Erratum: Approximating minimum-cost connectivity problems via uncrossable bifamilies

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There are two errors in our paper "Approximating minimum-cost connectivity problems via uncrossable bifamilies" (ACM Transactions on Algorithms (TALG), 9(1), Article No. 1, 2012). In that paper we consider the (undirected) Survivable Network problem. The input consists of a graph $G=(V,E)$ with edge-costs, a set $T \subseteq V$ of terminals, and connectivity demands $\{r_{st}>0 : st \in D \subseteq T \times T\}$. The goal is to find a minimum cost subgraph H of G that for all $st \in D$ contains r_{st} pairwise internally-disjoint st-paths. We claimed ratios $O(k \ln k)$ for rooted demands when the set D of demand pairs forms a star, where $k=\max_{st \in D} r_{st}$ is the maximum demand. This ratio is correct when the requirements are $r_{st}=k$ for all $t \in T \setminus \{s\}$, but for general rooted demands our paper implies only ratio $O(k^2)$ (which however is still the currently best known ratio for the problem). We also obtained various ratios for the node-weighted version of the problem. These results are valid, but the proof needs a correction described here.

Categories and Subject Descriptors: F.2.2 [**Nonnumerical Algorithms and Problems**]: Computations on discrete structures; G.2.2 [**Discrete Mathematics**]: Graph Algorithms

General Terms: Approximation Algorithms, Graph Connectivity, Rooted Connectivity, Edge-Costs, Node-Weights

1. BACKGROUND

All graphs here are assumed to be undirected, unless stated otherwise. For a graph $H=(V,J)$ and $Q \subseteq V$, the **Q-connectivity** $\lambda_H^Q(s,t)$ of a node pair s,t is the maximum number of st-paths in H such that no two of them have an edge or a node in $Q \setminus \{s,t\}$ in common. Then $Q=\emptyset$ is the case of **edge-connectivity**, and $Q=V$ is the case of **node-connectivity** for which we use the notation $\kappa_G(s,t) := \lambda_G^V(s,t)$. Given positive integral **connectivity demands** $r=\{r_{st} \geq 1 : st \in D\}$ over a set $D \subseteq V \times V$ of **demand pairs** we say that H **satisfies** r if $\lambda_H^Q(s,t) \geq r_{st}$ for all $st \in D$. In our paper [Nutov 2012a] we consider variants of the following problem:

Survivable Network
Input: A graph $G=(V,E)$ with edge-costs $\{c_e : e \in E\}$, $Q \subseteq V$, and connectivity demands $\{r_{st}>0 : st \in D\}$.
Output: A minimum cost subgraph H of G that satisfies r.

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A node is a **terminal** if it belongs to some demand pair. Let T denote the set of terminals and $k = \max_{st \in D} r_{st}$ the maximum demand. We claimed ratio $O(k \ln k)$ for **rooted demands** – when the set D of demand pairs forms a star with center s. A correct results is:

THEOREM 1.1. SURVIVABLE NETWORK *with rooted demands admits ratio* $O(k^2)$*; for rooted demands* $r_{st} = k$ *for all* $t \in T \setminus \{s\}$ *the problem admits ratio* $O(k \ln k)$.

An important type of demands are **element-connectivity demands**, when $Q \subseteq V \setminus T$. We claimed the following ratios for NODE-WEIGHTED SURVIVABLE NETWORK problems, in which instead of edge-costs we are given node-weights $\{w_v : v \in V\}$ and seek a minimum weight subgraph that satisfies r.

THEOREM 1.2. NODE-WEIGHTED SURVIVABLE NETWORK *admits ratio* $O(\ln |T|)$ *for element-connectivity demands and ratio* $O(k^2 \ln |T|)$ *for rooted demands.*

Theorem 1.2 is correct, but its proof in [Nutov 2012a] relies on an erroneous analysis of an approximation algorithm for the problem of finding a minimum node-weight edge-cover of an uncrossable biset family. A related paper of the author [Nutov 2013] that claims the same ratios for the more general ACTIVATION SURVIVABLE NETWORK problem, has the same error. Recently, [Fukunaga 2015] showed by a non-trivial analysis, that for this problem the algorithm in [Nutov 2013] has ratios k times larger than the ones given in Theorem 1.2. However, as was observed earlier by [Vakilian 2013], for node-weighted problems, a slight modification of our algorithm from [Nutov 2012a] enables to achieve the same ratios as in Theorem 1.2.

2. BISET FAMILIES AND THE ERRORS

To indicate the errors in our paper [Nutov 2012a] we need some definitions.

DEFINITION 2.1. *An ordered pair* $\mathbb{A} = (A, A^+)$ *of subsets of* V *with* $A \subseteq A^+$ *is called a* **biset***;* A *is the* **inner part** *and* A^+ *is the* **outer part** *of* $\mathbb{A}$*, and* $\partial\mathbb{A} = A^+ \setminus A$ *is the* **boundary** *of* $\mathbb{A}$*. We will also use the notation* $A^* = V \setminus A^+$.

DEFINITION 2.2. *An* **edge covers a biset** $\mathbb{A}$ *if it goes from* A^* *to* A*. For an edge-set/graph* J *let* $\delta_J(\mathbb{A})$ *denote the set of edges in* J *covering* $\mathbb{A}$*. We say that* J **covers** *a biset family* $\mathcal{F}$*, or that* J *is an* $\mathcal{F}$**-cover***, if* $\delta_J(\mathbb{A}) \neq \emptyset$ *for all* $\mathbb{A} \in \mathcal{F}$*. The* **residual family** *of* $\mathcal{F}$ *w.r.t.* J *is defined by* $\mathcal{F}^J = \{\mathbb{A} \in \mathcal{F} : \delta_J(\mathbb{A}) = \emptyset\}$.

In [Nutov 2012a] we considered the following generic problem:

BISET-FAMILY EDGE-COVER
Input: A graph $G = (V, E)$ with edge costs $\{c_e : e \in E\}$ and a biset family $\mathcal{F}$.
Output: A minimum cost edge-set $J \subseteq E$ that covers $\mathcal{F}$.

A standard LP-relaxation for the problems is:

$$\textbf{Biset-LP} \quad \min \left\{ \sum_{e \in E} c_e x_e : \sum_{e \in \delta_E(\mathbb{A})} x_e \geq 1 \ \forall \mathbb{A} \in \mathcal{F}, x_e \geq 0 \ \forall e \in E \right\} .$$

Definition 2.3. *The* **intersection** *and the* **union** *of two bisets* $\mathbb{A}, \mathbb{B}$ *are defined by* $\mathbb{A} \cap \mathbb{B} = (A \cap B, A^+ \cap B^+)$ *and* $\mathbb{A} \cup \mathbb{B} = (A \cup B, A^+ \cup B^+)$. *The biset* $\mathbb{A} \setminus \mathbb{B}$ *is defined by* $\mathbb{A} \setminus \mathbb{B} = (A \setminus B^+, A^+ \setminus B)$.

Definition 2.4. *A biset-family* $\mathcal{F}$ *is* **uncrossable** *if* $\mathbb{A} \cap \mathbb{B}, \mathbb{A} \cap \mathbb{B} \in \mathcal{F}$ *or if* $\mathbb{A} \setminus \mathbb{B}, \mathbb{B} \setminus \mathbb{A} \in \mathcal{F}$ *for all* $\mathbb{A}, \mathbb{B} \in \mathcal{F}$. *We say that bisets* $\mathbb{A}, \mathbb{B}$*:* T**-intersect** *if* $A \cap B \cap T \neq \emptyset$, *and* T**-co-cross** *if* $A \cap B^* \cap T \neq \emptyset$ *and* $B \cap A^* \cap T \neq \emptyset$. *A biset family* $\mathcal{F}$ *is* T**-uncrossable** *if* $A \cap T \neq \emptyset$ *for all* $\mathbb{A} \in \mathcal{F}$ *and if for any* $\mathbb{A}, \mathbb{B} \in \mathcal{F}$ *holds:* $\mathbb{A} \cap \mathbb{B}, \mathbb{A} \cup \mathbb{B} \in \mathcal{F}$ *if* $\mathbb{A}, \mathbb{B}$ T*-intersect, and* $\mathbb{A} \setminus \mathbb{B}, \mathbb{B} \setminus \mathbb{A} \in \mathcal{F}$ *if* $\mathbb{A}, \mathbb{B}$ T*-co-cross.*

Definition 2.5. *We say that* $\mathbb{B}$ **contains** $\mathbb{A}$ *and write* $\mathbb{A} \subseteq \mathbb{B}$ *if* $A \subseteq B$ *and* $A^+ \subseteq B^+$. *Inclusionwise minimal members of a biset family* $\mathcal{F}$ *are called* $\mathcal{F}$**-cores**, *or simply* **cores**, *if* $\mathcal{F}$ *is clear from the context. Let* $\mathcal{C}(\mathcal{F})$ *denote the family of* $\mathcal{F}$*-cores. For an* $\mathcal{F}$*-core* $\mathbb{C} \in \mathcal{C}(\mathcal{F})$, *the* **halo-family** $\mathcal{F}(\mathbb{C})$ *of* $\mathbb{C}$ *is the family of those members of* $\mathcal{F}$ *that contain* $\mathbb{C}$ *and contain no* $\mathcal{F}$*-core distinct from* $\mathbb{C}$. $\mathcal{F}$ *is a* **simple biset family** *if it is a union of its halo-families.*

It is known that if $\mathcal{F}$ is uncrossable or T-uncrossable, then so is the residual family $\mathcal{F}^J$ of $\mathcal{F}$, for any edge-set J. Let us say that an instance of the Biset-Family Edge-Cover problem admits **LP-ratio** ρ if there exists a polynomial time algorithm that computes an $\mathcal{F}$-cover of cost at most ρ times the optimal value of the Biset-LP. Let α and β denote the best known LP-ratios for Biset-Family Edge-Cover with simple uncrossable $\mathcal{F}$ and with uncrossable $\mathcal{F}$, respectively; currently $\alpha = 4/3$ [Fukunaga 2016] and $\beta = 2$ [Fleischer et al. 2006] (see also a simple combinatorial algorithm in [Nutov 2009]). A main results in [Nutov 2012a] is:

Theorem 2.1. *There exists a polynomial time algorithm that given a* T*-uncrossable biset family* $\mathcal{F}$ *sequentially finds* $\ell + \frac{4\gamma^2}{\gamma+1}$ *simple uncrossable subfamilies and one uncrossable subfamily of* $\mathcal{F}$, *such that the union of covers of these subfamilies covers* $\mathcal{F}$, *where* $\gamma = \max_{\mathbb{A},\mathbb{B} \in \mathcal{F}} |\partial\mathbb{A} \cap B \cap T|$ *and* ℓ *is the least integer such that* $2^\ell \geq \gamma + 1$.

The proof of Theorem 2.1 in [Nutov 2012a] is correct, but to remove any doubts we provide a short proof in Section 3. Here let us show that Theorem 2.1 implies Theorem 1.1. For simplicity of exposition we consider the node-connectivity case.

Let us say that a graph H is k**-**(T,s)**-connected** if $\kappa_H(t,s) \geq k$ for all $t \in T$. In the k-(T,s)-Connectivity Augmentation problem the goal is to augment a k-(T,s)-connected graph H by a minimum cost edge-set J such that $H \cup J$ is $(k+1)$-(T,s)-connected. We say that a biset $\mathbb{A}$ is a (T,s)**-biset** if $A \cap T \neq \emptyset$ and $s \in A^*$, and call $\mathbb{A}$ **tight** if $\psi_H(\mathbb{A}) := |\partial\mathbb{A}| + |\delta_H(\mathbb{A})| = k$. From Menger's Theorem we get that H is k-(T,s)-connected if and only if $\psi_H(\mathbb{A}) \geq k$ for every (T,s)-biset $\mathbb{A}$. Thus J is a feasible solution to the k-(T,s)-Connectivity Augmentation problem if and only if J covers the family of tight bisets.

Lemma 2.2. *The family of tight bisets of a* k*-*(T,s)*-connected graph is* T*-uncrossable.*

Proof. Note that for any two bisets $\mathbb{A}$ and $\mathbb{B}$ in any graph H we have

$$\psi_H(\mathbb{A})+\psi_H(\mathbb{B}) \geq \psi_H(\mathbb{A}\cap\mathbb{B})+\psi_H(\mathbb{A}\cup\mathbb{B}) \quad \text{and} \quad \psi_H(\mathbb{A})+\psi_H(\mathbb{B}) \geq \psi_H(\mathbb{A}\setminus\mathbb{B})+\psi_H(\mathbb{B}\setminus\mathbb{A}) \,.$$

Now let $\mathbb{A}, \mathbb{B}$ be tight bisets in a k-(T,s)-connected graph H. If $\mathbb{A}, \mathbb{B}$ T-intersect then $\mathbb{A} \cap \mathbb{B}, \mathbb{A} \cup \mathbb{B}$ are both (T,s)-bisets, and since H is k-(T,s)-connected we have

$\psi_H(\mathbb{A} \cap \mathbb{B}) \geq k$ and $\psi_H(\mathbb{A} \cup \mathbb{B}) \geq k$. This implies $k + k = \psi_H(\mathbb{A}) + \psi_H(\mathbb{B}) \geq \psi_H(\mathbb{A} \cap \mathbb{B}) + \psi_H(\mathbb{A} \cup \mathbb{B}) \geq k + k$. Hence equality holds everywhere, and thus $\mathbb{A}\cap\mathbb{B}, \mathbb{A}\cup\mathbb{B}$ are both tight. The proof of the case when $\mathbb{A}, \mathbb{B}$ T-co-cross is similar. □

Observing that $|\partial\mathbb{A}| \leq k$ for any tight biset $\mathbb{A}$ we get from Theorem 2.1 and Lemma 2.2 that the k-(T, s)-CONNECTIVITY AUGMENTATION problem admits LP-ratio $O(k)$. We compute a solution to SURVIVABLE NETWORK with rooted demands in k iterations, where at iteration $i = 1, \ldots, k$ we increase the (T, s)-connectivity from $i - 1$ to i. For general rooted demands we get ratio $\sum_{i=1}^{k} O(i) = O(k^2)$. For rooted uniform demands $r_{st} = k$ for all $t \in T$, this is equivalent to the so called "backward augmentation method" [Goemans et al. 1994]. It can be shown that in this case the cost of the solution computed at iteration i is $O\left(\frac{i}{k-i+1}\right)$ times the optimal solution value for the SURVIVABLE NETWORK with rooted uniform demands instance, so we get ratio $O(k \ln k)$ in this case.

In [Nutov 2012a] we considered a more general augmentation problem, when $\kappa_{H\cup J}(t, s) \geq \kappa_H(t, s) + 1$ should hold for all $t \in T$. A (t, s)-biset was called tight if $\psi_H(\mathbb{A}) = \kappa_H(s, t)$. It was claimed that this family of tight bisets is T-uncrossable. If this was so, then we could apply the backward augmentation method and get ratio $O(k \ln k)$ for arbitrary rooted demands. This family has some "uncrossing" properties, c.f. [Nutov 2012b; 2016], but it is *not* T-uncrossable. To see this, consider the following example. Let H have node-set $\{s, a, b, t\}$, edge-set $\{sa, sb, ab\}$, and let $T = \{a, b, t\}$. Then $\kappa_H(a, s) = \kappa_H(b, s) = 2$ and $\kappa_H(t, s) = 0$. Consider the bisets $\mathbb{A}, \mathbb{B}$ where $A = \{a, t\}$, $\partial\mathbb{A} = \{b\}$, $B = \{b, t\}$, and $\partial\mathbb{B} = \{a\}$. Note that:
(i) $\psi_H(\mathbb{A}) = \psi_H(\mathbb{B}) = 0$, hence by the definition in [Nutov 2012a] $\mathbb{A}, \mathbb{B}$ are tight;
(ii) $A \cap B \cap T = \{t\}$, hence $\mathbb{A}, \mathbb{B}$ T-intersect.
However, the biset $\mathbb{A} \cap \mathbb{B}$ is not tight, since $\partial(\mathbb{A} \cap \mathbb{B}) = |\{a, b\}| = 2 > 0 = \kappa_H(t, s)$.

Another error in [Nutov 2012a] is ratio $O(\ln |\mathcal{C}(\mathcal{F})|)$ for NODE-WEIGHTED BISET-FAMILY EDGE-COVER with uncrossable $\mathcal{F}$; here instead of edge-costs we are given node-weights $\{w_v : v \in V\}$ and seek to minimize the **node-weight** $w(V(J))$ of a cover J of $\mathcal{F}$, where $V(J)$ denotes the set of end-nodes of the edges in J. As was observed by [Vakilian 2013] and [Fukunaga 2015], the proof has an error. Recently [Fukunaga 2015] showed that the algorithm in [Nutov 2012a] achieves ratio $O(\max_{\mathbb{A}\in\mathcal{F}} |\partial\mathbb{A}| \cdot \ln |\mathcal{C}(\mathcal{F})|)$, which is the currently best known ratio for the problem. This gives ratios by a factor of k larger than the ones in Theorem 1.2. A correct result (proved in Section 4) that enables to obtain the ratios in Theorem 1.2 is:

THEOREM 2.3. NODE-WEIGHTED BISET-FAMILY EDGE-COVER *with uncrossable biset family* $\mathcal{F}$ *admits ratio* $O(\ln |\mathcal{C}(\mathcal{F})|)$*, provided that* $w(\partial\mathbb{A}) = 0$ *for all* $\mathbb{A} \in \mathcal{F}$.

In [Nutov 2012a] we claimed the same ratio without the condition "$w(\partial\mathbb{A}) = 0$ for all $\mathbb{A} \in \mathcal{F}$", but the proof has an error. The algorithm in [Nutov 2012a] imitates the approach of [Klein and Ravi 1995] for the NODE-WEIGHTED STEINER FOREST problem. The algorithm starts with $J = \emptyset$ and repeatedly adds to J an edge-set S such that $\frac{w(V(S))}{\nu(J)-\nu(J\cup S)} = O\left(\frac{\mathsf{opt}}{\nu(\emptyset)}\right)$, where we use the notation $\nu(S) = |\mathcal{C}(\mathcal{F}^S)|$; note that $\nu(\emptyset) = |\mathcal{C}(\mathcal{F})|$. To indicate the error in [Nutov 2012a] let us state a correct statement, that is also needed for the proof of Theorem 2.3. For an $\mathcal{F}$-core $\mathbb{C} \in \mathcal{C}(\mathcal{F})$ and $h \in V$ let us use the notation $\mathcal{F}(h, \mathbb{C}) = \{A \in \mathcal{F}(\mathbb{C}) : h \in A^*\}$.

Lemma 2.4. *Let $\mathcal{F}$ be an uncrossable biset-family, let $\emptyset \neq \mathcal{C} \subseteq \mathcal{C}(\mathcal{F})$, and let S be an edge-set with the following property: if $\mathcal{C} = \{\mathbb{C}\}$ then S covers $\mathcal{F}(\mathbb{C})$, and if $|\mathcal{C}| \geq 2$ then S covers $\mathcal{F}(h, \mathbb{C})$ for all $\mathbb{C} \in \mathcal{C}$ for some node h that belongs to no boundary of a biset in $\bigcup_{\mathbb{C} \in \mathcal{C}} \mathcal{F}(\mathbb{C})$. Then $\nu(S) \leq \nu(\emptyset) - |\mathcal{C}|/3$.*

Proof. Each $\mathcal{F}^S$-core contains some $\mathcal{F}$-core. Let $\mathbb{A}$ be an $\mathcal{F}^S$-core that contains some $\mathbb{C} \in \mathcal{C}$. We claim that $\mathbb{A}$ contains at least two $\mathcal{F}$-cores or $|\mathcal{C}| \geq 2$ and $h \in A$. Otherwise, $\mathbb{A} \in \mathcal{F}(\mathbb{C})$ (since $\mathbb{A}$ contains no $\mathcal{F}$-core distinct from $\mathbb{C}$) and $h \in A^*$ if $|\mathcal{C}| \geq 2$ (since h belongs to no boundary of a biset in $\mathcal{F}(\mathbb{C})$); hence $\mathbb{A} \in \mathcal{F}(\mathbb{C})$ if $|\mathcal{C}| = 1$ and $\mathbb{A} \in \mathcal{F}(h, \mathbb{C})$ if $|\mathcal{C}| \geq 2$, contradicting the definition of S. Now let p be the number of $\mathcal{F}^S$-cores that contain at least two $\mathcal{F}$-cores. The inner parts of the $\mathcal{F}^S$-cores are pairwise disjoint, since $\mathcal{F}^S$ is uncrossable; thus h belongs to at most one inner part of them, and every $\mathcal{F}$-core is contained in at most one $\mathcal{F}^S$-core. From this we get that $\nu(S) \leq \nu(\emptyset) - p$, and that $p \geq 1$ if $|\mathcal{C}| = 1$ and $p \geq \lceil (|\mathcal{C}| - 1)/2 \rceil$ if $|\mathcal{C}| \geq 2$. In both cases we have $p \geq |\mathcal{C}|/3$, and the lemma follows. □

In [Nutov 2012a] Lemma 2.4 was stated without the condition on h, which is not correct. To see this, consider the following example from [Fukunaga 2015]. Let $V = \{h, u_1, \ldots, u_n\}$ and $\mathcal{F} = \{\mathbb{C}_1, \ldots, \mathbb{C}_n\}$ where $\mathbb{C}_i = (u_i, \{u_i, h\})$. Let $S = \{hu_1, \ldots, hu_n\}$. Then S covers $\mathcal{F}(h, \mathbb{C}_i)$ for every i, but $\mathcal{F}^S = \mathcal{F}$ and hence $\nu(S) = \nu(\emptyset)$.

Theorem 2.3 is proved in Section 4. Here let us show that Theorem 2.3 implies Theorem 1.2. Consider the case of rooted demands. As in edge-costs case, at iteration $i = 1, \ldots, k$ we increase the (T, s)-connectivity from $i - 1$ to i. Iteration i starts with a subgraph $H_{i-1} = (V_{i-1}, E_{i-1})$ with all nodes in V_{i-1} already included in the solution graph, so we set their weight to be 0 at the beginning of the iteration. At iteration i we compute an edge-set J_i that covers the family $\mathcal{F}_{i-1}$ of tight bisets of H_{i-1}. Since nodes in V_{i-1} have zero weight, $w(\partial\mathbb{A}) = 0$ for all $\mathbb{A} \in \mathcal{F}_{i-1}$. We then use Theorem 2.1 to decompose the problem of covering $\mathcal{F}_{i-1}$ into $O(i)$ Node-Weighted Biset-Family Edge-Cover problems, each with an uncrossable biset family $\mathcal{F}$. Each such $\mathcal{F}$ is a subfamily of $\mathcal{F}_{i-1}$ and thus satisfies the condition in Theorem 2.3; furthermore, $\mathcal{F}$ has at most $|T|$ cores. Thus the algorithm from Theorem 2.3 produces an $O(\ln |T|)$ approximate solution. As we cover $O(k^2)$ uncrossable families, ratio $O(k^2 \ln |T|)$ for rooted demands follows.

In the case of element-connectivity demands the problem is also decomposed into a sequence of k augmentation problems. Let $D_i = \{st \in D : r_{st} \geq i\}$, $i = 1, \ldots k$. Iteration i starts with a subgraph $H_{i-1} = (V_{i-1}, E_{i-1})$ of G such that $\lambda^Q_{H_{i-1}}(s,t) \geq i - 1$ for all $st \in D_{i-1}$; all nodes in V_{i-1} are already included in the solution graph, so we set their weight to be 0 at the beginning of the iteration. During iteration i we compute an edge-set J_i such that the graph $H_i = H_{i-1} \cup J_i$ satisfies $\lambda^Q_{H_i}(s,t) \geq i$ for all $st \in D_i$. Given such H_i we call a biset $\mathbb{A}$ on V tight if there exists $st \in D_i$ such that $|A \cap \{s,t\}| = |A^* \cap \{s,t\}| = 1$, $\partial\mathbb{A} \subseteq Q$, and $\psi_{H_{i-1}}(\mathbb{A}) = i - 1$. Then J_i is a feasible solution to the above augmentation problem if and only if J_i covers the family of tight bisets. It is known that this family is uncrossable. Furthermore, for any tight biset $\mathbb{A}$ we have $\partial\mathbb{A} \subseteq V_{i-1}$, and thus $w(\partial\mathbb{A}) = 0$ at iteration i, since all nodes in V_{i-1} have zero weight. Consequently, by the same argument as in the case of rooted demands we get ratio $O(\ln |T|)$ for the augmentation problem, and overall ratio $O(k \ln |T|)$.

In what follows, we use the following property halo families c.f. [Nutov 2012a].

Lemma 2.5. *Let $\mathcal{F}$ be an uncrossable or a T-uncrossable biset family and let $\mathbb{A} \in \mathcal{F}(\mathbb{C})$ and $\mathbb{B} \in \mathcal{F}(\mathbb{C}')$ for some $\mathbb{C}, \mathbb{C}' \in \mathcal{C}(\mathcal{F})$. If $\mathbb{C} = \mathbb{C}'$ then $\mathbb{A} \cap \mathbb{B}, \mathbb{A} \cup \mathbb{B} \in \mathcal{F}(\mathbb{C})$ and thus $\mathcal{F}(\mathbb{C})$ has a unique maximal member $\mathbb{M}_{\mathbb{C}}$ – the union of all bisets in $\mathcal{F}(\mathbb{C})$. If $\mathbb{C} \neq \mathbb{C}'$ then $\mathbb{A} \setminus \mathbb{B} \in \mathcal{F}(\mathbb{C})$ and $\mathbb{B} \setminus \mathbb{A} \in \mathcal{F}(\mathbb{C}')$ if $\mathcal{F}$ is uncrossable, or if $\mathcal{F}$ is T-uncrossable and $\mathbb{A}, \mathbb{B}$ T-co-cross.*

3. A SHORT PROOF OF THEOREM 2.1

The proof of Theorem 2.1 relies on the following key lemma.

Lemma 3.1. *Let $\mathcal{F}$ be a T-uncrossable biset family and let $p = \min_{\mathbb{A} \in \mathcal{F}} |A \cap T|$. Then there exists a polynomial time algorithm that computes a partition Π of $\mathcal{C}(\mathcal{F})$ with at most $2\lfloor \gamma/p \rfloor + 1$ parts such that for each $\mathcal{C} \in \Pi$ the family $\bigcup_{\mathbb{C} \in \mathcal{C}} \mathcal{F}(\mathbb{C})$ is uncrossable, $\gamma = \max_{\mathbb{A}, \mathbb{B} \in \mathcal{F}} |\partial \mathbb{A} \cap B \cap T|$. Furthermore, if $p \geq \gamma + 1$ then $\mathcal{F}$ is uncrossable.*

Proof. If $p \geq \gamma + 1$ then any $\mathbb{A}, \mathbb{B} \in \mathcal{F}$ must T-intersect or T-co-cross; thus $\mathcal{F}$ is uncrossable in this case. We prove the first statement. For $\mathbb{C}_i \in \mathcal{C}(\mathcal{F})$ let $\mathbb{M}_i$ be the inclusionwise maximal biset in $\mathcal{F}(\mathbb{C}_i)$. Since $\mathcal{F}$ is T-uncrossable, and by Lemma 2.5, for any $\mathbb{A}_i \in \mathcal{F}(\mathbb{C}_i)$ and $\mathbb{A}_j \in \mathcal{F}(\mathbb{C}_j)$ we have:

(i) $\mathbb{A}_i, \mathbb{A}_j$ T-intersect if and only if $i = j$.

(ii) If $C_i \cap M_j^* \cap T$ and $C_j \cap M_i^* \cap T$ are both nonempty then $\mathbb{A}_i, \mathbb{A}_j$ T-co-cross.

Construct an auxiliary directed graph $\mathcal{J}$ that has node-set $\mathcal{C}(\mathcal{F})$ and arc-set $\{\mathbb{C}_i\mathbb{C}_j : C_i \cap T \subseteq \partial \mathbb{M}_j\}$. The indegree of every node in $\mathcal{J}$ is at most $\lfloor \gamma/p \rfloor$, by (i). This implies that every subgraph of the underlying graph of $\mathcal{J}$ has a node of degree $\leq 2\lfloor \gamma/p \rfloor$. Hence the underlying graph of $\mathcal{J}$ is $(2\lfloor \gamma/p \rfloor + 1)$-colorable, and such a coloring can be computed in polynomial time. Consequently, we can compute in polynomial time a partition Π of $\mathcal{C}(\mathcal{F})$ into at most $2\lfloor \gamma/p \rfloor + 1$ independent sets. For each independent set $\mathcal{C} \in \Pi$, the family $\bigcup_{\mathbb{C} \in \mathcal{C}} \mathcal{F}(\mathbb{C})$ is uncrossable, by (ii). $\square$

Now consider the following algorithm.

Algorithm 1: T-Uncrossable Biset-Family Edge-Cover$(G, c, \mathcal{F})$

$J \leftarrow \emptyset$
while $p := \min_{\mathbb{A} \in \mathcal{C}(\mathcal{F}^J)} |A \cap T| \leq \gamma$ **do**
  find a partition Π of $\mathcal{C}(\mathcal{F}^J)$ as in Lemma 3.1 with at most $2\lfloor \gamma/p \rfloor + 1$ parts
  for every $\mathcal{C} \in \Pi$ find an α-approximate cover $J_{\mathcal{C}}$ of $\bigcup_{\mathbb{C} \in \mathcal{C}} \mathcal{F}^J(\mathbb{C})$
  for every $\mathcal{C} \in \Pi$ do: $J \leftarrow J \cup J_{\mathcal{C}}$
find a β-approximate cover of J' of $\mathcal{F}^J$ and add J' to J
return J

Let p_i denote the value of p at the beginning of iteration i in the while loop. Initially, $p_1 \geq 1$. Note that for any T-uncrossable family $\mathcal{F}$, if an $\mathcal{F}$-core $\mathbb{C}$ and $\mathbb{A} \in \mathcal{F}$ T-intersect then $\mathbb{C} \subseteq \mathbb{A}$; this implies that if J covers all halo families of

$\mathcal{F}$ then every $\mathcal{F}^J$-core $\mathbb{A}$ contains at least two $\mathcal{F}$-cores. From this it follows that $p_i \geq 2p_{i-1}$ for all i. Thus the number of iterations in the while loop is at most $\ell - 1$, where ℓ is the least integer such that $2^\ell \geq \gamma + 1$. Consequently, the number of simple uncrossable biset families covered in the while loop is bounded by

$$\sum_{i=0}^{\ell-1} (2\lfloor \gamma/2^i \rfloor + 1) \leq \ell + 2\gamma \sum_{i=0}^{\ell-1} (1/2)^i = \ell + 4\gamma(1 - 1/2^\ell) \leq \ell + \frac{4\gamma^2}{\gamma+1} \ .$$

4. PROOF OF THEOREM 2.3

For an edge-set S let $V(S)$ denote the set of endnodes of the edges in S. Given a node-set R we say that edge-sets S_1 and S_2 are **R-disjoint** if $V(S_1) \cap V(S_2) \cap R = \emptyset$. A **spider** is a non-empty union S of paths (**legs** of S) that start at the same node h (the **head** of S) such that no two of them have other node in common. Note that if S has legs $S_1, \ldots S_d$ then $w(V(S)) = w_h + \sum_{i=1}^d w(V(S_i) \setminus \{h\})$. We extend this definition to an **R-spider** by requiring that the legs of S are only $(R \setminus \{h\})$-disjoint; if $w_v = 0$ for all $v \in V \setminus R$ then we still have $w(V(S)) = w_h + \sum_{i=1}^d w(V(S_i) \setminus \{h\})$. The following definition extends in a similar way "$\mathcal{F}$-spiders" from [Nutov 2012a].

Definition 4.1. *Let $\mathcal{F}$ be a biset family on V and let $R \subseteq V$. An $(\mathcal{F}, R)$***-spider** *is a pair $(S, \mathcal{C})$, where $\emptyset \neq \mathcal{C} \subseteq \mathcal{C}(\mathcal{F})$ is the set of* **cores hit by** *the $(\mathcal{F}, R)$-spider, and S is an edge-set such that: if $\mathcal{C} = \{\mathbb{C}\}$ then S covers $\mathcal{F}(\mathbb{C})$, and if $|\mathcal{C}| \geq 2$ then S is a union of (possibly empty) pairwise $(R \setminus \{h\})$-disjoint $\mathcal{F}(h, \mathbb{C})$-covers $\{S_{\mathbb{C}} : \mathbb{C} \in \mathcal{C}\}$ (***legs** *of the $(\mathcal{F}, R)$-spider) for some $h \in R$ (a* **head** *of the $(\mathcal{F}, R)$-spider).*

We often denote an $(\mathcal{F}, R)$-spider just by S, meaning that the set $\mathcal{C} = \mathcal{C}_S$ of cores hit by S is clear from the context. Later, we will prove the following.

Theorem 4.1. *For any $R \subseteq V$, any cover J of an uncrossable biset family $\mathcal{F}$ contains a family $\mathcal{S}$ of R-disjoint $(\mathcal{F}, R)$-spiders that collectively hit at least $\frac{2}{3}|\mathcal{C}(\mathcal{F})|$ distinct $\mathcal{F}$-cores.*

In [Nutov 2012a], Theorem 4.1 was stated and proved for the case $R = V$. The proof is correct, but the case $R = V$ is not sufficient for proving Theorem 2.3. However, the proof of Theorem 4.1 is a minor modification of the proof of the case $R = V$ in [Nutov 2012a].

Let us briefly describe how Theorem 4.1 and Lemma 2.4 imply Theorem 2.3. Let R be the set of nodes in V that belong to no boundary of a biset in $\bigcup_{\mathbb{C} \in \mathcal{C}} \mathcal{F}(\mathbb{C})$. Note that in the setting of Theorem 2.3 all nodes in $V \setminus R$ have weight zero.

Lemma 4.2. *Let $\mathcal{S}$ be a family of $(\mathcal{F}, R)$-spiders as in Theorem 4.1 for an optimal cover of an uncrossable biset family $\mathcal{F}$. There is an $(\mathcal{F}, R)$-spider $(S^*, \mathcal{C}^*)$ in $\mathcal{S}$ such that $\frac{w(V(S^*))}{|\mathcal{C}^*|/3} \leq \frac{9}{2} \cdot \frac{\mathsf{opt}}{\nu(\emptyset)}$.*

Proof. Let $\mathcal{C}_S$ denote the set of $\mathcal{F}$-cores hit by a spider $S \in \mathcal{S}$. Since the spiders in $\mathcal{S}$ are R-disjoint $\sum_{S \in \mathcal{S}} w(V(S)) \leq \mathsf{opt}$. Since the spiders in $\mathcal{S}$ hit at least $\frac{2}{3}\nu(\emptyset)$ distinct $\mathcal{F}$-cores $\sum_{S \in \mathcal{S}} |\mathcal{C}_S| \geq \frac{2}{3}\nu(\emptyset)$. Thus $\sum_{S \in \mathcal{S}} |\mathcal{C}_S|/3 \geq \frac{2}{9}\nu(\emptyset)$. Consequently, by an averaging argument, there is $(S^*, \mathcal{C}^*) \in \mathcal{S}$ as required. $\square$

LEMMA 4.3. *There exists a polynomial time algorithm that given an instance of* NODE-WEIGHTED BISET-FAMILY EDGE-COVER *with uncrossable $\mathcal{F}$ and $w_v = 0$ for all $v \in V \setminus R$, finds an edge-set $S \subseteq E$ such that $\frac{w(V(S))}{\nu(\emptyset)-\nu(S)} \leq 9 \cdot \frac{\mathsf{opt}}{\nu(\emptyset)}$.*

PROOF. Let $(S^*, \mathcal{C}^*)$ be an $(\mathcal{F}, R)$-spider as in Lemma 4.2. Assume that we know the number $d = |\mathcal{C}^*|$, and that if $d \geq 2$ then we know a head h of $(S^*, \mathcal{C}^*)$ and whether $\delta_{S^*}(h) \neq \emptyset$. There is a polynomial number of choices, so we can try all choices and return the best outcome (guessing d can be avoided, by a slightly more complicated algorithm). We note that given $\mathbb{C}$ and h the problem of finding a minimum node-weight cover of $\mathcal{F}(\mathbb{C})$ or of $\mathcal{F}(h, \mathbb{C})$ admits ratio 2. If $d = 1$ then for each $\mathbb{C} \in \mathcal{C}(\mathcal{F})$ we compute a 2-approximate $\mathcal{F}(\mathbb{C})$-cover and return the lightest one. If $d \geq 2$ then we temporarily set $w_h = 0$ if $\delta_{S^*}(h) \neq \emptyset$ or $w_h = \infty$ if $\delta_{S^*}(h) = \emptyset$; then for each $\mathbb{C} \in \mathcal{C}(\mathcal{F})$ we compute a 2-approximate $\mathcal{F}(h, \mathbb{C})$-cover $S_{\mathbb{C}}$ and return the union S of d lightest sets $S_{\mathbb{C}}$. Then $w(V(S)) \leq 2w(V(S^*))$, since legs of S^* are pairwise $(R \setminus \{h\})$-disjoint and since $w_v = 0$ for all $v \in V \setminus R$. Thus from Lemma 2.4 and our choice of S^* we get $\frac{w(V(S))}{\nu(\emptyset)-\nu(S)} \leq 2\frac{w(V(S^*))}{|\mathcal{C}^*|/3} \leq 9\frac{\mathsf{opt}}{\nu(\emptyset)}$. $\square$

The overall algorithm starts with $J = \emptyset$ and while $\nu(J) \geq 1$ repeatedly adds to J an edge-set S such that $\frac{w(V(S))}{\nu(J)-\nu(J \cup S)} \leq 9\frac{\mathsf{opt}}{\nu(\emptyset)}$. Such an algorithm has ratio $9(\ln \nu(\emptyset) + 1) = 9(\ln |\mathcal{C}(\mathcal{F})| + 1)$; if $\mathcal{F}$ is symmetric (namely, if $(V \setminus A^+, V \setminus A) \in \mathcal{F}$ whenever $\mathbb{A} \in \mathcal{F}$), then the ratio is in fact $9 \ln |\mathcal{C}(\mathcal{F})|$, see [Klein and Ravi 1995]. Furthermore, for biset families arising from SURVIVABLE NETWORK problems, the problem of finding a minimum node-weight cover of $\mathcal{F}(\mathbb{C})$ or of $\mathcal{F}(h, \mathbb{C})$ admits a polynomial time algorithm, and the ratio can be further reduced to $\frac{9}{2} \ln |\mathcal{C}(\mathcal{F})|$.

In the rest of this section we prove Theorem 4.1. A biset family is a **ring** if it is is closed under intersection and union. To prove Theorem 4.1 the only properties of $\mathcal{F}$ that we need are that the inner parts of the $\mathcal{F}$-cores are pairwise-disjoint and that $\mathcal{F}(\mathbb{C})$ is a ring for any $\mathcal{F}$-core $\mathbb{C}$ (this is so by Lemma 2.5); it is not hard to see that then $\mathcal{F}(h, \mathbb{C})$ is a ring for any $h \in V$. Note that any ring has a unique core. We need the following property of rings, c.f. [Nutov 2012a].

LEMMA 4.4. *Let J be an inclusionwise minimal cover of a ring $\mathcal{F}$ with core $\mathbb{C}$. Then there is an ordering $e_1, \ldots, e_q$ of J and bisets $\mathbb{C}_1 \subseteq \cdots \subseteq \mathbb{C}_q$ in $\mathcal{F}$ where $\mathbb{C}_1 = \mathbb{C}$, such that $\delta_J(\mathbb{C}_i) = \{e_i\}$, and if $e_i = v_i u_i$ where $u_i \in C_i$, then $\{e_1, \ldots, e_i\}$ covers $\mathcal{F}(h, \mathbb{C})$ for $h \in \{v_i, u_{i+1}\}$.*

The following definition extends the concept of R-spiders introduced earlier, and as we shall see it is also closely related to $(\mathcal{F}, R)$-spiders in Definition 4.1.

DEFINITION 4.2. *Let $\mathcal{P} = \{P_u : u \in U(\mathcal{P})\}$ be a family of simple directed paths on V with a set $U(\mathcal{P})$ of distinct ends, where each P_u ends at u, and let $R \subseteq V$. An R-spider S with head h is called a* $(\mathcal{P}, R)$**-spider** *if S is a union of subpaths (one may be of length 0) $\{S_u : u \in U\}$ of the paths in $\mathcal{P}$ for some $\emptyset \neq U \subseteq U(\mathcal{P})$ (the set of* **ends hit by** *S), where each S_u is an hu-subpath of P_u, such that if $|U| = 1$ then $S \in \mathcal{P}$ and if $|U| \geq 2$ then $h \in R$.*

LEMMA 4.5. *Let $\mathcal{P}$ be a family of simple directed paths on V with a set $U(\mathcal{P})$ of distinct endnodes and let $R \subseteq V$. Then there is a family $\mathcal{S}$ of pairwise R-disjoint $(\mathcal{P}, R)$-spiders that collectively hit at least $\frac{2}{3}|U(\mathcal{P})|$ distinct nodes in $U(\mathcal{P})$.*

PROOF. The case $R = V$ was proved in [Chuzhoy and Khanna 2008]. Hence there exists a family $\mathcal{S}$ of pairwise node-disjoint $(\mathcal{P}, V)$-spiders that hit at least $\frac{2}{3}|U(\mathcal{P})|$ nodes in $U(\mathcal{P})$. Since any $(\mathcal{P}, V)$-spider is also a $(\mathcal{P}, R)$-spider, this family satisfies all the requirements except of one: there can be $S \in \mathcal{S}$ that hits at least 2 ends with head $h \in V \setminus R$. We resolve this by an elementary construction that makes the paths in $\mathcal{P}$ to be $(V \setminus R)$-disjoint: for every path P and every $v \in V(P) \setminus R$ that is not an end of P, make a copy v_P of v and let P go through v_P instead of v. Note that this operation does not affect the ends of the paths, hence their number remains the same. Then the paths in $\mathcal{P}$ become pairwise $(V \setminus R)$-disjoint; hence the [Chuzhoy and Khanna 2008] result gives a family $\mathcal{S}$ as required, since now every $(\mathcal{P}, V)$-spider in $\mathcal{S}$ that hits at least 2 ends has head in R. The lemma follows since shrinking the nodes v_P back into v keeps the required properties: each spider remains a $(\mathcal{P}, R)$-spider since its legs remain pairwise $(R \setminus \{h\})$-disjoint, and any two $(\mathcal{P}, R)$-spiders remain R-disjoint. $\square$

Now we use Lemmas 4.4 and 4.5 to prove Theorem 4.1. The proof essentially coincides with the proof in [Nutov 2012a] for the case $R = V$. Define a family $\mathcal{P}$ of directed paths in a complete directed graph on V as follows. For every $\mathbb{C} \in \mathcal{C}(\mathcal{F})$ fix some inclusionwise-minimal $\mathcal{F}(\mathbb{C})$-cover $J_{\mathbb{C}} \subseteq J$. By lemma 2.5, $\mathcal{F}(\mathbb{C})$ is a ring. Let $e_1, \ldots, e_q$ be an ordering of $J_{\mathbb{C}}$ and $\mathbb{C}_1 \subset \cdots \subset \mathbb{C}_q$ bisets in $\mathcal{F}(\mathbb{C})$ as in Lemma 4.4, where $e_i = v_i u_i$ is as in the lemma. Obtain a directed path $P_{\mathbb{C}}$ by taking for each edge e_i the arc $v_i u_i$ and for every $i = q, \ldots, 2$ the **dummy arc** $u_i v_{i-1}$, if $u_i \neq v_{i-1}$; e.g., if $u_i \neq v_{i-1}$ for all i, then the node sequence of $P_{\mathbb{C}}$ is $(v_q, u_q, v_{q-1}, u_{q-1}, \ldots, v_1, u_1)$. Denote $u_{\mathbb{C}} = u_1$ and note that $u_{\mathbb{C}} \in C$.

Let $\mathcal{P} = \{P_{\mathbb{C}} : \mathbb{C} \in \mathcal{C}(\mathcal{F})\}$. Since the sets $\{C : \mathbb{C} \in \mathcal{C}(\mathcal{F})\}$ are pairwise-disjoint (by Lemma 2.5), any two paths in $\mathcal{P}$ have distinct ends. Hence Lemma 4.5 applies, and there exists a family $\tilde{\mathcal{S}}$ of node-disjoint $(\mathcal{P}, R)$-spiders that hits at least $\frac{2}{3}|U(\mathcal{P})|$ nodes in $U(\mathcal{P}) = \{u_{\mathbb{C}} : \mathbb{C} \in \mathcal{C}(\mathcal{F})\}$.

For any $(\mathcal{P}, R)$-spider $\tilde{S} \in \tilde{\mathcal{S}}$ and the set U of nodes in $U(\mathcal{P})$ hit by $\tilde{S}$ naturally corresponds a pair $(S, \mathcal{C})$, where $S \subseteq J$ is defined by the non-dummy arcs in $\tilde{S}$ and $\mathcal{C} = \{\mathbb{C} \in \mathcal{C}(\mathcal{F}) : u_{\mathbb{C}} \in U\}$. We show that $(S, \mathcal{C})$ is an $(\mathcal{F}, R)$-spider. For $\mathbb{C} \in \mathcal{C}$ let $\tilde{S}_{\mathbb{C}}$ be the $hu_{\mathbb{C}}$-path in $\tilde{S}$ and let $S_{\mathbb{C}}$ be the corresponding subset of S. If $\mathcal{C} = \{\mathbb{C}\}$ then $\tilde{S}_{\mathbb{C}} = P_{\mathbb{C}}$; thus in this case $S = S_{\mathbb{C}} = J_{\mathbb{C}}$, and since $J_{\mathbb{C}}$ covers $\mathcal{F}(\mathbb{C})$ the pair $(S, \{\mathbb{C}\})$ is an $(\mathcal{F}, R)$-spider. Assume that $|\mathcal{C}| \geq 2$ and let h be the head of $\tilde{S}$. Since $\tilde{S}$ is a $(\mathcal{P}, R)$-spider, the edge-sets $\{S_{\mathbb{C}} : \mathbb{C} \in \mathcal{C}\}$ are pairwise $(R \setminus \{h\})$-disjoint. By Lemma 4.4 and the construction, each $S_{\mathbb{C}}$ is an $\mathcal{F}(h, \mathbb{C})$-cover. Thus $(S, \mathcal{C})$ is an $(\mathcal{F}, R)$-spider in this case as well.

Now let $\mathcal{S}$ be the family $(\mathcal{F}, R)$-spiders corresponding to the $(\mathcal{P}, R)$-spiders in $\tilde{\mathcal{S}}$. Since the arc-sets in $\tilde{S}$ are node-disjoint, so are the edge-sets in $\mathcal{S}$. Since $\tilde{\mathcal{S}}$ hits at least $\frac{2}{3}|U(\mathcal{P})|$ nodes in $U(\mathcal{P})$ and since $|\mathcal{C}(\mathcal{F})| = |U(\mathcal{P})|$, $\mathcal{S}$ hits at least $\frac{2}{3}|\mathcal{C}(\mathcal{F})|$ cores in $\mathcal{C}(\mathcal{F})$. Thus $\mathcal{S}$ is as required, and the proof of Theorem 4.1 is complete.

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