On Finite Closures of Homogenized Solutions of Nonlinear Hyperbolic Equations

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Abstract

We study nonlinear equations subject to oscillatory initial data. The oscillatory solution of such problems tends to a homogenized weak limit that is characterized by the corresponding homogenized equations. Those equations usually involve an additional independent variable, so that the weak limit is an average of infinitely many functions. In certain cases, however, there is an alternative description to the weak limit via a closed finite system of equations that the weak limit and some of its moments satisfy. We study the question of an existence of such finite closures in the context of semilinear Boltzmann type equations and the quasilinear Euler equations and show that, in most cases, finite closures do not exist.

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1 Introduction

We study nonlinear hyperbolic equations which are subject to oscillatory initial data. The weak limit of the solution to such problems is characterized by the corresponding *homogenized* equations. Since the effect of homogenization is *averaging* with respect to the oscillatory variable, the usual description of the homogenized weak limit involves the introduction of a new independent variable. For instance, if the problem takes the form

$$\mathcal{H}(t, \mathbf{x}, \partial_t, \nabla_{\mathbf{X}})\mathbf{u} = 0 \quad , \quad \mathbf{u}(\mathbf{x}, 0) = \mathbf{u}^0(\mathbf{x}, \mathbf{x}/\varepsilon) \tag{1.1}$$

where $\mathbf{x} \in \mathcal{R}^d$, $\mathbf{u} = \mathbf{u}(\mathbf{x}, t) \in \mathcal{R}^m$, $\mathcal{H}$ is the nonlinear hyperbolic operator and $\mathbf{u}^0(\mathbf{x}, \mathbf{x}/\varepsilon)$ is the oscillatory initial data with $\mathbf{u}^0(\mathbf{x}, \mathbf{y})$ being a function of $\mathbf{x}$ and $\mathbf{y} \in T^d$ – the d-dimensional unit torus, then the usual description of the homogenized weak limit is

$$\mathbf{u}(\mathbf{x}, t) \underset{\varepsilon\to 0}{\rightharpoonup} \overline{\mathbf{U}}(\mathbf{x}, t) = \int_{T^d} \mathbf{U}(\mathbf{x}, \mathbf{y}, t) d\mathbf{y} \ ,$$

where $\mathbf{U}(\mathbf{x}, \mathbf{y}, t)$ is the solution of the homogenized problem

$$\tilde{\mathcal{H}}(t, \mathbf{x}, \mathbf{y}, \partial_t, \nabla_{\mathbf{X}}, \int d\mathbf{y})\mathbf{U} = 0 \quad , \quad \mathbf{U}(\mathbf{x}, \mathbf{y}, 0) = \mathbf{u}^0(\mathbf{x}, \mathbf{y}) \ . \tag{1.2}$$

Namely, in order to describe the weak limit $\overline{\mathbf{U}}$ of $\mathbf{u}$ – both are functions only of $\mathbf{x}$ and t, we are forced to introduce a new independent variable, $\mathbf{y}$, that encodes the nature of the oscillations, and a corresponding continuum of new unknown functions, $\mathbf{U}(\mathbf{x}, \mathbf{y}, t)$.

The question which we address here is the following: is the introduction of the new independent variable essential? In other words, can we find an alternative description of the weak limit $\overline{\mathbf{U}}(\mathbf{x}, t)$ which is *finite dimensional* in the sense that it employs only a finite number of additional unknown functions? This question may be formulated precisely in the following manner: can we find a finite number of unknown functions, $\mathbf{z}(\mathbf{x}, t) = (z_1(\mathbf{x}, t), \ldots, z_n(\mathbf{x}, t))$, $n \geq m$ (m being the original number of unknowns), which take the typical form

$$z_i(\mathbf{x}, t) = \int_{T^d} f_i(\mathbf{U}(\mathbf{x}, \mathbf{y}, t)) d\mathbf{y} \qquad 1 \leq i \leq n \ , \tag{1.3}$$

where $f_i(\mathbf{U}) = U_i$ for $1 \leq i \leq m$ (so that $(z_1, ..., z_m) = \overline{\mathbf{U}}$) such that $\mathbf{z}$ satisfies a closed system of equations,

$$\hat{\mathcal{H}}(t, \mathbf{x}, \partial_t, \nabla_{\mathbf{X}})\mathbf{z} = 0 \quad , \quad z_i(\mathbf{x}, 0) = \int_{T^d} f_i(\mathbf{u}^0(\mathbf{x}, \mathbf{y})) d\mathbf{y} \qquad 1 \leq i \leq n \ ? \tag{1.4}$$

(throughout this paper, bold faced regular mode letters denote vectors while the corresponding indexed letters denote their components). The vector $\mathbf{z}$, if exists, is referred to as a *finite closure* of $\overline{\mathbf{U}}$.

There are several reasons why this question is interesting:

- First, in the context of semilinear hyperbolic equations there are certain exceptional cases where such a finite closure exists. However, those exceptional cases do not include any of the physically interesting Boltzmann type equations. In §3 we study those equations in order to derive necessary conditions, as well as sufficient conditions, for the existence of a finite closure.

- In the context of the quasilinear equations of fluid dynamics, McLaughlin et al. derived effective equations for the weak limit of oscillatory solutions to Euler equations [5]. Using asymptotic methods, they arrived at a seemingly finite closure consisting of six unknown functions, but they were unable to prove that this augmented system is indeed satisfied by the weak homogenized limit. Moreover, their system was not a genuinely finite dimensional closure, as it involved coefficients that were determined by external evolution equations that depended on the periodic variable y. Hence, there is a will to have a finite closure in this context, but such a finite closure was never found. The question about the existence of such remains.

- The question about the existence of finite closures is interesting also from a computational point of view. Solutions of certain equations are sometimes approximated by obtaining equations for certain moments, as in (1.3), and then truncating those equations in order to obtain a closed system of equations [1, 4]. In view of such closure methods, it is natural to ask whether an *exact* closure may be obtained by a clever choice of moment functions.

We consider here two types of problems: semilinear hyperbolic systems of Boltzmann type and quasilinear equations of fluid dynamics. We show that the answer to the above question for those types of problems is, in general, negative. Our analysis, in both contexts, is based on some fundamental principles which we present in §2. Those principles are interesting for their own sake and are much more general than the questions which they help to answer in the subsequent sections.

In §3 we deal with semilinear hyperbolic equations. We start in §3.1 with the simple ordinary differential equation $u_t - p(u) = 0$. We characterize the set of functions $p(u)$ for which the corresponding initial value problem with oscillatory initial datum has a finite closure. That set does not include $p(u) = -u^2$ that corresponds to the so-called Riccati equation. In §3.2 we consider the generalized Carleman model of the discrete Boltzmann equations; this model is similar to the classical Carleman model but it has a general collision matrix. We show that for almost all collision matrices (and, in particular, the one which corresponds to the classical Carleman model) there is no finite closure. For the complement set of collision matrices (which is of zero measure) a finite closure *does* exist: two additional unknowns may be added to the original two components of the homogenized limit so that the resulting 4-dimensional vector satisfies a closed system of PDEs that are independent of the oscillatory variable. Finally, in §3.3, we extend our discussion to a general 1D model of the Boltzmann equations. We first characterize the weak limit of such equations, Theorem 3.4, thus extending previous results for particular cases such as the Broadwell model. Then, we derive a necessary condition for the existence of a finite closure, Theorem 3.5: if at least one of the equations has a self nonlinear term, then there is no finite closure. The usual physical models fail to satisfy this condition and, consequently, they do not posses a finite closure. Proposition 3.1 provides a sufficient condition for the existence of a finite closure. This sufficient condition is stricter than the necessary condition, hence it remains to deal with the intermediate set of collision matrices. It is not clear whether there is a finite closure for systems having such collision matrices; however, we show that if it does exist, it must take a more general form than simple moments as in (1.3).

In §4 we extend our discussion to equations of fluid dynamics. We concentrate on the Euler equations for three-dimensional inviscid incompressible flows and, using different principles from those used for the semilinear equations, we prove the non-existence of a finite closure in this context as well.

2 Fundamental principles

Our main results in this section are Theorems 2.1-2.3 which play a central role in the non-existence proofs in the subsequent sections.

We start with a basic lemma which is used in proving both Theorems 2.1 and 2.2 and is interesting for its own sake:

Lemma 2.1 *Let $f_k(x)$, $1 \leq k \leq n$, be locally linearly independent on $\Omega \subset \mathcal{R}$; i.e., $\sum_{k=1}^n c_k f_k(x)$ vanishes on an Ω-subset of positive measure only if $c_k = 0$, $1 \leq k \leq n$. Then*

$$V(x_1, ..., x_n) := det(f_i(x_j))_{i,j=1}^n \neq 0 \qquad a.e. \ in \ \Omega^n \ . \tag{2.1}$$

Proof. By induction: if $n = 1$ then indeed $V(x) = f_1(x) \neq 0$ a.e. in Ω by local linear independence. Assume that

$$V(x_1, ..., x_{n-1}) \neq 0 \qquad \text{in} \ \ \Omega^{n-1} \setminus \mathcal{N} \tag{2.2}$$

where $\mathcal{N}$ is of zero measure in Ω^{n-1}. Fix $(x_1, ..., x_{n-1}) \in \Omega^{n-1} \setminus \mathcal{N}$ and consider

$$V(x_1, ..., x_{n-1}, x) = \sum_{k=1}^{n} c_k f_k(x) \ . \tag{2.3}$$

Since $c_n = V(x_1, ..., x_{n-1}) \neq 0$, the sum in (2.3) is a nontrivial linear combination of $\{f_k(x)\}_{1 \leq k \leq n}$. Therefore, by the local linear independence, $V(x_1, ..., x_{n-1}, x) \neq 0$ for all $x \in \Omega \setminus \mathcal{M}$ where $\mathcal{M}$ is of zero measure in Ω. This proves that $V(x_1, ..., x_n) \neq 0$ whenever $(x_1, ..., x_{n-1}) \in \Omega^{n-1} \setminus \mathcal{N}$ and $x_n \in \Omega \setminus \mathcal{M}$ (where $\mathcal{M}$ depends on $(x_1, ..., x_{n-1})$). Hence, since $\mathcal{N}$ and $\mathcal{M}$ are null sets in Ω^{n-1} and Ω, respectively, we conclude that $V \neq 0$ a.e. in Ω^n. $\square$

Example. If $f_k(x) = x^{k-1}$, $1 \le k \le n$, we get that $V(x_1, ..., x_n) = \det(x_j^{i-1})_{i,j=1}^n \neq 0$ a.e. in $\mathcal{R}^n$. Indeed, this is just the Vandermonde determinant which is nonzero everywhere in $\mathcal{R}^n$ apart from the null set of points where $\prod_{i>j}(x_i - x_j) = 0$.

Corollary 2.1 *Assume that the assumptions of Lemma 2.1 hold and that $m > n$. For each point $\mathbf{x} = (x_1, ..., x_m) \in \Omega^m$, let*

$$\mathbf{f}_k(\mathbf{x}) := (f_k(x_1), ..., f_k(x_m)) \ . \tag{2.4}$$

Let $\mathcal{N}$ be the set of points $\mathbf{x} \in \Omega^m$ for which the n vectors $\{\mathbf{f}_k(\mathbf{x})\}_{1 \le k \le n}$ are linearly dependent in $\mathcal{R}^m$. Then $\mathcal{N}$ is of zero measure in Ω^m.

Proof. We can find $m - n$ additional functions, $f_k(x)$, $n < k \le m$, such that $\{f_k(x)\}_{1 \le k \le m}$ are locally linearly independent in Ω. By Lemma 2.1,

$$V(x_1, ..., x_m) = \det(f_i(x_j))_{i,j=1}^m \neq 0 \quad \text{a.e. in } \Omega^m \ .$$

Hence, for almost every $\mathbf{x} \in \Omega^m$, the m vectors $\mathbf{f}_k(\mathbf{x})$, $1 \le k \le m$, are linearly independent and, consequently, so are the first n vectors. $\Box$

Theorem 2.1 *Let $f_k(x) \in C^1(\mathcal{R})$, $1 \le k \le n$, be such that $\{f_k'(x)\}_{1 \le k \le n}$ are locally linearly independent. For a given function $a(y) \in L^\infty[0, 1]$, let $\mathbf{P}_a$ denote the following vector in $\mathcal{R}^n$,*

$$\mathbf{P}_a := \left(\int_0^1 f_k(a(y)) dy \right)_{k=1}^n \ ,$$

and let

$$\mathcal{A} := \{\mathbf{P}_a \ : \quad a \in L^\infty[0, 1]\} \ .$$

Assume that $G : \mathcal{R}^n \to \mathcal{R}$ is a Lipschitz continuous function which vanishes in $\mathcal{A}$, namely,

$$G(\mathbf{P}_a) = 0 \quad \forall a \in L^\infty[0, 1] \ .$$

Then all its first order derivatives, $\partial_k G$, $1 \le k \le n$, exist in $\mathcal{A}$ and equal zero there.

Proof. We first assume that G is a C^1-smooth function. In order to show that $\partial_k G = 0$ in $\mathcal{A}$, we show that $\partial_k G = 0$ in

$$\mathcal{B} := \{\mathbf{P}_a \ : \quad a \in \mathcal{S}\} \ , \tag{2.5}$$

where

$$\mathcal{S} := \left\{ a(y) = \sum_{j=1}^m a_j \chi_{[\frac{j-1}{m}, \frac{j}{m})}(y) \ : \quad m > n \text{ and } (a_1, ..., a_m) \in \mathcal{R}^m \setminus \mathcal{N} \right\} \ , \tag{2.6}$$

$\mathcal{N}$ being the null set such that the n vectors

$$\mathbf{f}_k{}'(\mathbf{x}) = (f_k'(x_1), ..., f_k'(x_m)) \qquad 1 \le k \le n$$

are linearly independent whenever $\mathbf{x} \in \mathcal{R}^m \setminus \mathcal{N}$ (Corollary 2.1). Since $\mathcal{B}$ is dense in $\mathcal{A}$ and $\partial_k G$ are continuous, it will follow that $\partial_k G$ vanishes in $\mathcal{A}$, as required.

Let $a(y)$ be a function in $\mathcal{S}$, let m denote its number of steps and let $b(y)$ be any piecewise constant function with m steps,

$$b(y) = \sum_{j=1}^{m} b_j \chi_{[\frac{j-1}{m}, \frac{j}{m})}(y) \ .$$

Consider the function $g(t) = G(\mathbf{P}_{a+tb})$. Since $g(t) = 0$ for all t, we conclude that

$$g'(t=0) = \sum_{i=1}^{n} \partial_i G(\mathbf{P}_a) \cdot \int_0^1 f_i'(a(y))b(y)dy = 0 \ . \tag{2.7}$$

Since both $a(y)$ and $b(y)$ are piecewise constant functions,

$$\int_0^1 f_i'(a(y))b(y)dy = \frac{1}{m}\mathbf{f}_i{}'(\mathbf{a}) \cdot \mathbf{b} \ ,$$

where $\mathbf{f}_i{}'(\mathbf{a}) = (f_i'(a_j))_{j=1}^m$ and $\mathbf{b} = (b_j)_{j=1}^m$. But $\mathbf{f}_i{}'(\mathbf{a})$, $1 \le i \le n$, are linearly independent and, therefore, for every k, $1 \le k \le n$, we can choose $\mathbf{b}$ so that $\mathbf{f}_i{}'(\mathbf{a}) \cdot \mathbf{b} = 0$ for all $i \ne k$ and $\mathbf{f}_k{}'(\mathbf{a}) \cdot \mathbf{b} \ne 0$. This choice of $\mathbf{b}$ shows that $\partial_k G(\mathbf{P}_a) = 0$. That concludes the proof in case G is assumed to be C^1-smooth.

The proof when G is assumed to be only Lipschitz continuous goes along the same lines: in (2.7), instead of taking the derivative $g'(t=0)$ we consider $\lim_{t\to 0} \frac{g(t)-g(0)}{t}$. Using the chain rule for differences (rather than derivatives) of composite functions, the Lipschitz continuity and the same choices for the test functions $b(y)$, we get that each of the first order derivatives $\partial_k G$ exists in $\mathcal{B}$ and equal zero there. Since $\mathcal{B}$ is dense in $\mathcal{A}$, we conclude that the same holds in $\mathcal{A}$ as well. $\square$

Corollary 2.2 *If in Theorem 2.1 we assume that G is real-analytic, then all its derivatives of any order vanish in $\mathcal{A}$ and, consequently, $G \equiv 0$.*

While Theorem 2.1 is necessary in the context of semilinear equations §3, the next two theorems are essential for our results in the case of quasi-linear equations, §4.

Theorem 2.2 *Given $f_k(x) \in C^1(\mathcal{R})$, $1 \le k \le n$, and an interval I, there exist two distinct monotonically increasing functions on I, $a(\cdot)$ and $b(\cdot)$, for which*

$$\int_I f_k(a(y))dy = \int_I f_k(b(y))dy \qquad 1 \le k \le n \ . \tag{2.8}$$

Proof. Assume, for the sake of simplicity, that $I = [0,1]$. We look for the functions a and b in the class of step functions which take the form

$$u(y) = \sum_{j=1}^{n+1} u_j \chi_{[\frac{j-1}{n+1}, \frac{j}{n+1})}(y) \ .$$

Hence, we need to find two distinct vectors in $\mathcal{R}^{n+1} = \mathcal{R}^n \times \mathcal{R}$, $(\mathbf{a}, \alpha)$ and $(\mathbf{b}, \beta)$, for which

$$\mathbf{F}(\mathbf{a}, \alpha) = \mathbf{F}(\mathbf{b}, \beta) \quad \text{where} \quad \mathbf{F} = (F_1, ..., F_n) \quad \text{and} \quad F_k(u_1, ..., u_{n+1}) = \sum_{j=1}^{n+1} f_k(u_j) \ . \tag{2.9}$$

First, we assume that $\{f_k'(x)\}_{1 \le k \le n}$ are locally linearly independent. Then, by Lemma 2.1,

$$\det\left(\frac{\partial \mathbf{F}}{\partial \mathbf{a}}\right) = \det\Big(f_k'(a_j)\Big)_{k,j=1}^n \neq 0 \quad \text{for almost all } (\mathbf{a}, \alpha) \in \mathcal{R}^{n+1} \ . \tag{2.10}$$

We fix $(\mathbf{a}, \alpha)$ to be a vector for which the inequality in (2.10) holds. Then, by The Implicit Function Theorem, (2.9) defines $\mathbf{b}$ as a function of β in a neighborhood of $(\mathbf{a}, \alpha)$. We may now pick $\beta \neq \alpha$ in that neighborhood such that $(\mathbf{b}(\beta), \beta)$ is a vector in $\mathcal{R}^{n+1}$, different from $(\mathbf{a}, \alpha)$, for which (2.9) holds. Finally, we may rearrange the vectors $(\mathbf{a}, \alpha)$ and $(\mathbf{b}, \beta)$ to be monotonically increasing.

If $\{f_k'(x)\}_{1 \le k \le n}$ are not locally linearly independent, there exists a subset of those functions, say $\{f_k'(x)\}_{1 \le k \le m}$, $m < n$, which are locally linearly independent in some interval Ω and, for $m < k \le n$,

$$f_k(x) = C_k + \sum_{j=1}^{m} c_{k,j} f_j(x) \qquad \text{in } \Omega \ . \tag{2.11}$$

Hence, we may find functions a and b for which (2.8) holds for $1 \le k \le m$ and consequently, by (2.11), also for $k > m$. $\Box$

Theorem 2.2 is actually implied by The Hahn-Banach Extension Theorem. One of the consequences of that theorem states that given a normed linear space, X, a sequence of elements $\{x_k\} \subset X$, a sequence of scalars $\{\alpha_k \in \mathcal{C}\}$ and $\gamma > 0$ such that

$$\left|\sum_{k=1}^{n} \beta_k \alpha_k\right| \le \gamma \left\|\sum_{k=1}^{n} \beta_k x_k\right\| \qquad \text{for all} \quad n \in \mathcal{N} \quad \text{and} \quad \beta_1, \ldots, \beta_n \in \mathcal{C} \ ,$$

there exists a continuous linear functional ϕ on X such that $\phi(x_k) = \alpha_k$ for all $k \in \mathcal{N}$; see [11, Chapter IV, §5, Theorem 2]. To apply this theorem to our case we take $X = L_2(\Omega)$, Ω being a bounded interval in $\mathcal{R}$,

$$x_k = \begin{cases} f_k & 1 \le k \le n \\ g & k = n+1 \\ 0 & k > n+1 \end{cases} \qquad \text{where} \quad g \perp \mathrm{Span}\{f_1, \ldots, f_n\} \qquad ,$$

$\alpha_k = 0$ for all $k \neq n+1$ while $\alpha_{n+1} = 1$ and finally, $\gamma = 1/\|g\|$. Then, there exists a nonnegative Baïre measure $d\mu$ such that

$$\int f_k(t)d\mu(t) = 0 \quad 1 \leq k \leq n \quad \text{and} \quad \int g(t)d\mu(t) = 1 \; .$$

Writing $d\mu$ as a difference of two nonnegative measures, $d\mu = d\mu_a - d\mu_b$, we get that

$$\int f_k(t)d\mu_a(t) = \int f_k(t)d\mu_b(t) \quad 1 \leq k \leq n \quad \text{while} \quad \int g(t)d\mu_a(t) \neq \int g(t)d\mu_a(t) \; .$$

Finally, those two measures are just the Young measures of the functions $a = \mu_a^{-1}$ and $b = \mu_b^{-1}$. Those two distinct and monotonically increasing functions satisfy (2.8).

Theorem 2.3 *Let a_1, a_2 be measurable functions on $[0,1]$ and let a_1^*, a_2^* denote their monotonically increasing rearrangements. Then if $a_1^* \neq a_2^*$, there exists a smooth function b such that $\int_0^1 b(a_1(y))dy \neq \int_0^1 b(a_2(y))dy$.*

Proof. Without loss of generality, we assume that a_1 and a_2 are monotonic increasing. For the sake of simplicity, we assume that they are also continuous.

Let $R_i = [r_i, s_i]$ denote the range of a_i on $[0,1]$, $i = 1,2$. If the two ranges differ, say $R_1 \setminus R_2 \neq \emptyset$, any $0 < b \in C_0^1(R_1 \setminus R_2)$ will yield the desired inequality. If the two ranges coincide, there must be a subinterval $(y_l, y_r) \subset [0,1]$ on which, say, $a_1 > a_2$ and $a_1(y_l) = a_2(y_l) = \alpha_l$, $a_1(y_r) = a_2(y_r) = \alpha_r$. Here, we take b to be a smooth approximation of $b(\alpha) = \alpha \cdot \chi_{[\alpha_l,\alpha_r]}(\alpha)$. $\Box$

3 Semilinear hyperbolic equations

3.1 Riccati-type equations

Let $u(x,t)$ be the solution of the Riccati-type equation

$$u_t - p(u) = 0 \quad , \quad u(x,0) = a(\frac{x}{\varepsilon}) \; , \tag{3.1}$$

where a is a 1-periodic function. The classical Riccati equation corresponds to $p(u) = -u^2$. The solution to (3.1) is given by

$$u(x,t) = P^{-1}(t + P(a)) \quad \text{where} \quad P(u) := \int \frac{du}{p(u)}$$

and, when $\varepsilon \downarrow 0$, it tends in the weak sense to

$$\overline{U}(t) = \int_0^1 U(t,y)dy \quad , \quad U(t,y) = P^{-1}(t + P(a(y))) \tag{3.2}$$

(here and henceforth, the overbar notation indicates the average with respect to the periodic variable). Hence, $\overline{U}(t)$ is an average of a continuum of solutions of the Riccati-type equation, $\{U(t,y)\}_{y\in[0,1]}$. Our question about an existence of a finite closure to the homogenized limit takes the following form in this case: can we introduce a finite number of unknowns,

$$z_i(t) = \int_0^1 f_i(U(t,y))dy \qquad 1 \le i \le n \ , \tag{3.3}$$

where $f_1 = id$ so that $z_1(t) = \overline{U}(t)$, such that $\mathbf{z}(t) = (z_1(t), ..., z_n(t))$ satisfies a closed system of ODEs, the form of which **does not depend** on $a(y)$?

Note that it suffices to consider systems of **the first order**,

$$z_i' = F_i(t, \mathbf{z}) \qquad 1 \le i \le n \ , \tag{3.4}$$

because all the derivatives of $z_i(t)$, (3.3), take a similar form of moments of $U(t,y)$, i.e.,

$$z_i^{(r)}(t) = \int_0^1 f_{i,r}(U(t,y))dy \ .$$

For the sake of well posedness of the corresponding initial value problem, we must assume that all F_i are continuous with respect to t and Lipschitz continuous with respect to $\mathbf{z}$. We may assume that $\{f_i'\}_{1\le i\le n}$ are linearly independent, because, otherwise, we could extract a maximal linearly independent subset $\{f_i'\}_{i\in I\subset\{1,\ldots,n\}}$ and then write a closed system of ODEs for $\{z_i\}_{i\in I}$. Finally, we define

$$\langle p\rangle = \mathrm{Span}\{p_k'\}_{k\in\mathcal{N}} \qquad \text{where } \ p_1 = id \ \text{ and } \ p_k = p_{k-1}'\cdot p \ \ \forall k > 1 \ .$$

Theorem 3.1 *Consider the initial value problem (3.1) whose homogenized weak limit is given by (3.2). This homogenized limit has a finite closure, (3.3)-(3.4), iff $\langle p\rangle$ is finite dimensional. Moreover, if dim$\langle p\rangle = n < \infty$, the minimal finite closure is of dimension n as well.*

Proof. Assume that $\dim\langle p\rangle = \infty$ and that there is a closure of a finite dimension n, (3.4). Then the equality in (3.4) takes the following form when $t = 0$, in view of (3.2)–(3.3):

$$\int_0^1 f_i'(a(y))\cdot p(a(y))dy = F_i\left(0, \int_0^1 \mathbf{f}(a(y))dy\right) \quad 1 \le i \le n \qquad \text{where } \mathbf{f} = (f_k)_{k=1}^n \ . \tag{3.5}$$

Since $f_1' \equiv 1$, equality (3.5) reads when $i = 1$

$$G\left(\int_0^1 \mathbf{f}(a(y))dy, \int_0^1 f_{n+1}(a(y))dy\right) = 0 \quad \forall a \in L^\infty[0,1] \ , \tag{3.6}$$

where

$$f_{n+1} = p = p_2 \ \text{ and } \ G(\mathbf{z}, z_{n+1}) = F_1(0,\mathbf{z}) - z_{n+1} \ . \tag{3.7}$$

Since $\partial_{n+1}G = -1$, namely, the first order derivative $\partial_{n+1}G$ never vanishes, equality (3.6) can hold only if $f_1', ..., f_n', f_{n+1}' = p_2'$ are linearly dependent, as implied by Theorem 2.1. But since the first n functions are linearly independent, we conclude that

$$p_2' \in \mathrm{Span}\{f_i'\}_{1\le i\le n} \ . \tag{3.8}$$

Assume that $p_2' = \sum_{i=1}^n c_i f_i'$. Taking the corresponding linear combination of the equations (3.5), we get that

$$\int_0^1 p_2'(a(y)) \cdot p(a(y))dy = \sum_{i=1}^n c_i F_i \left(0, \int_0^1 \mathbf{f}(a(y))dy\right) \ , \tag{3.9}$$

or, equivalently,

$$G\left(\int_0^1 \mathbf{f}(a(y))dy, \int_0^1 f_{n+1}(a(y))dy\right) = 0 \quad \forall a \in L^\infty[0,1] \ , \tag{3.10}$$

where

$$f_{n+1} = p_2' p = p_3 \ \text{ and } \ G(\mathbf{z}, z_{n+1}) = \sum_{i=1}^n c_i F_i(0, \mathbf{z}) - z_{n+1} \ . \tag{3.11}$$

Since $\partial_{n+1}G = -1$, we conclude, arguing along the same lines as before, that

$$p_3' \in \mathrm{Span}\{f_i'\}_{1\le i\le n} \ . \tag{3.12}$$

Repeating the above arguments, we arrive at the conclusion that all functions p_k' , $k \in \mathcal{N}$, lie in the finite dimensional space $\mathrm{Span}\{f_i'(x)\}_{1\le i\le n}$. That contradicts our assumption that $\dim\langle p\rangle = \infty$.

Assume next that $\dim\langle p\rangle$ is finite-dimensional, namely, there exists n such that for all $\ell > n$,

$$p_\ell = c_0^\ell + \sum_{i=1}^n c_i^\ell p_i \ . \tag{3.13}$$

It can be easily verified that by taking in (3.3) the same value of n as above and $f_i = p_i$ for $1 \le i \le n$, we get that $\mathbf{z}(t)$ satisfies the following closed system:

$$z_i' = z_{i+1} \quad 1 \le i < n \ \text{ and } \ z_n' = c_0^{n+1} + \sum_{i=1}^n c_i^{n+1} z_i \ . \tag{3.14}$$

Moreover, if the actual dimension of $\langle p\rangle$ is $k < n$, the closure system (3.14) may also be reduced to include k equations in k unknowns.

Hence, in the first part of the proof we saw that any closure of the system must have a dimension greater than or equal to $\dim\langle p\rangle$. In the second part we saw that when the latter is finite, there is at least one closure having the same dimension. That concludes the proof. □

Examples.

1. When $p(u) = Cu^r$, the sequence of functions $p_k(u)$ is given by $p_1(u) = u$ and $p_k(u) = \mathrm{Const} \cdot u^{(k-1)r-(k-2)}$ for all $k > 1$. When $r = (n-1)/n$ for any value of $n \in \mathcal{N}$, p_k becomes identically zero for all $k \geq n+2$. Hence, in that case a finite closure exists. Another case when a finite closure exists is $r = 1$, because then all p_k are linear in u. Apart from those two cases, equation (3.1) with $p(u) = Cu^r$ has no finite closure. This includes the regular Ricatti equation that corresponds to $p(u) = -u^2$.

2. When $p(u) = \sqrt{C + u^2}$, the sequence $p_k(u)$ becomes periodic as $p_3(u) = p_1(u) = u$. Hence, there exists a closure of dimension 2 with $z_1 = \overline{U}$ and $z_2 = \overline{p(U)}$.

3.2 The generalized Carleman equations

Consider the generalized Carleman equations

$$u_t + u_x + \alpha u^2 - \beta v^2 = 0 \ , \tag{3.15}$$

$$v_t - v_x + \gamma v^2 - \delta u^2 = 0 \ , \tag{3.16}$$

where $\alpha, \beta, \gamma, \delta$ are nonnegative constants. The classical Carleman equations, which serve as a simple model for the nonlinear discrete Boltzmann equations, correspond to $\alpha = \beta = \gamma = \delta = 1$. Assume that u and v are subject to oscillatory initial data,

$$u(x,0) = u_0(x, \frac{x}{\varepsilon}) \quad , \quad v(x,0) = v_0(x, \frac{x}{\varepsilon}) \ , \tag{3.17}$$

where $u_0(x,y)$ and $v_0(x,y)$ are smooth, non-negative and 1-periodic with respect to y. Then the following holds:

Theorem 3.2 *Let $u = u(x,t)$ and $v = v(x,t)$ be the solution of (3.15)–(3.17) and let $U = U(x,y,t)$ and $V = V(x,y,t)$ be the solution of the corresponding homogenized equations*

$$U_t + U_x + \alpha U^2 - \beta \int_0^1 V^2(x,y,t)dy = 0 \ , \tag{3.18}$$

$$V_t - V_x + \gamma V^2 - \delta \int_0^1 U^2(x,y,t)dy = 0 \ , \tag{3.19}$$

subject to the initial data

$$U(x,y,0) = u_0(x,y) \quad , \quad V(x,y,0) = v_0(x,y) \ . \tag{3.20}$$

Then

$$\left| u(x,t) - U\left(x, \frac{x-t}{\varepsilon}, t\right) \right| \underset{\varepsilon \to 0}{\longrightarrow} 0 \quad \textit{and} \quad \left| v(x,t) - V\left(x, \frac{x+t}{\varepsilon}, t\right) \right| \underset{\varepsilon \to 0}{\longrightarrow} 0 \ .$$

Theorem 3.2 is a special case of Theorem 3.4 which we prove in the next section. This theorem implies that in the weak $W^{-1,\infty}$-sense

$$u(\cdot,t) \rightharpoonup \overline{U}(\cdot,t) = \int_0^1 U(\cdot,y,t)dy \quad \text{and} \quad v(\cdot,t) \rightharpoonup \overline{V}(\cdot,t) = \int_0^1 V(\cdot,y,t)dy$$

when $\varepsilon \downarrow 0$ (consult [7, Lemma 2.1]).

A finite closure to the system (3.15)-(3.17) is a collection of $n \geq 2$ unknowns,

$$\mathbf{z} = (z_1, ..., z_n) \quad , \quad z_i = z_i(x,t) = \int_0^1 f_i(U,V)dy \qquad 1 \leq i \leq n \ , \tag{3.21}$$

where $f_1(U,V) = U$ and $f_2(U,V) = V$, so that $z_1 = \overline{U}$ and $z_2 = \overline{V}$, that satisfy a closed system of equations,

$$\partial_t z_i = F_i(x, t, \mathbf{z}, \partial_x \mathbf{z}, \partial_x^2 \mathbf{z}, ...) \qquad 1 \leq i \leq n \ , \tag{3.22}$$

the form of which does not depend on $u_0(x,y)$ and $v_0(x,y)$. Our main result in this case is as follows:

Theorem 3.3 *The system (3.15)-(3.17) has a finite dimensional closure, (3.21)-(3.22), if and only if $\alpha = \gamma = 0$.*

Proof. First, if $\alpha = \gamma = 0$ we may take

$$z_1 = \overline{U} \ , \ z_2 = \overline{V} \ , \ z_3 = \overline{U^2} \ \text{ and } \ z_4 = \overline{V^2} \ ,$$

and find that these four unknown functions of (x,t), the first two of which are the sought-after weak limits, satisfy a closed system of equations:

$$\partial_t z_1 + \partial_x z_1 - \beta z_4 = 0 \ ,$$

$$\partial_t z_2 - \partial_x z_2 - \delta z_3 = 0 \ ,$$

$$\partial_t z_3 + \partial_x z_3 - 2\beta z_1 z_4 = 0 \ ,$$

$$\partial_t z_4 - \partial_x z_4 - 2\delta z_2 z_3 = 0 \ .$$

Assume next that both α and γ are nonzero and that a finite closure of the form (3.21)-(3.22) does exist (later on, we shall prove our statement also in the case where only one of those two parameters is nonzero). To arrive at contradiction, it suffices to concentrate on the following initial functions,

$$u_0(x,y) = a(y) \quad , \quad v_0(x,y) = b \ , \tag{3.23}$$

where $a(y)$ is 1-periodic and the constant b is selected so that

$$\gamma \cdot b^2 - \delta \cdot \overline{a(y)^2} = 0 \ . \tag{3.24}$$

We evaluate the equality (3.22) at time $t = 0$:

$$z_i\Big|_{t=0} = \int_0^1 f_i(a(y), b)dy = \int_0^1 g_i(a(y))dy \qquad \text{where} \;\; g_i(\cdot) := f_i(\cdot, b) \; . \tag{3.25}$$

Since at $t = 0$ z_i do not depend on x, all of their spatial derivatives are zero at that time:

$$\partial_x^k z_i\Big|_{t=0} = 0 \qquad \forall k \geq 1 \; . \tag{3.26}$$

In order to evaluate $\partial_t z_i$ at $t = 0$ we first find, using (3.18)–(3.20) and (3.23), that

$$U_t\Big|_{t=0} = -U_x - \alpha U^2 + \beta\overline{V^2}\Big|_{t=0} = -\alpha \cdot a(y)^2 + \beta \cdot b^2 \; , \tag{3.27}$$

and, in view of (3.24),

$$V_t\Big|_{t=0} = V_x - \gamma V^2 + \delta\overline{U^2}\Big|_{t=0} = -\gamma \cdot b^2 + \delta \cdot \overline{a(y)^2} = 0 \; . \tag{3.28}$$

The last two equalities imply that

$$\partial_t z_i\Big|_{t=0} = \int_0^1 g_i'(a(y)) \cdot (-\alpha a(y)^2 + \beta b^2)dy \; . \tag{3.29}$$

Hence, in view of (3.25), (3.26) and (3.29), the equations of (3.22) read as follows when $t = 0$:

$$\int_0^1 g_i'(a(y)) \cdot (-\alpha a(y)^2 + \beta b^2)dy = \hat{F}_i \left(\int_0^1 g_1(a(y))dy, ..., \int_0^1 g_n(a(y))dy \right) \; , \tag{3.30}$$

where $\hat{F}_i(\mathbf{z}) = F_i(x, 0, \mathbf{z}, 0, 0, ...)$ (x is viewed here as a parameter).
The proof from this point on goes along the lines of the proof of Theorem 3.1 and, therefore, we shall outline it briefly: Since $g_1 = id$, (3.30) reads as follows for $i = 1$,

$$\int_0^1 (-\alpha a(y)^2 + \beta b^2)dy = \hat{F}_1 \left(\int_0^1 g_1(a(y))dy, ..., \int_0^1 g_n(a(y))dy \right) \; . \tag{3.31}$$

As in the proof of Theorem 3.1, we may assume that the functions $\{g_i'\}_{1 \leq i \leq n}$ are linearly independent. Hence, since $a(y)$ is an arbitrary $L^\infty[0, 1]$-function, we conclude by (3.31) that

$$(-\alpha\xi^2 + \beta b^2)' \in Span\{g_i'(\xi)\}_{1 \leq i \leq n} \; ,$$

or, equivalently, since $\alpha \neq 0$,

$$\xi \in Span\{g_i'(\xi)\}_{1 \leq i \leq n} \; . \tag{3.32}$$

Consequently, using (3.32) and induction, we arrive at the absurd conclusion that

$$\xi^\ell \in Span\{g_i'(\xi)\}_{1 \leq i \leq n} \qquad \forall \ell \in \mathcal{N} \; . \tag{3.33}$$

Finally, we handle the case where $\alpha \neq 0$ and $\gamma = 0$. In the previous case, we chose the initial data to be as in (3.23)+(3.24) in order to have

$$V_t\Big|_{t=0} = 0 \ . \tag{3.34}$$

We can not make this choice in this case since $\gamma = 0$. However, we can still arrange for (3.34) to hold by taking

$$u_0(x,y) = a(y) \quad \text{and} \quad v_0(x,y) = -\delta \cdot \overline{a(y)^2} \cdot x \ . \tag{3.35}$$

With this choice, the initial values of U_t and V_t are:

$$U_t\Big|_{t=0} = -\alpha \cdot a(y)^2 + \beta \cdot \left(\delta \cdot \overline{a(y)^2} \cdot x\right)^2 \quad \text{and} \quad V_t\Big|_{t=0} = 0 \ ,$$

and, hence, we may proceed with the proof as before. □

3.3 A general model for the 1D discrete Boltzmann equations

In the Carleman model it is assumed that all particles move in the x-direction with velocities ± 1. In the Broadwell model, the possible velocities of the particles are ± 1 or 0. Letting u, v, w denote the density numbers of particles with velocity $1, -1, 0$, respectively, we obtain the following equations, [2]:

$$u_t + u_x + uv - w^2 = 0 \ , \tag{3.36}$$

$$v_t - v_x + uv - w^2 = 0 \ , \tag{3.37}$$

$$w_t - uv + w^2 = 0 \ . \tag{3.38}$$

We would like to consider here a general 1D setting that includes both the Carleman and the Broadwell models. Assume that all particles are moving in the x-direction with velocities that may take one of the values $\{c_i\}_{1 \leq i \leq m}$. Denoting the corresponding density numbers by $u_i = u_i(x,t)$, the resulting equations will be of the form

$$\partial_t u_i + c_i \partial_x u_i = \mathrm{Q}^i(\mathbf{u}) \ , \tag{3.39}$$

where $\mathbf{u} = (u_1, ..., u_m)$ and $\mathrm{Q}^i(\mathbf{u}) = \mathbf{u} \cdot Q^i \mathbf{u}$, Q^i being a symmetric collision matrix [8].

We would like to derive the homogenized equations that describe the weak limit of solutions of (3.39) which are subject to oscillatory initial data. To this end, we define the following:

Definition 3.1 *Let* $\mathbf{a} = (a_1, ..., a_m)$ *be a vector in* $\mathcal{R}^m$*. Then:*
(D1) $\mathcal{M}(\mathbf{a})$ *denotes the* $\mathcal{Z}$*-module of vectors in* $\mathcal{Z}^m$ *that are orthogonal to* $\mathbf{a}$*, i.e.,*

$$\mathcal{M}(\mathbf{a}) = \{(k_1, ..., k_m) \in \mathcal{Z}^m \ : \ \sum_i k_i a_i = 0\} \ ;$$

(D2) $\mathcal{M}_{\mathcal{R}}(\mathbf{a})$ *denotes the* $\mathcal{R}$*-subspace of* $\mathcal{R}^m$ *spanned by the vectors of* $\mathcal{M}(\mathbf{a})$ *and* $\mathcal{M}_{\mathcal{R}}(\mathbf{a})^\perp$ *is its orthogonal complement in* $\mathcal{R}^m$*;*
(D3) $\mathcal{PM}_{\mathcal{R}}(\mathbf{a})^\perp$ *is the projection modulo 1 of* $\mathcal{M}_{\mathcal{R}}(\mathbf{a})^\perp$ *on the* m*-dimensional unit torus* T^m*.*

Since $\mathcal{M}_{\mathcal{R}}(\mathbf{a})$ has a base of vectors in $\mathcal{Z}^m$, so does $\mathcal{M}_{\mathcal{R}}(\mathbf{a})^{\perp}$. Let $\{\mathbf{v}^{\ell}\}_{1\leq\ell\leq k} \subset \mathcal{Z}^m$ be such a base. Then the projection modulo 1 of $\mathcal{M}_{\mathcal{R}}(\mathbf{a})^{\perp}$ on the m-dimensional unit torus T^m is:

$$\mathcal{P}\mathcal{M}_{\mathcal{R}}(\mathbf{a})^{\perp} = \left\{ \sum_{\ell=1}^{k} s_\ell \mathbf{v}^{\ell} \quad : \quad s = (s_1, \dots, s_k) \in T^k \right\} \ .$$

Next, let $\mathbf{c} = (c_1, ..., c_m)$ denote the vector of velocities and let $\mathbf{c}^i$ be defined as follows,

$$\mathbf{c}^i = (c_i - c_1, c_i - c_2, ..., c_i - c_m) \ . \tag{3.40}$$

In Theorem 3.4 below, a main role is reserved for the manifolds $\mathcal{P}\mathcal{M}_{\mathcal{R}}(\mathbf{c}^i)^{\perp}$, $1 \leq i \leq m$. We let k^i denote the dimension of $\mathcal{P}\mathcal{M}_{\mathcal{R}}(\mathbf{c}^i)^{\perp}$ and $\{\mathbf{v}^{i,\ell}\}_{1\leq\ell\leq k^i}$ denote the corresponding base, i.e.,

$$\mathcal{P}\mathcal{M}_{\mathcal{R}}(\mathbf{c}^i)^{\perp} = \left\{ \mathbf{z}^i(\mathbf{s}) = \sum_{\ell=1}^{k^i} s_\ell \mathbf{v}^{i,\ell} \quad : \quad \mathbf{s} = (s_1, \dots, s_{k^i}) \in T^{k^i} \right\} \ . \tag{3.41}$$

With these definitions and notations, we may state the following generalization of Theorem 3.2 and [3, Theorem 2.1]:

Theorem 3.4 *Let $\mathbf{u} = \mathbf{u}(x,t)$ be the solution of (3.39) subject to the oscillatory initial data*

$$\mathbf{u}(x, 0) = \mathbf{u}^0(x, \frac{x}{\varepsilon}) \ , \tag{3.42}$$

where $\mathbf{u}^0(x, y)$ is smooth, non-negative and 1-periodic in y. Let $\mathbf{U} = \mathbf{U}(x, y, t)$, $\mathbf{U} = (U_1, ..., U_m)$, be the solution of the corresponding homogenized equations

$$\partial_t U_i + c_i \partial_x U_i = \int_{T^{k^i}} \mathrm{Q}^i(\mathbf{w}^i) d\mathbf{s} \quad \textit{where} \ \ w^i_j = U_j(x, y + z^i_j(\mathbf{s}), t) \qquad 1 \leq j \leq m \ , \tag{3.43}$$

$$\mathbf{U}(x, y, 0) = \mathbf{u}^0(x, y) \ , \tag{3.44}$$

where k^i and $\mathbf{z}^i(\mathbf{s})$ are as in (3.41). Then

$$E(x,t) := \sum_{i=1}^{m} \left| u_i(x,t) - U_i\left(x, \frac{x - c_i t}{\varepsilon}, t\right) \right| \leq \nu(\varepsilon) \underset{\varepsilon \to 0}{\longrightarrow} 0 \ . \tag{3.45}$$

<u>Remarks.</u>

1. We assume that both (3.39)+(3.42) and (3.43)+(3.44) have a smooth and bounded solution up to some time T. Under this assumption, (3.45) holds for all $t \leq T$.

2. The order of magnitude of $\nu(\varepsilon)$ depends on c_i, $1 \leq i \leq m$; more specifically, it depends on some measure of their linear dependence over the integers. When c_i are all rational it turns out that $\nu(\varepsilon) = \mathcal{O}(\varepsilon)$. In general, $\nu(\varepsilon) \geq \mathcal{O}(\varepsilon)$. The reader is referred to [10] for more details.

Proof. Integrating (3.39) along the characteristic curve

$$\mathcal{C}_i = \{ \ (x_i(\tau), \tau) \ : \ 0 \le \tau \le t \ , \text{ where } x_i(\tau) = x - c_i(t-\tau) \text{ and } t \le T \ \} \ , \tag{3.46}$$

we get

$$u_i(x,t) - u_i(x - c_i t, 0) = \int_{\mathcal{C}_i} \mathrm{Q}^i(\mathbf{u}) d\tau \ . \tag{3.47}$$

We do the same for the homogenized equation: in view of (3.43), the function

$$\tilde{U}_i(x,t) = U_i\left(x, \frac{x - c_i t}{\varepsilon}, t\right) \tag{3.48}$$

satisfies

$$\partial_t \tilde{U}_i + c_i \partial_x \tilde{U}_i = \int_{T^{k^i}} \mathrm{Q}^i(\tilde{\mathbf{w}}^i) d\mathbf{s} \quad \text{where } \tilde{w}^i_j = U_j\left(x, \frac{x - c_i t}{\varepsilon} + z^i_j(\mathbf{s}), t\right) \quad 1 \le j \le m \ . \tag{3.49}$$

Integrating (3.49) along $\mathcal{C}_i$ we get

$$\tilde{U}_i(x,t) - \tilde{U}_i(x - c_i t, 0) = \int_{\mathcal{C}_i} \int_{T^{k^i}} \mathrm{Q}^i(\tilde{\mathbf{w}}^i) d\mathbf{s} d\tau \ . \tag{3.50}$$

Subtracting (3.50) from (3.47) and denoting $\tilde{\mathbf{U}} = (\tilde{U}_1, ..., \tilde{U}_m)$ we get that

$$u_i(x,t) - \tilde{U}_i(x,t) = \int_{\mathcal{C}_i} \Big(\mathrm{Q}^i(\mathbf{u}) - \mathrm{Q}^i(\tilde{\mathbf{U}})\Big) d\tau + \int_{\mathcal{C}_i} \Big(\mathrm{Q}^i(\tilde{\mathbf{U}}) - \int_{T^{k^i}} \mathrm{Q}^i(\tilde{\mathbf{w}}^i) d\mathbf{s}\Big) d\tau = E_1 + E_2 \ . \tag{3.51}$$

By the boundedness of u_i and $\tilde{U}_i$, $1 \le i \le m$, there exists a constant K such that

$$|E_1| \le K \cdot \int_{\mathcal{C}_i} E d\tau \ , \tag{3.52}$$

where E is as in (3.45). As for E_2, we define the following function of $m+1$ variables:

$$G^i(\tau, \xi_j) = \mathrm{Q}^i(U_j(x_i(\tau), \xi_j, \tau)) \quad \text{where} \quad 1 \le j \le m \ . \tag{3.53}$$

In (3.53) and henceforth we adopt a notation agreement where a j-indexed term stands for a vector of dimension m whose jth component equals the given term. With those notation agreements, we note, using (3.46), that

$$\int_{\mathcal{C}_i} \mathrm{Q}^i(\tilde{\mathbf{U}}) d\tau = \int_0^t G^i\Big(\tau, \frac{(c_i - c_j)\tau}{\varepsilon} + \frac{x - c_i t}{\varepsilon}\Big) d\tau \ , \tag{3.54}$$

and

$$\int_{\mathcal{C}_i} \int_{T^{k^i}} \mathrm{Q}^i(\tilde{\mathbf{w}}^i) d\mathbf{s} d\tau = \int_0^t \int_{T^{k^i}} G^i\Big(\tau, z^i_j(\mathbf{s}) + \frac{x - c_i t}{\varepsilon}\Big) d\mathbf{s} d\tau \ . \tag{3.55}$$

As $U_j(x,y,t)$ are 1-periodic in y, G^i is 1-periodic with respect to ξ_j, $1 \le j \le m$. Hence, by [10, Theorem 3.5],

$$G^i\Big(\tau, \frac{(c_i - c_j)\tau}{\varepsilon} + \frac{x - c_i t}{\varepsilon}\Big) \underset{\varepsilon \to 0}{\rightharpoonup} \int_{T^{k^i}} G^i\Big(\tau, z^i_j(\mathbf{s}) + \frac{x - c_i t}{\varepsilon}\Big) d\mathbf{s} \quad \text{in } W^{-1,\infty} \ , \tag{3.56}$$

where the $W^{-1,\infty}$ rate of convergence, $\nu(\varepsilon)$, is determined by a measure of the linear dependence of c_i over the integers. Therefore, in view of (3.54)-(3.56),

$$|E_2| \leq \nu(\varepsilon) \underset{\varepsilon \to 0}{\longrightarrow} 0 \ . \tag{3.57}$$

Using (3.52) and (3.57) in (3.51) and taking the sum with respect to $1 \leq i \leq m$, we arrive at the conclusion that

$$E(x,t) \leq K \cdot \sum_{i=1}^{m} \int_{\mathcal{C}_i} E d\tau + \nu(\varepsilon) \ . \tag{3.58}$$

This implies that $\hat{E}(t) := \sup E(\cdot, t)$ satisfies

$$\hat{E}(t) \leq mK \int_0^t \hat{E}(\tau) d\tau + \nu(\varepsilon) \ . \tag{3.59}$$

Since by (3.42), (3.44) and (3.48) $\hat{E}(0) = 0$, (3.59) and Gronwall's inequality imply (3.45). $\square$

Example. If all the velocities, c_i, are commensurate over the integers, we take them to be integral by rescaling x. This is the situation in both the Carleman and the Broadwell models. In this case, the linear manifold in the torus T^m over which the quadratic form Q^i is integrated, (3.43), is a one-dimensional curve:

$$\mathcal{PM}_{\mathcal{R}}(\mathbf{c}^i)^{\perp} = \{\mathbf{c}^i s \ : \quad s \in T^1 = [0,1)\} \ .$$

Hence, the homogenized equations (3.43) read in this case

$$\partial_t U_i + c_i \partial_x U_i = \int_0^1 \mathrm{Q}^i(\mathbf{w}^i) ds \quad \text{where} \ \ w_j^i = U_j(x, y + (c_i - c_j)s, t) \qquad 1 \leq j \leq m \ . \tag{3.60}$$

Next, we aim at showing that the weak limit

$$\mathbf{u} \rightharpoonup \overline{\mathbf{U}} = \int_0^1 \mathbf{U}(x,y,t) dy \tag{3.61}$$

can not be described by a closed system in the unknowns

$$z_i(x,t) = \int_0^1 f_i(\mathbf{U}(x,y,t)) dy \qquad 1 \leq i \leq n \ , \quad n \geq m \ , \tag{3.62}$$

where

$$f_i(\mathbf{U}) = U_i \qquad 1 \leq i \leq m \ . \tag{3.63}$$

To avoid cumbersome notations we restrict ourselves to the above mentioned case where all the velocities are integral. Similar results may be obtained for the more general case.

Theorem 3.5 *Consider the system (3.39) subject to the oscillatory initial data (3.42). Assume that at least one of the equations has a self nonlinear term, namely,*

$$Q^i_{i,i} \neq 0 \qquad \textit{for some } \; 1 \leq i \leq m \; . \tag{3.64}$$

Then the homogenized weak limit (3.61) does not have a finite closure (3.62)-(3.63) that satisfies a closed system (3.22).

Before proving this theorem, we note that both the generalized Carleman equations, (3.15)–(3.16), with $|\alpha|+|\gamma|>0$, and the Broadwell equations, (3.36)–(3.38), satisfy condition (3.64).

Proof. Without loss of generality we may assume that condition (3.64) holds for $k=m$. We assume the existence of a finite closure and arrive at a contradiction, using similar techniques to those presented in previous sections. Let $a(y)$ be an arbitrary 1-periodic function and assume that we may find constants $\{b_i\}_{1\leq i\leq m-1}$ that satisfy

$$\sum_{j,k=1}^{m-1} Q^i_{j,k} b_j b_k + 2\overline{a} \sum_{j=1}^{m-1} Q^i_{j,m} b_j + \overline{a^2} Q^i_{m,m} = 0 \qquad , \quad 1 \leq i \leq m-1 \; . \tag{3.65}$$

Then, taking the initial value

$$\mathbf{u}^0(x,y) = (b_1, ..., b_{m-1}, a(y)) \; , \tag{3.66}$$

we find that:

$$z_i\Big|_{t=0} = \int_0^1 g_i(a(y))dy \qquad \text{where } \; g_i(\cdot) := f_i(b_1,...,b_{m-1},\cdot) \; , \tag{3.67}$$

and

$$\partial_x^k z_i\Big|_{t=0} = 0 \qquad \forall k \geq 1 \; . \tag{3.68}$$

In order to evaluate $\partial_t z_i$ at $t=0$ we compute $\partial_t U_i(x,y,0)$. Using (3.43), we get that

$$\partial_t U_i\Big|_{t=0} = \int_{T^{k^i}} \mathrm{Q}^i\Big(b_1,...,b_{m-1},a(y+z^i_m(\mathbf{s}))\Big)d\mathbf{s} \; . \tag{3.69}$$

Let us now distinguish between two cases. If $i=m$, then the manifold of integration, $\mathcal{PM}_{\mathcal{R}}(\mathbf{c}^m)^\perp$, (3.41), is embedded in $T^{m-1}\times\{0\}$, since the mth entry in $\mathbf{c}^m$ is zero, see (3.40). Therefore, $z^m_m(\mathbf{s})=0$ and, hence, the integrand in (3.69) does not depend on the integration variables $\mathbf{s}$. Consequently,

$$\partial_t U_m\Big|_{t=0} = Q^m_{m,m} a(y)^2 + q_1 a(y) + q_2 \tag{3.70}$$

where q_1, q_2 are some constants and $Q^m_{m,m}$ is nonzero as assumed. If, on the other hand, $i \neq m$, then by (3.41),

$$\mathbf{z}^i_m(\mathbf{s}) = \sum_{\ell=1}^{k^i} v^{i,\ell}_m s_\ell$$

where all of the coefficients $v_m^{i,\ell}$ are integers and at least one of them is nonzero (this is because the mth entry in $\mathbf{c}^i$ is nonzero). Consequently, as $a(\cdot)$ is 1-periodic, it may be easily verified that the k^i-dimensional integral in (3.69) boils down to a one-dimensional integral,

$$\partial_t U_i\Big|_{t=0} = \int_0^1 \mathrm{Q}^i\Big(b_1, ..., b_{m-1}, a(y+s)\Big)ds \qquad 1 \le i \le m-1 \ , \tag{3.71}$$

and, in view of (3.65), it vanishes,

$$\partial_t U_i\Big|_{t=0} = 0 \qquad 1 \le i \le m-1 \ . \tag{3.72}$$

Finally, by (3.62), (3.67), (3.70) and (3.72), we get that

$$\partial_t z_i\Big|_{t=0} = \int_0^1 g_i'(a(y)) \cdot (Q_{m,m}^m a(y)^2 + q_1 a(y) + q_2) dy \ . \tag{3.73}$$

The proof proceeds from this point on along the same lines as in the proofs presented previously.

It may happen that the coefficient matrices will be such that we could not make a selection of constant parameters $\{b_i\}_{1\le i\le m-1}$ so that (3.65) holds. This was the case, for example, with the generalized Carleman equations when $\gamma = 0$. In that case, we could select initial values that depend on x in order to have all the temporal derivatives vanish at $t = 0$ for $1 \le i \le m-1$, (3.72), while the mth temporal derivative is quadratic in the periodic function $a(y)$, (3.70), just as we did in the proof of Theorem 3.3. We omit further details. That completes the proof.□

Theorem 3.5 provides a necessary condition for the existence of a finite closure: there should be no self-nonlinear terms, namely, the evolution equation for u_i must not include the term u_i^2, for all $1 \le i \le m$. It is desirable to derive also a sufficient condition for the existence of a finite closure for systems of the form (3.39). The following proposition provides such a sufficient condition.

Proposition 3.1 *Assume that the coefficient matrices Q^i satisfy:*

$$Q_{i,i}^i = 0 \qquad \forall i \tag{3.74}$$

and

$$Q_{j,k}^i = 0 \qquad \forall i,j,k \quad \textit{such that} \quad i \ne j \ , \ j \ne k \quad \textit{and} \quad k \ne i \ . \tag{3.75}$$

In other words, all entries of Q^i outside the diagonal and outside its i^{th} row and i^{th} column are zero, and also the term $Q_{i,i}^i$ is zero. Then the system (3.39) has the finite closure $\mathbf{z} = (z_1, \dots, z_m, \zeta_1, \dots, \zeta_m)$ where $z_i = \int_0^1 U_i dy$ and $\zeta_i = \int_0^1 U_i^2 dy$, $1 \le i \le m$.

Proof. By averaging equation (3.43) with respect to y we get an evolution equation for z_i. By multiplying (3.43) by U_i and then integrating it with respect to y, we get an evolution equation for ζ_i. It may be easily verified that the above assumptions about the coefficient matrices guarantee that those $2m$ equations are closed, in the sense that their right hand sides include only the terms z_i and ζ_i. □

Let us summarize: according to Theorem 3.5, (3.74) is a necessary condition for the existence of a finite closure; according to Proposition 3.1, the conjunction of (3.74) and (3.75) is a sufficient condition for a finite closure to exist. It is desirable to close the gap between the necessary condition and the sufficient one, namely, to determine what happens with systems that satisfy (3.74) but violate (3.75). An example for such a system is the following one,

$$u_t = vw \quad , \quad v_t - v_x = 0 \quad , \quad w_t + w_x = 0 \ . \tag{3.76}$$

In view of Theorem 3.5, this system *could* have a finite closure, but it is not determined by Proposition 3.1. The homogenized solution in this case is given by averages of the following equations, see (3.60),

$$U_t = \int_0^1 V(x,y+s,t)W(x,y-s,t)ds \quad , \quad V_t - V_x = 0 \quad , \quad W_t + W_x = 0 \ . \tag{3.77}$$

Denoting the weak limits by $z_1 = \overline{U}$, $z_2 = \overline{V}$ and $z_3 = \overline{W}$, we see, using compensated compactness, that they satisfy a closed set of equations (no additional functions are necessary):

$$\partial_t z_1 = z_2 z_3 \quad , \quad \partial_t z_2 - \partial_x z_2 = 0 \quad , \quad \partial_t z_3 + \partial_x z_3 = 0 \ .$$

This example shows that the sufficient condition of Proposition 3.1 could be relaxed in order to include systems like (3.76). Another possibility that should be investigated is whether the necessary condition of Theorem 3.5 could be strengthen.

4 Equations of Fluid Dynamics

Here, we address the question of a finite closure in the context of the equations of fluid dynamics. We concentrate on the three-dimensional inviscid and incompressible Euler equations

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla \mathbf{p} = 0 \quad , \quad \nabla \cdot \mathbf{u} = 0 \qquad (\mathbf{x}, t) \in \mathcal{R}^3 \times \mathcal{R}^+ \ . \tag{4.1}$$

We consider solutions of (4.1) which are subject to initial oscillations,

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}^0(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}) \ , \tag{4.2}$$

$\mathbf{u}^0(\mathbf{x}, \mathbf{y})$ being 1-periodic in y_i, $1 \le i \le 3$. We prove that $\mathbf{v}(\mathbf{x}, t)$, the homogenized weak limit,

$$\mathbf{u}(\mathbf{x}, t) \underset{\varepsilon \to 0}{\rightharpoonup} \mathbf{v}(\mathbf{x}, t) \ ,$$

does not have a finite closure, i.e., it cannot be augmented into a vector of $n > 3$ unknowns, $\mathbf{z} \in \mathcal{R}^n$, which satisfies a well-posed problem of the following form:

$$\mathbf{z}(\mathbf{x}, 0) = \int_{T^3} \mathbf{f}(\mathbf{u}^0(\mathbf{x}, \mathbf{y}))d\mathbf{y} \ , \tag{4.3}$$

$$\mathcal{F}(t, \mathbf{x}, \partial_t, \nabla_{\mathbf{X}})\mathbf{z} = 0 \ . \tag{4.4}$$

In this problem, $\mathbf{z}$ is determined initially by moments of $\mathbf{u}^0(\mathbf{x}, \cdot)$, (4.3), and then it evolves according to (4.4). This form is much more general than the one considered previously for semilinear systems: all we assume about the equations of evolution, (4.4), is that they do not depend on $\mathbf{u}^0$ and that the corresponding initial value problem is well posed; no other assumption is made regarding the order of the equations or even their nature – they can be differential equations, integro-differential or even functional-differential equations. Problem (4.3)+(4.4) is more general than problem (3.62)+(3.22) which we considered in the previous section for the semilinear hyperbolic equations in another respect as well: if there we assumed that z_i equal moments of the fluctuating field for all $t \geq 0$, (3.62), here we assume this relation only at $t = 0$, (4.3).

On the other hand, we do need to make here one restriction that was not necessary before. We must assume that the functions $\mathbf{f} = (f_1, ..., f_n)$ in (4.3) are separable functions of (u_1^0, u_2^0, u_3^0) in the sense that they take the form

$$f_i(u_1^0, u_2^0, u_3^0) = \sum_{j=1}^{m_i} f_{i,1}^j(u_1^0) f_{i,2}^j(u_2^0) f_{i,3}^j(u_3^0) \qquad 1 \leq i \leq n \tag{4.5}$$

(e.g., polynomials).

Our proof in this quasi-linear setting is entirely different from the proofs presented for semilinear equations: instead of relying on Theorem 2.1, we arrive at our conclusion using Theorems 2.2 and 2.3. The key ingredient in our proof is the fact that the following is a solution of the Euler equations (4.1), with $\mathbf{p} \equiv Const$, for any choice of C^1-functions a and b,

$$\mathbf{u}_{a,b}(\mathbf{x}, t) = \left(\ a\left(\frac{x_2}{\varepsilon}\right) \ , \ 0 \ , \ b\left(x_1 - ta\left(\frac{x_2}{\varepsilon}\right)\right) \ \right) \ . \tag{4.6}$$

This solution equals initially to

$$\mathbf{u}_{a,b}(\mathbf{x}, 0) = \mathbf{u}_{a,b}^0\left(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}\right) \qquad \text{where} \ \ \mathbf{u}_{a,b}^0(\mathbf{x}, \mathbf{y}) = (\ a(y_2) \ , \ 0 \ , \ b(x_1) \ ) \ , \tag{4.7}$$

and, when $\varepsilon \downarrow 0$, it tends weakly to

$$\mathbf{v}_{a,b}(\mathbf{x}, t) = \int_0^1 (\ a(y_2) \ , \ 0 \ , \ b(x_1 - ta(y_2)) \ )dy_2 \ . \tag{4.8}$$

Assume that a closure of the form (4.3)+(4.4) did exist. Then, by (4.3) and (4.7), the initial value of $\mathbf{z}$ would be in this case

$$\mathbf{z}_{a,b}(\mathbf{x}, 0) = \int_0^1 \mathbf{f}(a(y), 0, b(x_1))dy \ . \tag{4.9}$$

By Theorem 2.2, we may find two distinct and monotonically increasing functions of $y \in [0,1]$, $a_1(y)$ and $a_2(y)$, such that

$$\int_0^1 f^j_{i,1}(a_1(y))dy = \int_0^1 f^j_{i,1}(a_2(y))dy \qquad \text{for } \ 1 \le i \le n \ , \ 1 \le j \le m_i \tag{4.10}$$

and consequently, by (4.5) and (4.9),

$$\mathbf{z}_{a_1,b}(\mathbf{x},0) = \mathbf{z}_{a_2,b}(\mathbf{x},0) \ .$$

Hence, by the well posedness of the problem and the independence of (4.4) of the initial data,

$$\mathbf{z}_{a_1,b}(\mathbf{x},t) = \mathbf{z}_{a_2,b}(\mathbf{x},t) \qquad \forall t \ge 0 \ . \tag{4.11}$$

Since $\mathbf{z} = (\mathbf{v}, z_4, ..., z_n)$, we conclude by (4.11) and (4.8) that

$$\int_0^1 b(x_1 - ta_1(y))dy = \int_0^1 b(x_1 - ta_2(y))dy$$

for all choices of b and for all x_1 and t. However, this is impossible in view of Theorem 2.3. That concludes the proof that a finite closure of the form (4.3)+(4.4) does not exist. $\Box$

In [5], McLaughlin et al. considered oscillatory solutions of (4.1)+(4.2) and aimed at obtaining effective equations for the weak limit $\mathbf{v}$ using an asymptotic method. In order to obtain a closed system for $\mathbf{v}$, it was augmented into $\mathbf{z} = (\mathbf{v}, \theta, q, r)$, where the three new unknown functions were:

- θ, the Lagrangian coordinate associated with the mean flow (see [5, (3.8)]);
- q, the mean kinetic energy (see (3.28) there); and
- r, the mean helicity (see (3.29) there).

Then, they considered an Euler-like equation in an unknown field $\tilde{w}$, depending on a temporal variable τ and the periodic variable y, [5, (3.17)], and looked for solutions of it that have kinetic energy q, helicity r and mean zero, see (3.30) there. If such solutions exist, they derived a closed system of equations for $\mathbf{z} = (\mathbf{v}, \theta, q, r)$, [5, (3.34)-(3.38)], without proving it. That system, however, is not a genuine finite dimensional closure, since the coefficients of its equations depend on averages of $\tilde{w}$ and those averages are determined by the evolution equations for $\tilde{w}$ that involve the periodic variable y.

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