On Integrality, Stability and Composition of Dicycle Packings and Covers

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Abstract

Given a digraph D, the *minimum integral dicycle cover problem* (known also as *the minimum feedback arc set problem*) is to find a minimum set of arcs that intersects every dicycle; the *maximum integral dicycle packing problem* is to find a maximum set of pairwise arc disjoint dicycles. These two problems are NP-complete.

Assume D has a 2-vertex cut. We show how to derive a minimum dicycle cover (a maximum dicycle packing) for D, by composing certain covers (packings) of the corresponding pieces. The composition of the covers is simple and was partially considered in the literature before. The main contribution of this paper is to the packing problem. Let τ be the value of a minimum integral dicycle cover, and ν^* (ν) the value of a maximum (integral) dicycle packing. We show that if $\tau = \nu$ then a simple composition, similar to that of the covers, is valid for the packing problem. We use these compositions to extend an $O(n^3)$ (resp., $O(n^4)$)algorithm for finding a minimum integral dicycle cover (resp., packing) from planar digraphs to $K_{3,3}$-free digraphs (i.e., digraphs not containing any subdivision of $K_{3,3}$).

However, if $\tau \neq \nu$, then such a simple composition for the packing problem is not valid. We show, that if the pieces satisfy, what we call, the stability property, then a

simple composition does work. We prove that if $\nu = \nu^*$ holds for each piece, then the stability property holds as well. Further, we use the stability property to show that if $\nu = \nu^*$ holds for each piece, then $\nu = \nu^*$ holds for D as well.

Key words: Graph, integral dicycle cover, integral dicycle packing, 3-connected component, composition, $K_{3,3}$-free digraph.

1 Introduction

Given a weighted digraph, with $w : A \to Z_+$ a weight function, the **minimum integral dicycle cover problem** (known also as the **minimum feedback arc set problem**) is to find a minimum weight set of arcs that intersects every dicycle. This problem has attracted a lot of attention and has many applications in areas such as economics, mathematical psychology, scheduling problems, and VLSI [14, 23, 19, 6]. The minimum integral dicycle cover problem is NP-complete in general [15]. The problem is polynomially solvable for some classes of digraphs such as planar digraphs [21, 22, 16, 7, 11, 8], reducible flow graphs [26] and weakly acyclic digraphs [10, 25]. For the general case, the best known approximation ratio is $O(\min\{\log \tau^* \log\log \tau^*, \log |V| \log\log |V|\})$, where τ^* is the value of an optimal solution of the linear relaxation of the problem [27, 5].

The dual problem to the minimum dicycle cover problem is the **maximum dicycle packing problem**. Its integral version is the **maximum integral dicycle packing problem**, which is to find a maximum family of dicycles, such that each arc a occurs in at most $w(a)$ members of the family. As mentioned in [11], the maximum integral dicycle packing problem is NP-complete. The problem is polynomially solvable for the class of planar digraphs [21, 22, 16, 7, 11].

Let τ be the value of a minimum integral dicycle cover and ν that of a maximum integral dicycle packing. It is well known that in general $\tau \neq \nu$. It was proved by Lucchesi and Younger [22] that $\tau = \nu$ holds for planar digraphs (for a simpler proof see [20]). This result was extended to $K_{3,3}$-free digraphs (i.e., digraphs not containing any subdivision of $K_{3,3}$) [3], see also [25]. For other extensions of results and algorithms from planar graphs to $K_{3,3}$-free ones, see, for example, [1, 2, 9, 18, 24, 28].

For more complete exposition and for the comparison with the packing composition, we present our results concerning the cover problem although these results are somewhat marginal. Given a digraph D that has a separation pair, we show how to derive a minimum integral dicycle cover in D by composing certain covers of its pieces. Our composition of minimum integral dicycle covers uses some ideas from [3] and from [25]. We combine our composition with a theorem of Wagner [29] (see also [12]) which states that each 3-connected component of a $K_{3,3}$-free graph is either planar or K_5, to extend an $O(n^3)$ algorithm of Gabow [8] for finding a minimum integral dicycle cover in planar graphs to $K_{3,3}$-free digraphs. Our algorithm is a combinatorial one, and is more efficient than the one presented in [10] which is based on the ellipsoid method, the one presented in [25] which reduces the problem to a polynomial size linear program, and the one presented in [3] which is based on a decomposition of an appropriate polytope.

The main results of this paper concern the packing problem. We show that a simple composition, similar to that of the covers, is in general not valid for the packing problem. However, if each piece satisfies the stability property (defined below), then a simple composition does work. A packing consisting of dicycles and dipaths from a vertex u to a vertex v in a digraph D is called a $\boldsymbol{(u,v)}$**-packing**. A digraph D is said to be $\boldsymbol{\{u,v\}}$**-stable** if there is a maximum integral dicycle packing in D that can be extended to maximum (u,v)- and (v,u)-packings. Let $\{u,v\}$ be a separation pair of D. We show that if each piece of D is $\{u,v\}$-stable, then a maximum integral dicycle packing of D can be derived by gluing together certain packings of the pieces. We further show that if $\nu = \nu^*$ holds for each such a piece, then D is $\{u,v\}$-stable. We also observe that if $\tau = \nu$ holds for each piece of D, then there is a simpler proof for the validity of the above composition. Combining this with Wagner theorem [29] and Lucchesi-Younger theorem [22] we extend an $O(n^4)$ algorithm of Frank [7] for finding a maximum integral dicycle packing in planar digraphs to $K_{3,3}$-free digraphs. To the best of our knowledge, no polynomial time algorithm for finding a maximum integral dicycle packing in $K_{3,3}$-free digraphs was known before.

In [3] it was shown that if $\tau = \nu$ holds for each piece, then $\tau = \nu$ holds for D as well. Here we prove a refinement of the above result. Namely, we show that if $\nu = \nu^*$ holds for each piece of D, then $\nu = \nu^*$ holds for D as well. We note that our result is not obtained from the results in [3].

We turn now to introduce some preliminaries and definitions. Let A be a finite set, w a non-negative *integral* weight function on the elements of A, and let $\mathcal{A}$ be a family of subsets of A. If no weight function is given, then we assume that $w(a) = 1$ for all $a \in A$. For $A' \subseteq A$ let $w(A') = \sum\{w(a) : a \in A'\}$ denote the weight of A'.

An $\boldsymbol{\mathcal{A}}$**-cover** (resp., **integral** $\boldsymbol{\mathcal{A}}$**-cover**) is a function $f : A \longrightarrow [0,1]$ (resp., $f : A \longrightarrow \{0,1\}$) such that for every $A' \in \mathcal{A}$, $\sum\{f(a) : a \in A'\} \geq 1$. The **value of an** $\boldsymbol{\mathcal{A}}$**-cover** f is $\sum\{f(a)w(a) : a \in A\}$. A **minimum** (resp., **integral**) $\boldsymbol{\mathcal{A}}$**-cover** is one with the smallest value among all (resp., integral) $\mathcal{A}$-covers. An $\boldsymbol{\mathcal{A}}$**-packing** (resp., **integral** $\boldsymbol{\mathcal{A}}$**-packing**) is a function $h : \mathcal{A}' \longrightarrow R_+$ (resp., $h : \mathcal{A}' \longrightarrow Z_+,$), where $\mathcal{A}' \subseteq \mathcal{A}$, such that for every $a \in A$, $\sum\{h(A') : a \in A', A' \in \mathcal{A}'\} \leq w(a)$. The definition of h can be extended to $\mathcal{A}$ by defining

$h(A') = 0$ for every $A' \in \mathcal{A} - \mathcal{A}'$. The **value of an $\mathcal{A}$-packing** h is $\sum\{h(A') : A' \in \mathcal{A}'\}$. A **maximum** (resp., **integral**) **$\mathcal{A}$-packing** is one with the largest value among all (resp., integral) $\mathcal{A}$-packings. In what follows we identify an integral $\{0,1\}$ $\mathcal{A}$-cover f with its corresponding subset $F = \{a \in A : f(a) = 1\}$ of A, and an integral $\{0,1\}$ $\mathcal{A}$-packing h with its corresponding subfamily $\mathcal{H} = \{A' \in \mathcal{A} : h(A') = 1\}$ of $\mathcal{A}$.

Let A be the arc set of a loopless digraph $D = (V, A)$ on a vertex set V, with $|V| = n$ and $|A| = m$, and let w be a non-negative weight function on A. For $u, v \in V$, a **(u,v)-arc** (resp., **(u,v)-dipath**) is an arc (resp., dipath) from u to v. If $\mathcal{A}$ is the set of all the dicycles in D, then an $\mathcal{A}$-cover (resp., $\mathcal{A}$-packing) is a **dicycle cover** (resp., **dicycle packing**). If $\mathcal{A}$ is the set of all the dicycles and the (u,v)-dipaths in D, then an $\mathcal{A}$-cover (resp., $\mathcal{A}$-packing) is called a **(u,v)-cover** (resp., **(u,v)-packing**). We denote by τ^*_D (resp., τ_D) the value of a minimum (resp., integral) dicycle cover in D, and by ν^*_D (resp., ν_D) the value of a maximum (resp., integral) dicycle packing in D. For (u,v), an ordered pair of distinct vertices of D, τ^{*uv}_D (resp., τ^{uv}_D) denotes the value of a minimum (resp., integral) (u,v)-cover in D, and ν^{*uv}_D (resp., ν^{uv}_D) denotes the value of a maximum (resp., integral) (u,v)-packing in D. Also, let $\Delta\tau^{uv}_D = \tau^{uv}_D - \tau^{vu}_D$, and $\Delta\nu^{uv}_D = \nu^{uv}_D - \nu^{vu}_D$.

A complete graph on n nodes is denoted by K_n. A **$K_{3,3}$-free graph** is a graph not containing $K_{3,3}$ or any of its subdivision as a subgraph. A subset of k vertices of G is a **k-vertex-cut** of G if its deletion results in a disconnected graph. We refer also to a 2-vertex cut as a **separation pair**. A graph is **k-vertex-connected** if it is K_{k+1} or if it has at least $k+2$ vertices and has no vertex-cut of cardinality less then k. Similar definitions are used for digraphs by referring to their underlying graphs, e.g., we say that D is k-connected if its underlying graph is k-connected.

In the rest of the paper we assume that D is 2-connected. The results extend naturally to disconnected or 1-connected digraphs. For simplicity, some of the combinatorial results in this paper are proved for the unweighted case. However, by elementary constructions (i.e., replacing a weighted (u,v)-arc a by $w(a)$ unweighted (u,v)-arcs and vice versa) these proofs can be generalized to any integral non-negative weight function w. These constructions are used only in the proofs of the combinatorial results and they do not affect the running times of our algorithms.

Let $D = (V, A)$ be a 2-connected digraph and $\{u, v\}$ a separation pair of D. Let $D_1 = (V_1, A_1)$ and $D_2 = (V_2, A_2)$ be two **$\{u,v\}$-separation digraphs** obtained from D by **separating** D at $\{u, v\}$. Namely, D_1, D_2 are subgraphs of D such that

$$\begin{aligned} V &= V_1 \cup V_2\ , \quad V_1 \cap V_2 = \{u, v\} \\ A &= A_1 \cup A_2\ , \quad A_1 \cap A_2 = \emptyset\ , \quad |A_1|\ ,\ |A_2| \geq 2 \end{aligned}$$

and there is no arc with one end in $V_1 \setminus \{u,v\}$ and the other in $V_2 \setminus \{u,v\}$. Two $\{u,v\}$-separation digraphs can be **merged** by identifying the copies of u and v in these digraphs and the resulting digraph is termed a **$\{u,v\}$-merged digraph** (of D_1 and D_2). In what follows, we identify an arc of a separation digraph with its corresponding arc in D (similarly we do for vertices, dicycles and dipaths etc.). A dicycle C in D is said to be **separated** by D_1 and D_2 if both $C \cap A_1$ and $C \cap A_2$ are not empty. Let $\mathcal{H}_1$ be an integral (u,v)-packing in D_1, and $\mathcal{H}_2$ an integral (v,u)-packing in D_2. Then, the union of the dicycles in $\mathcal{H}_1$ and $\mathcal{H}_2$, together with a maximum set of separated dicycles, formed by the (u,v)-dipaths of $\mathcal{H}_1$ and the (v,u)-dipaths of $\mathcal{H}_2$ is called a **merged dicycle packing**. For weighted digraphs, a merged dicycle packing h can be recursively obtained from h_1 and h_2 in the following way. For $i = 1,2$, let P_i be a dipath for which $h_i(P_i)$ is maximum, and let $t = \min\{h_1(P_1), h_2(P_2)\}$. If $t = 0$, then h is the natural extension of h_1 and h_2. Otherwise, let $C = P_1 \cup P_2$, set $h(C) = t$, reduce $h_i(P_i)$ by t, and recursively apply the same process. For $i = 1,2$, let c_i (resp., p_i) be the number of dicycles (resp., dipaths) in D_i with $h_i > 0$, and let $c = c_1 + c_2$ and $p = p_1 + p_2$. Note that given h_1, h_2, such a merging can be executed in $c + p \log p$ time, using basic data structures (e.g., a heap), and in the resulting packing, the number of dicycles with $h(C) > 0$ is at most $c + p$.

Let $D = (V, A)$ be a digraph and $u, v \in V$. A digraph $\tilde{D}$ is said to be a **(u,v)-augmented digraph** of D if it is obtained from D by adding any number of (u,v)-arcs; $\tilde{D}$ is a **$\{u,v\}$-augmented digraph** if it is a (u,v)- or a (v,u)- augmented digraph. Assume D has a decomposition into k 3-connected components. The 3-connected components of D are obtained by initially separating D at one of its separation pairs, then adding to each separation digraph thus obtained a new arc between the vertices of this separation pair, and recursively applying the same process to the remaining digraphs. Remove the newly added arcs from each 3-connected component to obtain $D_1 = (V_1, E_1), \ldots, D_k = (V_k, E_k)$. A **multiple–augmented digraph** of D_i is a digraph obtained from D_i by adding any number of parallel (u,v)- or (v,u)-arcs (or a single (u,v)- or (v,u)-arc of the corresponding integral weight) for any separation pair $\{u,v\}$ contained in V_i.

This paper is organized as follows. Section 2 studies composition of dicycle covers as well as efficient algorithms for the dicycle cover and the dicycle packing problems in $K_{3,3}$-free digraphs. Section 3 is devoted to stability and composition of dicycle packings. Section 4 is dedicated to the connection between integrality and stability. Section 5 contains conclusions and open problems.

2 Integral Dicycle Covers and $K_{3,3}$-Free Digraphs

Herein we show a simple method for constructing minimum integral dicycle covers by gluing together the covers in the separation digraphs. We note that Lemma 2.2 and Theorem 2.5 to follow, were stated in [3] in terms of acyclic subdigraphs and appropriate polytopes. However, for more a complete exposition, for the results concerning $K_{3,3}$-free digraphs, and for the comparison with the packing problem, we present these results here as well in combinatorial graph terms.

Observation 2.1 *Let D_1 and D_2 be two $\{u,v\}$-separation digraphs of D. An arc set F is an integral dicycle cover in D if and only if F is a union of integral (u,v)- or (v,u)-covers in D_1 and in D_2. Thus, $\tau_D = \min\{\tau^{uv}_{D_1} + \tau^{uv}_{D_2}, \tau^{vu}_{D_1} + \tau^{vu}_{D_2}\}$.*

Lemma 2.2 below, which is the basis of our recursive algorithm for the cover problem, can be derived from Theorem 4.2 in [3]. Recall that $\Delta\tau^{uv}_D := \tau^{uv}_D - \tau^{vu}_D$ indicates the additional weight (which might be negative) of a minimum integral (u,v)-cover over a minimum integral (v,u)-cover. Note that if $\Delta\tau^{uv}_{D_1} \geq 0$ and one wishes to construct a minimum integral dicycle cover in D as a union of minimum integral (u,v)-covers rather than (v,u)-covers, then one has an overhead of $\Delta\tau^{uv}_{D_1}$ over $\tau^{vu}_{D_1}$ relative to D_1. We transfer this overhead to D_2 by adding a (u,v)-arc of a weight $\Delta\tau^{uv}_{D_1}$ to D_2. In fact, to have an indicator for the case $\Delta\tau^{uv}_{D_1} = 0$, we augment D_2 by a (u,v)-arc of the weight $\Delta\tau^{uv}_{D_1} + 1$ and a single (v,u)-arc. Also, note that one can calculate a minimum (u,v)-cover in D by calculating a minimum integral dicycle cover in a (v,u)-augmented digraph of D obtained by adding a large number, say $|A|+1$, of (v,u)-arcs (or a (v,u)-arc of weight $w(A)+1$).

Lemma 2.2 ([3]) *Let D_1 and D_2 be $\{u,v\}$-separation digraphs of D, and assume that $\Delta\tau^{uv}_{D_1} \geq 0$. Let $\tilde{D}_2$ be obtained from D_2 by adding a set $\tilde{A}$ of two arcs: a (u,v)-arc of weight $\Delta\tau^{uv}_{D_1} + 1$ and a (v,u)-arc of weight 1. Let $\tilde{F}_2$ be a minimum integral dicycle cover in $\tilde{D}_2$. If $\tilde{F}_2$ contains the newly added (v,u)-arc, then $\tilde{F}_2 \setminus \tilde{A}$ together with any minimum integral (v,u)-cover of D_1 is a minimum integral dicycle cover in D; otherwise, $\tilde{F}_2 \setminus \tilde{A}$ together with any minimum integral (u,v)-cover of D_1 is the one.*

Lemma 2.2 suggests the following theorem.

Theorem 2.3 *Let $D_1,\ldots,D_k$ be a decomposition of D into k 3-connected components. If there is a polynomial time algorithm for finding a minimum integral dicycle cover in each multiple augmented digraph of D_i, $i=1,\ldots,k$, then there is a polynomial time algorithm for finding a minimum integral dicycle cover in D.*

Proof: The algorithm contains two phases. First, decompose D into its 3-connected components using Hopcroft and Tarjan $O(n+m)$ algorithm [13]. Second, find recursively minimum integral dicycle covers in the modified pieces by using the following method. Let D_1 be a 3-connected component from the above decomposition containing exactly one separation pair, say $\{u, v\}$, and let D_2 be the remaining separation digraph of D. Theorem 2.3 is clearly true if D is 3-connected, thus we assume that such D_1 exists. Assume $\tau_{D_1}^{uv} \geq \tau_{D_1}^{vu}$ (the case $\tau_{D_1}^{uv} < \tau_{D_1}^{vu}$ can be treated similarly), and set $\tilde{D}_2$ and $\tilde{F}_2$ to be as in Lemma 2.2. Now, using Lemma 2.2, one can solve recursively the minimum integral dicycle cover problem in $\tilde{D}_2$. Clearly, one has to calculate $O(k)$ times a minimum integral dicycle cover, each time in a digraph of a total weight of at most $2w(D) + 1$, and the proof is complete. We note that in the theorem, one can use strongly polynomial time algorithm instead of polynomial time algorithm and the theorem will be still valid. □

We turn now to consider $K_{3,3}$-free graphs. In particular, if $\tau = \nu$ holds for every multiple augmented 3-connected component, then we can make use of our results on the cover problem to obtain similar results for the packing one. Then we use the obtained results to derive an $O(n^3)$ algorithm for the minimum integral dicycle cover problem, and an $O(n^4)$ algorithm for the maximum integral dicycle packing problem in $K_{3,3}$-free digraphs. However, as demonstrated in Section 3, if the requirement $\tau = \nu$ does not hold, then a simple composition, similar to the cover one, does not necessarily hold for the packing problem. Theorem 4.7 in Section 4 is a generalization of Lemma 2.4 below. As the proof of Lemma 2.4 is much simpler than the proof of the theorem, we have chosen to present here a sketch of the proof.

Lemma 2.4 *Let D_1 and D_2 be two $\{u, v\}$-separation digraphs of D. Assume that $\tau = \nu$ holds for any $\{u, v\}$-augmented digraph of D_1 and of D_2, and that $\Delta = \Delta\nu_{D_1}^{uv} \geq 0$. Let $\tilde{D}_1$ (resp., $\tilde{D}_2$) be the digraph obtained from D_1 (resp., D_2) by adding Δ (v, u)-arcs (resp., (u, v)-arcs). Let $\tilde{\mathcal{H}}_i$ be a maximum integral dicycle packing in $\tilde{D}_i$, and let $\mathcal{H}_i$ be its counterpart packing of dicycles and dipaths in D_i, $i = 1, 2$. Then any merged dicycle packing $\mathcal{H}$ of $\mathcal{H}_1$ and $\mathcal{H}_2$, is a maximum integral dicycle packing in D.*

Sketch of the proof: Let $\tilde{D}$ be the $\{u, v\}$-merged digraph of $\tilde{D}_1$ and $\tilde{D}_2$, and let $\tilde{\mathcal{H}}$ be a merged dicycle packing of $\tilde{\mathcal{H}}_1$ and $\tilde{\mathcal{H}}_2$. Clearly, $|\mathcal{H}| \geq |\tilde{\mathcal{H}}| - \Delta$ and $|\tilde{\mathcal{H}}| \geq \nu_{\tilde{D}_1} + \nu_{\tilde{D}_2}$. This together with the assumption that $\nu_{\tilde{D}_i} = \tau_{\tilde{D}_i}$, $i = 1, 2$, imply that

$$|\mathcal{H}| \geq \tau_{\tilde{D}_1} + \tau_{\tilde{D}_2} - \Delta.$$

Now, the equations

$$\tau_D = \tau_{\tilde{D}} - \Delta = \tau_{\tilde{D}_1} + \tau_{\tilde{D}_2} - \Delta$$

follow from Observation 2.1 stated for $\tilde{D}$. Also, the fact that $\Delta = \tau^{uv}_{\tilde{D}_1} - \tau^{vu}_{\tilde{D}_1} \geq 0$ implies that $\tau^{uv}_{\tilde{D}_1} = \tau^{vu}_{\tilde{D}_1}$. In addition, one can verify that for any digraph D and any pair of vertices $\{u, v\}$, $\tau_D = \min\{\tau^{uv}_D, \tau^{vu}_D\}$. Thus, $\tau_{\tilde{D}} = \min\{\tau^{uv}_{\tilde{D}_1} + \tau^{uv}_{\tilde{D}_2}, \tau^{vu}_{\tilde{D}_1} + \tau^{vu}_{\tilde{D}_2}\} = \tau_{\tilde{D}_1} + \min\{\tau^{uv}_{\tilde{D}_2}, \tau^{vu}_{\tilde{D}_2}\} = \tau_{\tilde{D}_1} + \tau_{\tilde{D}_2}$. Hence, $|\mathcal{H}| \geq \tau_D$, and by LP-duality $\mathcal{H}$ is a maximum integral dicycle packing in D. □

We need the following theorem due to Barahona, Fonlupt and Mahjoub [3] (see also [25]).

Theorem 2.5 ([3]) *Let $D_1, \ldots, D_k$ be a decomposition of D into k 3-connected components. If $\tau = \nu$ holds for each multiple augmented digraph of D_i, $i = 1, \ldots, k$, then $\tau_D = \nu_D$ as well.*

Now, using similar recursive method as in the cover case, together with Lemma 2.4, Theorem 2.5 and the method for merging dicycle packings, as defined in Section 2, we obtain the following theorem.

Theorem 2.6 *Let $D_1, \ldots, D_k$ be a decomposition of D into k 3-connected components, and suppose that for any multiple augmented digraph $\tilde{D}_i$ of D_i, $i = 1, \ldots, k$, the following holds: (i) $\tau_{\tilde{D}_i} = \nu_{\tilde{D}_i}$, and (ii) there is a polynomial time algorithm for finding a maximum integral dicycle packing in $\tilde{D}_i$. Then there is a polynomial time algorithm for finding a maximum integral dicycle packing in D.*

Theorems 2.7–2.11 to follow, imply that $K_{3,3}$-free digraphs satisfy the conditions of Theorems 2.3 and 2.6.

Theorem 2.7 ([29]) *Let G be a $K_{3,3}$-free graph. Then each 3-connected component of G is either a planar graph or K_5.*

Theorem 2.8 ([22]) *$\tau = \nu$ holds for any integral weighted planar digraph.*

Theorem 2.9 ([3]) *$\tau = \nu$ holds for any integral weighted digraph on 5 vertices.*

Theorem 2.10 ([8]) *A minimum integral dicycle cover in a weighted planar digraph can be found in $O(n^3)$ time.*

Theorem 2.11 ([7]) *In a weighted planar digraph, a maximum integral dicycle packing of $O(n)$ dicycles, can be found in $O(n^4)$ time.*

Observe that based on Theorem 2.7, one can check in linear time if a given digraph D is $K_{3,3}$-free. This is done by decomposing D into its 3-connected components using Hopcroft and Tarjan $O(n+m)$ algorithm [13], and then checking each component for being K_5, or being planar by using Booth and Lueker planarity testing $O(n)$ algorithm [4].

Now, let us consider the remaining time complexity of our algorithms. Let $D_1, \dots, D_k$ be the 3-connected components of D with $|V_i| = n_i$ and with $\sum_{i=1}^{k} n_i = n + 2(k-1) \leq 3n$. Assume our algorithm is of time complexity of $O({n_i}^l)$ for each D_i. Then the overall time complexity of the algorithm in D, excluding the packing merging procedure, is $O(\sum_{i=1}^{k} {n_i}^l) \leq O((3n)^l)$. In a similar way, one can verify that the number of dicycles in the packing in D and in the D_is, is of the same order. It is not hard to see that finding a minimum integral dicycle cover or a maximum integral dicycle packing in a digraph on 5 vertices, can be done in $O(1)$ time. Also, the overall time required for all the mergings is $O(\sum_{i=1}^{k} n_i \log n_i) \leq O(n \log n)$.

Thus, the above observations together with Theorems 2.3, 2.6, as well as Theorems 2.7–2.11 imply Theorem 2.12.

Theorem 2.12 *Let D be a $K_{3,3}$-free weighted digraph. Then a minimum integral dicycle cover in D can be found in $O(n^3)$ time, and a maximum integral dicycle packing of $O(n)$ dicycles, in $O(n^4)$ time.*

3 Stability and Composition of Dicycle Packings

As mentioned before, Lemma 2.2 suggests a simple method for constructing an optimal dicycle cover by gluing together two covers in the separation digraphs. Unfortunately, as shown below, a similar approach does not work in general for the packing case. In this section we show that under the stability condition a similar approach does work.

Recall that Lemma 2.2 indicates that if D is a $\{u,v\}$-merged digraph of D_1 and D_2, then every minimum integral dicycle cover in D is a union of minimum integral (u,v)- or (v,u)-covers in D_1 and in D_2. One would hope that a similar result would hold for the packing case, namely, that a maximum integral dicycle packing in D can be constructed by merging a maximum integral (u,v)-packing in D_1 (or in D_2) with a maximum integral (v,u)-packing in D_2 (or in D_1). Unfortunately, this is not always true, as demonstrated by the example below. Let D be a $\{u,v\}$-merged digraph of D_1 and D_2, where D_1 and D_2 are as in Figure 1. The unique maximum (u,v)-packing in D_1 is a set of 5 disjoint (u,v)-dipaths, and the unique maximum (v,u)-packing in D_2 is a set of 6 disjoint (v,u)-dipaths. Merging these two packings results in a dicycle packing of value 5, while the maximum dicycle packing in D consists of C_1, C_3, C_4 and 3 separated dicycles, and is of value 6.

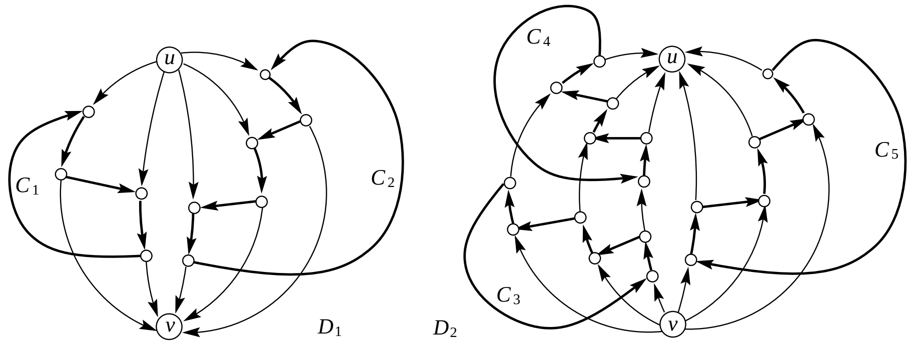

Figure 1: D_1 and D_2 are two separation digraphs of a $\{u,v\}$-merged digraph D, where the heavily lined dicycles form the maximum sets of disjoint dicycles in each digraph.

As an alternative approach, one may suggest merging the following two packings. Let ν_i be the value of a maximum integral dicycle packing in D_i, $i = 1, 2$. Among all the (u,v)- and (v,u)-packings in D_i with ν_i dicycles $i = 1, 2$, let us consider those with the largest value. One would hope to obtain a maximum integral dicycle packing in D by merging such a (u,v)-packing either in D_1 or in D_2, with such a (v,u)-packing in the other subgraph. Fig. 1 also shows that this approach does not work as well. Indeed, $\{C_1, C_2\}$ (resp., $\{C_3, C_4, C_5\}$) is the unique maximum integral dicycle packing in D_1 (resp., D_2), and there is no (u,v)- and no (v,u)-dipath disjoint to these dicycles. However, merging these two packings results in a packing of value 5, which is again, not an optimal one.

As we have just shown, simple composition methods for the packing problem do not seem to work for the general case. On the other hand, as shown in Lemma 2.4, such a method exists, provided $\tau = \nu$ hold for the appropriate digraphs. Here we show further sufficient conditions, in fact weaker ones (as is shown in Section 4), under which the simple composition mentioned in Lemma 4.1 for the packing problem holds. For that we need the following definition which was motivated by the previous discussion.

Definition 3.1 *A maximum integral dicycle packing is said to be* $\boldsymbol{(u,v)}$**-stable** *if it can be extended to a maximum integral (u,v)-packing by adding to it a (possibly empty) set of (u,v)-dipaths; it is* $\boldsymbol{\{u,v\}}$**-stable** *if it is both (u,v)- and (v,u)-stable. A digraph D is said to be* $\boldsymbol{(u,v)}$**-stable** *(resp.,* $\boldsymbol{\{u,v\}}$**-stable***) if it has a (u,v)-stable (resp., $\{u,v\}$-stable) dicycle packing.*

In general, the fact that D is (u,v)- and (v,u)-stable does not imply that D is $\{u,v\}$-stable. This is since $\{u,v\}$-stability requires the existence of a maximum dicycle packing which is (u,v)-stable as well as (v,u)-stable. Such a situation is demonstrated in Fig. 2. In this example, C_1 together with the simple (u,v)-dipath is the unique maximum integral

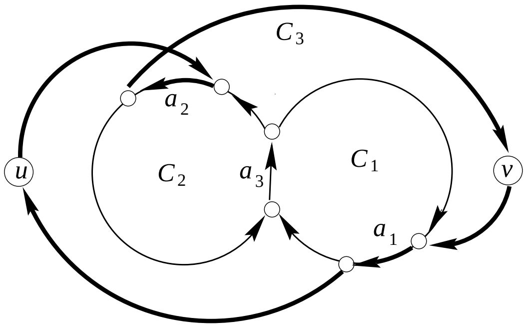

Figure 2: An example of a digraph which is (u, v)- and (v, u)-stable, but not $\{u, v\}$-stable.

(u, v)-packing, and C_2 together with the simple (v, u)-dipath is the unique maximum integral (v, u)-packing, but the dicycle sets of these two packings are distinct. The following lemma gives a necessary and sufficient condition for (u, v)- and (v, u)-stability to imply $\{u, v\}$-stability.

Lemma 3.1 *A digraph D is $\{u, v\}$-stable if and only if it is (u, v)- (or (v, u)-)stable and $\nu_D^{vu} = \nu_D$ (or $\nu_D^{uv} = \nu_D$).*

Proof: Clearly, if D is $\{u, v\}$-stable, then it is both (u, v)- and (v, u)-stable. Assume on the contrary that $\nu_D^{uv}, \nu_D^{vu} > \nu_D$. Let $\mathcal{H}$ be a $\{u, v\}$-stable packing in D. Our assumption implies that there is a (u, v)-dipath and a (v, u)-dipath in D, both disjoint to every dicycle in $\mathcal{H}$. Since the union of these two dipaths contains a dicycle we derive a contradiction to the maximality of $\mathcal{H}$.

Now, assume D is (u, v)-stable and that $\nu_D^{vu} = \nu_D$ (the other case can be treated similarly). Let $\mathcal{H}$ be a (u, v)-stable packing. Then the fact that $\nu_D^{vu} = \nu_D$ implies that $\mathcal{H}$ is also a (v, u)-stable packing, and thus a $\{u, v\}$-stable packing. $\square$

Lemma 3.2 *Let D be a digraph, and let $\tilde{D}$ be a (v, u)-augmented digraph obtained from D by adding $k = \nu_D^{uv} - \nu_D$ (v, u)-arcs. Then D is (u, v)-stable if and only if $\nu_D^{uv} = \nu_{\tilde{D}}$.*

Proof: If D is (u, v)-stable, then clearly $\nu_{\tilde{D}} = \nu_D + (\nu_D^{uv} - \nu_D) = \nu_D^{uv}$. Assume now that $\nu_D^{uv} = \nu_{\tilde{D}}$, and let $\tilde{\mathcal{H}}$ be a maximum packing in $\tilde{D}$. Let $\mathcal{H}$ be the packing formed by those dicycles of $\tilde{\mathcal{H}}$ not containing any of the k added arcs, and let $\mathcal{H}^{uv}$ be the (u, v)-packing induced by $\tilde{\mathcal{H}}$ in D. Clearly,

$$|\mathcal{H}| \geq |\tilde{\mathcal{H}}| - k = \nu_{\tilde{D}} - k = \nu_D^{uv} - (\nu_D^{uv} - \nu_D) = \nu_D.$$

Thus $\mathcal{H}$ is a maximum dicycle packing in D. Also, $|\mathcal{H}^{uv}| = |\tilde{\mathcal{H}}| = \nu_{\tilde{D}} = \nu_D^{uv}$. Hence, $\mathcal{H}^{uv}$ is a maximum (u,v)-packing in D. Now, as the dicycles of $\mathcal{H}$ and of $\mathcal{H}^{uv}$ coincide, the proof is complete. $\square$

The following lemma is the basis of our recursive algorithm. Note that the only difference between Lemma 2.4 and Lemma 3.3 is in the sufficient conditions where Lemma 2.4 requires that $\tau = \nu$ holds for any $\{u,v\}$-augmented separation digraph while Lemma 3.3 requires the $\{u,v\}$-stability for at least one of the separation digraphs. In what follows, note that if D is $\{u,v\}$ stable, and if $\Delta_D^{uv} = \nu_D^{uv} - \nu_D^{vu} \geq 0$, then (by Lemma 3.1) $\Delta_D^{uv} = \nu_D^{uv} - \nu_D$.

Lemma 3.3 *Let D_1 and D_2 be two $\{u,v\}$-separation digraphs of D and suppose that D_1 is $\{u,v\}$-stable and that $\Delta = \Delta_{D_1}^{uv} \geq 0$ (hence, $\nu_{D_1} = \nu_{D_1}^{vu}$). Let $\mathcal{H}_1$ be a maximum (u,v)-packing in D_1 with ν_{D_1} dicycles. Let $\tilde{D}_2$ be the digraph obtained from D_2 by adding Δ (u,v)-arcs, $\tilde{\mathcal{H}}_2$ a maximum integral dicycle packing in $\tilde{D}_2$, and let $\mathcal{H}_2$ be its counterpart packing of dicycles and dipaths in D_2. Then any merged dicycle packing $\mathcal{H}$, of $\mathcal{H}_1$ and $\mathcal{H}_2$, has a value of $\nu_{D_1} + \nu_{\tilde{D}_2}$, and $\mathcal{H}$ is a maximum integral dicycle packing in D.*

Proof: Obviously, $|\mathcal{H}| = \nu_{D_1} + \nu_{\tilde{D}_2}$. We will prove that $\mathcal{H}$ is a maximum packing in D. Among all maximum packings in D, let $\hat{\mathcal{H}}$ be the one with a minimum number of separated dicycles. Clearly, the number of dicycles in $\hat{\mathcal{H}}$ that are contained in D_1 is at most ν_{D_1}. Let $\hat{\mathcal{H}}_2$ be a family of dicycles in $\hat{\mathcal{H}}$ which are not contained in D_1 (i.e., the separated ones and the ones contained in D_2). We will show that $|\hat{\mathcal{H}}_2| \leq \nu_{\tilde{D}_2}$. Note that, by the above minimality assumption, any separated dicycle in $\hat{\mathcal{H}}_2$ is a union of a (u,v)-dipath in D_1 and a (v,u)-dipath in D_2. This implies that the number of separated dicycles in $\hat{\mathcal{H}}_2$ is at most Δ. Using the natural correspondence between separated dicycles in $\hat{\mathcal{H}}_2$ and dicycles in $\tilde{D}_2$ containing (u,v)-arcs, we conclude that $|\hat{\mathcal{H}}_2| \leq \nu_{\tilde{D}_2}$, and the proof is complete. $\square$

Remark: Note that Lemmas 3.1 and 3.2 imply that one can check if a digraph D is $\{u,v\}$-stable by calculating ν in its $\{u,v\}$-augmented digraphs in the following way. Let $M \geq w(D)+1$ be an integer and let $\hat{D}$ be a (u,v)-augmented digraph obtained from D by adding a (u,v)-arc of weight M. Then $\nu_D^{vu} = \nu_{\hat{D}}$. Clearly, ν_D^{uv} can be calculated in a similar way. Now, if $\nu_D^{uv}, \nu_D^{vu} \neq \nu_D$, then, by Lemma 3.1, D is not $\{u,v\}$-stable. Otherwise, assume that $\nu_D^{vu} = \nu_D$ (the other case can be treated similarly), and let $\tilde{D}$ be the (v,u)-augmented digraph obtained from D by adding a (v,u)-arc of weight $\Delta\nu_D^{uv}$. Then, by Lemmas 3.1 and 3.2, D is $\{u,v\}$-stable if and only if $\nu_{\tilde{D}} = \nu_D^{uv}$. Moreover, if $\tilde{\mathcal{H}}$ is a maximum integral dicycle packing in $\tilde{D}$, then its counterpart of dicycles and dipaths in D is a maximum (u,v)-packing, and its restriction to the dicycle packing in D, is of value ν_D.

Lemma 3.3 together with the above remark suggest the following algorithm. Decompose D into its 3-connected components, using Hopcroft and Tarjan $O(n+m)$ algorithm [13]. Assume that there is a polynomial time algorithm for finding a maximum integral dicycle packing in each multiple augmented 3-connected component of D. If D is 3-connected, we are done. Otherwise, there are at least two 3-connected component in this decomposition containing exactly one separation pair. We check whether at least one of them is $\{u,v\}$-stable, as described above. If so, let D_1 be a separation digraph corresponding to this 3-connected component, $\{u,v\}$ the separation pair contained in it, and D_2 the second separation digraph. Recall that by Lemma 3.1 either $\nu_{D_1}^{vu} = \nu_{D_1}$ or $\nu_{D_1}^{uv} = \nu_{D_1}$. Assume that $\nu_{D_1}^{vu} = \nu_{D_1}$, so $\Delta = \Delta_{D_1}^{uv} \geq 0$ (the other case can be treated similarly), and let $\tilde{D}_2$ be the digraph obtained from D_2 by adding a (u,v)-arc of weight Δ. So $\tilde{D}_2$ is as in Lemma 3.3, and by the remark above, computing a maximum (u,v)-packing as in Lemma 3.3, can be done by computing a maximum dicycle packing in $\tilde{D}_2$. Now, by Lemma 3.3, a maximum integral dicycle packing in D can be derived from maximum integral dicycle packings in D_1 and $\tilde{D}_2$ (using the merging procedure described in the introduction). Using the same method as before, we recursively find either a maximum integral dicycle packing in $\tilde{D}_2$, or observe that the corresponding digraph is not $\{u,v\}$-stable. Note that the total weight of any augmented digraph obtained by the algorithm is bounded by $2w(D)+1$. Observe further, that in the process described above, if D is a $\{u,v\}$-merged digraph of D_1 and D_2, then for the packing algorithm to hold, it is sufficient that at least one of D_1, D_2 is $\{u,v\}$-stable. Thus, in the following theorem, the $\{u,v\}$-stability is required for all components, except of one.

Theorem 3.4 *Let $D_1, \ldots, D_k$ be a decomposition of D into k 3-connected components of D. If for every $i = 1, \ldots, k$, except of may be one, any multiple augmented digraph of D_i is $\{u,v\}$-stable for every separation pair $\{u,v\}$ that belongs to D_i, and if there is a polynomial time algorithm for finding a maximum integral dicycle packing in each multiple augmented digraph of D_i, $i = 1, \ldots, k$, then there is a polynomial time algorithm for finding a maximum integral dicycle packing in D.*

Note that in the two algorithms, the one for the cover problem and the one for the packing problem, the desired sets can be found simultaneously in all the 3-connected components containing a single separation pair.

4 Integrality and Stability

The main results of this section are Theorems 4.4 and 4.7. Theorem 4.4 is similar to Theorem 2.6, where the sufficient condition $\tau = \nu$ is replaced by a weaker one, namely, $\nu = \nu^*$.

The second result, Theorem 4.7, states that if $\nu = \nu^*$ holds for each multiple augmented 3-connected component of D, then $\nu_D^* = \nu_D$ as well. This result is a refinement of a result of [3], stating that if $\tau = \nu$ holds for each multiple augmented 3-connected component of a digraph D, then $\tau_D = \nu_D$ as well (see also [25]). It should be noted that our result can not be obtained from the results of [3] and [25].

Recall that the two conditions, the $\{u, v\}$-stability and the requirement that $\tau = \nu$ in any $\{u, v\}$-augmented digraphs, enable us to derive a simple composition method for the packing problem. The following example and Theorem 4.3 to follow, show that the first condition is strictly weaker than the second one.

Example: Let D be the graph of Fig. 3. Here D is a digraph on $2k$ vertices, $k \geq 3$, where each arc has a weight (i.e., multiplicity) of $k-1$. Set $u = s_2$ and $v = t_3$. Let $\tilde{D}$ be a (u, v) augmented digraph obtained from D by adding a weighted (u, v)-arc. Let us define a set of dicycles in $\tilde{D}$ as follows: for $i = 2, \dots, k$, $C_i = (s_1, t_i, s_i, t_1, s_1)$, and $\hat{C} = (t_2, s_2, t_3, s_3, \dots, t_k, s_k, t_2)$. If $w(u, v) \geq k-1$, then

$$\tilde{h}(C) = \begin{cases} 1 & \text{if } C = C_i \quad i = 2, \dots, k \\ k-2 & \text{if } C = \hat{C} \\ 0 & \text{otherwise} \end{cases}$$

is an integral dicycle packing in $\tilde{D}$ of value $2k-3$. Note that this packing is optimal since

$$\tilde{f}(a) = \begin{cases} \frac{k-2}{k-1} & \text{if } a = (t_1, s_1) \\ \frac{1}{k-1} & \text{if } a = (t_i, s_i) \quad i = 2, \dots k \\ 0 & \text{otherwise} \end{cases}$$

is a dicycle cover in $\tilde{D}$ of the same value. Now, for every $a \in \hat{C} \setminus (u, v)$, $\{a, (t_1, s_1)\}$ is an integral dicycle cover in $\tilde{D}$ of value $2k-2$ and it is an optimal one since no single arc of $\tilde{D}$ intersects all the dicycles in D. Thus $2k-3 = \nu = \tau^* < \tau = 2k-2$ holds for $\tilde{D}$. Note that this gap can be made arbitrarily large by multiplying the weight of every arc of $\tilde{D}$ by some integral r, since then $r(2k-3) = \nu = \tau^* < \tau = r(2k-2)$, implying $\nu - \tau = r$.

Now, we show that D is $\{u, v\}$-stable. Let h (resp., h^{vu}) be the integral dicycle packing (resp., (v, u)-packing) induced by $\tilde{h}$ on D. Then h is an integral dicycle packing in D of value $k-1$; it is an optimal dicycle packing as well as an optimal (u, v)-packing in D since $\{(t_1, s_1)\}$ is an integral (u, v)-cover of the same value. Now, since h^{vu} is a maximum integral (v, u)-packing in D, and its restriction to the dicycles coincides with h, we obtain that D is $\{u, v\}$-stable.

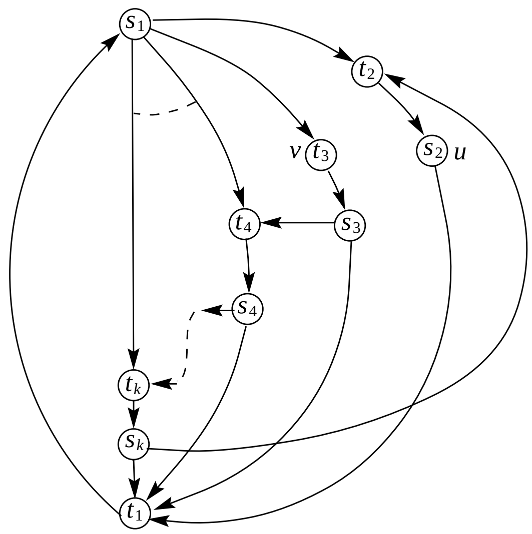

Figure 3: An example of a $\{u, v\}$-stable digraph with a $\{u, v\}$-augmented digraph for which $\nu < \tau$.

Observe however that $\nu = \nu^*$ holds for any $\{u, v\}$-augmented digraph of D. For a (u, v)-augmented digraph with $w(u, v) \geq k - 1$ this was already indicated above. One can verify from Fig. 3 that if $w(u, v) \leq k - 2$, then $\tau = \nu = w(u, v) + k - 1$ holds for any (u, v)-augmented digraph of D, while for any (v, u)-augmented digraph of it $\tau = \nu = k - 1$ holds. This motivates Lemma 4.1 to follow, which is the basis for the results presented in this section.

Lemma 4.1 *Let D be a digraph satisfying $\nu = \nu^*$ for every (v, u)-augmented digraph of it. Then D is (u, v)-stable.*

Proof: Assume on the contrary that D is not (u, v)-stable. Among all the integral (u, v)-packings with ν_D dicycles let $\mathcal{H}'$ be one of maximum value. Our assumption implies that there is an integral (u, v)-packing of value greater than that of $\mathcal{H}'$. Among those packings let $\mathcal{H}''$ be one with the maximum number of dicycles. Let p' and c' (resp., p'' and c'') be the number of (u, v)-dipaths and dicycles in $\mathcal{H}'$ (resp., $\mathcal{H}''$), respectively. Let $r = c' - c''$. Clearly, by our assumption, $r \geq 1$ and $p'' \geq p' + r + 1$.

Let $\hat{D}$ be a (v, u)-augmented digraph obtained from D by adding t (v, u)-arcs, where $t = p' + \lfloor \frac{r}{2} \rfloor + 1$. Let $\hat{\mathcal{H}}$ be any integral dicycle packing in $\hat{D}$ and let $\mathcal{H}$ be the integral (u, v)-packing obtained from $\hat{\mathcal{H}}$ by deleting the newly added (v, u)-arcs; the dicycles in $\mathcal{H}$ are the dicycles of $\hat{\mathcal{H}}$ not containing any added arc, and the (u, v)-dipaths in $\mathcal{H}$ are the remainders of the other dicycles of $\hat{\mathcal{H}}$. Let p and c be the number of (u, v)-dipaths and dicycles in $\mathcal{H}$, respectively. Clearly, $|\hat{\mathcal{H}}| = |\mathcal{H}| = c + p$.

Observe that $|\hat{\mathcal{H}}| \leq \max\{c' + p', c'' + t\}$, since if $|\hat{\mathcal{H}}| > c' + p'$, then $c + p > c' + p'$ which by the definition of $\mathcal{H}''$ implies $c \leq c''$. Therefore, $|\hat{\mathcal{H}}| = c + p \leq c + t \leq c'' + t$. Now, using the fact that $r \geq 1$, one obtains that

$$\begin{aligned} \nu_{\hat{D}} &\leq \max\left\{c' + p'\,,\, c'' + p' + \left\lfloor \frac{r}{2} \right\rfloor + 1\right\} = \\ &= \max\left\{c' + p'\,,\, c' + p' - \left(r - \left\lfloor \frac{r}{2} \right\rfloor - 1\right)\right\} = c' + p'\ . \end{aligned}$$

Note that if a digraph has a collection of $2k+1$ dicycles, not necessarily distinct, but such that any arc is contained in at most two dicycles of the collection, then this digraph has a dicycle packing of value $k+1/2$. We now build such a collection $\hat{\mathcal{C}}$ in $\hat{D}$ with $k = c' + p' \geq \nu_{\hat{D}}$. This will give a contradiction, since then $\nu^*_{\hat{D}} \geq \nu_{\hat{D}} + 1/2 > \nu_{\hat{D}}$.

We form the above collection $\hat{\mathcal{C}}$ from two families of colored dicycles in $\hat{D}$ Let us merge $\mathcal{H}'$ with p' newly added (v, u)-arcs. We thus obtain a family of **blue** dicycles. A family of **red** dicycles is a union of two families: the first is obtained by merging $\mathcal{H}''$ with all the newly added (v, u)-arcs, and the second is obtained by merging the remaining (u, v)-dipaths in $\mathcal{H}''$ with those newly added (v, u)-arcs which are not contained in any blue dicycle.

Note that, by the construction, the blue dicycles are disjoint. Also, an arc is contained in at most two red dicycles, and if an arc is contained in two red dicycles, then this arc is a newly added one that is not contained in any blue dicycle. Thus, an arc is contained in at most two dicycles of the collection.

We now calculate the number of the dicycles in $\hat{\mathcal{C}}$. Clearly, the number of blue dicycles is $|\mathcal{H}'| = c' + p'$, and the number of red dicycles is $(c'' + t) + \min\{t - p', p''\} = c'' + 2t - p'$. Thus

$$\begin{aligned} |\hat{\mathcal{C}}| &= c' + c'' + 2t = c' + c'' + 2(p' + \lfloor \frac{r}{2} \rfloor + 1) \geq \\ &\geq c' + c'' + 2p' + r + 1 = 2(c' + p') + 1 = 2\nu_{\hat{D}} + 1. \end{aligned}$$

This implies $\nu^*_{\hat{D}} \geq \nu_{\hat{D}} + \frac{1}{2}$, which contradicts our assumption that $\nu^*_{\hat{D}} = \nu_{\hat{D}}$. □

Lemma 4.2 *Assume $\nu = \nu^*$ holds for D. If D is (u, v)- as well as (v, u)-stable, then D is $\{u, v\}$-stable.*

Proof: Let $\mathcal{H}^{uv}$ (resp., $\mathcal{H}^{vu}$) be a maximum integral (u, v)-packing (resp., (v, u)-packing) in D containing ν_D dicycles. If there are no dipaths in $\mathcal{H}^{uv}$ or in $\mathcal{H}^{vu}$, then the statement is trivial. So, let C' be a merged dicycle which is composed by a (u, v)-dipath from $\mathcal{H}^{uv}$ with a (v,u)-dipath from $\mathcal{H}^{vu}$. Set h to be

$$h(C) = \begin{cases} 1 & \text{if } C \in \mathcal{H}^{uv} \cap \mathcal{H}^{vu} \\ \frac{1}{2} & \text{if } C \in [\mathcal{H}^{uv} \cup \mathcal{H}^{vu}] \setminus [\mathcal{H}^{uv} \cap \mathcal{H}^{vu}] \text{ or } C = C' \\ 0 & \text{otherwise.} \end{cases}$$

Then h is a dicycle packing in D of value $\nu_D + \frac{1}{2}$. Hence $\nu^*_D \geq \nu_D + \frac{1}{2} > \nu_D$, contradicting our assumption that $\nu_D = \nu^*_D$. □

Now, combining Lemmas 4.1 and 4.2 we obtain

Theorem 4.3 *If $\nu = \nu^*$ holds for every $\{u,v\}$-augmented digraph of D, then D is $\{u,v\}$-stable.*

Note that the inverse is usually not true. Indeed, let us take any digraph satisfying $\nu < \nu^*$ and add to it a new vertex u together with an arc (u,v), where v is any vertex of the digraph. Then, clearly, the resulting digraph is $\{u,v\}$-stable, but for any of its $\{u,v\}$-augmented digraph $\nu < \nu^*$ holds. Theorems 3.4 and 4.3 imply the following theorem.

Theorem 4.4 *Let $D_1, \ldots, D_k$ be a decomposition of D into k 3-connected components. If $\nu_{D_i} = \nu^*_{D_i}$ for all $i = 1, \ldots, k$, except for may be one, and if there is a polynomial time algorithm for finding a maximum integral dicycle packing in each multiple-augmented digraph of D_i, $i = 1, \ldots, k$, then there is a polynomial time algorithm for finding a maximum integral dicycle packing in D.*

We further use Theorem 4.3 to show that if $\nu = \nu^*$ holds for every multiple augmented 3-connected component of a digraph D, then $\nu = \nu^*$ holds for D as well.

Lemma 4.5 *If $\nu = \nu^*$ holds for every (v,u)-augmented digraph of D, then $\nu^{uv} = \nu^{uv*}$ holds as well.*

Proof: Assume on the contrary that $\nu^{uv} < \nu^{uv*}$ in D. Let h be a maximum (u,v)-packing in D. Consider the (v,u)-augmented digraph $\hat{D}$ obtained from D by adding $\lceil p_h \rceil$ (v,u)-arcs, where p_h is the value of the restriction of h to the (u,v)-dipaths. One can verify that $\nu^*_{\hat{D}} = {\nu_D}^{uv*} > \nu^{uv}_D = \nu_{\hat{D}}$, contradicting our assumption that $\nu = \nu^*$ holds for every (v,u)-augmented digraph of D. □

Lemma 4.6 *Let D_1 and D_2 be two $\{u,v\}$-separation digraphs of D. If $\nu = \nu^*$ holds for every $\{u,v\}$-augmented digraph of D_1 and of D_2, then $\nu = \nu^*$ holds for D as well.*

Proof: Clearly,

$$\nu^*_D \leq \max\{\min\{{\nu_{D_1}}^{uv*} + \nu^*_{D_2}, {\nu_{D_2}}^{vu*} + \nu^*_{D_1}\}, \min\{{\nu_{D_1}}^{vu*} + \nu^*_{D_2}, {\nu_{D_2}}^{uv*} + \nu^*_{D_1}\}\}.$$

By Theorem 4.3, D_1 and D_2 are both $\{u, v\}$-stable, implying

$$\nu_D = \max\{\min\{\nu^{uv}_{D_1} + \nu_{D_2}, \nu^{vu}_{D_2} + \nu_{D_1}\}, \min\{\nu^{vu}_{D_1} + \nu_{D_2}, \nu^{uv}_{D_2} + \nu_{D_1}\}\}.$$

Now, by our assumption, $\nu_{D_i} = \nu^*_{D_i}$, and thus by Lemma 4.5, ${\nu_{D_i}}^{uv} = {\nu_{D_i}}^{uv*}$ and ${\nu_{D_i}}^{vu} = {\nu_{D_i}}^{vu*}$ for $i = 1, 2$. Combining all these we obtain $\nu^*_D \leq \nu_D$, which implies $\nu_D = \nu^*_D$. □

Lemma 4.6 suggests the following theorem.

Theorem 4.7 *If $\nu = \nu^*$ holds for every multiple augmented 3-connected component of D, then $\nu = \nu^*$ holds for D as well.*

5 Conclusions

In this paper we addressed some structural and computational aspects of the minimum (integral) dicycle cover problem and the maximum (integral) dicycle packing one. Our main contribution is to the packing problem. Assume D has a 2-vertex cut, thus D is a 2-sum of some pieces. We derived a simple procedure for composing a minimum cover in D from the covers of its pieces. We also derived an efficient algorithm for finding maximum packings in $K_{3,3}$-free digraphs. We demonstrated however that, in general, there is no simple procedure for computing a maximum packing from packings of the pieces. Nevertheless, we have shown that the stability property is suffice for the existence of a simple composition. We also exposed the connection between stability and integrality in the packing case, and showed that this connection plays a major role in our compositions.

We have demonstrated some $\{u, v\}$-stable digraphs for which the min-max relation, $\tau = \nu$, does not hold. It will be interesting to classify some classes of graphs for which the stability property holds, but the min-max relation does not.

The study presented in the literature on the cover problem yields the min-max relation in some classes of graphs. The study presented here yields some integrality results concerning the packing problem. In particular, we have shown that if $\nu = \nu^*$ for every multiple augmented 3-connected component of D, then $\nu_D = \nu^*_D$. Although we have tried to tackle the issue of whether a similar result holds for dicycle covers, namely, whether $\tau = \tau^*$ for every multiple augmented 3-connected component of D implies $\tau_D = \tau^*_D$, it remains an open problem.

Another interesting direction for generalizing the study presented in this paper, is to extend some of the results introduced here, from digraphs that are 2-sum of some pieces, to the ones that are 3-sum.

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