HOMOGENIZATION OF TWO DIMENSIONAL LINEAR FLOWS WITH INTEGRAL INVARIANCE[1]

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Abstract

We study the homogenization of 2D linear transport equations, $u_t + \vec{a}(\vec{x}/\varepsilon) \cdot \nabla_{\vec{x}} u = 0$, where $\vec{a}$ is a non-vanishing vector field with integral invariance on the torus T^2. When the underlying flow on T^2 is ergodic, we derive the efficient equation which is a linear transport equation with constant coefficients and quantify the pointwise convergence rate. This result unifies and illuminates the previously known results in the special cases of incompressible flows and shear flows. When the flow on T^2 is non-ergodic, the homogenized limit is an average, over T^1, of solutions of linear transport equations with constant coefficients; the convergence here is in the weak sense of $W^{-1,\infty}_{loc}(\mathbb{R}^1)$ and the sharp convergence rate is $\mathcal{O}(\varepsilon)$.

One of the main ingredients in our analysis is a classical theorem due to Kolmogorov, regarding flows with integral invariance on T^2, to which we present here an elementary and constructive proof.

Key Words. linear transport equations, homogenization, dynamical systems, ergodic theory

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1 Introduction

In this paper we study the homogenization of two-dimensional linear transport equations with oscillatory coefficients:

$$u_t + \vec{a}\left(\frac{\vec{x}}{\varepsilon}\right) \cdot \frac{\partial u}{\partial \vec{x}} = 0 \qquad \vec{x} \in \mathbb{R}^2 \ , \ t \in \mathbb{R}^+ \ , \ \varepsilon > 0 \ , \tag{1.1}$$

$$u(\vec{x}, 0) = u^0(\vec{x}) \ . \tag{1.2}$$

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Here, u^0 is a Lipschitz continuous function on $\mathbb{R}^2$ and $\vec{a}$, the transport vector field, is a smooth mapping from T^2, the two-dimensional unit torus, to $\mathbb{R}^2$, having no critical points,

$$\vec{a} \neq 0 \qquad \text{on } T^2 \ . \tag{1.3}$$

Arrowed bold faced letters will denote henceforth vectors and the corresponding indexed letters will denote their components, e.g., $\vec{a}(\vec{x}) = (a_1(x_1, x_2), a_2(x_1, x_2))$.

Equations of the above form serve as typical models for miscible displacement problems in the oil reservoir simulation; the unknown u corresponds to the concentration of the invading fluid (e.g. [1]).

In the one-dimensional case,

$$u_t + a(\frac{x}{\varepsilon})u_x = 0 \quad , \quad u(x,0) = u^0(x) \ , \tag{1.4}$$

a being a non-vanishing scalar function on T^1, the homogenized equation is easily derived, e.g. [1]: $u(x,t)$ converges pointwise to a limit, $v(x,t)$, which solves the homogenized equation

$$v_t + a^* v_x = 0 \quad , \quad v(x,0) = u^0(x) \ , \tag{1.5}$$

where a^* is the harmonic average of a over T^1,

$$a^* = \left(\int_0^1 \frac{dx}{a(x)} \right)^{-1} \ . \tag{1.6}$$

In the two-dimensional case, however, the problem of homogenization becomes more intricate. The effective equations depend sensitively on the topological structure and ergodicity of the flow on T^2 generated by the vector field $\vec{a}$, see [3, 4] and the references therein. Hou and Xin studied in [4] the homogenization of transport equations of the form (1.1) with divergence-free vector fields, $\nabla_{\vec{x}} \cdot \vec{a} = 0$, as a model problem for the incompressible Euler equations with oscillatory data,

$$\vec{u}_t + \vec{u} \cdot \frac{\partial \vec{u}}{\partial \vec{x}} = -\nabla p \quad , \quad \nabla_{\vec{x}} \cdot \vec{u} = 0 \quad ; \quad \vec{u}(\vec{x},0) = \vec{u}^0(\vec{x}, \frac{\vec{x}}{\varepsilon}) \quad , \quad \vec{u}^0(\vec{x},\vec{y}) \in C_0^1(\mathbb{R}^2 \times T^2) \ .$$

Using a weak L^2 formulation, they showed that when the rotation number of the flow which $\vec{a}$ generates on T^2 is irrational, the effective equation is

$$v_t + \overline{\vec{a}} \cdot \frac{\partial v}{\partial \vec{x}} = 0 \quad , \quad v(\vec{x},0) = u^0(\vec{x}) \ , \tag{1.7}$$

where $\overline{\vec{a}}$ stands for the arithmetic average of $\vec{a}$ over T^2. In case the rotation number is rational, the homogenized weak limit is characterized by an infinite symmetric hyperbolic system. Similar results were obtained by E in [3] for much more general incompressible flows.

The motivation for the present work came from the following three questions:

1. For incompressible flows with irrational rotation numbers, the effective equation, (1.7), is a linear transport equation with constant coefficients which are the **arithmetic** averages

over T^2 of the original transport vector field $\vec{a}$. On the other hand, for the one-dimensional case, (1.4), or for shear flows with an irrational rotation number γ,

$$u_t + a(\frac{\vec{x}}{\varepsilon})u_{x_1} + \gamma a(\frac{\vec{x}}{\varepsilon})u_{x_2} = 0 \quad , \quad u(\vec{x}, 0) = u^0(\vec{x}) \ , \tag{1.8}$$

a being a non-vanishing scalar function on T^2, the effective equation is a similar transport equation with the **harmonic** averages of the oscillatory field as coefficients (see §3). These results, put side by side, raise an interesting question: why when passing to the homogenized limit, divergence-free fields give rise to arithmetic averages, while shear fields yield harmonic averages? Is there an umbrella setup which unifies these two disjoint classes of vector fields[2], in which we can derive an effective equation (when the rotation number is irrational) that has the arithmetic averages as coefficients in the first case and the harmonic averages as coefficients in the second case?

2. When the rotation number is rational, the limit solution usually does not satisfy a transport equation (see §4). Instead, the effective equations are either non-local diffusion equations with memory terms or systems of hyperbolic equations [3, 4, 10]. In other words, homogenization, in case the underlying flow is non-ergodic, has a destructive effect on the simplicity of the original equations. Hence, as Tartar asks in [10], when possible, would it not be more reasonable to look for the limit solution itself rather than looking for the equation (or equations) that it satisfies? And does this limit solution retain, in some sense, the simple structure of linear transport?

3. What is the strongest topology in which the oscillatory solution converges to its homogenized limit and what is the convergence rate?

These questions are addressed in this paper.

It turns out that the appropriate class of vector fields which is large enough to include both divergence-free fields and shear ones is the following:

Definition 1.1 *The vector field $\vec{a} \in C^1(T^2)$ is in class $\mathcal{I}$, if it has an invariant measure density, i.e., a positive function $\mu \in C^1(T^2)$ such that $\int_{T^2} \mu dx_1 dx_2 = 1$ and $\nabla_{\vec{x}} \cdot \mu\vec{a} = 0$.*

A classical theorem, due to Kolmogorov (Theorem 2.1), relates any non-vanishing vector field $\vec{a} \in \mathcal{I}$ to a shear vector field,

$$(a(\cdot), \gamma a(\cdot)) \quad , \quad a : T^2 \to \mathbb{R} \ , \ \gamma = \text{Const} \ , \tag{1.9}$$

through a diffeomorphism of T^2; γ is called *the rotation number* and it determines the ergodicity of the flow on the torus T^2 which the dynamical system $d\vec{x}/dt = \vec{a}(\vec{x})$ generates [8]. §2 is devoted to this theorem which plays a central role in our analysis. The main result of that section is given in Theorem 2.2: the explicit diffeomorphism, shear form and rotation number are derived for vector fields which satisfy the restricted version of condition (1.3),

$$a_1 \neq 0 \qquad \text{on } T^2 \ . \tag{1.10}$$

[2]The intersection of these two classes of vector fields, when the rotation number is irrational, consists only of constant vectors.

This result, which also provides an elementary and straightforward proof of Kolmogorov's Theorem, is interesting for its own sake.

Using this theorem, we may reduce the general problem (1.1)–(1.2) to the simpler problem (1.8). In §3 we concentrate on such shear flows. By solving explicitly the equation with the oscillatory shear vector field, we are able to derive the homogenized limit and to determine the type and the rate of convergence, in the ergodic case, §3.1, and in the non-ergodic one, §3.2.

We are now ready to analyze the general problem, (1.1)–(1.2); this is done in §4. As in §3, we separate the discussion into two cases, according to the ergodicity of the underlying flow on the torus. In §4.1 we show that when the flow is ergodic (γ is irrational), the effective equation is a linear transport equation with a transport field which equals the harmonic average of the corresponding shear vector field (1.9). Namely, if a^* is the harmonic average over T^2 of $a(\cdot)$ in (1.9), then

$$u(\cdot,t) \to v(\cdot,t) \quad \text{where} \quad v_t + a^* v_{x_1} + \gamma a^* v_{x_2} = 0 \quad , \quad v(\cdot,0) = u^0 \; ; \tag{1.11}$$

in addition, we derive a pointwise convergence rate estimate (Theorem 4.1).

When the original vector field $\vec{a}$ is already in a shear form, Theorem 4.1 agrees with the result derived in §3, namely, the homogenized equation has the harmonic averages of $\vec{a}$ as coefficients. On the other hand, when $\vec{a}$ is divergence-free, Theorem 4.1 recovers the arithmetic average homogenized equation, (1.7); to show this, we prove in Theorem 4.2 that the harmonic average of the shear vector field (1.9) equals the arithmetic average of $\mu\vec{a}$, μ being the invariant measure density of the original field $\vec{a}$. This is done under assumption (1.10), using our explicit version of Kolmogorov's Theorem, namely, Theorem 2.2. In case $\vec{a}$ is divergence-free, $\mu \equiv 1$ and we therefore get equation (1.7). Hence, class $\mathcal{I}$ is the appropriate framework for a unified discussion of both divergence-free transport vector fields and shear ones, and Theorem 4.1 'interpolates' successfully between the two different homogenization results in these two disjoint cases.

In §4.2 we consider the non-ergodic case where the rotation number γ is rational. Here, instead of deriving the effective equations, we obtain an explicit expression for the limit solution itself, $\bar{v}$, and show that $u(\cdot,t) \rightharpoonup \bar{v}(\cdot,t)$ in $W^{-1,\infty}(\mathbb{R}^1)$ and the sharp convergence rate is $\mathcal{O}(\varepsilon)$, Theorem 4.3.

We show that in the general case $\bar{v}$ is not a solution of a linear transport equation; indeed, as mentioned earlier, the effective equations for $\bar{v}$ are of much greater complexity than the original linear transport equation. However, by deriving the explicit form of $\bar{v}$, we are able to see that it does retain the simple structure of linear transport: $\bar{v}(\vec{x},t) = \int_{T^1} v(\vec{x},t,\eta)d\eta$ where $v(\vec{x},t,\eta)$ satisfies a linear transport equation with η-dependent constant coefficients:

$$u(\cdot,t) \rightharpoonup \bar{v}(\cdot,t) = \int_{T^1} v(\cdot,t,\eta)d\eta \quad \text{where} \quad v_t + a^*_\eta v_{x_1} + \gamma a^*_\eta v_{x_2} = 0 \quad , \quad v(\cdot,0,\eta) = u^0 \; . \tag{1.12}$$

Hence, although $\bar{v}$ itself is not a solution of a linear transport equation, it is an average of such. Consequently, in case the coefficients $a^*_\eta \equiv a^* \; \forall \eta \in T^1$, (1.12) becomes the effective equation for $\bar{v}$.

We would like to mention in this context a proposition due to Brenier, [2, Proposition 1], which states that whenever $\vec{a}$ is divergence-free, there exists a probability space $(\Omega, d\eta)$

and a bounded measurable mapping $\vec{b} = \vec{b}(\eta) : \Omega \to \mathbb{R}^2$ such that

$$u(\cdot,t) \rightharpoonup \bar{v}(\cdot,t) = \int_\Omega v(\cdot,t,\eta)d\eta \quad \text{where} \quad v_t + \vec{b}(\eta)\cdot\nabla_{\vec{x}} v = 0 \quad , \quad v(\cdot,0,\eta) = u^0 \ . \tag{1.13}$$

The proof of this proposition (given in the multidimensional case) is based on Birkhoff's pointwise ergodic theorem which implies the existence of the vector field $\vec{b}(\eta)$, without constructing it. In §4 we prove this result for two-dimensional flows in the more general context of vector fields $\vec{a} \in \mathcal{I}$ and construct the explicit form of (1.13) in the ergodic case, (1.11), and in the the non-ergodic one, (1.12).

Finally, we would like to point out that our analysis may be extended to include oscillatory data, i.e.

$$u(\vec{x},0) = u^0(\vec{x},\frac{\vec{x}}{\varepsilon}) \qquad \text{where} \quad u^0(\vec{x},\vec{y}) \in Lip(\mathbb{R}^2 \times T^2) \ , \tag{1.14}$$

instead of (1.2) and also oscillatory forcing terms. The effect of homogenization on such oscillatory data is merely arithmetic averaging [3, 4, 9] and the convergence will be in the weaker sense of $W^{-1,\infty}_{loc}(\mathbb{R}^2)$.

2 Dynamical systems on the two-dimensional torus

Consider the dynamical system on T^2,

$$\frac{d\vec{x}}{dt} = \vec{a}(\vec{x}) \ , \tag{2.1}$$

where the non-vanishing vector field $\vec{a}(\vec{x})$ is in class $\mathcal{I}$ (see Definition 1.1). A classical theorem, due to Kolmogorov, relates flows generated by such vector fields to shear flows through a diffeomorphism, [5, 8]:

Theorem 2.1 *There exists a smooth change of variables on the torus, $\vec{y} = \vec{f}(\vec{x})$, under which (2.1) transforms into*

$$\frac{d\vec{y}}{dt} = \vec{p}\cdot a(\vec{y}) \quad , \quad \vec{p} = \begin{pmatrix} 1 \\ \gamma \end{pmatrix} \ , \tag{2.2}$$

where γ is a constant – the rotation number – and $a(\vec{y})$ is a non-vanishing smooth scalar function.

In the following proposition we show that the change of variables $\vec{y} = \vec{f}(\vec{x})$ may be normalized on the torus:

Proposition 2.1 *The change of variables in Theorem 2.1, $\vec{y} = \vec{f}(\vec{x})$, may be chosen so that*

$$f_i(\vec{x} + \vec{e}_j) = f_i(\vec{x}) + I_{i,j} \qquad 1 \le i,j \le 2 \ , \tag{2.3}$$

where $\vec{e}_1 = (1,0)$, $\vec{e}_2 = (0,1)$ and I stands for the 2×2 identity matrix.

Proof. Let $\vec{y} = \vec{f}(\vec{x})$ be the smooth change of variables of Theorem 2.1. Since $\vec{f}: T^2 \to T^2$ is continuous, there exists a 2×2 matrix L with integer entries $L_{i,j}$ such that

$$f_i(\vec{x} + \vec{e}_j) = f_i(\vec{x}) + L_{i,j} \qquad 1 \le i, j \le 2 \; . \tag{2.4}$$

The inverse transformation, $\vec{g} : T^2 \to T^2$, $\vec{g} = \vec{f}^{\,-1}$, being continuous as well, satisfies, similarly,

$$g_i(\vec{x} + \vec{e}_j) = g_i(\vec{x}) + M_{i,j} \qquad 1 \le i, j \le 2 \; , \tag{2.5}$$

where $M_{i,j}$ are integers. Clearly, $M = L^{-1}$.

We now consider the linear change of variables $\vec{z} = M\vec{y} = M\vec{f}(\vec{x})$. This transformation, which is smooth since $M_{i,j}$ are integers, leaves the system (2.2) in a shear form, and, by (2.4),

$$\vec{z}(\vec{x} + \vec{e}_j) = M\vec{f}(\vec{x} + \vec{e}_j) = M \cdot (\vec{f}(\vec{x}) + L_{\cdot,j}) = \vec{z}(\vec{x}) + I_{\cdot,j} \qquad j = 1, 2 \; .$$

That concludes the proof. $\square$

Kolmogorov's proof of Theorem 2.1 is not constructive. Namely, it does not provide us with neither the explicit change of variables, $\vec{y} = \vec{f}(\vec{x})$, nor with the value of the rotation number γ or the scalar function $a(\vec{y})$. We give below a more explicit version of Kolmogorov's Theorem followed by a proof which is both elementary and constructive.

As in [8], we replace the assumption of no critical points, (1.3), with the assumption that one of the components of the vector field, say $a_1(\vec{x})$, never vanishes, (1.10). The geometric implication of (1.3) is that there exists a smooth non-bounding cycle which is everywhere transversal to the flow. By replacing (1.3) with (1.10), we simply take that cycle to be $x_2 = 0$. This enables us to find the explicit change of variables and rotation number of the flow:

Theorem 2.2 *Assume that $\vec{a} \in \mathcal{I}$ and that (1.10) holds. Then there exists a smooth change of variables $\vec{x} \mapsto \vec{y}$ under which (2.1) transforms into (2.2) where*

$$\gamma = \frac{\overline{\mu a_2}}{\overline{\mu a_1}} \quad , \quad \overline{\mu a_i} = \int_{T^2} \mu(\vec{x}) a_i(\vec{x}) dx_1 dx_2 \quad i = 1, 2 \; , \tag{2.6}$$

and $a(\vec{y})$ is a non-vanishing smooth scalar function.

Proof. We start by assuming that both a_1 and a_2 do not vanish on T^2. Later, we remove the assumption that $a_2 \neq 0$.

Let $\vec{b}(\vec{x}) = \mu(\vec{x}) \cdot \vec{a}(\vec{x})$. Since $\nabla_{\vec{x}} \cdot \vec{b} = 0$, it follows that

$$\int_0^1 b_1(\vec{x}) dx_2 = \overline{b_1} \quad \forall x_1 \qquad \text{and} \qquad \int_0^1 b_2(\vec{x}) dx_1 = \overline{b_2} \quad \forall x_2 \; , \tag{2.7}$$

where the bar notation stands henceforth for the arithmetic average over T^2, i.e., $\overline{b_i} = \int_{T^2} b_i(\vec{x}) dx_1 dx_2$, $i = 1, 2$. We now introduce the new variables,

$$y_1 = f_1(\vec{x}) = \frac{1}{\overline{b_2}} \int_0^{x_1} b_2(\xi, 0) d\xi \quad , \quad y_2 = f_2(\vec{x}) = \frac{1}{\overline{b_1}} \int_0^{x_2} b_1(x_1, \xi) d\xi \; . \tag{2.8}$$

First, we note that (2.7) and the 1-periodicity of $b_i(\cdot,\cdot)$ imply that $\vec{f}$ satisfies (2.3) and, consequently, $\vec{y}=\vec{f}(\vec{x})$ is a smooth change of variables on T^2. Moreover, this transformation is invertible since its Jacobian is non-vanishing:

$$\left|\frac{\partial \vec{y}}{\partial \vec{x}}\right| = \frac{1}{\overline{b_1}\cdot\overline{b_2}} b_1(x_1,x_2)b_2(x_1,0) \neq 0 \qquad \forall \vec{x}\in T^2 \ . \tag{2.9}$$

Denoting the inverse transformation by $\vec{x}=\vec{g}(\vec{y})$, we show below that (2.8) transforms (2.1) into (2.2) with

$$a(\vec{y}) = \frac{1}{\overline{b_2}} b_2(g_1(\vec{y}),0)\cdot a_1(g_1(\vec{y}),g_2(\vec{y})) \neq 0 \qquad \text{and} \qquad \gamma = \frac{\overline{b_2}}{\overline{b_1}} \ . \tag{2.10}$$

Indeed,

$$\frac{dy_1}{dt} = \frac{1}{\overline{b_2}} b_2(x_1,0)\cdot a_1(x_1,x_2) = a(\vec{y}) \ ,$$

and

$$\begin{aligned}\frac{dy_2}{dt} &= \frac{1}{\overline{b_1}}\int_0^{x_2}\frac{\partial b_1}{\partial x_1}(x_1,\xi)d\xi\cdot a_1(x_1,x_2) + \frac{1}{\overline{b_1}} b_1(x_1,x_2)\cdot a_2(x_1,x_2) = \\ &\frac{1}{\overline{b_1}}\cdot\left(b_2(x_1,0)-b_2(x_1,x_2)\right)\cdot a_1(x_1,x_2) + \frac{1}{\overline{b_1}} b_1(x_1,x_2)\cdot a_2(x_1,x_2) = \\ &\frac{1}{\overline{b_1}} b_2(x_1,0)\cdot a_1(x_1,x_2) = \gamma\cdot a(\vec{y}) \ .\end{aligned}$$

This concludes the proof when both a_1 and a_2 are non-vanishing. Next, we handle the case where only a_1 does not vanish. In that case, we can always find an integer $k>0$ so large such that

$$a_2 + k a_1 \neq 0 \qquad \text{on } T^2 \ . \tag{2.11}$$

Hence, introducing the new variables

$$\vec{x}' = K\vec{x} \quad , \quad K = \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix} \ , \tag{2.12}$$

system (2.2) transforms into a system where both components of the vector field are non-vanishing:

$$\frac{d\vec{x}'}{dt} = \vec{a}'(K^{-1}\vec{x}') \quad , \quad \vec{a}' = K\vec{a} \ . \tag{2.13}$$

This new system has the same invariant measure as the original one, (2.1), namely $\mu(K^{-1}\vec{x}')$:

$$\nabla_{\vec{x}'}\cdot\mu\vec{a}' = \left(K^{-1}\right)^T\nabla_{\vec{x}}\cdot\mu K\vec{a} = \nabla_{\vec{x}}\cdot\mu\vec{a} = 0 \ .$$

We may, therefore, proceed and apply the change of variables (2.8),

$$y_1' = f_1'(\vec{x}') = \frac{1}{\overline{b_2'}}\int_0^{x_1'} b_2'(\xi,0)d\xi \quad , \quad y_2' = f_2'(\vec{x}') = \frac{1}{\overline{b_1'}}\int_0^{x_2'} b_1'(x_1',\xi)d\xi \ ,$$

where $\vec{b}' = \mu\vec{a}'$, in order to get the system

$$\frac{d\vec{y}'}{dt} = \vec{p}' \cdot a'(\vec{y}') \quad , \quad \vec{p}' = \begin{pmatrix} 1 \\ \gamma' \end{pmatrix} ,$$

with

$$\gamma' = \frac{\overline{\mu a_2'}}{\overline{\mu a_1'}} = \frac{\overline{\mu(a_2 + k a_1)}}{\overline{\mu a_1}} = \frac{\overline{\mu a_2}}{\overline{\mu a_1}} + k$$

(note that the transformation $\vec{x} \mapsto \vec{x}' = K\vec{x}$ is average-preserving). Finally, applying the additional transformation $\vec{y} = K^{-1}\vec{y}'$, we get the system (2.2) with the value of γ as in (2.6). □

3 Shear flows

Here, we concentrate on shear flows,

$$u_t + a\left(\frac{\vec{x}}{\varepsilon}\right) \vec{p} \cdot \frac{\partial u}{\partial \vec{x}} = 0 \quad , \quad \vec{p} = \begin{pmatrix} 1 \\ \gamma \end{pmatrix} \qquad , \qquad u(\vec{x}, 0) = u^0(\vec{x}) \ , \tag{3.1}$$

where γ is a constant – the rotation number – and $a(\vec{x})$ is a non-vanishing scalar function on T^2. Applying the method of characteristics, we find that the solution of (3.1) is

$$u(\vec{x}, t) = u^0(\varepsilon r, \varepsilon(\gamma r + \eta)) \ , \tag{3.2}$$

where

$$r = r(\vec{x}, t) := A_\eta^{-1}\left(-\frac{t}{\varepsilon} + A_\eta(\frac{x_1}{\varepsilon})\right) \quad , \quad \eta = \frac{x_2 - \gamma x_1}{\varepsilon} \tag{3.3}$$

and

$$A_\eta(x) := \int_0^x \frac{dy}{a(y, \gamma y + \eta)} \ . \tag{3.4}$$

We now study the behavior of $u(\vec{x}, t)$ when $\varepsilon \downarrow 0$. The discussion is separated into two cases:

3.1 Case 1: Irrational rotation numbers

We start by obtaining an asymptotic approximation for $A_\eta(x)$, (3.4), for large values of x. Denoting $\delta = \frac{1}{x}$, we find that

$$A_\eta(x) = \int_0^{\frac{1}{\delta}} \frac{dy}{a(y, \gamma y + \eta)} = \frac{1}{\delta}\int_0^1 \frac{dy}{a(\frac{y}{\delta}, \gamma\frac{y}{\delta} + \eta)} = \frac{1}{\delta}\int_0^1 b(\frac{y}{\delta_1}, \frac{y}{\delta_2})dy \ , \tag{3.5}$$

where $b(z_1, z_2) = 1/a(z_1, z_2 + \eta)$, $\delta_1 = \delta$ and $\delta_2 = \delta/\gamma$. Let a^* denote the harmonic average of a over T^2,

$$a^* = \left(\int_{T^2} \frac{1}{a(\vec{y})} dy_1 dy_2\right)^{-1} \ . \tag{3.6}$$

The arithmetic average of b over T^2 equals $1/a^*$, **regardless** of the value of η, thanks to the periodicity of a. Hence, since the ratio between δ_1 and δ_2 is irrational, we conclude by [11, Theorem 3.1] that

$$\int_0^1 b(\frac{y}{\delta_1}, \frac{y}{\delta_2})dy = \frac{1}{a^*} + \nu(\delta) \tag{3.7}$$

where $\nu(\delta)$ is (here and henceforth) an order of magnitude which vanishes when $\delta \to 0$.

Remark. The order of magnitude of $\nu(\delta)$ in (3.7) depends on the type of irrationality of γ and on the smoothness of b; if, for instance, γ has a finite type (which means that there exists $\sigma \geq 1$ such that $\mathrm{dist}(\gamma n, \mathbb{Z}) \geq \mathcal{O}(n^{-\sigma})$ for all $n \in \mathbb{N}$) and b is sufficiently smooth, $\nu(\delta) = \mathcal{O}(\delta)$. For a thorough discussion of this matter, please consult [11, §3].

Multiplying (3.7) by $x = 1/\delta$, we get by, in view of (3.7), that

$$A_\eta(x) = \frac{x}{a^*} \cdot \left(1 + \nu(\frac{1}{x})\right) \qquad |x| \gg 1 \ . \tag{3.8}$$

Estimate (3.8) implies that

$$A_\eta^{-1}(x) = a^* x \cdot \left(1 + \nu(\frac{1}{x})\right) \qquad |x| \gg 1 \ . \tag{3.9}$$

Applying the asymptotic estimates (3.8) and (3.9) in (3.3), we find that for fixed x_1 and t

$$\varepsilon r = \varepsilon A^{-1}_{\frac{x_2-\gamma x_1}{\varepsilon}} \left(-\frac{t}{\varepsilon} + A_{\frac{x_2-\gamma x_1}{\varepsilon}}(\frac{x_1}{\varepsilon})\right) = \varepsilon A^{-1}_{\frac{x_2-\gamma x_1}{\varepsilon}} \left(-\frac{t}{\varepsilon} + \frac{x_1}{a^*\varepsilon} \cdot (1+\nu(\varepsilon))\right) = -a^* t + x_1 + \nu(\varepsilon) \ .$$

Using this in (3.2) we find that for fixed $\vec{x}$ and t,

$$u(\vec{x}, t) = u^0(-a^* \vec{p} t + \vec{x} + \nu(\varepsilon)) \ .$$

Since u^0 is Lipschitz continuous, we conclude the following:

Proposition 3.1 *The solution $u(\vec{x}, t)$ of the transport equation (3.1), when γ is irrational, converges pointwise to*

$$v(\vec{x}, t) = u^0(-a^* \vec{p} t + \vec{x}) \ , \tag{3.10}$$

the solution of the homogenized equation

$$v_t + a^* \vec{p} \cdot \frac{\partial v}{\partial \vec{x}} = 0 \quad , \quad v(\vec{x}, 0) = u^0(\vec{x}) \ , \tag{3.11}$$

where a^* is the harmonic average of $a(\vec{x})$ over T^2, (3.6). Moreover, the following pointwise error estimate holds,

$$|u(\vec{x}, t) - v(\vec{x}, t)| \leq \nu(\varepsilon) \underset{\varepsilon \to 0}{\longrightarrow} 0 \ , \tag{3.12}$$

where the order of magnitude of $\nu(\varepsilon)$ depends on the smoothness of $a(\cdot)$ and on γ.

3.2 Case 2: Rational rotation numbers

Assume that $\gamma = \frac{m}{n} \in \mathbb{Q}$. Since $a(\vec{x})$ is 1-periodic in its variables, $\frac{d}{dx} A_\eta(x) = 1/a(x, \gamma x + \eta)$ is n-periodic in x and 1-periodic in η. Letting a_η^* denote the harmonic average of $a(\cdot, \gamma \cdot + \eta)$ over its period,

$$a_\eta^* = \left(\frac{1}{n} \int_0^n \frac{dx}{a(x, \gamma x + \eta)} \right)^{-1} , \tag{3.13}$$

we conclude that

$$A_\eta(x + n) = A_\eta(x) + n(a_\eta^*)^{-1} . \tag{3.14}$$

Hence, for all $x \in \mathbb{R}$,

$$A_\eta(x) = \frac{x}{a_\eta^*} + d_1 \qquad \text{where} \quad |d_1| = |d_1(x)| \le \max_{0 \le x \le n} \left| A_\eta(x) - \frac{x}{a_\eta^*} \right| . \tag{3.15}$$

Equality (3.15) provides us with a linear asymptotic approximation to $A_\eta(x)$ for large values of x. In view of (3.14), it follows that

$$A_\eta^{-1}(x + n(a_\eta^*)^{-1}) = A_\eta^{-1}(x) + n . \tag{3.16}$$

Consequently,

$$A_\eta^{-1}(x) = a_\eta^* x + d_2 \qquad \text{where} \quad |d_2| = |d_2(x)| \le \max_{0 \le x \le n(a_\eta^*)^{-1}} |A_\eta^{-1}(x) - a_\eta^* x| . \tag{3.17}$$

Using (3.15) and (3.17) in (3.3) we get:

$$r = A_\eta^{-1} \left(-\frac{t}{\varepsilon} + \frac{x_1}{\varepsilon a_\eta^*} + d_1 \right) = a_\eta^* \cdot \left(-\frac{t}{\varepsilon} + \frac{x_1}{\varepsilon a_\eta^*} + d_1 \right) + d_2 = \frac{-a_\eta^* t + x_1 + \mathcal{O}(\varepsilon)}{\varepsilon} .$$

Hence, in view of (3.2), the solution of (3.1) equals

$$u(\vec{x}, t) = u^0(-a_\eta^* \vec{p} t + \vec{x} + \mathcal{O}(\varepsilon)) \quad , \quad \eta = \frac{x_2 - \gamma x_1}{\varepsilon} . \tag{3.18}$$

Since u^0 is Lipschitz continuous with respect to its variables, (3.18) implies that

$$\|u(\vec{x}, t) - w(\vec{x}, t)\|_{L^\infty} \le \mathcal{O}(\varepsilon) \qquad \text{where} \quad w(\vec{x}, t) = u^0(-a_\eta^* \vec{p} t + \vec{x}) . \tag{3.19}$$

Let Γ be a curve in $\mathbb{R}^2$ which is nowhere parallel to $\vec{p}$ and, hence, parameterizable by $z = x_2 - \gamma x_1$. Along such curves

$$w = v(\vec{x}(z), t, \frac{z}{\varepsilon}) \qquad \text{where} \quad v(\vec{x}, t, \eta) = u^0(-a_\eta^* \vec{p} t + \vec{x}) . \tag{3.20}$$

Since v is 1-periodic in η, [9, Lemma 2.1] implies that for fixed t,

$$\|v(\vec{x}(z), t, \frac{z}{\varepsilon}) - \bar{v}(\vec{x}(z), t)\|_{W_{loc}^{-1,\infty}} \le \mathcal{O}(\varepsilon) \qquad \text{where} \quad \bar{v}(\vec{x}, t) = \int_{T^1} v(\vec{x}, t, \eta) d\eta . \tag{3.21}$$

Combining (3.19)–(3.21) we arrive at:

Proposition 3.2 *The solution $u(\vec{x},t)$ of the transport equation (3.1), where $\gamma = \frac{m}{n} \in \mathbb{Q}$, converges weakly in $W^{-1,\infty}_{loc}(\mathbb{R}^1)$ to*

$$\bar{v}(\vec{x},t) = \int_{T^1} u^0(-a^*_\eta \vec{p}t + \vec{x})d\eta \ , \tag{3.22}$$

where a^*_η is given in (3.13). Moreover, the following error estimate holds for any fixed t along any curve $\vec{x} = \vec{x}(z)$ not parallel to $\vec{p}$:

$$\|u(\vec{x}(\cdot),t) - \bar{v}(\vec{x}(\cdot),t)\|_{W^{-1,\infty}_{loc}} \leq \mathcal{O}(\varepsilon) \ . \tag{3.23}$$

In the special case where $a^*_\eta \equiv a^* \ \forall \eta \in [0,1]$, the limit is $\bar{v}(\vec{x},t) = u^0(-a^*\vec{p}t + \vec{x})$, the solution of the homogenized equation

$$\bar{v}_t + a^*\vec{p} \cdot \frac{\partial \bar{v}}{\partial \vec{x}} = 0 \quad , \quad \bar{v}(\vec{x},0) = u^0(\vec{x}) \ , \tag{3.24}$$

and the convergence is in the strong sense:

$$\|u(\vec{x},t) - \bar{v}(\vec{x},t)\|_{L^\infty} \leq \mathcal{O}(\varepsilon) \ . \tag{3.25}$$

Remark. The one-dimensional transport equation,

$$u_t + a(\frac{x}{\varepsilon})u_x = 0 \quad ; \quad u(x,0) = u^0(x) \qquad x \in \mathbb{R}^1 \ , \ t \in \mathbb{R}^+ \ , \tag{3.26}$$

$a(x)$ being a non-vanishing function on T^1, is a special case of (3.1), where $a(\vec{x})$ and $u^0(\vec{x})$ depend solely on x_1 and $\gamma = 0$. Since in this case $a^*_\eta \equiv a^*$, the harmonic average of $a(x)$ over T^1, we get, in view of (3.24) and (3.25), that $u(x,t)$ converges strongly to $\bar{v}(x,t)$, the solution of the homogenized equation

$$\bar{v}_t + a^*\bar{v}_x = 0 \quad ; \quad \bar{v}(x,0) = u^0(x) \ .$$

4 General flows

We now extend our discussion to general vector fields in class $\mathcal{I}$. The question of homogenization of transport equations (1.1) with vector fields in that class, may be reduced to homogenization of an equation of type (3.1), thanks to Theorem 2.1. Using this theorem, we shall construct the explicit solution to (1.1)-(1.2). To do that, we need to solve the characteristic equations.

As a first step, we find the semigroup S_t which is associated with the dynamical system

$$\frac{d\vec{x}}{dt} = \vec{a}(\vec{x}) \quad , \quad \vec{x}(t=0) = \vec{x}^0 \ ; \tag{4.1}$$

i.e., the mapping $S_t : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\vec{x}(t) = S_t(\vec{x}^0)$ is the solution of (4.1). Then, if S_t^{-1} is the inverse semigroup, the solution of (1.1)–(1.2) is given by:

$$u(\vec{x},t) = u^0\left(\varepsilon S^{-1}_{\frac{t}{\varepsilon}}\left(\frac{\vec{x}}{\varepsilon}\right)\right) \ . \tag{4.2}$$

Applying Kolmogorov's Theorem, we change variables in (4.1) to $\vec{y} = \vec{f}(\vec{x})$ and obtain the system

$$\frac{d\vec{y}}{dt} = \vec{p} \cdot a(\vec{y}) \quad , \quad \vec{p} = \begin{pmatrix} 1 \\ \gamma \end{pmatrix} \quad ; \quad \vec{y}(t=0) = \vec{y}^0 = \vec{f}(\vec{x}^0) \ . \tag{4.3}$$

Since $\eta = y_2 - \gamma y_1$ is an integral of (4.3), this system can be easily solved and its solution is:

$$\vec{y} = A_\eta^{-1}(t + A_\eta(y_1^0)) \ \vec{p} + \eta \vec{e}_2 \quad , \quad \eta = y_2^0 - \gamma y_1^0 \ ,$$

where $\vec{e}_2 = (0,1)$ and $A_\eta(x)$ is given in (3.4). Changing variables back to $\vec{x} = \vec{g}(\vec{y})$ ($\vec{g} = \vec{f}^{\,-1}$), we get that

$$\vec{x} = S_t(\vec{x}^0) = \vec{g}(\ A_\eta^{-1}(t + A_\eta(f_1(\vec{x}^0))) \ \vec{p} + \eta \vec{e}_2 \ ) \quad , \quad \eta = f_2(\vec{x}^0) - \gamma f_1(\vec{x}^0) \ . \tag{4.4}$$

By eliminating $\vec{f}(\vec{x}^0)$ from (4.4) we get that

$$\vec{f}(\vec{x}^0) = A_\eta^{-1}(-t + A_\eta(f_1(\vec{x}))) \ \vec{p} + \eta \vec{e}_2 \quad , \quad \eta = f_2(\vec{x}) - \gamma f_1(\vec{x}) \ .$$

Applying the inverse transformation $\vec{g}$, we conclude that

$$\vec{x}^0 = S_t^{-1}(\vec{x}) = \vec{g}(\ A_\eta^{-1}(-t + A_\eta(f_1(\vec{x}))) \ \vec{p} + \eta \vec{e}_2 \ ) \quad , \quad \eta = f_2(\vec{x}) - \gamma f_1(\vec{x}) \ . \tag{4.5}$$

Finally, using (4.2) and (4.5), we arrive at the explicit solution of (1.1)–(1.2):

Proposition 4.1 *The solution of (1.1)–(1.2) is given by*

$$u(\vec{x}, t) = u^0(\varepsilon \vec{g}(r, \gamma r + \eta)) \ , \tag{4.6}$$

where

$$r = r(\vec{x}, t) := A_\eta^{-1}\left(-\frac{t}{\varepsilon} + A_\eta \left(f_1 \left(\frac{\vec{x}}{\varepsilon} \right) \right) \right) \quad , \quad \eta = f_2 \left(\frac{\vec{x}}{\varepsilon} \right) - \gamma f_1 \left(\frac{\vec{x}}{\varepsilon} \right) \tag{4.7}$$

and A_η is defined in (3.4). Here, $\vec{f}$ is the mapping from T^2 to T^2 which Theorem 2.1 associates with the system (4.1) and $\vec{g}$ is its inverse.

Next, we find the limit of the solution when $\varepsilon \to 0$. In doing so, we shall make use of the following asymptotic estimates for $\vec{f}$ and its inverse $\vec{g}$, which are a direct consequence of Proposition 2.1:

$$\vec{f}(\vec{x}) = \vec{x} + \vec{d}_f(\vec{x}) \ \text{ where } \ |\vec{d}_f(\vec{x})| \le \max_{\vec{x} \in [0,1]^2} |\vec{f}(\vec{x}) - \vec{x}| \qquad \forall \vec{x} \in \mathbb{R}^2 \ , \tag{4.8}$$

$$\vec{g}(\vec{x}) = \vec{x} + \vec{d}_g(\vec{x}) \ \text{ where } \ |\vec{d}_g(\vec{x})| \le \max_{\vec{x} \in [0,1]^2} |\vec{g}(\vec{x}) - \vec{x}| \qquad \forall \vec{x} \in \mathbb{R}^2 \ . \tag{4.9}$$

4.1 Case 1: Ergodic flows

Here, we deal with the case where the rotation number γ of the system (4.1) is irrational. The asymptotic estimates with which we are equipped are (3.8)+(3.9) for A_η and A_η^{-1}, and (4.8)+(4.9) for $\vec{f}$ and $\vec{g}$.

Fixing a point, $(\vec{x}, t)$, we now aim at finding the limit of $u(\vec{x}, t)$ when $\varepsilon \to 0$. In view of (4.8) and (3.8),

$$A_\eta\left(f_1\left(\frac{\vec{x}}{\varepsilon}\right)\right) = A_\eta\left(\frac{x_1 + \mathcal{O}(\varepsilon)}{\varepsilon}\right) = \frac{x_1 + \mathcal{O}(\varepsilon) + \nu(\varepsilon)}{a^*\varepsilon} \ .$$

Therefore, by (4.7) and (3.9),

$$r = A_\eta^{-1}\left(\frac{-a^*t + x_1 + \mathcal{O}(\varepsilon) + \nu(\varepsilon)}{a^*\varepsilon}\right) = \frac{-a^*t + x_1 + \mathcal{O}(\varepsilon) + \nu(\varepsilon)}{\varepsilon} \ . \tag{4.10}$$

Next, we estimate η, (4.7), using (4.8):

$$\eta = f_2\left(\frac{\vec{x}}{\varepsilon}\right) - \gamma f_1\left(\frac{\vec{x}}{\varepsilon}\right) = \frac{x_2 - \gamma x_1 + \mathcal{O}(\varepsilon)}{\varepsilon} \ . \tag{4.11}$$

The last two equalities imply that

$$\gamma r + \eta = \frac{-\gamma a^* t + x_2 + \mathcal{O}(\varepsilon) + \nu(\varepsilon)}{\varepsilon} \ . \tag{4.12}$$

Using (4.9), we conclude by (4.10) and (4.12) that

$$\varepsilon\vec{g}(r, \gamma r + \eta) = \varepsilon\begin{pmatrix} r \\ \gamma r + \eta \end{pmatrix} + \mathcal{O}(\varepsilon) = -a^*t\vec{p} + \vec{x} + \mathcal{O}(\varepsilon) + \nu(\varepsilon) \ . \tag{4.13}$$

Therefore, by (4.6), (4.13) and the Lipschitz continuity of u^0, we arrive at:

Theorem 4.1 *Let $\vec{a}(\vec{x})$ be a vector field with a shear flow form $(a(\vec{y}), \gamma a(\vec{y}))$ and let $u(\vec{x}, t)$ be the solution of (1.1)-(1.2). Then if the rotation number γ is irrational, $u(\vec{x}, t)$ converges pointwise to*

$$v(\vec{x}, t) = u^0(-a^*t\vec{p} + \vec{x}) \quad , \quad \vec{p} = \begin{pmatrix} 1 \\ \gamma \end{pmatrix} \ , \tag{4.14}$$

the solution of the homogenized equation

$$v_t + a^*\vec{p} \cdot \frac{\partial v}{\partial \vec{x}} = 0 \quad , \quad v(\vec{x}, 0) = u^0(\vec{x}) \ , \tag{4.15}$$

where a^* is the harmonic average of $a(\vec{y})$ over T^2, (3.6). Moreover, the following pointwise error estimate holds,

$$|u(\vec{x}, t) - v(\vec{x}, t)| \leq \mathcal{O}(\varepsilon) + \nu(\varepsilon) \underset{\varepsilon \to 0}{\longrightarrow} 0 \ , \tag{4.16}$$

where the order of magnitude of $\nu(\varepsilon)$ depends on the smoothness of $a(\cdot)$ and on γ.

In other words, Theorem 4.1 states that given a transport equation, (1.1), with a non-vanishing vector field $\vec{a} \in \mathcal{I}$, its homogenized equation, (4.15), is a transport equation with a constant vector field that equals *the harmonic average* of *the shear vector field* associated with $\vec{a}$. Hence, it is a natural generalization of Proposition 3.1.

Thanks to Theorem 2.2, we may derive the explicit form of the homogenized equation (4.15) when $a_1 \neq 0$.

We concentrate on the case where both a_1 and a_2 do not vanish on T^2; the case where only a_1 is non-vanishing is treated similarly, along the lines of the second part of the proof of Theorem 2.2.

We start by computing a^* – the harmonic average of $a(\vec{y})$ which is given in (2.10):

$$\frac{1}{a^*} = \int_{T^2} \frac{\overline{b_2}}{b_2(g_1(\vec{y}),0) \cdot a_1(g_1(\vec{y}), g_2(\vec{y}))} dy_1 dy_2 \ .$$

Changing variables to $\vec{x} = \vec{g}(\vec{y})$, we get, using (2.9), that

$$\frac{1}{a^*} = \int_{T^2} \frac{\mu(\vec{x})}{\overline{b_1}} dx_1 dx_2 = \frac{1}{\overline{b_1}} \ . \tag{4.17}$$

Hence, by (4.17) and (2.10),

$$a^* \vec{p} = \overline{b_1} \cdot \begin{pmatrix} 1 \\ \\ \frac{\overline{b_2}}{\overline{b_1}} \end{pmatrix} = \begin{pmatrix} \overline{b_1} \\ \\ \overline{b_2} \end{pmatrix} = \begin{pmatrix} \overline{\mu a_1} \\ \\ \overline{\mu a_2} \end{pmatrix} \ . \tag{4.18}$$

This proves the following:

Theorem 4.2 *Under the assumptions of Theorem 4.1, if $a_1 \neq 0$, $u(\vec{x}, t)$ converges pointwise to*

$$v(\vec{x}, t) = u^0(-\overline{\mu \vec{a}} \cdot t + \vec{x}) \ , \tag{4.19}$$

the solution of the homogenized equation

$$v_t + \overline{\mu \vec{a}} \cdot \frac{\partial v}{\partial \vec{x}} = 0 \quad , \quad v(\vec{x}, 0) = u^0(\vec{x}) \ , \tag{4.20}$$

where μ is the invariant measure of the vector field $\vec{a}$ and $\overline{\mu \vec{a}}$ is the arithmetic average of $\mu \vec{a}$ over T^2.

Examples.

1. When $\nabla_{\vec{x}} \cdot \vec{a} = 0$, $\mu \equiv 1$ and then equation (4.20) coincides with (1.7).

2. Assume that $\vec{a} = a(\vec{x})\vec{p}$ where $\vec{p} = (1, \gamma)$ and γ is irrational. The invariant measure in this case is $\mu(\vec{x}) = a^*/a(\vec{x})$, where a^* is the harmonic average of a over T^2, (3.6). Hence, $\mu \vec{a} = a^* \vec{p}$, and, consequently, equation (4.20) coincides with (3.11).

4.2 Case 2: Non-ergodic flows

Here, $\gamma = \frac{m}{n}$ is rational. We still use (4.8)+(4.9) for $\vec{f}$ and $\vec{g}$; A_η and A_η^{-1}, however, are now estimated by (3.15)+(3.17), rather than (3.8)+(3.9) as it was in the previous case. Here, in fact, lies the difference between the two cases: while, for ergodic flows, the asymptotic behavior of A_η is **independent** of the oscillatory term η (and, therefore, enables **strong** convergence of the solution), in non-ergodic flows, it depends on η (and, consequently, the solution converges only in a **weak** sense).

Arguing along the same lines as in the previous case, we get the equivalent estimate to (4.13), which holds uniformly for all $\vec{x}$ and t:

$$\varepsilon\vec{g}(r, \gamma r + \eta) = -a_\eta^* t\vec{p} + \vec{x} + \mathcal{O}(\varepsilon) \ . \tag{4.21}$$

Hence, by (4.6) and the Lipschitz continuity of u^0, we conclude that

$$\|u(\vec{x}, t) - w(\vec{x}, t)\|_{L^\infty} \leq \mathcal{O}(\varepsilon) \qquad \text{where} \quad w(\vec{x}, t) = u^0(-a_\eta^* t\vec{p} + \vec{x}) \ . \tag{4.22}$$

Denoting $z = x_2 - \gamma x_1$, we get, in view of (4.22) and (4.11), that along curves which are nowhere parallel to $\vec{p}$ (and, hence, parameterizable by z),

$$w = v(\vec{x}(z), t, \frac{z + \mathcal{O}(\varepsilon)}{\varepsilon}) \qquad \text{where} \quad v(\vec{x}, t, \eta) = u^0(-a_\eta^* t\vec{p} + \vec{x}) \ . \tag{4.23}$$

We now invoke the following fundamental lemma which is a straightforward generalization of [9, Lemma 2.1]:

Lemma 4.1 *Let $f(x, y)$ be a bounded function of $x \in \mathbb{R}$ and $y \in T^1$, and let $r(x)$ be a smooth function on $\mathbb{R}$ with a non-vanishing derivative, $r'(x) \neq 0$. Then if $f^\varepsilon(x) = f(x, \frac{r(x)}{\varepsilon})$ and $\bar{f}(x) = \int_{T^1} f(x, y)dy$,*

$$\|f^\varepsilon(x) - \bar{f}(x)\|_{W_{loc}^{-1,\infty}} \leq \mathcal{O}(\varepsilon) \quad \text{when} \ \ \varepsilon \downarrow 0 \ . \tag{4.24}$$

We use this lemma for the function $v(\vec{x}(z), t, \eta = \frac{z+\mathcal{O}(\varepsilon)}{\varepsilon})$ for fixed values of t; note that v is 1-periodic in η and that $r(z) = z + \mathcal{O}(\varepsilon)$ is invertible when ε is sufficiently small. Therefore, we conclude that for fixed t

$$\|v(\vec{x}(z), t, \frac{z + \mathcal{O}(\varepsilon)}{\varepsilon}) - \bar{v}(\vec{x}(z), t)\|_{W_{loc}^{-1,\infty}} \leq \mathcal{O}(\varepsilon) \qquad \text{where} \quad \bar{v}(\vec{x}, t) = \int_{T^1} v(\vec{x}, t, \eta)d\eta \ . \tag{4.25}$$

Combining (4.22)–(4.25) we arrive at:

Theorem 4.3 *Let $\vec{a}(\vec{x})$ be a vector field with a shear flow form $a(\vec{y})\vec{p}$, $\vec{p} = (1, \gamma)$, and let $u(\vec{x}, t)$ be the solution of (1.1)-(1.2). Then if the rotation number is rational, $\gamma = \frac{m}{n}$, $u(\vec{x}, t)$ converges weakly in $W_{loc}^{-1,\infty}(\mathbb{R}^1)$ to*

$$\bar{v}(\vec{x}, t) = \int_{T^1} u^0(-a_\eta^* \vec{p}t + \vec{x})d\eta \ ,$$

where a^*_η is given in (3.13). Moreover, the following error estimate holds for any fixed t along any curve $\vec{x} = \vec{x}(z)$ not parallel to $\vec{p}$:

$$\|u(\vec{x}(\cdot),t) - \bar{v}(\vec{x}(\cdot),t)\|_{W^{-1,\infty}_{loc}} \leq \mathcal{O}(\varepsilon) \ . \tag{4.26}$$

In the special case where $a^*_\eta \equiv a^*$ $\forall \eta \in [0,1]$, the limit is $\bar{v}(\vec{x},t) = u^0(-a^*\vec{p}t + \vec{x})$, the solution of the homogenized equation

$$\bar{v}_t + a^*\vec{p} \cdot \frac{\partial \bar{v}}{\partial \vec{x}} = 0 \quad , \quad \bar{v}(\vec{x},0) = u^0(\vec{x}) \ , \tag{4.27}$$

and the convergence is in the strong sense:

$$\|u(\vec{x},t) - \bar{v}(\vec{x},t)\|_{L^\infty} \leq \mathcal{O}(\varepsilon) \ . \tag{4.28}$$

Remark. We mention in this context The Averaging Lemma due to Lions, Perthame and Tadmor [6], which states that if $u(\vec{x},t)$ is an integral of solutions of linear transport equations,

$$u(\vec{x},t) = \int_{\mathbb{R}^1} v(\vec{x},t,\eta)d\eta \quad \text{where} \quad v_t + \vec{a}(\eta) \cdot \frac{\partial v}{\partial \vec{x}} = \frac{\partial m}{\partial \eta} \ ,$$

m being a nonnegative bounded measure on $\mathbb{R}^N_x \times \mathbb{R}^+_t \times \mathbb{R}^1_\eta$, and the velocity field $\vec{a}(\eta)$ is non-degenerate in the sense that it does not "stay" in any hyper-plane of positive co-dimension in $\mathbb{R}^N$, $u(\cdot,t)$ is smoother than $v(\cdot,t,\eta)$. Hence, averaging, which serves as the lifting machinery from the microscopic to the macroscopic level, yields a regularizing effect under the non-degeneracy assumption.

In our case, the homogenized solution, $\bar{v}(\vec{x},t)$, is the average over T^1 of the "density functions" $v(\vec{x},t,\eta) = u^0(-a^*_\eta t\vec{p} + \vec{x})$ which satisfy

$$v_t + a^*_\eta \vec{p} \cdot \frac{\partial v}{\partial \vec{x}} = 0 \quad , \quad v(\vec{x},0,\eta) = u^0(\vec{x}) \ .$$

Since the velocity field, $\vec{a}(\eta) = a^*_\eta \vec{p}$, does degenerate – we have no gain of regularity, as expected in the linear regime. It is interesting to note, however, that the only direction in which there is no weak convergence of u to its limit $\bar{v}$ is $\vec{p}$ – the direction to which the velocity field degenerates.

Since the weak limit $\bar{v}(\vec{x},t)$ is an average of solutions of linear transport equations (which depend on a parameter $\eta \in T^1$), a natural question which arises is whether $\bar{v}(\vec{x},t)$ itself satisfies a similar transport equation. Namely, is there a vector field $\vec{b} = \vec{b}(\vec{x})$, or possibly $\vec{b} = \vec{b}(\vec{x},t)$, which depends only on the original vector field $\vec{a}(\vec{x})$, such that the homogenized solution, $\bar{v}$, is a solution of

$$\bar{v}_t + \vec{b} \cdot \frac{\partial \bar{v}}{\partial \vec{x}} = 0 \ ? \tag{4.29}$$

In the special case where $a^*_\eta \equiv a^*$ $\forall \eta \in [0,1]$, the answer is positive, as stated in the second part of Theorem 4.3; the homogenized equation is then given in (4.27). In the general case, however, the answer is negative: since

$$\bar{v}(\vec{x},t) = \int_0^1 u^0 \left(-a^*_\eta t + x_1, -\gamma a^*_\eta t + x_2\right) d\eta \ , \tag{4.30}$$

we get that

$$\bar{v}_t = -\int_0^1 (u_1^0 + \gamma u_2^0) a_\eta^* d\eta \ , \quad \bar{v}_{x_1} = \int_0^1 u_1^0 d\eta \ , \quad \bar{v}_{x_2} = \int_0^1 u_2^0 d\eta \ , \tag{4.31}$$

where u_1^0 and u_2^0 are the partial derivatives of u^0 evaluated at $(x_1 - a_\eta^* t, x_2 - \gamma a_\eta^* t)$. We now take $u^0(\vec{x})$ to be a function only of x_1. Hence, in view of (4.29) and (4.31), the following equality must hold for all $\vec{x} \in \mathbb{R}^2$, $t \in \mathbb{R}^+$ and any function $u_1^0(\cdot)$:

$$\int_0^1 u_1^0(-a_\eta^* t + x_1) \cdot \Big(b_1(\vec{x}, t) - a_\eta^* \Big) \, d\eta = 0 \ . \tag{4.32}$$

The continuous dependence of a_η^* on η implies that for a given point $(\vec{x}, t)$ and a given choice of $b_1(\vec{x}, t)$, there exists an interval $I = [\eta_0, \eta_1] \subset [0, 1]$ such that $b_1(\vec{x}, t) - a_\eta^* \neq 0 \ \forall \eta \in I$. Let a_m^* and a_M^* denote, respectively, the minimum and the maximum of a_η^* in I. Then, by choosing $u^0(x_1)$ so that $u_1^0 = \frac{du^0}{dx_1}$ is positive in the interval $(x_1 - a_M^* t, x_1 - a_m^* t)$ and zero elsewhere, we get that equality (4.32) cannot hold. This proves that in the general case, the homogenized solution $\bar{v}(\vec{x}, t)$ may not be represented as a solution of a linear transport equation.

Concluding remarks. We have proved in this section the convergence of $u(\cdot, t)$ to its homogenized limit in the strongest possible topology. In the ergodic case, the convergence is strong in $L^\infty_{loc}(\mathbb{R}^2)$ (compare to [3, Theorem 5.1] where the corresponding convergence result is in $L^2_{loc}(\mathbb{R}^2)$; see also [7] where strong convergence in L^∞_{loc} is proved in the *one*-dimensional case). In the nonergodic case, the convergence is in $W^{-1,\infty}(\mathbb{R}^1)$ rather than weak in $L^2(\mathbb{R}^2)$ as proved in [3, 4]; namely, if $\bar{v}(\cdot, t)$ is the homogenized limit of $u(\cdot, t)$, then already *line* integrals of $u(\cdot, t)$ converge to the corresponding *line* integrals of $\bar{v}(\cdot, t)$ (rather than convergence of *double* integrals of $u(\cdot, t)$ to those of $\bar{v}(\cdot, t)$).

We would like to point out that in the presence of oscillatory initial data, (1.14), the convergence in both cases will be in the weaker sense of $W^{-1,\infty}(\mathbb{R}^2)$.

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