M_{ji} - M_{j\ell} - N(S_j)$. This means there is no edge between S_ℓ and $S_i - N(S_j)$ (since Lemma 6(4),(5) imply $S_\ell \subseteq M_{j\ell} - M_{ji}$ and $S_i \subseteq M_{ji} - M_{j\ell}$) and so

$$N(S_\ell) \cap S_i \subseteq N(S_j) \cap S_i.$$

The above argument applies to any two of the three sets $M_{ij}, M_{i\ell}, M_{j\ell}$. We apply it to $M_{\ell i}$ and $M_{\ell j}$ and conclude that there is no edge between $M_{\ell j} - M_{\ell i} \supseteq S_j$ and $M_{\ell i} - M_{\ell j} - N(M_{\ell i} \cap M_{\ell j}) \supseteq S_i - N(S_\ell)$. Hence,

$$N(S_j) \cap S_i \subseteq N(S_\ell) \cap S_i.$$

This completes the proof. $\square$

Lemma 10 *Let $A, B \subseteq V - r$ satisfy $A \cap B = \emptyset$, $g(B) = k$, and $B - N(A) \neq \emptyset$. Then*

$$|N(A) \cap B| \geq |N(B) \cap A|.$$

Proof: Let $W = B - N(A) \neq \emptyset$. Note that $g(W) \geq k$ by (2), $g(B) = k$, and $W \cap R \subseteq B \cap R$. Thus $|N(B)| \leq |N(W)|$. The lemma follows from the following inequalities:

$$\begin{aligned}&|N(B) \cap A| + |N(B) - A| = |N(B)| \leq \\ &\quad |N(W)| = |N(W) \cap N(B)| + |N(W) - N(B)| \leq |N(B) - A| + |N(A) \cap B|,\end{aligned}$$

where the last inequality holds since $N(W) \cap N(B) \subseteq N(B) - A$ and $N(W) - N(B) \subseteq N(A) \cap B$. $\square$

In what follows we assume that every pair ru, rv of edges is illegal. From this we shall deduce $|S_x \cap R| = 1$ for each $x \in R$. Using this fact, a short argument will finish the proof by showing that property (T) holds for G.

Lemma 11 *Suppose that every pair of edges incident to the root is illegal. Let i, j be a pair of nodes in R such that there is no set M_{ij} (that is, there exists no dangerous set X with $i, j \in X$). Then:*

(1) Either $(S_i \cap R) \subseteq N(S_j)$, or $(S_j \cap R) \subseteq N(S_i)$.

(2) If $|S_i \cap R| \geq |S_j \cap R|$, then $|N(S_i) \cap S_j| \geq |S_j \cap R|$.

Proof: First, note that $S_i \neq S_j$, otherwise S_i is a dangerous set with $i, j \in S_i$.
(1) If part (1) of the lemma fails, then there is a node $w \in (S_i \cap R) - N(S_j)$ and a node $z \in (S_j \cap R) - N(S_i)$. The edge pair rw, rz is illegal, so one of the cases (i),(ii),(iii) for illegal edge pairs (stated after Lemma 5) must apply. It is easily seen that cases (ii),(iii) do not apply to rw, rz, since every critical set X including w (respectively, z) is contained in S_i (respectively, S_j). Hence, case (i) must apply to rw, rz, so there is a dangerous set X with $w, z \in X$. But then X is a dangerous set intersecting both S_i and S_j, so Lemma 6(4) gives $S_i \cup S_j \subseteq X$. This contradicts the assumption of the lemma.
(2) Either $(S_j \cap R) \subseteq N(S_i)$, in which case part (2) follows directly, or $S_j - N(S_i)$ is nonempty and by part (1) of the lemma, $|N(S_j) \cap S_i| \geq |S_i \cap R| \geq |S_j \cap R|$, in which case part (2) follows since $|N(S_i) \cap S_j| \geq |N(S_j) \cap S_i|$ by Lemma 10 (with $A = S_i$, $B = S_j$). $\square$

Let us fix a node $x \in R$ such that $|S_x \cap R|$ is maximum.

Claim 12 *There exists a node $y \in R - S_x$ such that M_{xy} exists.*

Proof: Suppose that for each $y \in R - S_x$ there is no set M_{xy}. Then Lemma 11 implies $|N(S_x) \cap S_y| \geq |S_y \cap R|$. Thus each distinct S_y ($y \in R - S_x$) contributes at least $|S_y \cap R|$ nodes to $N(S_x)$, giving the contradiction $g(S_x) = |S_x \cap R| + |N(S_x)| \geq |R| \geq k + 2$. □

Now, fix a $y \in R - S_x$ such that M_{xy} exists and subject to this $|S_y \cap R|$ is maximum.

Claim 13 *There exists a node $z \in R - M_{xy}$ such that both M_{xz} and M_{yz} exist and are distinct, and moreover, $|N(M_{xy}) \cap S_z| < |S_z \cap R|$.*

Proof: Note that $R - M_{xy} \neq \emptyset$ since $|M_{xy} \cap R| \leq k + 1 < |R|$. Suppose that each $z \in R - M_{xy}$ violates the inequality in the claim, and so satisfies $|N(M_{xy}) \cap S_z| \geq |S_z \cap R|$. Then each distinct S_z ($z \in R - M_{xy}$) contributes at least $|S_z \cap R|$ nodes to $N(M_{xy})$, giving the contradiction $g(M_{xy}) = |M_{xy} \cap R| + |N(M_{xy})| \geq |R| \geq k + 2$. Hence, there is a $z \in R - M_{xy}$ that satisfies the inequality in the claim; let us fix this z. We claim that both M_{xz} and M_{yz} exist. Suppose that M_{xz} does not exist. Then Lemma 11 implies $|N(S_x) \cap S_z| \geq |S_z \cap R|$. Since S_z is disjoint from M_{xy} by Lemma 6(4) and $S_x \subseteq M_{xy}$, we have $N(S_x) \cap S_z \subseteq N(M_{xy}) \cap S_z$, and this gives the contradiction $|N(M_{xy}) \cap S_z| \geq |N(S_x) \cap S_z| \geq |S_z \cap R|$. Hence, M_{xz} exists. Note that $|S_y \cap R| \geq |S_z \cap R|$, by our choice of y, z. It can be seen that M_{yz} exists, otherwise Lemma 11 gives the contradiction $|N(M_{xy}) \cap S_z| \geq |N(S_y) \cap S_z| \geq |S_z \cap R|$. Finally, note that M_{xz} and M_{yz} are distinct, otherwise, this set contains $S_x \cup S_y$, and so $M_{xz} \cap M_{xy}$ contains $S_x \cup S_y$, a contradiction to Lemma 6(5) (with $D_1 = M_{xy}, D_2 = M_{xz}$). □

Let us pick a $z \in R - M_{xy}$ that satisfies the properties verified in Claim 13.

Claim 14 *Each of the three pairs of sets S_x, S_y or S_x, S_z or S_y, S_z is nonadjacent.*

Proof: Clearly, the sets S_x, S_y, S_z are distinct, and all three sets M_{xy}, M_{xz}, M_{yz} exist and are distinct, so these sets satisfy the conditions of Lemma 9. Applying Lemma 9 three times we get $N(S_y) \cap S_x = N(S_z) \cap S_x$, $N(S_x) \cap S_y = N(S_z) \cap S_y$, $N(S_x) \cap S_z = N(S_y) \cap S_z$. Let n_x, n_y, n_z denote the cardinalities of these three sets, respectively. Since z satisfies the inequality in Claim 13, $S_z - N(S_x) = S_z - N(S_y) \supseteq S_z - N(M_{xy})$ is nonempty. Then Lemma 10 (with $A = S_x \cup S_y$, $B = S_z$) implies

$$n_z = |N(S_x) \cap S_z| = |N(S_x \cup S_y) \cap S_z| \geq |N(S_z) \cap (S_x \cup S_y)| = |N(S_z) \cap S_x| + |N(S_z) \cap S_y| = n_x + n_y.$$

Moreover, $S_x - N(S_z)$ is nonempty, since otherwise we have

$$|N(M_{xy}) \cap S_z| \geq n_z = |N(S_x) \cap S_z| \geq n_x = |N(S_z) \cap S_x| \geq |S_x| \geq |S_x \cap R| \geq |S_z \cap R|,$$

which contradicts the inequality in Claim 13. Applying Lemma 10 (with $A = S_y \cup S_z$, $B = S_x$) implies

$$n_x = |N(S_z) \cap S_x| = |N(S_y \cup S_z) \cap S_x| \geq |N(S_x) \cap (S_y \cup S_z)| = n_y + n_z.$$

Hence, $n_x \geq n_x + 2n_y$, and so $n_y = 0$. Therefore, S_x and S_y are nonadjacent. Then Lemma 9 implies that each of the three pairs of sets S_x, S_y or S_x, S_z or S_y, S_z is nonadjacent. □

Claim 15 $N(S_x) = N(S_y) = N(S_z)$ *and this set has cardinality $k - 1$. Consequently, $|S_x \cap R| = 1$.*

Proof: The claim follows by applying Lemma 7 twice, first with $i = x, j = y, \ell = z$ and then with $i = x, j = z, \ell = y$. Hence, $|N(S_x)| = k - 1$. Since S_x is a critical set, we must have $|S_x \cap R| = 1$, because $k = g(S_x) = |S_x \cap R| + |N(S_x)| = |S_x \cap R| + (k - 1)$. □

This concludes the first part of the proof of Theorem 3: we must have $|S_i \cap R| = 1$ for all $i \in R$, since $1 \le |S_i \cap R| \le |S_x \cap R| = 1$. In other words, $S_i \ne S_j$ for every pair of (distinct) nodes $i, j \in R$. (This property is much simpler to deduce when G is k-connected, as in Theorem 4.)

First suppose that for every pair of nodes $u, v \in R$, S_u and S_v are nonadjacent. Then for every pair $u, v \in R$, Lemma 8 applies, and this implies that Property (T) holds: we take $T = N(S_u) \cup \{r\}$, and note that T is a separator of G with $r \in T$ and $|T| = k$. Let us show the following: $V = (\cup_{i \in R} S_i) \cup T$, G is k-connected, and the number of components of $G - T$ equals $\deg(r)$. If there is a node $v \in V - (\cup_{i \in R} S_i) - T$, then note that by k-outconnectivity v has a path to r in $G - (T - \{r\})$, and so for the neighbour ℓ of r in this path we must have $v \in S_\ell$, which is a contradiction. Clearly, $G - T$ has $|R| = \deg(r)$ components. G is k-connected since it has $\ge |R| \ge k + 2$ openly disjoint paths between every pair of nodes in T, and for every node $v \in V - T$, G has k openly disjoint paths between v and T (by Menger's theorem and (2)). Thus the proof of the theorem is complete when S_u, S_v are nonadjacent, for each pair $u, v \in R$.

Lemma 16 *If S_x, S_y are adjacent for some $x, y \in R$ then $S_y \subseteq N(S_x)$ (and similarly, $S_x \subseteq N(S_y)$).*

Proof: Let Q denote $N(S_x)$ and for a contradiction suppose that $S_y - Q \ne \emptyset$. Let $R_a \subseteq R - x$ be the set of nodes $z \in R$ such that S_z, S_x are adjacent, and let $a = |R_a|$. Let $b = |R| - a$. Notice that for any set S_i with $i \in R - R_a$ we have $N(S_i) = Q$ by Lemma 8. Also, S_i, S_y are adjacent. Applying Lemma 10 with $A = \bigcup_{i \in R - R_a} S_i$ and $B = S_y$ (so $B - N(A) = S_y - Q \ne \emptyset$) gives

$$|N(A) \cap B| = |Q \cap S_y| \ge |N(B) \cap A| = |N(S_y) \cap (\bigcup_{i \in R - R_a} S_i)| \ge b.$$

Thus S_y contributes at least b nodes to Q. Clearly, every other set S_z which is adjacent to S_x contributes at least one node to Q. Since these sets are pairwise disjoint, we get $k - 1 \ge |Q| \ge b + a - 1 = |R| - 1 \ge k + 1$, a contradiction. □

In what follows we show that adjacent pairs S_u, S_v ($u, v \in R$) do not exist when $|V| \ge 2k$. For a contradiction, suppose that there is a pair $u, v \in R$ such that S_u and S_v are adjacent. Let $Q = N(S_u)$, $P = N(S_v)$. Clearly, $P \ne Q$. Lemma 16 implies that Q contains every set S_i, $i \in R$ which is adjacent to S_u. Moreover, every set S_j, $j \in R$ that is not adjacent to S_u has $N(S_j) = Q$ (by Lemma 8). Hence, each such S_j is adjacent to S_v, and therefore P contains S_j. Finally, observe that $V = P \cup Q \cup \{r\}$. Otherwise, if there is a node $v \in V - (P \cup Q)$, $v \ne r$, then note that by k-outconnectivity v has a path to r in $G - Q$; for the neighbour ℓ of r in this path note that S_ℓ is nonadjacent to S_u and so we have $N(S_\ell) = Q$ and $v \in S_\ell \subseteq P$. Consequently, $|V| = |P \cup Q| + 1 \le 2k - 1$ (since $|P|, |Q| \le k - 1$), and this contradicts the assumption $|V| \ge 2k$. This completes the proof of Theorem 3. □

If we drop the condition $|V| \ge 2k$ in Theorem 3, then we obtain a weaker result (see Theorem 17 below). Our approximation algorithm in Section 5.2 uses the weaker result rather than Theorem 3. Unfortunately, a direct proof of the weaker result is not significantly shorter or simpler than the proof of Theorem 3. (A direct proof of the weaker result uses Lemmas 6–11 and Claims 12–15, as well as some additional steps.)

Theorem 17 *Let $G = (V, E)$ be a graph which is k-outconnected from a root node $r \in V$ and suppose that $\deg(r) \geq k+2$ and every edge incident to r is critical with respect to k-outconnectivity from r. Suppose that none of the edge pairs incident to r can be split off preserving k-outconnectivity. Then G is k-connected.*

Proof: Let us use the notation in the proof of Theorem 3. Note that the condition $|V| \geq 2k$ is used only in the last paragraph of that proof.

Suppose that G is not k-connected and let C be a separator with $|C| < k$. Since $|R| \geq k+2$, and $|S_i \cap R| = 1$ for every $i \in R$ (by Claim 15), there exists an $x \in R$ with $S_x \cap C = \emptyset$. Let $Q = N(S_x)$ and let A be the set of those neighbours y of r for which S_x and S_y are nonadjacent. By Lemma 8 we have $N(S_y) = Q$ for each $y \in A$. Let $W = \bigcup_{y \in A} S_y \cup S_x$. It is easy to see that $V = Q \cup W \cup r$.

For each node $v \in S_i$, $i = x$ or $i \in A$, there exist k openly disjoint paths from v to $Q \cup r$ in the subgraph of G induced by $S_i \cup Q \cup r$ (by Menger's theorem and (2)). Focus on $G - C$. Since $S_x \cap C = \emptyset$ and $(Q \cup r) - C \neq \emptyset$, there exists a component B (of $G - C$) that contains $S_x \cup ((Q \cup r) - C)$. Moreover, (in $G - C$) each node $v \in S_i - C$, $i \in A$, has at least one path to $(Q \cup r) - C$, hence, B contains $W - C$. Thus $V = Q \cup r \cup W \subseteq C \cup B$, and this contradicts our assumption that C is a separator of G. $\Box$

To illustrate that the problem in Theorem 3 is more general than the problem in Theorem 4, consider the following strengthening: if there exists an admissible edge pair incident to r in Theorem 4 and $\deg(r) \geq 2k-1$, then any fixed edge rv is part of an admissible edge pair. This fact was deduced in [8], where a related augmentation problem was considered. Here is an example showing that such a strengthening of Theorem 3 fails even if we assume $\deg(r) \geq k^2 - 2k + 2$: take $k-1$ disjoint copies of a k-connected graph and two additional nodes r, x. Connect r to each copy by $k-1$ edges each, and connect x to each copy by one edge. Also, add the edge rx. This graph is k-outconnected from r, $\deg(r) = k^2 - 2k + 2$, every edge incident to r is critical, there exists an admissible edge pair incident to r, but rx is in no admissible edge pair.

In several applications of splitting-off theorems one may assume that the edges to be split off are critical with respect to the connectivity property to be preserved. This is the case when we apply Theorem 3 in Section 5 and also in [1]. However, for other applications, it may be useful to have a more general result when edges incident to the root r are not necessarily critical.

Given an integer $k \geq 1$, a graph H, and a specified node r of H, *Property (T')* is said to hold if

> H is k-connected, and there exists a node set T such that $|T| = k$, $r \in T$, the number of components of $H - T$ equals $\deg_H(r) - 1$ and there is an edge rt with $t \in T$.

Theorem 18 *Let $G = (V, E)$ be a graph with $|V| \geq 2k$ which is k-outconnected from a root node $r \in V$ and suppose that $\deg(r) \geq k+3$. Then either*

(a) G satisfies Property (T), or G satisfies Property (T'), or

(b) there exists a pair of edges incident to r that can be split off preserving k-outconnectivity.

Proof: If every edge incident to r is critical with respect to k-outconnectivity from r, then we are done by Theorem 3. Otherwise, there is an edge rv for which $G - rv$ is k-outconnected from r. If there is an edge rw in $G - rv$ for which $G - rv - rw$ is still k-outconnected from r, then clearly rv, rw is an admissible pair of edges in G. Otherwise, all the edges incident to r are critical in $G - rv$. Then Theorem 3 implies that either $G - rv$ has an admissible pair of edges rx, ry, or $G - rv$

satisfies property (T) with some separator T, $|T| = k$, $r \in T$. Now there are two possibilities. If $v \in T$ then property (T′) holds in G, and moreover, it can be seen that G has no admissible pair of edges. If $v \in V - T$, then v belongs to some component D of $V - T$. Let rz be an edge in G with $z \notin D$. We claim that splitting off the pair rv, rz preserves k-outconnectivity from r. This follows from Menger's theorem, since $G - rv$ satisfies property (T), so the graph obtained from G by splitting off rv, rz has k openly disjoint paths between every pair of nodes in T and has k openly disjoint paths from each node $y \in V - T$ to T. This proves the theorem. □

4 A rooted counterpart of Mader's theorem

In this section we prove a rooted version of Mader's theorem on "cycles of critical edges" in k-connected graphs. First we state Mader's [11] result and show an application we shall need later.

Theorem 19 ([11]) *Let G be a k-connected graph, and let C be a cycle of G such that each edge in C is critical with respect to k-connectivity. Then $deg_G(v) = k$ for some node $v \in C$.* □

The next lemma illustrates a typical application of Theorem 19.

Lemma 20 *Let $G = (V, E)$ be a graph that is k-outconnected from a root node r, and let R be the set of neighbours of r. Then G can be made k-connected by adding at most $|R| - 1$ new edges in such a way that each new edge has both ends in R.*

Proof: We start with two observations. For a connected graph G and an (inclusionwise) minimal separator S of G, note that any node in S has a neighbour in each component of $G - S$. Now consider a graph G that is k-outconnected from a root node r. If G is not k-connected, then every separator of cardinality less than k must contain r. Consequently, there are two neighbours v, w of r such that G has at most $k - 1$ openly disjoint paths between v and w.

Let $G' = G + F' = (V, E \cup F')$ where $F' = \{vw \mid v \in R, w \in R, vw \notin E\}$; that is, G' is obtained from G by adding new edges to ensure that R induces a complete subgraph. Then G' is k-connected, otherwise, by the previous observations, there is a minimal separator S with $|S| < k$, $r \in S$, that separates two neighbours of r, but this is not possible.

Take an inclusionwise minimal subset $\widetilde{F}$ of F' for which $\widetilde{G} = G + \widetilde{F} = (V, E \cup \widetilde{F})$ is k-connected. Clearly, every new edge $f \in \widetilde{F}$ is critical for the k-connectivity of $\widetilde{G}$. Thus Theorem 19 implies that $\widetilde{F}$ is a forest. To see this, suppose there is a cycle C whose edge set is contained in $\widetilde{F}$. Then each node v incident to C has degree $\geq k+2$ in $\widetilde{G}$, because v is incident to at least k edges of G and to two edges of C. This contradicts Theorem 19, and so $\widetilde{F}$ is a forest. Therefore $|\widetilde{F}| \leq |V(\widetilde{F})| - 1 \leq |R| - 1$. □

Our result, the rooted version of Mader's theorem, is the following.

Theorem 21 *Let G be a graph that is k-outconnected from a node r, and let C be a cycle of G such that each edge in C is critical with respect to k-outconnectivity from r. Then $\deg_G(v) = k$ for some node $v \in C$, $v \neq r$.*

The proof of Theorem 21 is based on several lemmas.

Lemma 22 *Let $G = (V, E)$ be k-outconnected from r. Let v be a neighbour of r with $\deg(v) \geq k+1$, and let $vw \neq vr$ be an edge. Then there are k openly disjoint paths between v and r in $G - vw$.*

Proof: For a contradiction suppose $G-vw$ has at most $k-1$ openly disjoint paths between v and r. Then $G-vw-vr$ has a (v,r)-separator $S\subset V$ with $|S|=k-2$. Since $\deg(v)\geq k+1$, v must have a neighbour b in $(G-vw-vr)-S$. Since there is no path between b and r in $(G-vw-vr)-S$, there is no path between b and r in $(G-(\{v\}\cup S))$ either. This contradicts the fact that G is k-outconnected from r. □

If H is a k-connected graph and $e=vw$ is a critical edge of H (with respect to k-connectivity), then it is clear that H has a separator S of cardinality $k-1$ such that $H-e-S$ has exactly two components, one containing v and the other containing w. This claim extends to a k-outconnected graph G and an edge $e=vw$ of G, but some care is needed since every node x with fewer than k openly disjoint paths between x and r in $G-e$ may be a neighbour of r. For example, take $k=2$, $G=K_3$, r to be any node of G, and e to be the edge of G disjoint from r. Then e is critical for k-outconnectivity from r, but $G-e$ has no cut node other than r.

Lemma 23 *Let $G=(V,E)$ be k-outconnected from r, and let $e=vw$ be a critical edge (with respect to k-outconnectivity from r) such that either*

(i) e is incident to r, or

(ii) for both ends v and w of e, if that end is a neighbour of r, then that end has degree $\geq k+1$.

Then in $G-e$ there is a separator $S_e\subset V-\{r,v,w\}$ with $|S_e|=(k-1)$ such that $G-e-S_e$ has exactly two components, one containing v and the other containing w.

Proof: Let $H=G-e$. We claim that either $\kappa_H(v,r)<k$ or $\kappa_H(w,r)<k$. To see this, note that H is not k-outconnected from r (since e is critical), so there exists a set $C\subseteq E(H)\cup V(H)$ with $|C|<k$ and $r\notin C$ such that $H-C$ has ≥ 2 components, but $G-C=(H-C)\cup\{e\}$ is connected, hence v and w must be in different components of $H-C$.

Focus on the end of e, say v, that has at most $k-1$ openly disjoint paths to r in H. If $e=vr$, then note that v and r are not adjacent in H, so by Menger's theorem the required separator $S_e\subseteq V-\{r,v\}$ exists. Otherwise, e is not incident to r. Then note that v cannot be a neighbour of r in G, otherwise by assumption (ii) v has degree $\geq k+1$ in G and so by Lemma 22 $G-e=H$ has k openly disjoint paths between v and r. Again, by Menger's theorem, H has a v,r separator S with $|S|=k-1$. Moreover, $w\notin S$ because $G-S=(H-S)\cup\{vw\}$ is connected. Thus in this case we take $S_e=S$. □

The next lemma is similar to the key lemma used by Mader in his proof of Theorem 19, though Mader does not discuss part (4). We include the proof, but skip some details in parts (1),(2) and refer the reader interested in the detailed proof to [11, Lemma 1] or to [2, Lemma 4.4].

Let $G=(V,E)$ be k-outconnected from r. For a critical edge $e=vw$ of G satisfying condition (i) or (ii) in Lemma 23, let S_e denote a separator of cardinality $k-1$ as in the lemma, and let the node sets of the two components of $G-e-S_e$ be denoted by $D_{v,e}$ and $D_{w,e}$, where $v\in D_{v,e}$ and $w\in D_{w,e}$.

Lemma 24 *Let $G=(V,E)$ be k-outconnected from r. Let $v\neq r$ be a node with $\deg(v)\geq(k+1)$. Let $e=vw$ and $f=vx$ be two edges that are critical with respect to k-outconnectivity from r, such that each satisfies condition (i) or (ii) in Lemma 23. Let $S_e, D_{v,e}, D_{w,e}$ and $S_f, D_{v,f}, D_{x,f}$ be as defined above. Let $Z=(S_e\cap D_{v,f})\cup(S_e\cap S_f)\cup(S_f\cap D_{v,e})$. Then the following hold:*

(1) $|Z|\geq k-1$.

(2) $D_{w,e} \cap D_{x,f} = \emptyset$.

(3) $|D_{w,e}| < |D_{v,f}|$.

(4) If $r \in D_{w,e}$*, then* $r \in D_{v,f}$.

Proof: (1) Suppose that $|Z| \leq k-2$. Since $\deg(v) \geq k+1$, there is a neighbour b of v such that $b \neq w$, $b \neq x$, and $b \notin Z$. Now, it can be seen that $(D_{v,e} \cap D_{v,f}) - \{v\}$ is nonempty, and hence $Z \cup \{v\}$ is a separator of G with cardinality $\leq k-1$, and also $r \notin Z \cup \{v\}$. This contradicts the fact that G is k-outconnected from r. Hence, $|Z| \geq k-1$.
(2) Let $Y = (S_e \cap D_{x,f}) \cup (S_e \cap S_f) \cup (S_f \cap D_{w,e})$. Suppose that $D_{w,e} \cap D_{x,f}$ is nonempty. Then note that $N(D_{w,e} \cap D_{x,f}) \subseteq Y$, and so $G - Y$ has two or more components. This is a contradiction since $r \notin Y$ and

$$|Y| = (|Y| + |Z|) - |Z| = (|S_e| + |S_f|) - |Z| = (2k-2) - |Z| \leq (2k-2) - (k-1) = (k-1),$$

where the inequality follows from part (1).
(3) First, we claim that $|S_e \cap D_{v,f}| \geq |S_f \cap D_{w,e}|$. To see this, let $Q = Z - (S_e \cap D_{v,f})$, i.e., $Q = (S_e \cap S_f) \cup (S_f \cap D_{v,e})$, and note that $Q = S_f - (S_f \cap D_{w,e})$. Then we have

$$|Q| + |S_f \cap D_{w,e}| = |S_f| = k-1 \leq |Z| = |Q| + |S_e \cap D_{v,f}|,$$

where the inequality follows from part (1). This proves the claim. Now, consider part (3). We have

$$\begin{aligned} |D_{w,e}| &= |D_{w,e} \cap D_{v,f}| + |D_{w,e} \cap S_f| + |D_{w,e} \cap D_{x,f}| \\ &\leq |D_{w,e} \cap D_{v,f}| + |D_{v,f} \cap S_e| = |D_{v,f}| - |D_{v,f} \cap D_{v,e}| \\ &\leq |D_{v,f}| - 1, \end{aligned}$$

where the first inequality follows from part (2) and the previous claim, and the second inequality follows since the node v is in $D_{v,f} \cap D_{v,e}$.
(4) For part (4), note that $D_{w,e} = (D_{w,e} \cap D_{v,f}) \cup (D_{w,e} \cap S_f) \cup (D_{w,e} \cap D_{x,f})$. If r is in $D_{w,e}$, then r is in $D_{w,e} \cap D_{v,f}$, because $r \notin S_f$ (by hypothesis), and $r \notin D_{w,e} \cap D_{x,f}$ (by part (2)). This completes the proof of the lemma. □

Proof: *(of Theorem 21)* The proof is by contradiction. Let $C = v_0, v_1, v_2, \ldots, v_p, v_0$ be a cycle of critical edges, and let every node in $V(C) - \{r\}$ have degree $\geq k+1$. Note that each edge in C satisfies condition (i) or (ii) in Lemma 23. For each edge $v_i v_{i+1}$ in C (taking $v_{p+1} = v_0$), let us revise our notation to $S_i' = S_{v_i v_{i+1}}$ and $D_i = D_{v_i, v_i v_{i+1}}$, that is, $S_i' \subseteq V - \{r, v_i, v_{i+1}\}$ has cardinality $k-1$, $G - v_i v_{i+1} - S_i'$ has two components, and D_i is the node set of the component containing v_i.

First suppose that C is incident to r. Let $r = v_0$. We claim that for each $i = 0, 1, 2, 3, \ldots, p$, the root $r = v_0$ is in D_i. This follows easily by induction on i, applying Lemma 24(4) by taking $v = v_i$, $i = 1, 2, \ldots, p$. The induction basis is immediate. (Note that Lemma 24 cannot be used with $v = v_0 = r$.) Thus our claim holds. This gives a contradiction, since the claim states that in $G - v_p v_0 - S_p'$ the root r is in the component of v_p, rather than in the component of $r = v_0$. Therefore if C is incident to r, then this proves the theorem.

Now, suppose that C is not incident to r. Then note that every node incident to C has degree at least $k+1$. By repeatedly using Lemma 24(3), taking $v = v_i$, $i = 1, 2, \ldots, p, 0$ we get $|D_0| < |D_1| < \ldots < |D_{p-1}| < |D_p| < |D_{p+1}| = |D_0|$, a contradiction. This proves the theorem if C is not incident to r. □

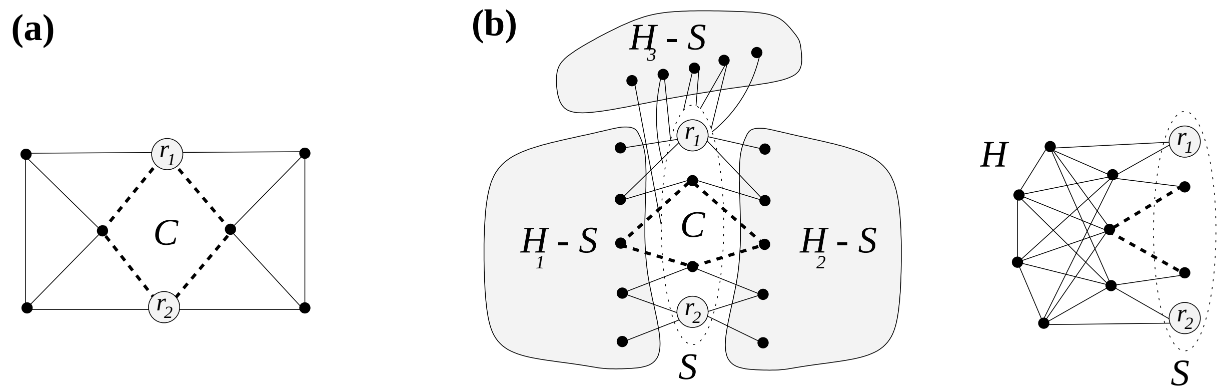

Figure 2: Examples showing that a multi-root outconnected graph may have a cycle of critical edges such that each incident node has degree $\geq k+1$, where k is the maximum connectivity requirement. **(a)** The graph is $(3, 3)$-outconnected from (r_1, r_2). Each edge is critical, either for 3-outconnectivity from r_1 or for 3-outconnectivity from r_2. The cycle C has the property stated above.
(b) The graph is obtained by taking 3 copies of the graph H, and identifying the 3 copies of each node in S. This graph is $(5, 5)$-outconnected from (r_1, r_2), and each edge is critical. The cycle C is disjoint from r_1 and r_2, and has the property stated above.

We remark that the method in the last paragraph of the proof, based on Lemma 24(3), does not suffice to prove Theorem 21, because the cycle C may be incident to the root r, but Lemma 24 cannot be applied with $v = r$, since the proof of part (1) fails ($Z \cup \{v\}$ does not separate r from another node).

Theorem 21 does not seem to have any obvious extension to multi-root outconnected graphs. Figure 2 has an example graph with two roots r_1, r_2 and $k = 3$ that is k-outconnected from each of r_1 and r_2 such that there is a cycle of critical edges such that each incident node has degree $\geq 4 > k = 3$. Also, Figure 2 has another example graph that is $(5, 5)$-outconnected from (r_1, r_2) such that the cycle of interest is incident to no root.

The following corollary of Theorem 21 gives new structural information for the setup where another condition is added to the hypothesis of Theorem 19. For an illustration of this corollary, consider the graph $P_4 + \overline{K_3}$ (this is the graph $H - S$ in Figure 2(b)), take $k = 4$, and take C to be a cycle of length 3 that has two nodes of degree $k + 1 = 5$.

Corollary 25 *Let G be a k-connected graph. Let C be a cycle of G such that each edge is critical with respect to k-connectivity, and C is incident to exactly one node r with $\deg(r) = k$ (so $\deg(v) \geq k + 1$ for all nodes $v \in V(C) - \{r\}$). Then there exists an edge e in C such that every separator S of $G - e$ with $|S| = (k - 1)$ has $r \in S$.*

Proof: Clearly, G is k-outconnected from r. If each edge in C is critical with respect to k-outconnectivity from r, then we have a contradiction to Theorem 21. Hence there is an edge e in C for which $G - e$ remains k-outconnected from r. Then for every separator S of $G - e$ with $|S| < k$ we must have $r \in S$. $\square$

5 Approximation algorithms

In this section we apply our structural results from the preceding sections to design approximation algorithms for Problem B. For the special case of Problem B with metric weights, we give an approximation algorithm that is based on Theorem 17 (which is a weaker version of Theorem 3). For the special case of Problem B with uniform weights, we give an approximation algorithm that is based on Theorem 21. Throughout this section, when discussing a problem, we use $G = (V, E)$ to denote the graph for the instance of Problem B, and *opt* to denote the optimal value of the problem. We also assume that the instance has a feasible solution. Let n denote $|V|$.

5.1 A $2q$-approximation algorithm for the multi-root problem with q roots

First, we discuss previous results and algorithmic questions related to Problem B. Consider the special case of Problem B where $q = 1$. Here there is only one root node r with positive node requirement c_r. Let $k := c_r$. We call this the *minimum-weight single-root k-outconnected subgraph problem*, or the *single-root problem.* This problem is NP-hard, even for $k = 2$ and uniform weights or metric weights. For uniform weights, this follows from the fact that a 2-outconnected subgraph of a graph G has at most $|V(G)|$ edges if and only if it is a Hamiltonian cycle in G. For metric weights, a similar reduction works by giving weight 1 to edges of G and weight 2 to edges of the complement.

Frank and Tardos [6] presented a polynomial-time algorithm for finding an optimal solution for the following directed version of the single-root problem.

Problem C: Given a directed graph, non-negative weights on the edges, a root node r, and a connectivity requirement k, find a minimum-weight subdigraph H such that there exist at least k openly disjoint directed paths from r to each node $v \neq r$ in H.

The Frank-Tardos algorithm provides a 2-approximation algorithm and a useful lower bound on *opt* for the undirected minimum-weight single-root k-outconnected subgraph problem as follows. We take the input graph G and create a directed graph $\vec{G}$ by replacing each undirected edge vw by a pair of antiparallel directed edges (v, w) and (w, v), where both directed edges have the same weight as vw. Then we apply the Frank-Tardos algorithm to find an optimal subdigraph $\vec{G}^*$ of weight c^* for Problem C, taking the root to be the same as in the undirected single-root problem. For the undirected problem, note that $opt \geq c^*/2$ since the directed version of the optimal subgraph is a feasible solution for the Frank-Tardos algorithm. Moreover, the undirected graph G^* obtained from $\vec{G}^*$ by replacing each directed edge by the corresponding undirected edge (and removing parallel edges) is a feasible solution to the undirected problem of weight $\leq c^* \leq 2opt$; also see Khuller and Raghavachari [10].

For the multi-root problem with q roots a $2q$-approximation algorithm follows by sequentially applying the above 2-approximation algorithm to each of the roots $r_1, \ldots, r_q$. Note that the approximation guarantee $2q$ of this algorithm for the multi-root problem is tight. To see this consider the following example; see Figure 3. Suppose $k \geq 2$ and take $q = k - 1$ roots $r_1, \ldots, r_q$ with node requirement k each. The graph G has a separator $R = \{r_1, \ldots, r_q\}$ that induces a complete subgraph. In $G - R$, there are two components D_1, D_2, and each is a complete subgraph on at least $k+1$ nodes. There is a matching of size q between R and each of D_1, D_2. All the above edges have zero weight. Each root r_i is incident to two edges r_is_1, r_is_2 of weight M, where s_1 is in D_1 and s_2 is in D_2. Finally, there is one edge of weight $M + \epsilon$ between D_1 and D_2. The optimal subgraph has all the zero-weight edges and the edge of weight $M + \epsilon$, thus the optimal solution has weight $M + \epsilon$. The first iteration of the Frank-Tardos algorithm adds all the zero-weight edges and the two edges

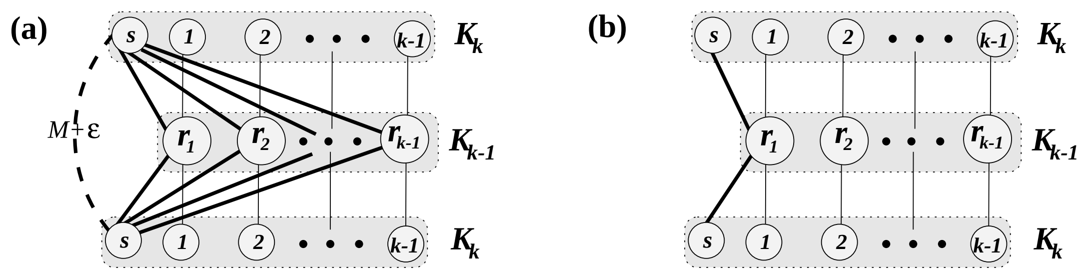

Figure 3: An example of Problem B where the $2q$ approximation guarantee is tight.
(a) The graph G. The dashed edge has weight $M + \epsilon$, the $2(k-1)$ thick edges each have weight M, and all other edges have zero weight.
(b) Each iteration of the Frank-Tardos algorithm with root r_i, $i = 2, \ldots, k-1$, adds the two thick edges incident to r_i; the first iteration adds all the zero-weight edges and the two thick edges incident to r_1.

of weight M incident to r_1. Each iteration with root r_i, $i = 2, \ldots, q$, adds the two edges of weight M incident to r_i. Hence, the subgraph found by the algorithm has weight $2qM$. In this example, the optimal subgraph turns out to be k-connected, but the example can be modified easily to avoid this.

5.2 Metric weights

We consider Problem B with the additional assumption that the edge weights satisfy the triangle inequality and we give a 3-approximation algorithm. Since the triangle inequality holds, G is assumed to be the complete graph. We start with an auxiliary result related to Lemma 20.

Lemma 26 *Let $G = (V, E)$ be a graph that is k-outconnected from a root node r and let R be the set of neighbours of r. Then G can be made p-connected for any $p \leq k$ by adding at most $(|R| + p - 1)/2$ new edges in such a way that each new edge has both ends in R.*

Proof: Let us delete edges incident to r from G as long as possible while maintaining p-outconnectivity from r. In the resulting graph G' every edge incident to r is critical for p-outconnectivity. Let R' be the set of neighbours of r in G'. Applying Theorem 17 to G' it follows that by splitting off appropriate pairs of edges incident to r we can obtain a graph G^* which is p-outconnected from r and which is either p-connected or has $\deg_{G^*}(r) \leq p + 1$. Let E' be the set of new edges obtained by these splitting off operations. Since $\deg_{G^*}(r) \geq p$, we have $|E'| \leq (|R'| - p)/2$. By Lemma 20 applied to G^*, there exists a set F of edges such that $G^* + F$ is p-connected, each F-edge has both ends in R', and $|F| \leq \deg_{G^*}(r) - 1 \leq p$. Clearly, $G + (E' \cup F)$ is p-connected. Since $\deg_{G'}(r) - \deg_{G^*}(r)$ is even and $R' \subseteq R$, we get $|E' \cup F| \leq (|R| - p - 1)/2 + p$, as required. $\square$

Theorem 27 *Consider instances of Problem B such that the edge weights are metric. There is a $(2.5 + \frac{k_s}{2k})$-approximation algorithm, where k and k_s denote the largest and second largest node requirement, respectively (note: $2.5 + \frac{k_s}{2k} \leq 3$).*

Proof: Let $G = (V, E)$ be the input graph and let $\vec{c} = (c_1, \ldots, c_q)$ and $\vec{B} = (r_1, \ldots, r_q)$ be the connectivity requirement vector and the vector of roots, respectively. We may assume $q \geq 2$ (and $c_1 \geq \ldots \geq c_q$). Let $k = c_1$ and $k_s = c_2$ denote the largest and the second largest node requirement, respectively. Let $r = r_1$ (thus r is a root node with the maximum node requirement k).

The algorithm starts by finding a subgraph H that is k-outconnected from r, with weight $w(H) \leq 2\,opt$. This can be done in polynomial time via the Frank-Tardos result, as mentioned earlier. In the graph H, we may assume (by deleting edges if necessary) that each edge incident to r is critical with respect to k-outconnectivity from r.

If $\deg_H(r) \geq k+2$, then by Theorem 17 either H is k-connected or there exists a pair of edges incident to r that can be split off while preserving k-outconnectivity from r. In the former case the algorithm outputs H. Clearly, H is $\vec{c}$-outconnected from $\vec{B}$, and has weight at most $2opt$, as required. In the latter case, the algorithm splits off admissible edge pairs as long as possible. Since w is a metric, splitting off an edge pair does not increase the weight of the subgraph. An admissible edge pair, if one exists, can be found in polynomial time by max-flow computations. If the resulting graph becomes k-connected after several iteration, we are done as above. When the algorithm stops splitting-off iterations, we may assume that $\deg_H(r) \leq k+1$ holds in the current subgraph H.

In the next step, the algorithm finds a set of new edges $\widetilde{F}$ for which $H + \widetilde{F}$ is k_s-connected and such that $|\widetilde{F}| \leq (k+k_s)/2$. By Lemma 26 (using that $\deg_H(r) \leq k+1$) such a set exists and can be found efficiently. The algorithm outputs $H'' := H + \widetilde{F}$ and terminates. Since H'' is k-outconnected from r and k_s-connected, the choice of k and k_s implies that H'' is $\vec{c}$-outconnected from $\vec{B}$.

We claim that every edge in $\widetilde{F}$ (in fact, every edge of the complete graph) has weight at most opt/k. To see this, observe that every feasible solution must be k-edge-connected and hence for any two nodes u, v, there exist k edge-disjoint paths between u and v; each of these paths has weight $\geq w(uv)$ by the triangle inequality. Thus $w(H'') \leq 2opt + ((k+k_s)/2)(opt/k) = (2.5 + \frac{k_s}{2k})opt \leq 3opt$, as required. $\square$

We remark that Theorem 27 is related to [10, Theorem 4.8], but neither result implies the other one. Khuller and Raghavachari [10] give an approximation guarantee of $(2 + 2(k-1)/n)$ for the minimum-weight k-connected subgraph problem, assuming metric weights. A by-product of Theorem 27 is a 3-approximation algorithm for the same problem.

Finally, we remark that our 3-approximation algorithm works for an even larger class of local node-connectivity requirements (provided w is metric). Namely, when there exists a node u for which $c(u,v) = k$ holds for every $v \in V - u$, where $k = \max\{c(x,y) : x, y \in V\}$.

5.3 Uniform weights

Here, we give approximation algorithms for Problem B assuming the edge weights are uniform. Our proofs are based on Theorem 21 and the following result of Cheriyan and Thurimella [3, Theorem 3.5].

Theorem 28 ([3]) *Let $G^* = (V, E^*)$ be a k-edge-connected graph ($k \geq 1$) on n nodes. Let $M^* \subseteq E^*$ be a minimum-size edge set such that every node $v \in V$ is incident to at least $k-1$ edges of M^*. Then $|E^*| \geq |M^*| + \lfloor n/2 \rfloor$.*

We shall present two independent approximation algorithms.

Theorem 29 *Consider instances of Problem B such that the edge weights are uniform. There is a $\min\{2, \frac{k+2q-1}{k}\}$-approximation algorithm, where k denotes the largest node requirement and q denotes the number of positive node requirements.*

Proof: Let $G = (V, E)$ be the input graph and let $\vec{c} = (c_1, \ldots, c_q)$ and $\vec{B} = (r_1, \ldots, r_q)$ be the connectivity requirement vector and the vector of roots, respectively. We use $k = c_1$ to denote the largest node requirement.

Our first algorithm simply finds a "sparse certificate" for local node connectivity in G. In detail, it employs the polynomial algorithm of Nagamochi and Ibaraki [12] to find k edge disjoint forests $F_1, \ldots, F_k$ of G such that in the graph $H = (V, F_1 \cup \ldots \cup F_k)$, we have $\kappa_H(u,v) \geq \min\{k, \kappa_G(u,v)\}$ for every two nodes u, v. This graph H has at most $k(n-1)$ edges, while the optimal subgraph has at least $nk/2$ edges, since it has minimum degree at least k. Furthermore, H has c_i openly disjoint paths between v and r_i for every $r_i \in \vec{B}$ and every $v \in V - \{r_i\}$, by the choice of k and since G has c_i openly disjoint paths between v and r_i. Consequently, H is $\vec{c}$-outconnected from $\vec{B}$, as required, and has size at most $2opt$. Thus this is a 2-approximation algorithm.

The second algorithm starts by finding a minimum-size subgraph (V, M) of minimum degree $(k-1)$ in G. This is essentially a matching problem and can be computed in polynomial time, see [3]. Then, sequentially for each of the roots r_i, $i = 1, \ldots, q$, it finds an inclusionwise minimal edge set $F_i \subseteq E(G)$ such that $H_i = (V, M \cup F_1 \cup \ldots \cup F_i)$ is c_i-outconnected from r_i and outputs $H = (V, M \cup F_1 \cup \ldots \cup F_q)$. Clearly, H is $\vec{c}$-outconnected from $\vec{B}$.

Note that every edge $f \in F_i$ is critical for c_i-outconnectivity in H_i. Thus we can apply Theorem 21 to H_i and F_i and conclude that F_i is a forest. Therefore each F_i $(i = 1, \ldots, q)$ has size at most $(n-1)$. Also, we have $|M| \leq opt - \lfloor n/2 \rfloor$ by Theorem 28. (Note that the optimal k-outconnected subgraph of G is k-edge connected.) Thus, using $opt \geq nk/2$, we get

$$|E(H)| = |M \cup F_1 \cup \ldots \cup F_q| = |M| + \sum_{i=1}^{q} |F_i| \leq (opt - \lfloor n/2 \rfloor) + q(n-1) \leq (k + 2q - 1)opt/k.$$

This proves Theorem 29. □

As we remarked, Proposition 2 shows we can assume $q \leq k$. Therefore $(k + 2q - 1)/k \leq 3$. Also note that in the case of the single-root problem, when $q = 1$, the approximation guarantee is $1 + \frac{1}{k}$.

References

[1] D.Bienstock, E.F.Brickell and C.L.Monma, "On the structure of minimum-weight k-connected spanning networks," *SIAM J. Discrete Math.* **3** (1990), 320–329.

[2] B.Bollobás, *Extremal Graph Theory*, Academic Press, London, 1978.

[3] J.Cheriyan and R.Thurimella, "Approximating minimum-size k-connected spanning subgraphs via matching," to appear in *SIAM J. Computing.* Preliminary version in *Proc. 37th IEEE F.O.C.S.* (1996), 292–301.

[4] J.Cheriyan and R.Thurimella, "Fast algorithms for k-shredders and k-node connectivity augmentation," *Journal of Algorithms* **33** (1999), 15–50.

[5] A.Frank, "Connectivity augmentation problems in network design," in *Mathematical Programming: State of the Art 1994*, (Eds. J. R. Birge and K. G. Murty), The University of Michigan, Ann Arbor, MI, 1994, 34–63.

[6] A.Frank and É.Tardos, "An application of submodular flows," *Linear Algebra and its Applications* **114/115** (1989), 320–348.

[7] G.L.Frederickson and J.Ja'Ja', "On the relationship between the biconnectivity augmentation and traveling salesman problems," *Theor. Comp. Sci.* **19** (1982), 189–201.

[8] T.Jordán, "On the optimal vertex-connectivity augmentation," *J. Combinatorial Theory, Series B* **63** (1995), 8–20.

[9] S.Khuller, "Approximation algorithms for finding highly connected subgraphs," in *Approximation algorithms for NP-hard problems*, Ed. D.S.Hochbaum, PWS publishing co., Boston, 1996.

[10] S.Khuller and B.Raghavachari, "Improved approximation algorithms for uniform connectivity problems," *Journal of Algorithms* **21** (1996), 434–450.

[11] W.Mader, "Ecken vom Grad n in minimalen n-fach zusammenhängenden Graphen," *Archive der Mathematik* **23** (1972), 219–224.

[12] H.Nagamochi and T.Ibaraki, "A linear-time algorithm for finding a sparse k-connected spanning subgraph of a k-connected graph," *Algorithmica* **7** (1992), 583–596.

[13] Z.Nutov, M.Penn and D.Sinreich, "On mobile robots flow in locally uniform networks," *Canadian Journal of Information Systems and Operational Research* **35** (1997), 197–208.

[14] M. Stoer, *Design of survivable networks*, Lecture Notes in Mathematics, 1531, Springer-Verlag, Berlin, 1992.