Cactus tree type models for families of bisections of a set

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Abstract

A bisection of a set U is a partition of U into two nonempty parts. Two bisections are parallel if they collectively partition U into three parts. A natural model for a family of pairwise parallel bisections is a tree. Given a bisection family F and a family T of pairwise parallel bisections, we suggest decomposition/composition tools for modeling F based on T. We introduce *plant models* resulting from such compositions. As an application, we obtain a simple characterization of bisection families that can be modeled by a cactus-tree (i.e., a tree-of-edges-and-cycles) and its variants. We use this characterization to derive several related results.

1 Introduction and Notation

Several types of subset families of a set, e.g.: laminar, crossing, intersecting families, and rings, arise often in graph connectivity problems. For example, in a digraph, the minimum (s,t)-cuts form a ring family, while the globally minimum cuts form a crossing family. Gabow [11] suggests a representation for intersecting and crossing families, which size is quadratic in the size of the groundset, with several applications to digraph connectivity problems. The representations mentioned can be applied to similar cut families of (undirected) graphs, via

*Up to 1990, E. A. Dinic, Moscow. This research was supported partly by the Fund for the Promotion of Research at the Technion, Israel.

†Part of this work was done as a part of the author's Ph.D. thesis at the Dept. of Mathematics, Technion, Haifa, Israel.

replacing every edge by the pair of anti-parallel arcs between its end-vertices. However, for such families of edge-cuts of undirected graphs, it is possible to find more simple and compact representations, often of a linear size. This enables, among others, to develop efficient algorithms for several problems; see, for example, [2, 3, 9, 8, 7, 12, 13, 14].

While an arc-cut of a digraph is defined by a subset of its vertices, a natural object to represent an edge-cut of a graph is an (unordered) *bisection* of the vertex set, i.e., its partition into two nonempty parts. Motivated by edge connectivity problems, we develop tools for modeling certain bisection families of a set. These tools generalize the techniques used in [6] for modeling the family of minimum edge cuts of a weighted graph. Our results form a basis for models and algorithms of [7] and, partly, of [8].

Two distinct bisections of a groundset U are called *parallel* if they collectively partition U into three nonempty parts. It is known that any parallel bisection family T (i.e., of mutually parallel bisections) has the naturally defined bijective *tree model* $(\mathcal{T}, \tau)$. $\mathcal{T}$ is a tree on a node set $\mathcal{U}$ with $|\mathcal{U}| \leq 2|U| - 3$, and $\tau : U \rightarrow \mathcal{U}$ is a mapping such that: for every bisection $\{\mathcal{X}, \mathcal{U} \setminus \mathcal{X}\}$ of $\mathcal{U}$ determined by deletion of an edge from $\mathcal{T}$, $\{\tau^{-1}(\mathcal{X}), \tau^{-1}(\mathcal{U} \setminus \mathcal{X})\}$ is a bisection in T, and every bisection in T arises in this way exactly once (see Fig. 1a,b).

A set of edges $C \subseteq E$ is an (*edge*)-*cut* of G if there exists a bisection $B = \{X, \bar{X}\}$ of V such that C is the set of edges in E having endnodes in distinct parts of B; in this case we say that B (or that X) *defines* C (*in* G). Note that in a connected graph, every cut is defined by a unique bisection, thus there is a bijective correspondence between edge-cuts of G and the bisections of its vertex set. Henceforth, unless stated otherwise, "cut" means edge cut. The weight (resp., cardinality) of a cut is defined as the sum of weights (resp., the number) of its edges. Cuts consisting of one edge are referred as *bridges*. Let $\lambda(G)$ denote the minimum weight (or the cardinality, if no weights are given) of a cut of G. Note that there can be $\Omega(n^2)$ minimum cuts, and thus the space required to list all of them can be $\Omega(n^3)$ if every cut is described as a bisection of V, or as a set of edges.

It is not hard to prove that in an integrally weighted graph with $\lambda(G)$ odd, the minimum cuts are pairwise parallel. In [6] it was shown that the minimum cuts of an arbitrary graph with nonnegative weights on its edges can be represented in a compact and simple way by the cactus-tree model $(\mathcal{H}, \varphi)$. $\mathcal{H}$ is a *cactus-tree*, that is a connected graph such that every its block is an edge or a cycle, and φ is a mapping from V to the node set of $\mathcal{H}$. The (inclusion) minimal cuts of a cactus-tree have a simple structure: any such cut is either a bridge, or a pair of edges belonging to the same cycle. For every minimal cut of $\mathcal{H}$ determined by a bisection $\{\mathcal{X}, \mathcal{U} \setminus \mathcal{X}\}$ of the node set $\mathcal{U}$ of $\mathcal{H}$, $\{\tau^{-1}(\mathcal{X}), \tau^{-1}(\mathcal{U} \setminus \mathcal{X})\}$ is a bisection that determines a minimum cut of G, and every minimum cut of G arises in this way. Under certain minimality assumptions, $\mathcal{H}$ is unique, $|\mathcal{U}| = O(|V|)$, and this representation is almost bijective, see [15]. For applications of the cactus tree model see, for example, [14, 13, 2, 8, 9]. Two naturally arising questions are:

(i) Is there a simple characterization of bisection families that can be modeled by a cactus tree (similar to that of families that can be modeled by a tree)?

(ii) Can the tools used in [6] be extended to model near minimum cuts of graphs?

In this paper, we answer the first question by giving such a characterization of the bisection families modeled by a cactus-tree as families with a simple pairwise relation on its bisections. We note that a half year after our result was published in [7, Theorem 4.2] a similar characterization was announced by Fleiner and Jordán, see [10, Theorem 3]. We also note, that this particular result can be deduced from works of Cunningham and Edmonds, see [4, 5]. Here we suggest more intuitive approach, and provide a short and direct proof.

In addition, generalizing modeling by a cactus-tree, we suggest a certain type of models which we call plant models. These models help proving the above characterization and are applied for modeling minimum and minimum+1 cuts of a multigraph in [7]; a similar approach is applied for near minimum cuts in [3].

Here are some definitions and notation used in the paper. Let U be a finite groundset. For a proper subset X of U, $\bar{X} = U \setminus X$ denotes the complementary set of X, and $B(X)$ the bisection $\{X, \bar{X}\}$. Two distinct bisections $B(X), B(Y)$ *cross* (or $B(X)$ *crosses* $B(Y)$) if they are not parallel, i.e., if all the four *corner sets* $X \cap Y, X \cap \bar{Y}, \bar{X} \cap Y, \bar{X} \cap \bar{Y}$ are nonempty. A bisection defined by a nonempty corner set, is called a *corner bisection.* If $B(X), B(Y)$ cross, the bisection $B(X \Delta Y)$ is called their *diagonal bisection*, where $X \Delta Y = (X \setminus Y) \cup (Y \setminus X)$ denotes the symmetric difference between X and Y. It is easy to see that if $B(X), B(Y)$ cross, then any bisection that crosses their corner bisection crosses at least one of $B(X), B(Y)$.

We say that a bisection $\{X, \bar{X}\}$ *divides* a subset S of U if both $X \cap S$ and $\bar{X} \cap S$ are nonempty. The equivalence classes of the relation "$x, y \in U$, $\{x, y\}$ is not divided by any bisection in F" are called F*-atoms.*

When U is the vertex set of a connected graph G, similar definition are used for cuts considering them as bisections of U. To *shrink* a subset S of U means to identify all elements in S to a single new element s. When U is the vertex set of a graph G, shrinking implies also deletion of all edges with both endpoints in S, and, for every edge with one endpoint in S, replacing this endpoint by s; an edge of a new graph is identified with its corresponding edge of G. For a given partition of U, the *quotient set (graph)* is defined to be the result of shrinking each its part; the corresponding mapping is called the *quotient mapping*.

We say that a graph is a *tree of graphs* of a certain type if every its block is of this type.

This paper is organized as follows. Section 2 describes plant modeling. In Section 3, we introduce crossing and semicrossing bisection families and show that they are exactly the families that can be modeled by a cactus-tree and by a tree-of-edges-and-cycles-and-complete-graphs, respectively. Section 4 shows several applications.

2 Plant models

Let U be a groundset, and $\psi : U \to \mathcal{U}$ a mapping. For a bisection $B = \{\mathcal{X}, \bar{\mathcal{X}}\}$ of $\mathcal{U}$ we define $\psi^{-1}(\mathcal{B}) = \{\psi^{-1}(\mathcal{X}), \psi^{-1}(\bar{\mathcal{X}})\}$, and for a family $\mathcal{F}$ of bisections of $\mathcal{U}$ let us denote $\psi^{-1}(\mathcal{F}) = \{\psi^{-1}(\mathcal{B}) : \mathcal{B} \in \mathcal{F}\}$. Given a family F of bisections of U, we say that the triple $(\mathcal{U}, \psi, \mathcal{F})$ is a *model for* F if $\psi^{-1}(\mathcal{F}) = F$. When $\mathcal{U}$ is the vertex set of a connected graph $\mathcal{G}$, $\mathcal{F}$ becomes a family of cuts of $\mathcal{G}$. In this case, we call $(\mathcal{G}, \psi, \mathcal{F})$ a *cut model* and the members of $\mathcal{F}$ *modeling cuts*.

We need a formal definition of the tree model to continue. Let T be a parallel family. We say that two sets X_1, X_2 with $B(X_1), B(X_2) \in T$ are *neighbors* if there is no bisection $\{Y, \bar{Y}\} \in T$ such that both $X_1 \subset Y$ and $X_2 \subset \bar{Y}$, where "$\subset$" means proper inclusion. It is not hard to see that the neighbor relation is an equivalence; we call its equivalence classes *neighbor groups*. The nodes of the tree $\mathcal{T}$ are the neighbor groups. For every bisection $\{X, \bar{X}\} \in T$, we put a structural edge $\varepsilon_X = \varepsilon_{\bar{X}}$ between the neighbor group containing X and the neighbor group containing $\bar{X}$. For a node $\mathcal{N}$ of $\mathcal{T}$, $\tau^{-1}(\mathcal{N})$ is defined to be $\bigcap_{X \in \mathcal{N}} \bar{X}$. Such subsets, if nonempty, are just the T-atoms. The described construction implies the following statement (proof is omitted).

Theorem 2.1 *The model $(\mathcal{T}, \tau,$ bridges of $\mathcal{T})$ as above is a bijective cut model for T.*

Let F be an arbitrary bisection family, and assume that a parallel family T consisting of bisections that do not cross any bisection in F is given (T and F might not be disjoint). In this section, we consider a decomposition of F w.r.t. such family T; provided that for each part of this decomposition a cut model is given, we show how those models can be "implanted" into $\mathcal{T}$ to obtain a cut model for the entire F. We decompose F relatively to the nodes of $\mathcal{T}$ as follows: a bisection $B \in F$ is assigned to a node $\mathcal{N}$ if it does not divide any set belonging to the neighbor group $\mathcal{N}$ (see fig. 1a). Denote by $F_{\mathcal{N}}$ the bisections in F assigned to a node $\mathcal{N}$. It is easy to see that every bisection in $F \setminus T$ is assigned to exactly

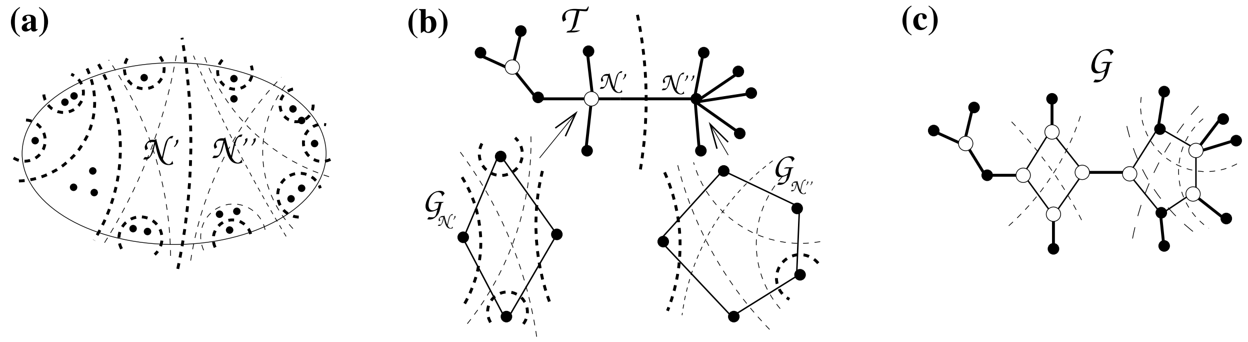

Figure 1: (a) decomposition w.r.t. a parallel family (shown by heavily dashed lines); (b) local models and implanting; (c) plant model.

one node of $\mathcal{T}$, while any bisection in $F \cap T$ is assigned to exactly two nodes (connected by the edge modeling this bisection).

Let us define a *local model at* $\mathcal{N}$ as a cut model $(\mathcal{G}_\mathcal{N}, \psi_\mathcal{N}, \mathcal{F}_\mathcal{N})$ for $F_\mathcal{N}$ with the following additional property: for every set $X \in \mathcal{N}$, $\psi_\mathcal{N}(X)$ is a single node of $\mathcal{G}_\mathcal{N}$ (see Fig 1b). Suppose that for every family $F_\mathcal{N}$ with $\mathcal{F}_\mathcal{N} \neq \emptyset$ a local model $(\mathcal{G}_\mathcal{N}, \psi_\mathcal{N}, \mathcal{F}_\mathcal{N})$ is given. Under this assumption, we can show how to construct a cut model for F. Let $\mathcal{N}$ be a node of $\mathcal{T}$. For $X \in \mathcal{N}$, let ε_X be the edge modeling the bisection $B(X)$. *Implanting* $\mathcal{G}_\mathcal{N}$ instead of $\mathcal{N}$ is defined as follows: every edge ε_X, $X \in \mathcal{N}$, is disattached from its endnode $\mathcal{N}$, and attached to the node $\psi_\mathcal{N}(X)$; then $\mathcal{N}$ is deleted (see Fig. 1c). Let $\mathcal{G} = (\mathcal{U}, \mathcal{E})$ be the graph resulting from implanting all the local models. The model mapping $\psi : U \to \mathcal{U}$ takes the elements in $\tau^{-1}(\mathcal{N})$ as the mapping $\psi_\mathcal{N}$ does in the case $\mathcal{N}$ has undergo implanting, and takes all of them to $\mathcal{N}$ otherwise. Since the sets $\tau^{-1}(\mathcal{N})$ with $\tau^{-1}(\mathcal{N}) \neq \emptyset$ partition U, ψ is well defined. Note that by the definition of implanting, $\mathcal{E}$ contains all the edges of the local models; the additional edges in $\mathcal{E}$ are those of $\mathcal{T}$. Assuming that a cut is represented by an edge set, the modeling family $\mathcal{F}$ of cuts of $\mathcal{G}$ is naturally derived from those of the implanted local models: $\mathcal{F}$ is the union of the families $\mathcal{F}_\mathcal{N}$ (see Fig. 1b,c).

Theorem 2.2 *Let F and T be bisection families such that no bisection in T crosses a bisection in F. Let $\mathcal{T}$ be a tree model for T, and suppose that for every node $\mathcal{N}$ of $\mathcal{T}$ with $F_\mathcal{N} \neq \emptyset$ a local model $(\mathcal{G}_\mathcal{N}, \psi_\mathcal{N}, \mathcal{F}_\mathcal{N})$ is given. Then the model $(\mathcal{G}, \psi, \mathcal{F})$ obtained by implanting these local models into $\mathcal{T}$ is a cut model for F.*

Proof: For simplicity, let us assume that every node $\mathcal{N}$ of $\mathcal{T}$ has undergo implanting; if for some family $F_\mathcal{N}$ no local model is given (so $F_\mathcal{N} = \emptyset$), then implanting the "trivial model", where $\mathcal{G}_\mathcal{N}$ consists of a single node, gives the same plant model.

It is sufficient to show that for any node $\mathcal{N}$ of $\mathcal{T}$ and any cut $\mathcal{C}$ of $\mathcal{G}_\mathcal{N}$ there exists a

proper subset $\mathcal{Y}$ of $\mathcal{U}$ such that:
(A) $\mathcal{Y}$ defines $\mathcal{C}$ in $\mathcal{G}$, and
(B) $\psi^{-1}(\mathcal{Y}) = \psi_{\mathcal{N}}^{-1}(\mathcal{Y}_{\mathcal{N}})$, where $\mathcal{Y}_{\mathcal{N}}$ defines $\mathcal{C}$ in $\mathcal{G}_{\mathcal{N}}$.

Focus on a specific node $\mathcal{M}$ of $\mathcal{T}$. Then, by the construction of $\mathcal{G}$, every (connected) component of $\mathcal{G} - \mathcal{G}_{\mathcal{M}}$ is attached by a bridge to a unique node of $\mathcal{G}_{\mathcal{M}}$. Thus every proper subset $\mathcal{Y}_{\mathcal{M}}$ of the nodes of $\mathcal{G}_{\mathcal{M}}$ naturally extends to a proper subset $\mathcal{Y}$ of nodes of $\mathcal{G}$, where $\mathcal{Y}$ is obtained by adding to $\mathcal{Y}_{\mathcal{M}}$ every components of $\mathcal{G} - \mathcal{G}_{\mathcal{M}}$ that is attached to a node in $\mathcal{Y}_{\mathcal{M}}$. This implies that the cut defined by $\mathcal{Y}$ in $\mathcal{G}$ coincides with the cut defined by $\mathcal{Y}_{\mathcal{M}}$ in $\mathcal{G}_{\mathcal{M}}$. This shows that (A) holds.

To prove that (B) holds for $\mathcal{Y}$ as above, it is sufficient to show that for any $u \in U$ either
(B1) $\psi(u) = \psi_{\mathcal{M}}(u)$ (if $u \in \tau^{-1}(\mathcal{M})$), or
(B2) $\psi(u)$ belongs to a component of $\mathcal{G} - \mathcal{G}_{\mathcal{M}}$ attached to $\psi_{\mathcal{M}}(u)$.
Then, by the definition of $\mathcal{Y}$, we would have $\psi_{\mathcal{M}}^{-1}(\mathcal{Y}_{\mathcal{M}}) = \psi^{-1}(\mathcal{Y})$, as required.

Note that by the construction of $\mathcal{G}$ and the definition of ψ we have:

Claim: Let $\mathcal{N}$ be an arbitrary node of $\mathcal{T}$. Then:
(i) For any $u \in U$ holds: $\psi(u) = \psi_{\mathcal{N}}(u)$ if, and only if, $u \in \tau^{-1}(\mathcal{N})$; in particular, $\psi(u)$ is a node of $\mathcal{G}_{\mathcal{N}}$ if, and only if, $u \in \tau^{-1}(\mathcal{N})$.
(ii) If $\mathcal{N}$ belongs to a component of $\mathcal{T} - \mathcal{M}$ attached to $\mathcal{M}$ by ε, then all nodes of $\mathcal{G}_{\mathcal{N}}$ belong to the component of $\mathcal{G} - \mathcal{G}_{\mathcal{M}}$ that is attached to $\mathcal{G}_{\mathcal{M}}$ by ε.

Let $u \in U$ be arbitrary. Claim (i) above implies immediately that if $u \in \tau^{-1}(\mathcal{M})$ then $\psi(u) = \psi_{\mathcal{M}}(u)$; thus (B1) holds. Henceforth assume that $\psi(u) \notin \tau^{-1}(\mathcal{M})$. Then $u \in X$ for some $X \in \mathcal{M}$. Consider the component $\mathcal{X}$ attached to $\mathcal{G}_{\mathcal{M}}$ by the bridge ε_X. By the construction of the plant model and the definition of a local model, the endnode of ε_X in $\mathcal{G}_{\mathcal{M}}$ is $\psi_{\mathcal{M}}(X) = \psi_{\mathcal{M}}(u)$. We claim that $\psi(u) \in \mathcal{X}$. By the definition of the tree model $\mathcal{T}$, $\mathcal{N} = \tau(u)$ belongs to the component of $\mathcal{T} - \mathcal{M}$ that is connected to $\mathcal{M}$ by ε_X. By Claim (i) above, $\psi(u)$ is a node of $\mathcal{G}_{\mathcal{N}}$. By Claim (ii) above, all the nodes of $\mathcal{G}_{\mathcal{N}}$ belong to $\mathcal{X}$. In particular, $\psi(u) \in \mathcal{X}$, and (B2) holds as well. □

Theorem 2.2 provides decomposition/composition tools for modeling bisection or cut families. The reader may wonder what is the advantage of the construction above, as the totality of the models $\mathcal{G}_{\mathcal{N}}$ already represents the entire family F. We can see two. First, note that the space required for the totality of the model mappings $\psi_{\mathcal{N}}$ can be $\Omega(|U|^2)$, while the model mapping of their plant model needs only $O(|U|)$ space. Second, for graphs, several connectivity problems can be reduced to a corresponding problem for a cut model, see, for example, [2, 3, 7, 9, 13, 14].

3 Crossing Families

Henceforth, we say that a family of bisections is modeled by a graph $\mathcal{G}$, implicitly meaning that the model mapping is given, and that the modeling family is the family of (inclusion) minimal cuts of $\mathcal{G}$.

Definition 3.1 *A bisection family F is called* crossing *if for any pair of crossing bisections in F holds:*

(C1) *their four corner bisections are in F;*

(C2) *the diagonal bisection is not in F.*

When only (C1) *holds always, the family is called* semicrossing.

Theorem 3.1 *A bisection family F is modeled by*

(i) *a cactus tree if and only if F is a crossing family;*

(ii) *a tree of edges, cycles, and complete graphs if and only if F is a semicrossing family.*

Proof: *The "only if" part.*

Let $B(X), B(Y)$ be a pair of crossing bisections in F. Let $\mathcal{C}_X, \mathcal{C}_Y$ be an arbitrary pair of cuts modeling $B(X), B(Y)$, respectively. Note that the following holds for an arbitrary cut model: if $\mathcal{C}_X, \mathcal{C}_Y$ are cuts modeling $B(X), B(Y)$ respectively, and if $B(X), B(Y)$ cross then: (a) $\mathcal{C}_X, \mathcal{C}_Y$ cross, and (b) the corner cuts and the diagonal cut of $\mathcal{C}_X, \mathcal{C}_Y$ model the corner bisections and the diagonal bisection of $B(X), B(Y)$, respectively. It is well known, and easy to prove, that a cut of a graph is minimal if and only if it is a cut of a (unique) block of the graph. Since two cuts that do not divide the same block are parallel, $\mathcal{C}_X, \mathcal{C}_Y$ are cuts of the same block $\mathcal{L}$, which must be either a cycle, or a complete graph, on at least 4 nodes.

Assume that $\mathcal{L}$ is a cycle. Then $\mathcal{C}_X$ consists of two edges of $\mathcal{L}$. Deletion of $\mathcal{C}_X$ partitions the remaining edges of $\mathcal{L}$ into two parts. Since $\mathcal{C}_X, \mathcal{C}_Y$ cross, $\mathcal{C}_Y$ consists of two edges belonging to distinct parts. One can easily verify that for $\mathcal{C}_X, \mathcal{C}_Y$: (1) their corner cuts are the four cut pairs $\{\{\varepsilon', \varepsilon''\} : \varepsilon' \in \mathcal{C}_X, \varepsilon'' \in \mathcal{C}_Y\}$, and thus each one of them belongs to the modeling family, and (2) their diagonal cut is $\mathcal{C}_X \cup \mathcal{C}_Y$, and thus it does not belong to the modeling family. Thus for $B(X), B(Y)$, each one of their corner cuts belongs to F, and their diagonal cut does not belong to F.

If $\mathcal{L}$ is a complete graph on at least 4 nodes, one can use similar arguments to show that the corner and the diagonal bisections of $B(X), B(Y)$ belong to F. In this case, F is semicrossing, but cannot be a crossing family.

The "if" part.

Let T be the family of all bisections in F that do not cross any other bisection in F. Clearly, T is parallel. Consider the family $F_{\mathcal{N}}$ of all bisections in F assigned to some node $\mathcal{N}$ of the tree model $\mathcal{T}$ for T. Let $\hat{F}_{\mathcal{N}} = F_{\mathcal{N}} \setminus \{B(X) : X \in \mathcal{N}\}$. By Theorem 2.2, to finish the proof of our Theorem, it is sufficient to show that, if $\hat{F}_{\mathcal{N}} \neq \emptyset$, there exists a local model at $\mathcal{N}$ which is a cycle, for the case (i), and a cycle or a complete graph, for the case (ii).

Let $\mathcal{N}$ be a node with $\hat{F}_{\mathcal{N}} \neq \emptyset$. In what follows, note the following: (a) if $B(X), B(Y) \in F_{\mathcal{N}}$ cross, then each one the conditions (C1),(C2) holds for $B(X), B(Y)$ and F if and only if it holds for $\psi_{\mathcal{N}}(B(X)), \psi_{\mathcal{N}}(B(Y))$ and $F_{\mathcal{N}}$; (b) for any $B(X) \in \hat{F}_{\mathcal{N}}$, there is $B(Y) \in \hat{F}_{\mathcal{N}}$ crossing $B(X)$.

Among all the subfamilies of $F_{\mathcal{N}}$ for which exists a quotient cut model of the type as in the Theorem, let F' be one with the maximal number t' of atoms. Observe, that F' is well defined and $t' \geq 4$. This is since there exists at least one model of this kind, as follows. By the assumption, $\hat{F}_{\mathcal{N}} \neq \emptyset$, so there is a pair of bisections in $\hat{F}_{\mathcal{N}}$ that cross. Moreover, in the case $F_{\mathcal{N}}$ is not a crossing family, among crossing pairs in $\hat{F}_{\mathcal{N}}$ there is at least one that does not fulfill (C2), i.e., its diagonal bisection belongs to $F_{\mathcal{N}}$. The family of the desired type is this pair of crossing bisections, their corner bisections, and, if $F_{\mathcal{N}}$ is not a crossing family, their diagonal bisection.

Let $(\mathcal{G}', \psi', \mathcal{F}')$ be a model as above for F'. First, let us prove that if $\mathcal{C}$ is a cut of $\mathcal{G}'$ such that ${\psi'}^{-1}(\mathcal{C}) \in F_{\mathcal{N}}$, then $\mathcal{C} \in \mathcal{F}'$. If $F_{\mathcal{N}}$ is not a crossing family, then $\mathcal{F}'$ is the complete set of cuts (bisections), and this is evident. If $F_{\mathcal{N}}$ is a crossing family, then $\mathcal{G}'$ is a cycle of length at least 4, and $\mathcal{F}'$ consists of its edge pairs. Assume, in negation, that there is a cut of $\mathcal{G}'$ such that ${\psi'}^{-1}(\mathcal{C}) \in F_{\mathcal{N}}$, but $\mathcal{C} \notin \mathcal{F}'$. Let $\mathcal{B}(\mathcal{X})$ be the bisection of the node set of the cycle $\mathcal{G}'$ that defines C'. Let us assign the black color to the nodes in $\mathcal{X}$ and the white color to the nodes in $\bar{\mathcal{X}}$. By the assumption, there are at least two inclusion-maximal black paths in the cycle $\mathcal{G}$. Let $\mathcal{A}_1$, $\mathcal{A}_0$, and $\mathcal{A}_2$ denote a triple of consequent black, white, and black inclusion-maximal paths of $\mathcal{G}'$. Observe that $\mathcal{B}(\mathcal{A}_1 \cup \mathcal{A}_2)$ either coincides with $\mathcal{B}$, or is one of the corner bisections of the pair $\mathcal{B}$ and $\mathcal{B}(\mathcal{A}_1 \cup \mathcal{A}_0 \cup \mathcal{A}_2)$, and thus belongs to $\mathcal{F}'$. The two paths $\mathcal{A}_1 \cup \mathcal{A}_0$ and $\mathcal{A}_2 \cup \mathcal{A}_0$ define bisections in $\mathcal{F}'$; they are crossing, and their diagonal bisection $\mathcal{B}(\mathcal{A}_1 \cup \mathcal{A}_2)$ belongs to $\mathcal{F}'$, a contradiction to (C2).

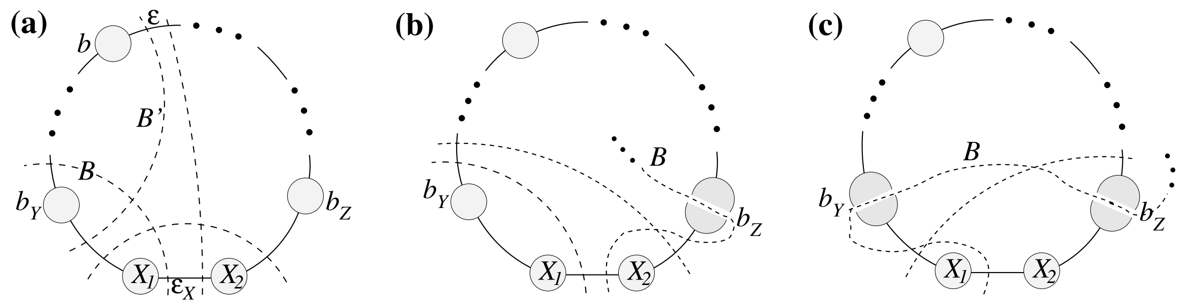

Figure 2: Illustration for the proof for a crossing family.

In order to complete the proof, we will show that the number t of $F_{\mathcal{N}}$-atoms equals t'. Let us denote a node of $\mathcal{G}'$ by b_V if $\psi'^{-1}(b_V) = V$. Notice that, for each such V, $B(V) \in F_{\mathcal{N}}$. Assume, in negation, that $t' < t$. Then, there is a node b_X of $\mathcal{G}'$, such that $B(X) \notin \mathcal{N}$. Thus, by assumptions of the Lemma, there is a bisection $B \in \hat{F}_{\mathcal{N}}$ crossing with $B(X)$; let B divide X into X_1 and X_2. By (C1), $B(X_1), B(X_2) \in \mathcal{F}_{\mathcal{N}}$.

Let us examine first the case (i) of our Theorem, when $F_{\mathcal{N}}$ is a crossing family and $\mathcal{G}'$ is a cycle on at least 4 nodes. Let b_Y and b_Z be the two nodes adjacent to b_X in $\mathcal{G}'$ (see Fig. 2).

Case 1: $B = B(Y \cup X_1)$ (see Fig. 2a). Let us define a new quotient model $(\mathcal{G}'', \psi'', \mathcal{F}'')$ as follows. The cycle $\mathcal{G}''$ with $t' + 1$ nodes is obtained from $\mathcal{G}'$ by splitting b_X into two new nodes b_{X_1} and b_{X_2}, connected by a structural edge ε_X, and replacing the edges (b_X, b_Y) and (b_X, b_Z) by the edges (b_{X_1}, b_Y) and (b_{X_2}, b_Z), respectively. The mapping ψ'' is the natural refinement of ψ' defined by $(\psi'')^{-1}(b_{X_i}) = X_i$, $i = 1, 2$, and $\mathcal{F}''$ is the set of minimal cuts of $\mathcal{G}''$. To obtain a contradiction, it suffices to show that $(\psi'')^{-1}(\mathcal{F}'') \subseteq F_{\mathcal{N}}$.

Let $\mathcal{C} \in \mathcal{F}''$. If $\varepsilon_X \notin \mathcal{C}$, the claim follows from the assumption on $\mathcal{G}'$. Assume now that $\varepsilon_X \in \mathcal{C}$, and let ε be the other edge of $\mathcal{C}$. Note that $B(X_1), B(X_2) \in F_{\mathcal{N}}$, since they are corner bisections of $B, B(X)$. Thus, if ε is incident to one of b_{X_1}, b_{X_2}, the claim follows. Else, let b be the endnode of ε which is closer to b_{X_1}. Let $\mathcal{C}' = \{(b_{X_1}, b_Y), \varepsilon\}$, and let $B' = (\psi'')^{-1}(\mathcal{C}')$. Note that $B' \in F_{\mathcal{N}}$. Clearly, B, B' cross. One of their corner bisections is $(\psi'')^{-1}(\mathcal{C})$; by (C1), it belongs to $F_{\mathcal{N}}$.

Case 2: *B does not divide at least one of Y and Z* (see Fig. 2b). We assume, w.l.o.g., that $B = B(V)$ where $Y, X_1 \subset V$. By the definition of $\mathcal{G}'$, the bisection $B(X \cup Y)$ belongs to $\mathcal{F}'$. If we are not in Case 1, B forms a crossing pair with $B(X \cup Y)$. Then, by (C1), $B(Y \cup X_1)$ belongs to $\mathcal{F}'$ as a corner bisection of their square, and we arrive at Case 1.

Case 3: *B divides both Y and Z.* (see Fig. 2c). Bisections B and $B(X \cup Z)$ form a crossing pair. The corner set of their square that contains X_2 defines a bisection which

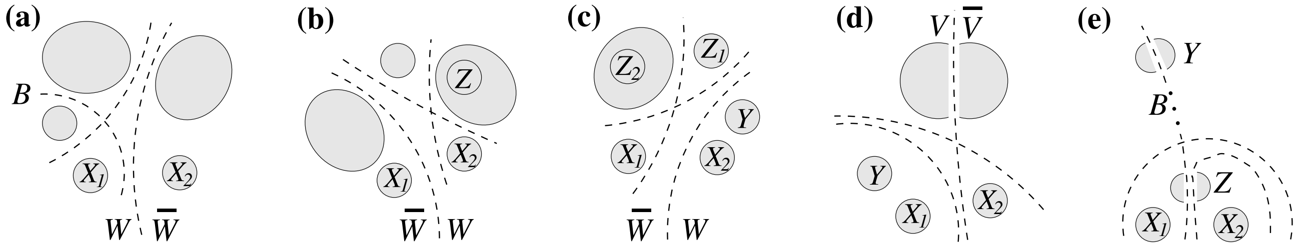

Figure 3: Illustration for the proof for a semicrossing family which is not crossing.

belongs to $\hat{F}$, crosses B, and does not divide Y. We arrive at Case 2.

This is the end of examining (i).

Let us now examine the case (ii), when $\mathcal{G}'$ is a complete graph on at least 4 nodes (see Fig. 3).

Case 1: $B = B(Y \cup X_1)$*, where* $b_Y \neq b_X$ *is a node of* $\mathcal{G}$ (see Fig. 3a,b,c). Let us define a new quotient model $(\mathcal{G}'', \psi'', \mathcal{F}'')$ similarly to the proof of the case (i): we form the complete graph $\mathcal{G}''$ with $t' + 1$ nodes by splitting the node b_X into two nodes b_{X_1} and b_{X_2}, adjusting the set of structural edges, and defining ψ'' and $\mathcal{F}''$ appropriately. Let $B(W)$ be a bisection of U such that for any node b_V of $\mathcal{G}''$, W does not divide V. To obtain a contradiction, it is sufficient to show that if $Y \subseteq W$, W properly contains exactly one of X_1, X_2, and $B(W) \neq B$, then $B(W) \in F_{\mathcal{N}}$.

Let us consider, first, the case $X_1 \subset W$ (see Fig. 3a). Let $W' = W \setminus (X_1 \cup Y)$. Since $B(W) \neq B$, $W' \neq \emptyset$. The bisections B and $B(Y \cup W')$ are crossing and both belong to $F_{\mathcal{N}}$: the former by the assumption of the case, and the latter since for any node b_V of $\mathcal{G}''$, W does not divide V. Since $B(W)$ is their corner bisection, by (C1) it belongs to $F_{\mathcal{N}}$.

Second, let us consider the case $X_2 \subset W$ (see Fig. 3b). Assume, first, that $W \neq X_2 \cup Y$, i.e., there exists a node b_Z of $\mathcal{G}'$ such that $Z \subset W$ (see Fig. 3c). Let us consider crossing bisections $B(W \setminus X_2)$ and $B(W \setminus Y)$. Note that both of them belong to $F_{\mathcal{N}}$. Thus $B(W)$ belongs to $F_{\mathcal{N}}$ as their corner bisection. It remains to show that $B(X_2 \cup Y)$ also belongs to $F_{\mathcal{N}}$. Since $\mathcal{G}''$ has at least five nodes, there are two nodes b_{Z_1} and b_{Z_2} distinct from b_{X_1}, b_{X_2}, and b_Y. Using similar considerations as previously, we deduce that the crossing bisections $B(X \cup Y)$ and $B(Z_1 \cup X_2 \cup Y)$ belong to $F_{\mathcal{N}}$; by (C1), their corner bisection $B(X_2 \cup Y)$ belongs to $F_{\mathcal{N}}$ as well.

Case 2: *There is a node* b_Y *of* $\mathcal{G}'$*, such that* B *does not divide* Y (see Fig. 3d). We assume, w.l.o.g., that $B = B(V)$, where $Y, X_1 \subset V$. By the definition of $\mathcal{G}'$, the bisection $B(X \cup Y)$ belongs to $\mathcal{F}'$. If we are not in Case 1, then $B, B(X \cup Y)$ is a crossing pair. Then,

$B(Y \cup X_1)$ belongs to $F_{\mathcal{N}}$ as their corner bisection, and we arrive at Case 1.

Case 3: *for every node b_V of $\mathcal{G}'$ B divides V.* (see Fig. 3e). Let $b_Y, b_Z \neq b_X$ be nodes of $\mathcal{G}'$. Bisections B and $B(X \cup Z)$ crossing, and the corner set of their square that contains X_2 defines a bisection which belongs to $F_{\mathcal{N}}$, crosses $B(X)$, and does not divide Y. We arrive at Case 2.

This is the end of examining (ii). □

Remark: Note that the number of nodes in the model for a bisection family F as in Theorem 3.1 is linear in $|U|$. Hence the total size of the model (that is, the number of nodes and edges in $\mathcal{G}$) for any crossing family is $O(|U|)$. The model for a semicrossing family can be implemented also in linear space, if we eliminate real implanting of complete graphs. Indeed, without loss of information, it is sufficient to mark, by a special sign, every node such that a complete graph should be implanted instead of it.

4 Related results and applications

4.1 Cactus and circumference models

It is easy to show that the minimum cuts of a graph obey conditions (C1),(C2), see [6]. Thus the main result of [6] — the cactus tree model for the family of minimum cuts of a graph — is an immediate consequence from Theorem 3.1(i).

Let a *circumference model* for a family F of bisections of U be a cut model where the structural graph is a cycle and every modeling cut is minimal (in geometrical terms: a mapping of U into a circumference such that every bisection in F is defined by a division of the circumference into exactly two arcs). Circumference Theorem [6], which establishes a circumference model for the minimum cuts of a graph, can be generalized as follows.

Theorem 4.1 (Circumference Theorem) *A family F of bisections has a circumference model if and only if it is a subfamily of a crossing family.*

Proof: *The "only if" part.* Let a circumference model for a family F of bisections of U be given. Let ψ be the model mapping, and let $\mathcal{F}'$ be the family of all minimal cuts of the structural graph. Then $F' = \psi^{-1}(\mathcal{F}')$ is a crossing family (by Theorem 3.1(i)), and, clearly, $F \subseteq F'$.

The "if" part. Clearly, it is sufficient to prove it for any crossing family. By Theorem 3.1, it is sufficient to prove that the set of minimal cuts of any cactus tree has a circumference model. Such a proof coincides, in fact, with the proof of Circumference Theorem [6]. □

Corollary 4.2 *Let F be a crossing family of bisections of a set U. Then $|F| \leq |U|(|U|-1)/2$.*

4.2 Ring families

Now, let G be a weighted graph and S be its distinguished vertex subset. Among all the cuts of G dividing S, let us consider those of the minimum weight; we call them *minimum S-cuts* and denote their weight by $\lambda(S)$. D. Naor and J. Westbrook stated independently that the family of bisections of S induced by the minimum S-cuts is modeled by a cactus tree, generalizing [6]. However, neither D. Naor, nor J. Westbrook presented any proof for their statement. We do not see for it any immediate generalization of the proof given in [6] and present a proof based on our characterization of families modeled by a cactus tree. In fact, we prove a more general result, stated in pure bisection terms.

Let S be a distinguished subset of a groundset U. For $X \subset U$, let $B_S(X) = \{X \cap S, \bar{X} \cap S\}$. If $B_S(X)$ is a bisection of S we say that $B(X)$ is an *S-bisection.* For a family F of S-bisections, let $F_S = \{B_S(X) : B(X) \in F\}$ denote the family of bisections of S *induced* by the bisections in F. Two bisections of U are called *S-crossing* if they partition S into 4 parts. Clearly, validity of the condition (C1) for every pair of S-crossing bisections in F implies that F_S is a semicrossing family. However, validity of conditions (C1),(C2) for every such pair, does not imply, in general, that F_S is a crossing family. Let us call F an *S-ring family* if for every $x, y \in S$ holds: if $B(X), B(Y) \in F$ is a pair of $\{x,y\}$-bisections with $x \in X$ and $y \in Y$ then $B(\bar{X} \cap Y), B(\bar{Y} \cap X) \in F$. It is immediate to show that if F is an S-ring family, then F_S satisfies the condition (C1), i.e., F_S is a semicrossing family; the following Theorem gives a necessary and sufficient condition for F_S to be a crossing family.

Theorem 4.3 *Let S be a subset of U, and F an S-ring family of bisections of U. Then F_S is a crossing family if and only if every pair of S-crossing bisections in F satisfies (C2).*

Proof: As stated above, F_S is a semicrossing family. We will show that (C2) holds for every crossing pair in F_S if and only if it holds for every pair of S-crossing bisections in F, and thus obtain validity of the statement via Theorem 3.1 (i).

Note that if $B(X), B(Y)$ is a pair of S-crossing bisections, then $B_S(X \Delta Y)$ is the diagonal bisection of the pair $B_S(X), B_S(Y)$. Thus, if (C2) does not hold for an S-crossing pair $B(X), B(Y) \in F$, it does not hold for $B_S(X), B_S(Y)$ as well.

Let us now show that if (C2) holds for every pair of S-crossing bisections in F, then it holds for every crossing pair in F_S. Consider an arbitrary crossing pair $B_S(X), B_S(Y) \in F_S$, where $B(X), B(Y) \in F$. Let $Z = X \Delta Y$, and let $B(W) \in F$ be such that $W_S = Z_S$. We claim that $B(W \setminus \bar{Z}) \in F$. Let $W' = W \setminus (X \cap Y)$. The bisections $B(W)$ and $B(X \cap Y)$ S-cross and both belong to F. One of their corner bisections is $B(W')$; thus it belongs to F. Let now $W'' = W' \setminus (\bar{X} \cap \bar{Y})$. By a similar argument, $B(W'') \in F$. Since $\bar{Z}$ is a disjoint union of $X \cap Y$ and $\bar{X} \cap \bar{Y}$, the claim follows.

Now, assume that (C2) holds for every pair of S-crossing bisections in F, and let X, Y, Z be as above. By the assumption, $B(Z) \notin F$. If condition (C2) does not hold for the pair $B_S(X), B_S(Y)$ and F_S, then there is $B(W) \in F$ such that $W_S = Z_S$. Let $R = W \setminus \bar{Z}$. By the claim above, $B(R) \in F$, and thus also $B(\bar{R} \setminus Z) \in F$. A contradiction to the assumption that $B(Z) \notin F$, since $\bar{R} \setminus Z = \bar{Z}$. □

The following simple Lemma shows that for a graph G and its vertex subset S, the family of minimum S-cuts satisfies the conditions of Theorem 4.3.

Lemma 4.4 *Let $G = (U, E)$ be a graph, and let S be a subset of U. Then the family of minimum S-cuts is an S-ring family, and the diagonal bisection of every pair of S-crossing minimum S-cuts is not a minimum S-cut.*

Proof: Let $x, y \in S$ and let $B(X), B(Y)$ be a pair of minimum S-cuts separating x from y, where $x \in X$ and $y \in Y$. For a subset Z, let us denote by $w(Z)$ the weight of the cut $B(Z)$. A straightforward computation shows that $w(\bar{X} \cap Y) + w(\bar{Y} \cap X) \leq w(X) + w(Y) = 2\lambda(S)$. Since clearly both $B(\bar{X} \cap Y), B(\bar{Y} \cap X)$ are S-cuts, it must be that $w(\bar{X} \cap Y) = w(\bar{Y} \cap X) = \lambda(S)$, which implies that the family of minimum S-cuts is an S-ring family.

Let now $B(X), B(Y)$ be a pair of bisections S-crossing cuts. The corner cuts of $B(X), B(Y)$ divide S, by the assumption. Simple computations, similar to those for proving Crossing Lemma [6], imply that the weight of $B(X \Delta Y)$ is $2\lambda(S) \neq \lambda(S)$, which finishes the proof. □

Corollary 4.5 *For a graph and its vertex subset S, the family of bisections of S induced by the minimum S-cuts is a crossing family.*

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