On (t, k)-shredders in k-connected graphs

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Abstract

Let $G = (V, E)$ be a k-connected graph. For $t \geq 3$ a subset $T \subset V$ is a (t, k)-*shredder* if $|T| = k$ and $G - T$ has at least t connected components. It is known that the number of (t, k)-shredders in a k-connected graph on n nodes is less than $2n/(2t - 3)$. We show a slightly better bound for the case $k \leq 2t - 3$.

1 Introduction

Let $G = (V, E)$ be a k-(node) connected graph, that is, G is is simple and there are k pairwise internally disjoint paths between every pair of its nodes. For $T \subseteq V$ the T-*components* are the connected components of $G - T$ and let $b(T)$ denote the number of T-components. T with $|T| = k$ is: a k-*separator* if $b(T) \geq 2$, a k-*shredder* if $b(T) \geq 3$, and a (t, k)-*shredder* if $b(T) \geq t \geq 3$. Let $B(t, k, G)$ denote number of (t, k)-shredders in G; note that $B(3, k, G)$ is just the number of k-shredders in G. Let $B(t, k, n) = \max B(t, k, G)$ where the maximum is taken over all k-connected graphs G on n nodes.

A motivation for studying shredders comes from the node-connectivity augmentation problem, see [3, 1, 5]. Cheriyan and Thurimella [1] showed that in a k-connected graph computing the number of k-separators (which may be roughly $2^k n^2/k^2$) is #-complete, while the number of k-shredders separating two given nodes r, s is $O(n)$ and that they all can be found using one max-flow computation. They also proved that $B(3, k, n) = O(n^2)$ and conjectured that $B(3, k, n) \leq n$. Jordán [4] proved this conjecture, and established a tight bound for $k \leq 3$: if $k \leq 3$ and if G is k-connected then $B(3, k, G) \leq (n - k - 1)/2$ unless $k = 3$ and $G = K_{3,3}$. For arbitrary k, Egawa [2] proved that $B(3, k, n) < 2n/3$ and that this bound is (asymptotically) the best possible. Liberman and Nutov [5], and independently the second author of this paper, considered (t, k)-shredders and proved that $B(t, k, n) < 2n/(2t-3)$.

Remark: The following simple example shows that the bound $B(t, k, n) < 2n/(2t-3)$ is asymptotically tight for $k \geq 2(t-1)$. Let t, q be integers. Let G be $(t-1)$-blow-up of a q-cycle, that is G is obtained from a cycle of length q by replacing every node a by a set V_a of $t-1$ nodes, and every edge ab by $(t-1)^2$ edges, so that $V_a \cup V_b$ induces a complete bipartite graph $K_{t-1,t-1}$. For $k = 2(t-1)$, G is k-connected and $n = qk/2 = q(t-1)$. Thus $2n/(2t-3) = 2q(t-1)/(2t-3) = q + q/(2t-3)$. On the other hand, $B(t, k, G) = q$. For $2t-3 = k-1 > q$, the above bound is tight. This example easily extends for the case $k > 2(t-1)$, by adding $k - 2(t-1)$ nodes to G and connecting by an edge every added node to all the other nodes.

We show a slightly better bound for the case $k \leq 2t-3$, and prove the following theorem:

Theorem Let $k \leq 2t-3$. Then $B(t, k, n) \leq (n-k-1)/(t-1)$ for $n \geq 2k+1$ and $B(t, k, n) < n/(t-1)$ for $n \leq 2k$.

Remark: Our bound generalizes the bound of Jordán [4] which states:
For $k \leq 3$ and $t = 3$, $B(t, k, G) \leq (n-k-1)/(t-1)$ unless $k = 3$ and $G = K_{3,3}$.
Indeed, let $t = 3$ and let $k \leq 3$. Then $k \leq 2t-3$ since $t = 3$. Our bound implies that $B(t, k, G) \leq (n-k-1)/(t-1)$ for $n \geq 2k+1$. For $n \leq 2k \leq 6$, an easy case analysis shows that this bound also holds, unless $k = 3$ and $G = K_{3,3}$.

The bound in the Theorem is sharp for $n \geq 2k+1$, in the sense that there are infinitely many graphs that attain this bound. Let p be an integer, and k, t be as in the Theorem. Define a graph $G = (V, E)$ with $n = |V| = k + t\sum_{1\leq i\leq p}(t-1)^{i-1}$ by:

$$
\begin{aligned}
V &= \{a\} \\
&\cup \{b_{i,j,h} : 1 \leq i \leq t, 1 \leq j \leq p, 1 \leq h \leq (t-1)^{j-1}\} \\
&\cup \{c_\ell : 1 \leq \ell \leq k-1\} \\
E &= \{ab_{i,1,1},\ b_{i,j,h}b_{i,j+1,\ell} | 1 \leq i \leq t,\ 1 \leq j \leq p-1, \\
&\quad\ 1 \leq h \leq (t-1)^{j-1},\ (h-1)(t-1)+1 \leq \ell \leq h(t-1)\} \\
&\cup \{c_i c_j | 1 \leq i < j \leq k-1\} \\
&\cup \{c_\ell a,\ c_\ell b_{i,j,h} | 1 \leq \ell \leq k-1,\ 1 \leq i \leq t,\ 1 \leq j \leq p, \\
&\quad 1 \leq h \leq (t-1)^{j-1}\}.
\end{aligned}
$$

Then G is k-connected and has $1+t\sum_{1\le i\le p-1}(t-1)^{i-1}$ (t,k)-shredders which are:

$$\{a, c_1, \ldots, c_{k-1}\}$$
$$\{b_{i,j,h}, c_1, \ldots, c_{k-1}\} \quad 1 \le i \le t, 1 \le j \le p-1, 1 \le h \le (t-1)^{j-1}.$$

Thus

$$\begin{aligned}
\frac{n-k-1}{t-1} &= \frac{1}{t-1}(k + t \sum_{1\le i\le p} (t-1)^{i-1} - k - 1) \\
&= \frac{1}{t-1}(t(t-1) \sum_{1\le i\le p} (t-1)^{i-2} - 1) \\
&= \frac{1}{t-1}(t(t-1) \sum_{2\le i\le p} (t-1)^{i-2} + t(t-1)(t-1)^{-1} - 1) \\
&= \frac{1}{t-1}(t(t-1) \sum_{1\le i\le p-1} (t-1)^{i-1} + t - 1) \\
&= 1 + t \sum_{1\le i\le p-1} (t-1)^{i-1} = B(t,k,G)
\end{aligned}$$

2 Properties of separators and shredders

Let $G=(V,E)$ be a k-connected graph. For $Y \subseteq V$ let $\Gamma(Y)$ denote the set of neighbors of Y in G, and let $Y^* = V - Y - \Gamma(Y)$. Y is *tight* if $|\Gamma(Y)| = k$ and $Y^* \neq \emptyset$. A separators S *meshes* a separator T if S intersects at least two T-components. As was mentioned in [1], if S meshes T, then each one of S,T intersects all the components of the other; thus "meshing" is a symmetric relation. The following statement is immediate.

Proposition 2.1 *Let S,T be distinct nonmeshing k-separators in a k-connected graph. Then there is an S-component X and a T-component Y so that $T \subset X \cup S$ and $S \subset Y \cup T$ holds; thus $Y^* \subset X$ and $X^* \subset Y$.*

Corollary 2.2 *Let $\mathcal{T}$ be a family of pairwise nonmeshing k-separators in a k-connected graph G. Then G has a node r not belonging to any member of $\mathcal{T}$.*

Proof: Let $\mathcal{C}$ be the family of tight sets obtained by picking the T-components for each $T \in \mathcal{T}$. Let X be a an inclusion minimal set in $\mathcal{C}$, and let $S = \Gamma(X)$. We claim that no member of $\mathcal{T}$ intersects X. Suppose this is not so, that is, there is $T \in \mathcal{T}$ intersecting X. Then $T \subset X \cup S$, since S,T are nonmeshing. By Proposition 2.1, there is a T-component strictly contained in X, contradicting the minimality of X. □

Lemma 2.3 *Let S,T be meshing k-separators in a k-connected graph $G=(V,E)$ so that $S \cup T \neq V$. Then $k \ge b(S) + b(T) - 2$.*

Proof: Let $t = b(T)$ and $s = b(S)$. Let Y be the union of T-components not containing r, and let Z be the union of S-components not containing r. Since S, T mesh, $|\Gamma(Z) \cap Y| \geq t-1, |\Gamma(Y) \cap Z| \geq s-1$. Let $W = Y^* \cap Z^*$. Then $r \in W^* \neq \emptyset$. Thus $|\Gamma(W)| \geq k$, since G is k-connected. Furthermore,

$$|\Gamma(W)| = |\Gamma(Y^* \cap Z^*)| \leq |\Gamma(Y^*)| + |\Gamma(Z^*)| - [|\Gamma(Y^*) \cap Z| + |\Gamma(Z^*) \cap Y|] \leq 2k - [(s-1)+(t-1)].$$

Thus we have $k \leq 2k - [(s-1)+(t-1)]$, that is $k \geq s+t-2$. □

For $r \in V$ let $B_r(t,k,G)$ be the number of (t,k)-shredders in G not containing r. The following statement follows from a simple averaging argument, e.g., see [5, Lemma 2.4].

Lemma 2.4 $B(t,k,G) \leq \frac{n}{n-k} \max_{r \in V} B_r(t,k,G)$. *If r is a node of G not contained in any (t,k)-shredder then $B(t,k,G) = B_r(t,k,G)$.*

Two intersecting sets X, Y are *crossing* (or Y *crosses* X) if none of them contains the other. We will use the following key statement (see [6, Lemma 3.14] and [5, Lemma 2.3]).

Lemma 2.5 ([6, 5]) *Let G be a k-connected graph, let T be a k-shredder in G, and let Y be a tight set in G so that Y^* intersects some T-component C. Then Y does not cross $V-T-C$ nor a T-component distinct from C.*

3 Proof of the Theorem

Let $r \in V$. Consider the family $\mathcal{L}$ obtained by picking for every (t,k)-shredders T the T-components that do not contain r and their union; color the former blue and the later red. Let U be the union of the sets in $\mathcal{L}$; note that $|U| \leq n - |\Gamma(r)| - 1 \leq n-k-1$. By Lemma 2.5, $\mathcal{L}$ is laminar (that is, if two sets in $\mathcal{L}$ intersect then one of them contains the other). Thus $\mathcal{L}$ can be represented by a forest $\mathcal{F}$ of rooted trees, if we order the sets in $\mathcal{L}$ by inclusion: X is a child of Y if X is the largest set in $\mathcal{L}$ properly contained in Y. Note that every red set is the union of its children. The forest $\mathcal{F}$ has the following properties:
(i) every member of $\mathcal{L}$ is either blue or red, but not both;
(ii) the children of every red set are blue, and there are at least $t-1$ of them;
(iii) every child of a blue set is red.

Claim 3.1 *If a blue set Z is the union of its children, then for every child Q of Z there exists a child R of Z so that $S = \Gamma(Q)$ and $T = \Gamma(R)$ are meshing. In particular, if Z has one child, then Z contains a node not contained in its children.*

Proof: Let Q be a child of Z. Since $S \neq \Gamma(Z)$ and $Q \subseteq Z$, and since Z is the union of its children, Q has a neighbor in some child R of Z. Consequently, Q has a child X and R has a child Y, so that there is an edge in G with one end in X and the other end in Y. This implies that S and T mesh. Otherwise, by Proposition 2.1, $Y^* \subset X$; this is a contradiction, since $r \in Y^* - X$. □

Claim 3.2 *If every blue set has a node not contained in any of its children then* $B_r(t,k,G) \leq (n-k-1)/(t-1)$.

Proof: Let ℓ be the number of blue sets. Then $\ell \leq |U| \leq n-k-1$, since every blue set has a node not contained in any of its children. We will show that the number of red sets (which equals $B_r(t,k,G)$) is at most $\ell/(t-1)$. We claim that in any tree $\mathcal{T}$ (and thus in any forest) that satisfies properties (i),(ii),(iii), the number of red nodes is at most $\ell/(t-1)$. If $\mathcal{T}$ has one red node, the statement is obvious. Otherwise, $\mathcal{T}$ has a blue node X so that every red descendant of X is a child of X. Let q be the number of children of X. By deleting the children of X and their descendants (which are all blue leaves) we get a tree with the same properties, and ℓ decreases by at least $q(t-1)$. The claim follows. □

Combining Corollary 2.2 and Lemma 2.4 with the two claims above, we get:

Corollary 3.3 *If no two* (t,k)*-shredders mesh, then* $B(t,k,G) \leq (n-k-1)/(t-1)$.

Proof of the Theorem By Lemma 2.3, if S,T are meshing (t,k)-shredders, then $S \cup T = V$ and thus $n \leq 2k$. Thus for $n \geq 2k+1$ no two (t,k)-shredders mesh, and Corollary 3.3 implies the bound $B(t,k,G) \leq (n-k-1)/(t-1)$.

Assume $n \leq 2k$. Let $r \in V$ and consider the corresponding forest $\mathcal{F}$. We claim that every blue set X has a node not contained in any of its children; thus by Claim 3.2 $B_r(t,k,G) \leq (n-k-1)/(t-1)$, implying (via Lemma 2.4) $B(t,k,G) < n/(t-1)$. Otherwise, by Claim 3.1, X has two (red) children Y,Z corresponding to meshing shredders. But then by Lemma 2.3 $k \geq 2t-2$, contradicting the assumption of the theorem.

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