Erratum: Approximating minimum-cost connectivity problems via uncrossable bifamilies

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There are two errors in our paper “Approximating minimum-cost connectivity problems via uncrossable bifamilies” (ACM Transactions on Algorithms (TALG), 9(1), Article No. 1, 2012). In that paper we consider the (undirected) SURVIVABLE NETWORK problem. The input consists of a graph \( G = (V, E) \) with edge-costs, a set \( T \subseteq V \) of terminals, and connectivity demands \( \{ r_{st} > 0 : st \in D \subseteq T \times T \} \). The goal is to find a minimum cost subgraph \( H \) of \( G \) that for all \( st \in D \) contains \( r_{st} \) pairwise internally-disjoint \( st \)-paths. We claimed ratios \( O(k \ln k) \) for rooted demands when the set \( D \) of demand pairs forms a star, where \( k = \max_{st \in D} r_{st} \) is the maximum demand. This ratio is correct when the requirements are \( r_{st} = k \) for all \( t \in T \setminus \{ s \} \), but for general rooted demands our paper implies only ratio \( O(k^2) \) (which however is still the currently best known ratio for the problem). We also obtained various ratios for the node-weighted version of the problem. These results are valid, but the proof needs a correction described here.

Categories and Subject Descriptors: F.2.2 [Nonnumerical Algorithms and Problems]: Computations on discrete structures; G.2.2 [Discrete Mathematics]: Graph Algorithms

General Terms: Approximation Algorithms, Graph Connectivity, Rooted Connectivity, Edge-Costs, Node-Weights

1. BACKGROUND

All graphs here are assumed to be undirected, unless stated otherwise. For a graph \( H = (V, J) \) and \( Q \subseteq V \), the \( Q \)-connectivity \( \lambda^Q_H(s, t) \) of a node pair \( s, t \) is the maximum number of \( st \)-paths in \( H \) such that no two of them have an edge or a node in \( Q \setminus \{ s, t \} \) in common. Then \( Q = \emptyset \) is the case of edge-connectivity, and \( Q = V \) is the case of node-connectivity for which we use the notation \( \kappa_G(s, t) := \lambda^V_G(s, t) \). Given positive integral connectivity demands \( r = \{ r_{st} \geq 1 : st \in D \} \) over a set \( D \subseteq V \times V \) of demand pairs we say that \( H \) satisfies \( r \) if \( \lambda^Q_H(s, t) \geq r_{st} \) for all \( st \in D \). In our paper [Nutov 2012a] we consider variants of the following problem:

**SURVIVABLE NETWORK**

**Input:** A graph \( G = (V, E) \) with edge-costs \( \{ c_e : e \in E \} \), \( Q \subseteq V \), and connectivity demands \( \{ r_{st} > 0 : st \in D \} \).

**Output:** A minimum cost subgraph \( H \) of \( G \) that satisfies \( r \).
A node is a terminal if it belongs to some demand pair. Let $T$ denote the set of terminals and $k = \max_{t \in D} r_{st}$ the maximum demand. We claimed ratio $O(k \ln k)$ for rooted demands – when the set $D$ of demand pairs forms a star with center $s$. A correct result is:

**Theorem 1.1.** Survivable Network with rooted demands admits ratio $O(k^2)$; for rooted demands $r_{st} = k$ for all $t \in T \setminus \{s\}$ the problem admits ratio $O(k \ln k)$.

An important type of demands are element-connectivity demands, when $Q \subseteq V \setminus T$. We claimed the following ratios for Node-Weighted Survivable Network problems, in which instead of edge-costs we are given node-weights $\{w_v : v \in V\}$ and seek a minimum weight subgraph that satisfies $r$.

**Theorem 1.2.** Node-Weighted Survivable Network admits ratio $O(\ln |T|)$ for element-connectivity demands and ratio $O(k^2 \ln |T|)$ for rooted demands.

Theorem 1.2 is correct, but its proof in [Nutov 2012a] relies on an erroneous analysis of an approximation algorithm for the problem of finding a minimum node-weight edge-cover of an uncrossable biset family. A related paper of the author [Nutov 2013] that claims the same ratios for the more general Activation Survivable Network problem, has the same error. Recently, [Fukunaga 2015] showed by a non-trivial analysis, that for this problem the algorithm in [Nutov 2013] has ratios $k$ times larger than the ones given in Theorem 1.2. However, as was observed earlier by [Vakilian 2013], for node-weighted problems, a slight modification of our algorithm from [Nutov 2012a] enables to achieve the same ratios as in Theorem 1.2.

2. **Biset Families and the Errors**

To indicate the errors in our paper [Nutov 2012a] we need some definitions.

**Definition 2.1.** An ordered pair $A = (A, A^+)$ of subsets of $V$ with $A \subseteq A^+$ is called a biset; $A$ is the inner part and $A^+$ is the outer part of $A$, and $\partial A = A^+ \setminus A$ is the boundary of $A$. We will also use the notation $A^* = V \setminus A^+$.

**Definition 2.2.** An edge covers a biset $A$ if it goes from $A^*$ to $A$. For an edge-set/graph $J$ let $\delta_J(A)$ denote the set of edges in $J$ covering $A$. We say that $J$ covers a biset family $F$, or that $J$ is an $F$-cover, if $\delta_J(A) \neq \emptyset$ for all $A \in F$. The residual family of $F$ w.r.t. $J$ is defined by $F^J = \{A \in F : \delta_J(A) = \emptyset\}$.

In [Nutov 2012a] we considered the following generic problem:

**Biset-Family Edge-Cover**

**Input:** A graph $G = (V, E)$ with edge costs $\{c_e : e \in E\}$ and a biset family $F$.

**Output:** A minimum cost edge-set $J \subseteq E$ that covers $F$.

A standard LP-relaxation for the problems is:

**Biset-LP**

\[
\min \left\{ \sum_{e \in E} c_e x_e : \sum_{e \in \delta_G(A)} x_e \geq 1 \ \forall A \in F, \ x_e \geq 0 \ \forall e \in E \right\}.
\]
Theorem 1.1. For simplicity of exposition we consider the node-connectivity case. We provide a short proof in Section 3. Here let us show that Theorem 2.1 implies

Definition 2.3. The intersection and the union of two bisets \(A, B\) are defined by \(A \cap B = (A \cap B^+, A^+ \cap B^+)\) and \(A \cup B = (A \cup B, A^+ \cup B^+)\). The biset \(A \setminus B\) is defined by \(A \setminus B = (A \setminus B^+, A^+ \setminus B)\).

Definition 2.4. A biset-family \(F\) is uncrossable if \(A \cap B, A \cap B \in F\) or if \(A \setminus B, B \setminus A \in F\) for all \(A, B \in F\). We say that bisets \(A, B\): T-intersect if \(A \cap B \cap T \neq \emptyset\), and T-co-cross if \(A \cap B^* \cap T \neq \emptyset\) and \(B \cap A^* \cap T \neq \emptyset\). A biset family \(F\) is T-uncrossable if \(A \cap T \neq \emptyset\) for all \(A \in F\) and if for any \(A, B \in F\) holds: \(A \cap B, A \cup B \in F\) if \(A, B\) T-intersect, and \(A \setminus B, B \setminus A \in F\) if \(A, B\) T-co-cross.

Definition 2.5. We say that \(B\) contains \(A\) and write \(A \subseteq B\) if \(A \subseteq B\) and \(A^+ \subseteq B^+\). Inclusionwise minimal members of a biset family \(F\) are called \(F\)-cores, or simply cores, if \(F\) is clear from the context. Let \(C(F)\) denote the family of \(F\)-cores. For an \(F\)-core \(C \in C(F)\), the halo-family \(F(C)\) of \(C\) is the family of those members of \(F\) that contain \(C\) and contain no \(F\)-core distinct from \(C\). \(F\) is a simple biset family if it is a union of its halo-families.

It is known that if \(F\) is uncrossable or T-uncrossable, then so is the residual family \(F^J\) of \(F\), for any edge-set \(J\). Let us say that an instance of the Biset-Family Edge-Cover problem admits LP-ratio \(\rho\) if there exists a polynomial time algorithm that computes an \(F\)-cover of cost at most \(\rho\) times the optimal value of the Biset-LP. Let \(\alpha\) and \(\beta\) denote the best known LP-ratios for Biset-Family Edge-Cover with simple uncrossable \(F\) and with uncrossable \(F\), respectively; currently \(\alpha = 4/3\) [Fukunaga 2016] and \(\beta = 2\) [Fleischer et al. 2006] (see also a simple combinatorial algorithm in [Nutov 2009]). A main results in [Nuto 2012a] is:

Theorem 2.1. There exists a polynomial time algorithm that given a T-uncrossable biset family \(F\) sequentially finds \(\gamma + \frac{4\gamma^2}{\gamma + 1}\) simple uncrossable subfamilies and one uncrossable subfamily of \(F\), such that the union of covers of these subfamilies covers \(F\), where \(\gamma = \max_{A, B \in F} |\partial A \cap B \cap T|\) and \(\ell\) is the least integer such that \(2^\ell \geq \gamma + 1\).

The proof of Theorem 2.1 in [Nuto 2012a] is correct, but to remove any doubts we provide a short proof in Section 3. Here let us show that Theorem 2.1 implies Theorem 1.1. For simplicity of exposition we consider the node-connectivity case.

Let us say that a graph \(H\) is \((k, t, s)\)-connected if \(\kappa_H(t, s) \geq k\) for all \(t \in T\). In the \((k, t, s)\)-Connectivity Augmentation problem the goal is to augment a \((k, t, s)\)-connected graph \(H\) by a minimum cost edge-set \(J\) such that \(H \cup J\) is \((k + 1)\)-\((t, s)\)-connected. We say that a biset \(A\) is a \((T, s)\)-biset if \(A \cap T \neq \emptyset\) and \(s \in A^+\), and call \(A\) tight if \(\psi_H(\delta A) := |\partial A| + |\delta_H(\delta A)| = k\). From Menger’s Theorem we get that \(H\) is \((k, t, s)\)-connected if and only if \(\psi_H(\delta A) \geq k\) for every \((T, s)\)-biset \(A\). Thus \(J\) is a feasible solution to the \((k, t, s)\)-Connectivity Augmentation problem if and only if \(J\) covers the family of tight bisets.

Lemma 2.2. The family of tight bisets of a \((k, t, s)\)-connected graph is T-uncrossable.

Proof. Note that for any two bisets \(A\) and \(B\) in any graph \(H\) we have

\[
\psi_H(\delta A) + \psi_H(\delta B) \geq \psi_H(\partial A) + \psi_H(\partial B) \quad \text{and} \quad \psi_H(\delta A) + \psi_H(\delta B) \geq \psi_H(\partial A) + \psi_H(\partial B).
\]

Now let \(A, B\) be tight bisets in a \((k, t, s)\)-connected graph \(H\). If \(A, B\) T-intersect then \(A \cap B, A \cup B\) are both \((T, s)\)-bisets, and since \(H\) is \((k, t, s)\)-connected we have...
$\psi_H(\mathcal{A} \cap \mathcal{B}) \geq k$ and $\psi_H(\mathcal{A} \cup \mathcal{B}) \geq k$. This implies $k + k = \psi_H(\mathcal{A}) + \psi_H(\mathcal{B}) \geq \psi_H(\mathcal{A} \cap \mathcal{B}) + \psi_H(\mathcal{A} \cup \mathcal{B}) \geq k + k$. Hence equality holds everywhere, and thus $\mathcal{A} \cap \mathcal{B}, \mathcal{A} \cup \mathcal{B}$ are both tight. The proof of the case when $\mathcal{A}, \mathcal{B}$ $T$-co-cross is similar. \hfill \square

Observing that $|\partial \mathcal{A}| \leq k$ for any tight biset $\mathcal{A}$ we get from Theorem 2.1 and Lemma 2.2 that the $k$-$(T, s)$-CONNECTIVITY AUGMENTATION problem admits LP-ratio $O(k)$. We compute a solution to SURVIVABLE NETWORK with rooted demands in $k$ iterations, where at iteration $i = 1, \ldots, k$ we increase the $(T, s)$-connectivity from $i - 1$ to $i$. For general rooted demands we get ratio $\sum_{i=1}^{k} O(i) = O(k^2)$. For rooted uniform demands $r_{st} = k$ for all $t \in T$, this is equivalent to the so called “backward augmentation method” [Goemans et al. 1994]. It can be shown that in this case the cost of the solution computed at iteration $i$ is $O \left( \frac{i}{k^{i+1}} \right)$ times the optimal solution value for the SURVIVABLE NETWORK with rooted uniform demands instance, so we get ratio $O(k \ln k)$ in this case.

In [Nutov 2012a] we considered a more general augmentation problem, when $\kappa_{H,U}(t,s) \geq \kappa_H(t,s) + 1$ should hold for all $t \in T$. A $(t,s)$-biset was called tight if $\psi_H(\mathcal{A}) = \kappa_H(s,t)$. It was claimed that this family of tight bisets is $T$-uncrossable. If this was so, then we could apply the backward augmentation method and get ratio $O(k \ln k)$ for arbitrary rooted demands. This family has some “uncrossing” properties, cf. [Nutov 2012b; 2016], but it is not $T$-uncrossable. To see this, consider the following example. Let $H$ have node-set $\{s,a,b,t\}$, edge-set $\{sa, sb, ab\}$, and let $T = \{a, b, t\}$. Then $\kappa_H(a, s) = \kappa_H(b, s) = 2$ and $\kappa_H(t, s) = 0$. Consider the bisets $\mathcal{A}, \mathcal{B}$ where $A = \{a, t\}$, $\partial A = \{b\}$, $B = \{b, t\}$, and $\partial B = \{a\}$. Note that:

(i) $\psi_H(\mathcal{A}) = \psi_H(\mathcal{B}) = 0$, hence by the definition in [Nutov 2012a] $\mathcal{A}, \mathcal{B}$ are tight;
(ii) $A \cap B \cap T = \{t\}$, hence $\mathcal{A}, \mathcal{B}$ $T$-intersect.

However, the biset $\mathcal{A} \cap \mathcal{B}$ is not tight, since $\partial(\mathcal{A} \cap \mathcal{B}) = |\{a, b\}| = 2 > 0 = \kappa_H(t, s)$.

Another error in [Nutov 2012a] is ratio $O(\ln |C(\mathcal{F})|)$ for NODE-WEIGHTED BISET-FAMILY EDGE-COVER with uncrossable $\mathcal{F}$; here instead of edge-costs we are given node-weights $\{w_v : v \in V\}$ and seek to minimize the node-weight $w(V(J))$ of a cover $J$ of $\mathcal{F}$, where $V(J)$ denotes the set of end-nodes of the edges in $J$. As was observed by [Vakilian 2013] and [Fukunaga 2015], the proof has an error. Recently [Fukunaga 2015] showed that the algorithm in [Nutov 2012a] achieves ratio $O(\max_{\mathcal{A} \in \mathcal{F}} |\partial \mathcal{A}| \cdot \ln |C(\mathcal{F})|)$, which is the currently best known ratio for the problem.

This gives ratios by a factor of $k$ larger than the ones in Theorem 1.2. A correct result (proved in Section 4) that enables to obtain the ratios in Theorem 1.2 is:

\textbf{Theorem 2.3. NODE-WEIGHTED BISET-FAMILY EDGE-COVER with uncrossable biset family $\mathcal{F}$ admits ratio $O(\ln |C(\mathcal{F})|)$, provided that $w(\partial \mathcal{A}) = 0$ for all $\mathcal{A} \in \mathcal{F}$.}

In [Nutov 2012a] we claimed the same ratio without the condition “$w(\partial \mathcal{A}) = 0$ for all $\mathcal{A} \in \mathcal{F}$”, but the proof has an error. The algorithm in [Nutov 2012a] imitates the approach of [Klein and Ravi 1995] for the NODE-WEIGHTED STEINER FOREST problem. The algorithm starts with $J = \emptyset$ and repeatedly adds to $J$ an edge-set $S$ such that $w(V(J \cup S)) = O \left( \frac{\text{opt}}{\nu(S)} \right)$, where we use the notation $\nu(S) = |C(\mathcal{F}^S)|$; note that $\nu(\emptyset) = |C(\mathcal{F})|$. To indicate the error in [Nutov 2012a] let us state a correct statement, that is also needed for the proof of Theorem 2.3. For an $\mathcal{F}$-core $\mathcal{C} \in C(\mathcal{F})$ and $h \in V$ let us use the notation $\mathcal{F}(h, \mathcal{C}) = \{A \in \mathcal{F}(\mathcal{C}) : h \in A^*\}$.
LEMMA 2.4. Let $F$ be an uncrossable biset-family, let $\emptyset \neq C \subseteq C(F)$, and let $S$ be an edge-set with the following property: if $C = \{C\}$ then $S$ covers $F(C)$, and if $|C| \geq 2$ then $S$ covers $F(h, C)$ for all $C \in C$ for some node $h$ that belongs to no boundary of a biset in $\bigcup_{C \in C} F(C)$. Then $\nu(S) \leq \nu(\emptyset) - |C|/3$.

PROOF. Each $F^S$-core contains some $F$-core. Let $A$ be an $F^S$-core that contains some $C \in C$. We claim that $A$ contains at least two $F$-cores or $|C| \geq 2$ and $h \in A$. Otherwise, $A \in F(C)$ (since $A$ contains no $F$-core distinct from $C$) and $h \in A^*$ if $|C| \geq 2$ (since $h$ belongs to no boundary of a biset in $F(C)$); hence $A \in F(C)$ if $|C| = 1$ and $A \in F(h, C)$ if $|C| \geq 2$, contradicting the definition of $S$. Now let $p$ be the number of $F^S$-cores that contain at least two $F$-cores. The inner parts of the $F^S$-cores are pairwise disjoint, since $F^S$ is uncrossable; thus $h$ belongs to at most one inner part of them, and every $F$-core is contained in at most one $F^S$-core. From this we get that $\nu(S) \leq \nu(\emptyset) - p$, and that $p \geq 1$ if $|C| = 1$ and $p \geq \lfloor (|C| - 1)/2 \rfloor$ if $|C| \geq 2$. In both cases we have $p \geq |C|/3$, and the lemma follows. \qed

In [Nutov 2012a] Lemma 2.4 was stated without the condition on $h$, which is not correct. To see this, consider the following example from [Fukunaga 2015]. Let $V = \{h, u_1, \ldots, u_n\}$ and $F = \{C_1, \ldots, C_n\}$ where $C_i = (u_i, \{u_i, h\})$. Let $S = \{hu_1, \ldots, hu_n\}$. Then $S$ covers $F(h, C_i)$ for every $i$, but $F^S = F$ and hence $\nu(S) = \nu(\emptyset)$. 

Theorem 2.3 is proved in Section 4. Here let us show that Theorem 2.3 implies Theorem 1.2. Consider the case of rooted demands. As in edge-costs case, at iteration $i = 1, \ldots, k$ we increase the $(T,s)$-connectivity from $i - 1$ to $i$. Iteration $i$ starts with a subgraph $H_{i-1} = (V_{i-1}, E_{i-1})$ with all nodes in $V_{i-1}$ already included in the solution graph, so we set their weight to be 0 at the beginning of the iteration. At iteration $i$ we compute an edge-set $J_i$ that covers the family $F_{i-1}$ of tight bisets of $H_{i-1}$. Since nodes in $V_{i-1}$ have zero weight, $w(\partial A) = 0$ for all $A \in F_{i-1}$. We then use Theorem 2.1 to decompose the problem of covering $F_{i-1}$ into $O(i)$ NODE-WEIGHTED BISET-FAMILY EDGE-COVER problems, each with an uncrossable biset family $F$. Each such $F$ is a subfamily of $F_{i-1}$ and thus satisfies the condition in Theorem 2.3; furthermore, $F$ has at most $|T|$ cores. Thus the algorithm from Theorem 2.3 produces an $O(\ln |T|)$ approximate solution. As we cover $O(k^2)$ uncrossable families, ratio $O(k^2 \ln |T|)$ for rooted demands follows.

In the case of element-connectivity demands the problem is also decomposed into a sequence of $k$ augmentation problems. Let $D_i = \{st \in D : r_{st} \geq i\}$, $i = 1, \ldots, k$. Iteration $i$ starts with a subgraph $H_{i-1} = (V_{i-1}, E_{i-1})$ of $G$ such that $\lambda^Q_{H_{i-1}}(s,t) \geq i - 1$ for all $st \in D_{i-1}$; all nodes in $V_{i-1}$ are already included in the solution graph, so we set their weight to be 0 at the beginning of the iteration. During iteration $i$ we compute an edge-set $J_i$ such that the graph $H_i = H_{i-1} \cup J_i$ satisfies $\lambda^Q_H(s,t) \geq i$ for all $st \in D_i$. Given such $H_i$, we call a biset $A$ on $V$ tight if there exists $st \in D_i$ such that $|A \cap \{s, t\}| = |A^* \cap \{s, t\}| = 1$, $\partial A \subseteq Q$, and $\psi_{H_{i-1}}(A) = i - 1$. Then $J_i$ is a feasible solution to the above augmentation problem if and only if $J_i$ covers the family of tight bisets. It is known that this family is uncrossable. Furthermore, for any tight biset $A$ we have $\partial A \subseteq V_{i-1}$, and thus $w(\partial A) = 0$ at iteration $i$, since all nodes in $V_{i-1}$ have zero weight. Consequently, by the same argument as in the case of rooted demands we get ratio $O(\ln |T|)$ for the augmentation problem, and overall ratio $O(k \ln |T|)$. 

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In what follows, we use the following property halo families c.f. [Nutov 2012a].

**Lemma 2.5.** Let $\mathcal{F}$ be an uncrossable or a $T$-uncrossable biset family and let $A \in \mathcal{F}(C)$ and $B \in \mathcal{F}(C')$ for some $C, C' \in \mathcal{C}(\mathcal{F})$. If $C = C'$ then $A \cap B, A \cup B \in \mathcal{F}(C)$ and thus $\mathcal{F}(C)$ has a unique maximal member $M_C$ - the union of all bisets in $\mathcal{F}(C)$. If $C \neq C'$ then $A \setminus B \in \mathcal{F}(C)$ and $B \setminus A \in \mathcal{F}(C')$ if $\mathcal{F}$ is uncrossable, or if $\mathcal{F}$ is $T$-uncrossable and $A, B$ $T$-co-cross.

3. A SHORT PROOF OF THEOREM 2.1

The proof of Theorem 2.1 relies on the following key lemma.

**Lemma 3.1.** Let $\mathcal{F}$ be a $T$-uncrossable biset family and let $p = \min_{A \in \mathcal{F}} |A \cap T|$. Then there exists a polynomial time algorithm that computes a partition $\Pi$ of $\mathcal{C}(\mathcal{F})$ with at most $2\lceil \gamma/p \rceil + 1$ parts such that for each $C \in \Pi$ the family $\bigcup_{C \in \mathcal{C}} \mathcal{F}(C)$ is uncrossable, $\gamma = \max_{A, B \in \mathcal{F}} |\partial A \cap B \cap T|$. Furthermore, if $p \geq \gamma + 1$ then $\mathcal{F}$ is uncrossable.

**Proof.** If $p \geq \gamma + 1$ then any $A, B \in \mathcal{F}$ must $T$-intersect or $T$-co-cross; thus $\mathcal{F}$ is uncrossable in this case. We prove the first statement. For $C_i \in \mathcal{C}(\mathcal{F})$ let $M_i$ be the inclusionwise maximal biset in $\mathcal{F}(C_i)$. Since $\mathcal{F}$ is $T$-uncrossable, and by Lemma 2.5, for any $A_i \in \mathcal{F}(C_i)$ and $A_j \in \mathcal{F}(C_j)$ we have:

(i) $A_i, A_j$ $T$-intersect if and only if $i = j$.
(ii) If $C_i \cap M_j^* \cap T$ and $C_j \cap M_i^* \cap T$ are both nonempty then $A_i, A_j$ $T$-co-cross.

Construct an auxiliary directed graph $J$ that has node-set $\mathcal{C}(\mathcal{F})$ and arc-set $\{C_i, C_j : C_i \cap T \subseteq \partial M_j\}$. The indegree of every node in $\mathcal{F}$ is at most $\lceil \gamma/p \rceil$, by (i). This implies that every subgraph of the underlying graph of $\mathcal{F}$ has a node of degree $\leq 2\lceil \gamma/p \rceil$. Hence the underlying graph of $\mathcal{F}$ is $(2\lceil \gamma/p \rceil + 1)$-colorable, and such a coloring can be computed in polynomial time. Consequently, we can compute in polynomial time a partition $\Pi$ of $\mathcal{C}(\mathcal{F})$ into at most $2\lceil \gamma/p \rceil + 1$ independent sets. For each independent set $C \in \Pi$, the family $\bigcup_{C \in \mathcal{C}} \mathcal{F}(C)$ is uncrossable, by (ii). \(\square\)

Now consider the following algorithm.

**Algorithm 1:** $T$-Uncrossable Biset-Family Edge-Cover($G, c, \mathcal{F}$)

1. $J \leftarrow \emptyset$
2. **while** $p := \min_{A \in \mathcal{C}(\mathcal{F})} |A \cap T| \leq \gamma$ **do**
3. **find** a partition $\Pi$ of $\mathcal{C}(\mathcal{F})$ as in Lemma 3.1 with at most $2\lceil \gamma/p \rceil + 1$ parts
4. **for every** $C \in \Pi$ **find** an $\alpha$-approximate cover $J_C$ of $\bigcup_{C \in \mathcal{C}} \mathcal{F}_C(\mathcal{C})$
5. **for every** $C \in \Pi$ **do:** $J \leftarrow J \cup J_C$
6. **find** a $\beta$-approximate cover of $J'$ of $\mathcal{F}$ and add $J'$ to $J$
7. return $J$

Let $p_i$ denote the value of $p$ at the beginning of iteration $i$ in the while loop. Initially, $p_1 \geq 1$. Note that for any $T$-uncrossable family $\mathcal{F}$, if an $\mathcal{F}$-core $C$ and $A \in \mathcal{F}$ $T$-intersect then $C \subseteq A$; this implies that if $J$ covers all halo families of

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\( F \) then every \( F^j \)-core \( A \) contains at least two \( F \)-cores. From this it follows that \( p_i \geq 2p_{i-1} \) for all \( i \). Thus the number of iterations in the while loop is at most \( \ell - 1 \), where \( \ell \) is the least integer such that \( 2^\ell \geq \gamma + 1 \). Consequently, the number of simple uncrossable biset families covered in the while loop is bounded by
\[
\sum_{i=0}^{\ell-1} (2^{\gamma/2^i} + 1) \leq \ell + 2^\gamma \sum_{i=0}^{\ell-1} (1/2)^i = \ell + 4\gamma(1 - 1/2^\ell) \leq \ell + 4\gamma^2/\gamma + 1.
\]

4. PROOF OF THEOREM 2.3

For an edge-set \( S \) let \( V(S) \) denote the set of endnodes of the edges in \( S \). Given a node-set \( R \) we say that edge-sets \( S_1 \) and \( S_2 \) are \q R\-disjoint if \( V(S_1) \cap V(S_2) \cap R = \emptyset \). A spider is a non-empty union \( S \) of paths \( (\text{legs of } S) \) that start at the same node \( h \) (the head of \( S \)) such that no two of them have other node in common. Note that if \( S \) has legs \( S_1, \ldots, S_d \) then \( w(V(S)) = w_h + \sum_{i=1}^d w(V(S_i) \setminus \{h\}) \). We extend this definition to an \( R \)-spider by requiring that the legs of \( S \) are only \((R \setminus \{h\})\)-disjoint; if \( w_v = 0 \) for all \( v \in V \setminus R \) then we still have \( w(V(S)) = w_h + \sum_{i=1}^d w(V(S_i) \setminus \{h\}) \).

The following definition extends in a similar way \( "F\)-spiders” from [Nutov 2012a].

**Definition 4.1.** Let \( F \) be a biset family on \( V \) and let \( R \subseteq V \). An \( (F,R)\)-spider is a pair \((S,C)\), where \( \emptyset \neq C \subseteq \mathcal{C}(F) \) is the set of cores hit by the \((F,R)\)-spider, and \( S \) is an edge-set such that: if \( C = \{\emptyset\} \) then \( S \) covers \( F(C) \), and if \( |C| \geq 2 \) then \( S \) is a union of (possibly empty) pairwise \((R \setminus \{h\})\)-disjoint \( F(h,C)\)-covers \( \{S_C : C \in \mathcal{C}\} \) (legs of the \((F,R)\)-spider) for some \( h \in R \) (a head of the \((F,R)\)-spider).

We often denote an \((F,R)\)-spider just by \( S \), meaning that the set \( C = C_S \) of cores hit by \( S \) is clear from the context. Later, we will prove the following.

**Theorem 4.1.** For any \( R \subseteq V \), any cover \( J \) of an uncrossable biset family \( F \) contains a family \( S \) of \( R \)-disjoint \((F,R)\)-spiders that collectively hit at least \( \frac{2}{3} |C(F)| \) distinct \( F \)-cores.

In [Nutov 2012a], Theorem 4.1 was stated and proved for the case \( R = V \). The proof is correct, but the case \( R = V \) is not sufficient for proving Theorem 2.3. However, the proof of Theorem 4.1 is a minor modification of the proof of the case \( R = V \) in [Nutov 2012a].

Let us briefly describe how Theorem 4.1 and Lemma 2.4 imply Theorem 2.3. Let \( R \) be the set of nodes in \( V \) that belong to no boundary of a biset in \( \bigcup_{C \in \mathcal{C}} F(C) \). Note that in the setting of Theorem 2.3 all nodes in \( V \setminus R \) have weight zero.

**Lemma 4.2.** Let \( S \) be a family of \((F,R)\)-spiders as in Theorem 4.1 for an optimal cover of an uncrossable biset family \( F \). There is an \((F,R)\)-spider \((S^*,C^*)\) in \( S \) such that \( \frac{w(V(S^*))}{|C^*/3\varepsilon|} \leq \frac{9}{2} \cdot \frac{\text{opt}}{\nu(\emptyset)} \).

**Proof.** Let \( C_S \) denote the set of \( F \)-cores hit by a spider \( S \in S \). Since the spiders in \( S \) are \( R \)-disjoint \( \sum_{S \in S} w(V(S)) \leq \text{opt} \). Since the spiders in \( S \) hit at least \( \frac{2}{3} \nu(\emptyset) \) distinct \( F \)-cores \( \sum_{S \in S} |C_S| \geq \frac{2}{3} \nu(\emptyset) \). Thus \( \sum_{S \in S} |C_S|/3 \geq \frac{2}{3} \nu(\emptyset) \). Consequently, by an averaging argument, there is \((S^*,C^*) \in S \) as required. \( \square \)
**Lemma 4.3.** There exists a polynomial time algorithm that given an instance of Node-Weighted Biset-Family Edge-Cover with uncrossable $F$ and $w_v = 0$ for all $v \in V \setminus R$, finds an edge-set $S \subseteq E$ such that $\frac{w(V(S))}{\nu(\emptyset) - \nu(S)} \leq 9 \frac{\text{opt}}{\nu(\emptyset)}$.

**Proof.** Let $(S^*, C^*)$ be an $(F, R)$-spider as in Lemma 4.2. Assume that we know the number $d = |C^*|$, and that if $d \geq 2$ then we know a head $h$ of $(S^*, C^*)$ and whether $\delta_{S^*}(h) \neq \emptyset$. There is a polynomial number of choices, so we can try all choices and return the best outcome (guessing $d$ can be avoided, by a slightly more complicated algorithm). We note that given $C$ and $h$ the problem of finding a minimum node-weight cover of $F(C)$ or of $F(h, C)$ admits ratio $2$. If $d = 1$ then for each $C \in C(F)$ we compute a $2$-approximate $F(C)$-cover and return the lightest one. If $d \geq 2$ then we temporarily set $w_h = 0$ if $\delta_{S^*}(h) \neq \emptyset$ or $w_h = \infty$ if $\delta_{S^*}(h) = \emptyset$; then for each $C \in C(F)$ we compute a $2$-approximate $F(h, C)$-cover $S_C$ and return the union $S$ of $d$ lightest sets $S_C$. Then $w(V(S)) \leq 2w(V(S^*))$, since legs of $S^*$ are pairwise $(R \setminus \{h\})$-disjoint and since $w_v = 0$ for all $v \in V \setminus R$. Thus from Lemma 2.4 and our choice of $S^*$ we get $\frac{w(V(S))}{\nu(\emptyset) - \nu(S)} \leq 2 \frac{w(V(S^*))}{\nu(\emptyset)} \leq 9 \frac{\text{opt}}{\nu(\emptyset)}$. \qed

The overall algorithm starts with $J = \emptyset$ and while $\nu(J) \geq 1$ repeatedly adds to $J$ an edge-set $S$ such that $\frac{w(V(S))}{\nu(J) - \nu(J \cup S)} \leq 9 \frac{\text{opt}}{\nu(\emptyset)}$. Such an algorithm has ratio $9(\ln \nu(\emptyset) + 1) = 9(\ln |C(F)| + 1)$; if $F$ is symmetric (namely, if $(V \setminus A^+) \cap (V \setminus A) \in F$ whenever $A \in F$), then the ratio is in fact $9 \ln |C(F)|$, see [Klein and Ravi 1995]. Furthermore, for biset families arising from Survivable Network problems, the problem of finding a minimum node-weight cover of $F(C)$ or of $F(h, C)$ admits a polynomial time algorithm, and the ratio can be further reduced to $\frac{3}{2} \ln |C(F)|$.

In the rest of this section we prove Theorem 4.1. A biset family is a **ring** if it is is closed under intersection and union. To prove Theorem 4.1 the only properties of $F$ that we need are that the inner parts of the $F$-cores are pairwise-disjoint and that $F(C)$ is a ring for any $F$-core $C$ (this is so by Lemma 2.5); it is not hard to see that then $F(h, C)$ is a ring for any $h \in V$. Note that any ring has a unique core. We need the following property of rings, c.f. [Nutov 2012a].

**Lemma 4.4.** Let $J$ be an inclusionwise minimal cover of a ring $F$ with core $C$. Then there is an ordering $e_1, \ldots, e_q$ of $J$ and bisets $C_1 \subseteq \cdots \subseteq C_q$ in $F$ where $C_1 = C$, such that $\delta_J(C_i) = \{e_i\}$, and if $e_i = v_i u_i$ where $u_i \in C_i$, then $\{e_1, \ldots, e_i\}$ covers $F(h, C)$ for $h \in \{v_i, u_i+1\}$.

The following definition extends the concept of $R$-spiders introduced earlier, and as we shall see it is also closely related to $(F, R)$-spiders in Definition 4.1.

**Definition 4.2.** Let $\mathcal{P} = \{P_u : u \in U(\mathcal{P})\}$ be a family of simple directed paths on $V$ with a set $U(\mathcal{P})$ of distinct ends, where each $P_u$ ends at $u$, and let $R \subseteq V$. An $R$-spider $S$ with head $h$ is called a $(\mathcal{P}, R)$-**spider** if $S$ is a union of subpaths (one may be of length 0) $\{S_u : u \in U\}$ of the paths in $\mathcal{P}$ for some $\emptyset \neq U \subseteq U(\mathcal{P})$ (the set of ends hit by $S$), where each $S_u$ is an $h$u-subpath of $P_u$, such that if $|U| = 1$ then $S \in \mathcal{P}$ and if $|U| \geq 2$ then $h \in R$.

**Lemma 4.5.** Let $\mathcal{P}$ be a family of simple directed paths on $V$ with a set $U(\mathcal{P})$ of distinct endnodes and let $R \subseteq V$. Then there is a family $S$ of pairwise $R$-disjoint $(\mathcal{P}, R)$-spiders that collectively hit at least $\frac{2}{3} |U(\mathcal{P})|$ distinct nodes in $U(\mathcal{P})$. 

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The case $R = V$ was proved in [Chuzhoy and Khanna 2008]. Hence there exists a family $S$ of pairwise node-disjoint $(P, V)$-spiders that hit at least $\frac{2}{3}|U(P)|$ nodes in $U(P)$. Since any $(P, V)$-spider is also a $(P, R)$-spider, this family satisfies all the requirements except of one: there can be $\overline{S} \in S$ that hits at least 2 ends with head $h \in V \setminus R$. We resolve this by an elementary construction that makes the paths in $P$ to be $(V \setminus R)$-disjoint: for every path $P$ and every $v \in V(P) \setminus R$ that is not an end of $P$, make a copy $v_P$ of $v$ and let $P$ go through $v_P$ instead of $v$. Note that this operation does not affect the ends of the paths, hence their number remains the same. Then the paths in $P$ become pairwise $(V \setminus R)$-disjoint; hence the [Chuzhoy and Khanna 2008] result gives a family $\overline{S}$ as required, since now every $(P, V)$-spider in $\overline{S}$ that hits at least 2 ends has head in $R$. The lemma follows since shrinking the nodes $v_P$ back into $v$ keeps the required properties: each spider remains a $(P, R)$-spider since its legs remain pairwise $(R \setminus \{h\})$-disjoint, and any two $(P, R)$-spiders remain $R$-disjoint.

Now we use Lemmas 4.4 and 4.5 to prove Theorem 4.1. The proof essentially coincides with the proof in [Nutov 2012a] for the case $R = V$. Define a family $P$ of directed paths in a complete directed graph on $V$ as follows. For every $C \in \mathcal{C}(F)$ fix some inclusion-wise-minimal $F(C)$-cover $J_C \subseteq J$. By Lemma 2.5, $F(C)$ is a ring. Let $e_1, \ldots, e_q$ be an ordering of $J_C$ and $C_1 \subseteq \cdots \subseteq C_q$ bisets in $F(C)$ as in Lemma 4.4, where $e_i = v_i u_i$ is as in the lemma. Obtain a directed path $P_C$ by taking for each edge $e_i$ the arc $v_i u_i$ and for every $i = q, \ldots, 2$ the dummy arc $u_i v_{i-1}$, if $u_i \neq v_{i-1}$; e.g., if $u_i \neq v_{i-1}$ for all $i$, then the node sequence of $P_C$ is $(v_q, u_q, v_{q-1}, u_{q-1}, \ldots, v_1, u_1)$. Denote $u_C = u_1$ and note that $u_C \in C$.

Let $P = \{P_C : C \in \mathcal{C}(F)\}$. Since the sets $\{C : C \in \mathcal{C}(F)\}$ are pairwise-disjoint (by Lemma 2.5), any two paths in $P$ have distinct ends. Hence Lemma 4.5 applies, and there exists a family $S$ of node-disjoint $(P, R)$-spiders that hits at least $\frac{2}{3}|U(P)|$ nodes in $U(P) = \{u_C : C \in \mathcal{C}(F)\}$.

For any $(P, R)$-spider $\overline{S} \in \overline{S}$ and the set $U$ of nodes in $U(P)$ hit by $\overline{S}$ naturally corresponds a pair $(\overline{S}, C)$, where $\overline{S} \subseteq J$ is defined by the non-dummy arcs in $\overline{S}$ and $C = \{C \in \mathcal{C}(F) : u_C \in U\}$. We show that $(\overline{S}, C)$ is an $(F, R)$-spider. For $C \in C$ let $\overline{S}_C$ be the $hu_C$-path in $\overline{S}$ and let $S_C$ be the corresponding subset of $S$. If $C = \{C\}$ then $S_C = P_C$; thus in this case $\overline{S} = S_C = J_C$, and since $J_C$ covers $F(C)$ the pair $(\overline{S}, C)$ is an $(F, R)$-spider. Assume that $|C| \geq 2$ and let $h$ be the head of $\overline{S}$. Since $\overline{S}$ is a $(P, R)$-spider, the edge-sets $\{S_C : C \in C\}$ are pairwise $(R \setminus \{h\})$-disjoint. By Lemma 4.4 and the construction, each $S_C$ is an $F(h, C)$-cover. Thus $(\overline{S}, C)$ is an $(F, R)$-spider in this case as well.

Now let $S$ be the family $(F, R)$-spiders corresponding to the $(P, R)$-spiders in $\overline{S}$. Since the arc-sets in $S$ are node-disjoint, so are the edge-sets in $S$. Since $\overline{S}$ hits at least $\frac{2}{3}|U(P)|$ nodes in $U(P)$ and since $|\mathcal{C}(F)| = |U(P)|$, $S$ hits at least $\frac{2}{3}|\mathcal{C}(F)|$ cores in $\mathcal{C}(F)$. Thus $S$ is as required, and the proof of Theorem 4.1 is complete.

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