REGULARITY OF WEAK SOLUTIONS OF THE NONLINEAR FOKKER-PLANCK EQUATION[1]

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Abstract

We study regularity properties of weak solutions of the degenerate parabolic equation $u_t + f(u)_x = K(u)_{xx}$, where $Q(u) := K'(u) > 0$ for all $u \neq 0$ and $Q(0) = 0$ (e.g., the porous media equation, $K(u) = |u|^{m-1}u$, $m > 1$). We show that whenever the solution u is nonnegative, $Q(u(\cdot, t))$ is uniformly Lipschitz continuous and $K(u(\cdot, t))$ is C^1-smooth and note that these global regularity results are optimal. Weak solutions with changing sign are proved to possess a weaker regularity – $K(u(\cdot, t))$, rather than $Q(u(\cdot, t))$, is uniformly Lipschitz continuous. This regularity is also optimal, as demonstrated by an example due to Barenblatt and Zeldovich.

1 Introduction

Consider the nonlinear parabolic equation

$$u_t + f(u)_x = K(u)_{xx} \quad , \quad (x, t) \in \mathbb{R} \times \mathbb{R}^+ \ , \tag{1.1}$$

subject to the Cauchy data

$$u(x, 0) = u_0(x) \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \ , \tag{1.2}$$

where f and K are smooth functions and K is strictly monotonic increasing. This equation is usually called the nonlinear Fokker-Planck equation due to its resemblance to the Fokker-Planck equation of statistical mechanics.

It is well known [8] that if (1.1) is uniformly parabolic, i.e., $Q(u) := K'(u) \geq \varepsilon > 0$, the Cauchy problem (1.1)–(1.2) admits a unique classical solution. We, on the other hand, are interested here in the degenerate case, where $Q(u)$ may vanish for some value of u, say at $u = 0$:

$$Q(u) > 0 \qquad \forall u \neq 0 \quad \text{and} \quad Q(0) = 0 \ . \tag{1.3}$$

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Such degenerate equations arise in the study of several diffusion-advection processes and the simplest example is the porous media equation,

$$u_t = (|u|^{m-1}u)_{xx} \quad , \quad m > 1 \ . \tag{1.4}$$

In the degenerate case classical solutions usually do not exist and, therefore, weak solutions in the sense of distributions are sought:

Definition 1.1 *A bounded function $u(x,t)$ is a weak solution of (1.1)–(1.2) if it satisfies the following equality for every test function $\phi \in C_0^\infty(\mathbb{R}^2)$:*

$$\iint_{\mathbb{R}\times\mathbb{R}^+} [u\phi_t + f(u)\phi_x + K(u)\phi_{xx}]dxdt = -\int_{\mathbb{R}} u_0\phi(\cdot,0)dx \tag{1.5}$$

The existence and uniqueness of weak solutions to the Cauchy problem (1.1)–(1.3), as well as the properties of these solutions, were studied in numerous manuscripts, e.g. [3], [4] and [11]. See also the summary paper of Kalashnikov, [6], and the references therein. In the present study we concentrate on the question of regularity of weak solutions.

Since in most practical applications u is nonnegative, a large part of the study of equation (1.1) concentrates on that case. The most recent results here are summarized below [4, Theorems 1, 4 & 7]:

Theorem 1.1 *(Gilding). Let $f, K \in C[0,\infty) \cap C^{2+\alpha}(0,\infty)$, $\alpha > 0$, and u_0 be nonnegative, bounded and continuous. Then the Cauchy problem (1.1)-(1.3) admits a unique weak solution, $u = u(x,t)$. Moreover, the derivative $K(u)_x$ exists, in the sense of distributions, and is uniformly bounded in $\mathbb{R} \times [\tau, T]$ for any $0 < \tau < T$; if, in addition, $K(u_0)$ is uniformly Lipschitz continuous, $K(u)_x$ is uniformly bounded in $\mathbb{R} \times [0,T]$ for any $T > 0$.*

We note that the regularity result in Theorem 1.1 which states that[2]

$$K(u(\cdot,t)) \in Lip \qquad \forall t > 0 \tag{1.6}$$

is not sharp. Indeed, nonnegative weak solutions of the porous media equation, (1.4), were proved by Aronson [1] to possess a better regularity, namely,

$$Q(u(\cdot,t)) \in Lip \quad \text{and} \quad K(u(\cdot,t)) \in C^1 \qquad \forall t > 0 \ . \tag{1.7}$$

The same type of regularity was established in [5] for nonnegative weak solutions of the equation

$$u_t - (u^n)_x = (u^m)_{xx} \qquad m, n > 1 \ , \tag{1.8}$$

which arises in the theory of infiltration. In §2 we revisit the question of regularity of nonnegative weak solutions of (1.1)+(1.3) and improve (1.6) to (1.7), under mild assumptions

[2] Lip denotes henceforth the space of functions which are uniformly Lipschitz continuous in $\mathbb{R}_x$

on $Q(\cdot)$. This global regularity is optimal in view of explicit examples of weak solutions given in [1] and [5].

The case of solutions with changing sign is essentially different from the case of one-signed solutions in more than one aspect. First, if one-signed solutions are uniquely determined by their initial data, Theorem 1.1, it is not known to be true for solutions with changing sign. To this end, entropy conditions are invoked in order to guarantee uniqueness [11]. The two cases differ also in the issue of regularity. In §3 we show that solutions with changing sign are regular in the sense of (1.6). An example due to Barenblatt and Zeldovich [2] demonstrates the sharpness of this regularity result, as well as the difference between the cases of one-signed and two-signed weak solutions.

2 Nonnegative solutions

Our objective in this section is to obtain improved and, in fact, optimal regularity for non-negative weak solutions of (1.1)+(1.3). We assume here that $f \in C^2$ and $Q \in C^3$ for $u > 0$.

We start with the following Lemma which we prove by using a well known technique due to Bernstein (e.g. [8]). In this Lemma we make the distinction between two cases:
- *Case 1:* $Q'(u) \downarrow 0$ when $u \downarrow 0$.
- *Case 2:* $Q'(u) \geq Const > 0$ when $u \downarrow 0$.

When $Q(u)$ behaves like a power for $u \downarrow 0$, i.e. $Q(u) \sim u^p$, $p > 0$, Case 1 corresponds to $p > 1$ and Case 2 corresponds to $0 < p \leq 1$.

Lemma 2.1 *Let $u = u(x,t)$ be a smooth positive classical solution of (1.1) in $R = (a,b) \times (0,T]$. Assume that for all $u \in (0,\mu]$, $\mu = \max_{\overline{R}} u$,*

$$Q'(u) > 0 \ , \tag{2.1}$$

$$\alpha \leq G(u) := \left(\frac{Q(u)}{Q'(u)}\right)' \leq \beta \ , \quad \text{for some constants } 0 < \alpha \leq \beta, \tag{2.2}$$

and

$$G'(u) \leq \theta \cdot \begin{cases} \frac{\alpha}{\mu} & \text{in Case 1} \\ \frac{1+\alpha}{2Q(\mu)} \cdot Q'(u) & \text{in Case 2} \end{cases} \ , \quad \text{for some constant } \theta \in [0,1) \ . \tag{2.3}$$

Then for any proper subrectangle of R, $R^* = (a_1, b_1) \times (\tau, T]$,

$$|Q(u)_x| \leq C \qquad \text{in } \ \overline{R^*} \ , \tag{2.4}$$

where the constant C depends on $f(\cdot), Q(\cdot), \mu, a_1 - a, b - b_1, \tau$ and is independent of the lower bound of u. If, in addition, $M := \max_{[a,b]} |Q(u_0)_x| < \infty$, then (2.4) holds for $R^* = (a_1, b_1) \times (0, T]$, where C depends on M instead of τ.

Proof. We first make the change of variables $u \mapsto v = Q(u)$. Due to assumption (2.1), this transformation is invertible and $u = q(v),\ q = Q^{-1}$. Equation (1.1) therefore translates to

$$v_t + f'(q(v))v_x = \left(\frac{q''(v)}{q'(v)}v + 1\right) v_x^2 + v v_{xx} \ . \tag{2.5}$$

Let $h(s)$ be defined as follows for $0 < s \leq \nu := Q(\mu)$,

$$h(s) = \begin{cases} q'(s) & \text{in Case 1} \\ 1 & \text{in Case 2} \end{cases} , \tag{2.6}$$

and $H(s) := \int_0^s h(\sigma)d\sigma$. Since, in view of (2.1),

$$h(s) > 0 \qquad \forall s > 0 \ , \tag{2.7}$$

$H(s)$ is positive and monotonically increasing for $s > 0$. Next, we define the function $r = r(\psi)$ by

$$r = \int_0^\psi (2H(\nu) - H(s))^{-1} ds \quad , \quad 0 \leq \psi \leq \nu \ .$$

Since $\frac{dr}{d\psi} = (2H(\nu) - H(\psi))^{-1} \geq (2H(\nu))^{-1} > 0$, the inverse function $\psi = \psi(r)$ exists and is smooth and monotonically increasing for $r \in [0, r(\nu)]$,

$$H(\nu) \leq \frac{d\psi}{dr} = 2H(\nu) - H(\psi) \leq 2H(\nu) \quad , \quad 0 \leq \psi \leq \nu \ . \tag{2.8}$$

Hence, since $0 < v \leq \nu$, the equation $v = \psi(w)$ defines a smooth function $w = w(x,t)$ which takes values in the interval $(0, r(\nu)]$. Substituting $v = \psi(w)$ in (2.5) yields the following equation for w:

$$w_t + f'(q)w_x = \left(\frac{q''}{q'}\psi + 1\right)\psi' w_x^2 + \psi\frac{\psi''}{\psi'}w_x^2 + \psi w_{xx} \ . \tag{2.9}$$

Here, $q^{(i)} = q^{(i)}(\psi(w))$ and $\psi^{(i)} = \psi^{(i)}(w),\ 0 \leq i \leq 2$. Differentiating (2.9) with respect to x and multiplying by $p = w_x$, we arrive at

$$\frac{1}{2}(p^2)_t - \psi p p_{xx} = F_1 \cdot p^4 + F_2 \cdot p^2 p_x - (f''(q)q'\psi' p^3 + f'(q)p p_x) \ , \tag{2.10}$$

where

$$F_1 = (\psi')^2 \cdot \left[\left(\frac{q''}{q'}\right)'\psi + \frac{q''}{q'}\right] + \psi'' \cdot \left[2 + \frac{q''}{q'}\psi\right] + \psi \cdot \left(\frac{\psi''}{\psi'}\right)' \ , \tag{2.11}$$

and

$$F_2 = \psi' \cdot \left[3 + 2\frac{q''}{q'}\psi\right] + 2\psi\frac{\psi''}{\psi'} \ . \tag{2.12}$$

Let $\eta = \eta(x,t)$ be a $C^2(\overline{R})$ function such that $\eta = 1$ on $\overline{R^*}$, $\eta = 0$ in a neighborhood of $x = a,\ x = b$ and $t = 0$, and $0 \leq \eta \leq 1$. Set $z = \eta^2 p^2$ and let $(x_0, t_0) \in R$ be the point in R where z attains its maximal value. Since $z_x = 0,\ z_{xx} \leq 0$ and $z_t \geq 0$ in that point, we conclude that

$$\eta p_x = -\eta_x p\Big|_{(x_0,t_0)} \ , \tag{2.13}$$

and (recall that $\psi \geq 0$)

$$\psi z_{xx} - z_t \leq 0\Big|_{(x_0,t_0)} \ . \tag{2.14}$$

Substituting $z = \eta^2 p^2$ into (2.14) and rearranging, we get that

$$\eta^2 \left\{ \frac{1}{2}(p^2)_t - \psi p p_{xx} \right\} \geq \psi \eta^2 p_x^2 + 4\psi \eta \eta_x p p_x + \psi \eta_x^2 p^2 + \psi \eta \eta_{xx} p^2 - \eta \eta_t p^2 \ . \tag{2.15}$$

Since $|4\psi\eta\eta_x p p_x| \leq \psi\eta^2 p_x^2 + 4\psi\eta_x^2 p^2$, (2.15) implies that

$$\eta^2 \left\{ \frac{1}{2}(p^2)_t - \psi p p_{xx} \right\} \geq -3\psi \eta_x^2 p^2 + \psi \eta \eta_{xx} p^2 - \eta \eta_t p^2 \ . \tag{2.16}$$

We may now conclude, in view of (2.10), (2.13) and (2.16), that the following inequality holds at (x_0, t_0):

$$-F_1 \eta^2 p^4 \leq -\{F_2 \eta_x + f''(q) q' \psi' \eta\} p^3 \eta + \{3\psi \eta_x^2 - \psi \eta \eta_{xx} + \eta \eta_t + f'(q) \eta \eta_x\} p^2 \ . \tag{2.17}$$

Since (2.8) and (2.7) imply that

$$\psi'' = -h(\psi)\psi' < 0 \ , \tag{2.18}$$

we may divide inequality (2.17) by $(-\psi'')$ and get

$$\tilde{F}_1 \eta^2 p^4 \leq \left\{ \tilde{F}_2 \eta_x - \frac{f''(q) q' \eta}{h(\psi)} \right\} p^3 \eta - \frac{1}{\psi''} \left\{ 3\psi \eta_x^2 - \psi \eta \eta_{xx} + \eta \eta_t + f'(q) \eta \eta_x \right\} p^2 \ , \tag{2.19}$$

where $\tilde{F}_i = F_i / \psi''$, $i = 1, 2$.

Our next step is estimating the coefficients in this inequality. We start with some straightforward identities: since $q'(v) = 1/Q'(u)$ and $q''(v)/q'(v) = -Q''(u)/Q'(u)^2$, we conclude, using the definition of $G(u)$, (2.2), that

$$1 + \frac{q''}{q'}\psi = 1 + \frac{q''(v)}{q'(v)} v = 1 - \frac{Q''(u)}{Q'(u)^2} Q(u) = G(u) \tag{2.20}$$

and

$$\left(\frac{q''}{q'} \right)' \psi + \frac{q''}{q'} = \left(\frac{q''(v)}{q'(v)} v \right)' = \frac{d}{dv} \left(-\frac{Q''(u) Q(u)}{Q'(u)^2} \right) = \frac{d}{dv}(G(u) - 1) = \frac{G'(u)}{Q'(u)} \ . \tag{2.21}$$

Furthermore, equality (2.18) implies that

$$\left(\frac{\psi''}{\psi'} \right)' = \psi'' \cdot \frac{h'}{h} \ . \tag{2.22}$$

Hence, in view of (2.11) and equalities (2.18), (2.21) and (2.22), we conclude that

$$\tilde{F}_1 = \frac{F_1}{\psi''} = -\frac{\psi'}{h} \cdot \frac{G'(u)}{Q'(u)} + 2 + \left(\frac{q''}{q'} + \frac{h'}{h} \right) \psi \ . \tag{2.23}$$

In Case 1, $h = q'$ and therefore, by (2.23) and (2.20),

$$\tilde{F}_1 = -\psi' G'(u) + 2 \cdot \left(1 + \frac{q''}{q'}\psi\right) = -\psi' G'(u) + 2G(u) \ . \tag{2.24}$$

Using (2.3) and (2.8) to lower bound the first term on the right hand side of (2.24) (note that $H(\nu) = q(Q(\mu)) = \mu$) and (2.2) to lower bound the second term, we conclude that

$$\tilde{F}_1 \geq 2\alpha(1-\theta) \tag{2.25}$$

in this case. In Case 2, $h = 1$ and therefore, by (2.23) and (2.20),

$$\tilde{F}_1 = -\frac{\psi' G'(u)}{Q'(u)} + 2 + \frac{q''}{q'}\psi = -\frac{\psi' G'(u)}{Q'(u)} + 1 + G(u) \ . \tag{2.26}$$

Using (2.3) and (2.8) to lower bound the first term on the right hand side of (2.26) (note that $H(\nu) = \nu = Q(\mu)$) and (2.2) to lower bound the last term, we conclude that

$$\tilde{F}_1 \geq (1+\alpha)(1-\theta) \tag{2.27}$$

in this case. Hence, we may summarize (2.25) and (2.27) as follows:

$$\tilde{F}_1 \geq \gamma := (1-\theta) \cdot \min(2\alpha, 1+\alpha) > 0 \ . \tag{2.28}$$

We now turn to estimate $\tilde{F}_2$. By (2.12) and (2.20),

$$|\tilde{F}_2| = \left|\frac{F_2}{\psi''}\right| \leq \left|\frac{\psi'}{\psi''}\right| \cdot (1 + 2G(u)) + 2\frac{\psi}{\psi'} \ .$$

By (2.18) and (2.7),

$$\left|\frac{\psi'}{\psi''}\right| = \frac{1}{h(\psi)} \ ;$$

hence, since definition (2.6) implies that

$$\frac{1}{h(\psi)} = \begin{cases} \frac{1}{q'(v)} = Q'(u) & \text{in Case 1} \\ 1 & \text{in Case 2} \end{cases} ,$$

we get that

$$\left|\frac{\psi'}{\psi''}\right| = \frac{1}{h(\psi)} \leq \kappa := \begin{cases} \sup_{0<u\leq\mu} Q'(u) < \infty & \text{in Case 1} \\ 1 & \text{in Case 2} \end{cases} . \tag{2.29}$$

Moreover, by (2.8),

$$0 < \frac{\psi}{\psi'} \leq \frac{\nu}{H(\nu)} \ .$$

Hence, using (2.2) and the above inequalities we conclude that

$$|\tilde{F}_2| \leq \kappa \cdot (1 + 2\beta) + \frac{2\nu}{H(\nu)} \ . \tag{2.30}$$

The last coefficient in (2.19) which needs special consideration is

$$\frac{f''(q)q'\eta}{h(\psi)} \ . \tag{2.31}$$

Once again, we consider separately the two cases in (2.6) and show that the term in (2.31) is uniformly bounded by a constant which depends on f, Q and μ, i.e.,

$$\left|\frac{f''(q)q'\eta}{h(\psi)}\right| \leq Const_{f,Q,\mu} \ . \tag{2.32}$$

Indeed, in Case 1 $h = q'$ and, therefore, $\left|\frac{f''(q)q'\eta}{h(\psi)}\right| = |f''(q)\eta|$ is uniformly bounded by $\sup_{0<u\leq\mu}|f''(u)|$; in Case 2 q' is uniformly bounded for $0 < v \leq \nu$, $h \equiv 1$ and, therefore, (2.32) holds in this case as well.

The rest of the coefficients in (2.19) are also uniformly bounded since, by (2.18), (2.8) and (2.29),

$$\left|\frac{1}{\psi''}\right| = \frac{1}{h(\psi)\psi'} \leq \frac{\kappa}{H(\nu)} \ . \tag{2.33}$$

Hence, returning to (2.19), we conclude by (2.28), (2.30), (2.32) and (2.33) that

$$\gamma\eta^2 p^2 \leq C_1 + \eta C_2|p| \tag{2.34}$$

at (x_0, t_0), where C_1 and C_2 depend on $f, Q, \mu, a_1 - a, b - b_1$ and τ. Using the simple quadratic inequality

$$\frac{2\eta C_2}{\gamma}|p| \leq \eta^2 p^2 + \frac{C_2^2}{\gamma^2} \ ,$$

we conclude by (2.34) that

$$\max_{\overline{R}} z(x,t) = \eta^2 p^2\Big|_{(x_0,t_0)} \leq C_3 := \frac{2C_1}{\gamma} + \frac{C_2^2}{\gamma^2} \ .$$

Hence, $\max_{\overline{R^*}}|w_x| \leq C_3^{\frac{1}{2}}$, and since $v_x = \psi'(w)w_x$ and $|\psi'| \leq 2H(\nu)$, we arrive at (2.4) with $C = 2H(\nu)C_3^{\frac{1}{2}}$.

This proves the first assertion of the Lemma. In order to prove the second assertion we take $\eta = \eta(x)$ to be a $C_0^2[a,b]$-function such that $\eta = 1$ on $[a_1, b_1]$ and $0 \leq \eta \leq 1$, and proceed in the same manner. $\square$

Since the local bound on $|Q(u)_x|$, given in Lemma 2.1, is independent of the lower bound of u, we may conclude the same for nonnegative weak solutions of (1.1)–(1.3) as well. To this end, we first state and prove the following:

Lemma 2.2 *Assume that $Q(u)$ vanishes algebraically fast when $u \downarrow 0$,*

$$Q(u) = cu^p + r(u) \qquad \textit{where} \ \ c > 0 \ , \ p > 0 \ \ \textit{and} \ \ r(u) = o(u^p) \ , \tag{2.35}$$

and that

$$\frac{r(u)}{u^{p+\min(p,1)}} \geq Const \qquad \textit{when} \ \ u \downarrow 0 \ . \tag{2.36}$$

Then there exists $\mu > 0$ such that (2.1)-(2.3) hold for $u \in (0, \mu]$.

Remarks.

1. $Q(u) = mu^{m-1}$, the viscosity coefficient of the porous media equations (1.4) and (1.8), satisfy the conditions of the Lemma with $p = m-1$ and $r(u) = 0$.

2. The above conditions are satisfied by any $Q(u)$ of the form $Q(u) = u^s Q_A(u)$ where $0 \le s < 1$ and $Q_A(u)$ is a real analytic function at $u = 0$.

Proof. Condition (2.1) is clearly satisfied by $Q(u)$ in (2.35) for sufficiently small $u > 0$. Since

$$G(u) = \left(\frac{Q(u)}{Q'(u)}\right)' = 1 - \frac{Q(u)Q''(u)}{Q'(u)^2} = 1 - \frac{c^2 p(p-1)u^{2p-2} + o(u^{2p-2})}{c^2p^2u^{2p-2} + o(u^{2p-2})} \quad ,$$

$\lim_{u\downarrow 0} G(u) = 1/p$ and, therefore, (2.2) holds near $u = 0$ with any $0 < \alpha < 1/p < \beta$. Hence, it remains to prove (2.3). For the sake of simplicity, we assume that $r(u)$ takes an algebraic form, namely $r(u) = du^q + o(u^q)$ where $q > p$. Then, by a simple calculation,

$$G'(u) = -\frac{(q-p)^2(q-p+1)}{p^2} \cdot \frac{d}{c} \cdot u^{q-p-1} + o(u^{q-p-1}) \ . \tag{2.37}$$

Assume that $d > 0$. Then (2.37) implies that $G'(u) < 0$ when $u \downarrow 0$ and, therefore, condition (2.3) is satisfied for small $u > 0$ with $\theta = 0$. If, on the other hand, $d < 0$, we deal separately with the two cases which we introduced earlier:

In Case 1 $p > 1$. Therefore, by (2.36), du^{q-p-1} is bounded from below for $u \downarrow 0$. As d is negative, we conclude that $q - p - 1 \ge 0$. Hence, in view of (2.37), $G'(u)$ remains bounded when $u \downarrow 0$. Since the bound on the right hand side of (2.3) tends to infinity when $\mu \downarrow 0$, we may choose $\mu > 0$ sufficiently small so that (2.3) holds for all $u \in (0, \mu]$.

In Case 2 $0 < p \le 1$. Since, by (2.36), du^{q-2p} is lower bounded for $u \downarrow 0$ and $d < 0$, we conclude that $q - 2p \ge 0$. Therefore, in view of (2.35) and (2.37), $G'(u)/Q'(u) \sim u^{q-2p}$ remains bounded when $u \downarrow 0$. Hence, a sufficiently small $\mu > 0$ may be chosen so that (2.3) will hold for all $u \in (0, \mu]$ in this case as well. $\square$

We may now state and prove the main result of this section:

Theorem 2.1 *Assume that $Q(u)$ satisfies the assumptions of Lemma 2.2 and let $u = u(x,t)$ be the weak solution of (1.1)–(1.3), where $u_0(x)$ is bounded and nonnegative. Then:*
(1) $u(x,t)$ is C^∞-smooth in the neighborhood of points in $\mathbb{R} \times (0,\infty)$ where it is positive;
(2) $Q(u(\cdot,t))$ is locally Lipschitz continuous for all $t > 0$;
(3) The derivative $K(u)_x$ exists and is continuous as a function of x for all $t > 0$; moreover, $K(u)_x = 0$ whenever $u = 0$.

Proof. As in [9], we let $u_\delta(x,t)$ denote the (classical) solution of (1.1)–(1.3) with the uniformly positive Cauchy data $u_\delta(x,0) = u_0(x) + \delta$, $\delta > 0$. By the maximum principle, this sequence of functions is uniformly lower and upper bounded,

$$0 < \delta \le u_\delta(x,t) \le 1 + \sup_{\mathbb{R}} u_0 \qquad \delta \downarrow 0 \ , \ (x,t) \in \mathbb{R} \times \mathbb{R}^+ \ .$$

Since this sequence of smooth functions is also monotonically decreasing, Dini's Theorem implies that it converges uniformly on compact domains to a continuous function $u(x,t)$, which is the weak solution of (1.1)–(1.3).

(1) Since, as argued above, u is continuous in $\mathbb{R}\times(0,\infty)$, each point (x,t), $t>0$, in which $u(x,t)>0$ has a neighborhood where $u>0$ and hence $Q>0$. In this neighborhood, equation (1.1) becomes uniformly parabolic and, therefore, u is C^∞-smooth there.

In view of Part *(1)* of the theorem, we restrict our attention in the proof of Parts *(2)* and *(3)* to points (x,t), $t>0$, where the parabolic equation degenerates, i.e. $u(x,t)=0$.

(2) In view of Lemma 2.2, there exists $\mu>0$ such that (2.1)–(2.3) hold for all $u\in(0,\mu]$. Let (x_0,t_0), $t_0>0$, be a point where $u=0$. Since u is continuous, there exists a rectangle $R\subset\mathbb{R}\times\mathbb{R}^+$, such that $(x_0,t_0)\in R$ and $\max_{\overline{R}} u\le\mu/2$. Hence, thanks to the locally uniform convergence of u_δ to u, we conclude that $0<u_\delta(x,t)\le\mu$ in $\overline{R}$ for sufficiently small δ, say $\delta\le\delta_0$. Applying Lemma 2.1 to u_δ, we conclude that for any proper subrectangle $\overline{R^*}\subset R$ there exists a constant C, which depends on μ but is independent of δ, such that

$$\max_{\overline{R^*}}|Q(u_\delta)_x|\le C \qquad \forall\delta\le\delta_0\ . \tag{2.38}$$

Letting $\delta\downarrow 0$, we find that $Q(u(\cdot,t))$ is Lipschitz continuous in (x_0,t_0) with a local Lipschitz constant less than or equal to C.

(3) Let μ, (x_0,t_0), R, δ_0 and R^* be as above. We fix $0<\delta\le\delta_0$. For any two points (x_1,t_0) and (x_2,t_0) in R^*, it holds

$$\frac{K(u_\delta(x_2,t_0))-K(u_\delta(x_1,t_0))}{x_2-x_1}=\frac{K(u_\delta(x_2,t_0))-K(u_\delta(x_1,t_0))}{Q(u_\delta(x_2,t_0))-Q(u_\delta(x_1,t_0))}\cdot\frac{Q(u_\delta(x_2,t_0))-Q(u_\delta(x_1,t_0))}{x_2-x_1}\ .$$

By (2.38),

$$\left|\frac{Q(u_\delta(x_2,t_0))-Q(u_\delta(x_1,t_0))}{x_2-x_1}\right|\le C\ ,$$

where C is independent of δ. Assumption (2.35) implies that

$$\left|\frac{K(u_\delta(x_2,t_0))-K(u_\delta(x_1,t_0))}{Q(u_\delta(x_2,t_0))-Q(u_\delta(x_1,t_0))}\right|\le\tilde{C}\cdot(u_\delta(x_1,t_0)+u_\delta(x_2,t_0))\ ,$$

where the constant $\tilde{C}$ depends only on the function K. Hence, we conclude in view of the above that

$$\left|\frac{K(u_\delta(x_2,t_0))-K(u_\delta(x_1,t_0))}{x_2-x_1}\right|\le Const\cdot(u_\delta(x_1,t_0)+u_\delta(x_2,t_0))\ ,$$

for all $(x_1,t_0),(x_2,t_0)\in R^*$ and $0<\delta\le\delta_0$. Letting $\delta\downarrow 0$, we conclude that the limit function u satisfies

$$\left|\frac{K(u(x_2,t_0))-K(u(x_1,t_0))}{x_2-x_1}\right|\le Const\cdot\sup_{R^*}u \qquad \forall(x_1,t_0),(x_2,t_0)\in R^*\ . \tag{2.39}$$

Since, by the continuity of u, $\sup_{R^*} u \downarrow 0$ when R^* shrinks to the point (x_0, t_0), inequality (2.39) implies that $K(u)_x$ exists and equals zero at (x_0, t_0). Moreover, since (2.39) holds for every two points in R^*, we get that

$$|K(u)_x| \leq Const \cdot \sup_{R^*} u \qquad \forall (x,t) \in R^*$$

which implies that $K(u)_x$ is a continuous function of x at (x_0, t_0). This concludes the proof. □

Remarks.

1. The Lipschitz continuity of $Q(u(\cdot, t))$ implies, in view of (2.35), that $u(\cdot, t)$ is Hölder continuous with exponent $\min\{\frac{1}{p}, 1\}$.
2. If $p < 1$, u_x exists and is continuous as a function of x for all $t > 0$ and $u_x = 0$ whenever $u = 0$. In order to show this, we observe that (2.38) implies that $|(u_\delta)_x| \leq C \cdot (Q'(u_\delta))^{-1}$ in R^* for $\delta \leq \delta_0$. Since $Q'(u)^{-1} \sim u^{1-p} \downarrow 0$ for $u \downarrow 0$, we may proceed along the lines of the proof of Part *(3)* in order to prove our assertion.
3. As in [5], the regularity of $u(x, t)$ with respect to x implies also regularity with respect to t. We omit further details.

In Theorem 2.1 we established *local* Lipschitz continuity for $Q(u(\cdot, t))$. In order to obtain a uniform estimate, $Q(u)$ must satisfy the conditions of Lemma 2.1 for all values of u and not only for small ones:

Theorem 2.2 *Let $u = u(x, t)$ be the weak solution of (1.1)–(1.3), where $u_0(x)$ is bounded and nonnegative. Assume that there exists $\mu_+ > \mu := \max u_0$ such that (2.1)–(2.3) are satisfied for all $u \in (0, \mu_+]$. Then $Q(u(\cdot, t))$ is uniformly Lipschitz continuous in any domain $\mathbb{R} \times [\tau, T]$, $0 < \tau < T$. If, in addition, $Q(u_0) \in Lip$ then $Q(u(\cdot, t))$ is uniformly Lipschitz continuous in $\mathbb{R} \times [0, T]$.*

Proof. We consider the sequence of classical solutions u_δ, defined in the proof of Theorem 2.1, which converges to the weak solution u as δ tends to 0. The maximum principle implies that for $\delta \leq \mu_+ - \mu$, $\delta \leq u_\delta \leq \mu_+$. Therefore, according to Lemma 2.1, for these values of δ, $Q(u_\delta(\cdot, t))$ are uniformly Lipschitz continuous in $\mathbb{R} \times [\tau, T]$, $0 < \tau < T$ (or in $\mathbb{R} \times [0, T]$ under the further assumption), with a Lipschitz constant which is independent of δ. By letting δ go to 0 we obtain the uniform Lipschitz continuity of $Q(u(\cdot, t))$. □

Example. Consider the general convective porous media equation,

$$u_t + f(u)_x = (u^m)_{xx} \qquad m > 1 \; . \tag{2.40}$$

Equations (1.4) and (1.8), which are special cases of that equation, were studied in [1] and [5]. It was shown there that nonnegative solutions of those equations are uniformly Hölder continuous (with respect to x) with exponent $\min\{\frac{1}{m-1}, 1\}$ in every strip $\mathbb{R} \times [\tau, T]$, $0 < \tau < T$;

moreover, if u_0 is Hölder continuous with the same exponent, then $u(\cdot,t)$ is uniformly Hölder continuous in $\mathbb{R}\times[0,T]$, $T>0$. In addition, $u^m(\cdot,t)$ was shown to be C^1-smooth for all $t>0$.

Our analysis implies that the same type of regularity is shared by nonnegative weak solutions of the more general equation (2.40). Indeed, for that equation $G(u)\equiv(m-1)^{-1}$ and, therefore, conditions (2.1)–(2.3) are satisfied for all $u>0$. Hence, by Theorem 2.2, $Q(u(\cdot,t))=mu(\cdot,t)^{m-1}$ is uniformly Lipschitz continuous (for $0<\tau\le t\le T$ or for $0\le t\le T$ if $Q(u_0)\in Lip$) and that implies the same type of Hölder continuity for $u(\cdot,t)$ as above. Moreover, by Theorem 2.1, $K(u(\cdot,t))=u^m(\cdot,t)$ is C^1-smooth for all $t>0$.

We refer the reader to [1] and [5] for examples of explicit solutions of (1.4) and (1.8) which demonstrate the sharpness of the above regularity results.

3 Solutions with changing sign

Here, we deal with weak solutions of (1.1) without any restriction on their sign; i.e., the weak solution may have a changing sign. When the nonnegativity assumption is removed, it is not known whether weak solutions of (1.1)+(1.3) are uniquely determined by their initial data. Hence, we consider the unique physically relevant weak solutions – these are the solutions which may be realized as a vanishing viscosity solution, $u=\lim_{\varepsilon\downarrow 0}u^\varepsilon$,

$$u_t^\varepsilon+f(u^\varepsilon)_x=K^\varepsilon(u^\varepsilon)_{xx},\qquad K^\varepsilon(u^\varepsilon):=K(u^\varepsilon)+\varepsilon u^\varepsilon\ , \tag{3.1}$$

$$u^\varepsilon(x,0)=u(x,0)=u_0(x)\ . \tag{3.2}$$

These *admissible* or *entropy* solutions are uniquely determined by their initial data (consult [11], where an alternative definition of these entropy solutions is presented). Our goal is to prove that the entropy solutions of (1.1)–(1.2) are regular in the sense of (1.6). In fact, to this end there is no need in the assumption on the mild nature of the degeneracy of the equation, (1.3); instead, we assume just that the viscosity coefficient is nonnegative,

$$Q(u)\ge 0\ , \tag{3.3}$$

thus extending the class of equations under consideration.

The main ingredient in proving the uniform Lipschitz continuity of $K(u(\cdot,t))$ is the following lemma, due to E. Tadmor [10]:

Lemma 3.1 *(Tadmor). Consider the uniformly parabolic equation,*

$$u_t+f(u)_x=K(u)_{xx},\qquad Q(u)=K'(u)\ge\varepsilon>0\ , \tag{3.4}$$

subject to the initial data

$$u(x,0)=u_0(x)\in L^1(\mathbb{R})\cap L^\infty(\mathbb{R})\ . \tag{3.5}$$

Then if $K(u_0)_x$ is uniformly bounded, there exists a constant C, independent of ε, such that

$$\|K(u)_x\|_{L^\infty(\mathbb{R}\times\mathbb{R}^+)}\le C\ . \tag{3.6}$$

Proof. We first recall that, thanks to the uniform parabolicity, (3.4)–(3.5) admits a unique classical solution. After differentiation of (3.4) with respect to t and integration with respect to x, we find that $w(x,t) := \int_{-\infty}^{x} u_t(\xi,t)d\xi$ satisfies

$$w_t + f'(u)w_x = (Q(u)w_x)_x \quad . \tag{3.7}$$

This is a uniformly parabolic linear equation in w and, therefore, by the maximum principle,

$$\|w\|_{L^\infty(\mathbb{R}\times\mathbb{R}^+)} \leq \|w(\cdot,0)\|_{L^\infty(\mathbb{R})} \ . \tag{3.8}$$

But, since equation (3.4) and the definition of w imply that $w = K(u)_x - f(u)$, we conclude by (3.8) and the maximum principle for (3.4) that (3.6) holds with

$$C = 2 \max_{|u|\leq\|u_0\|_{L^\infty}} |f(u)| + \|K(u_0)_x\|_{L^\infty} \ . \tag{3.9}$$

□

Since estimate (3.6) is independent of ε, a similar estimate may be obtained in the degenerate case as well:

Theorem 3.1 *Let u be the unique entropy solution of (1.1)–(1.3), where $u_0 \in W^{1,\infty} \cap BV$. Then the derivative $K(u)_x$ exists in the sense of distributions and it is uniformly bounded in $\mathbb{R}\times\mathbb{R}^+$.*

Remark. This theorem generalizes the regularity result of Theorem 1.1 in three aspects: (i) removing the restriction on the sign of the solution; (ii) allowing a more general type of degeneracy, i.e., (3.3) instead of (1.3); (iii) obtaining a uniform bound, independent of t, for $|K(u)_x|$.

Proof. Let $\{u^\varepsilon(x,t)\}_{\varepsilon>0}$ be the family of classical solutions of the corresponding uniformly parabolic problem (3.1)–(3.2). According to Lemma 3.1,

$$\|K^\varepsilon(u^\varepsilon)_x\|_{L^\infty(\mathbb{R}\times\mathbb{R}^+)} \leq C^\varepsilon \ ,$$

where

$$C^\varepsilon = 2 \max_{|u|\leq\|u_0\|_{L^\infty}} |f(u)| + \|K^\varepsilon(u_0)_x\|_{L^\infty} \ .$$

Since $u_0 \in W^{1,\infty}$,

$$\|K^\varepsilon(u_0)_x\|_{L^\infty} \leq \|K(u_0)_x\|_{L^\infty} + \varepsilon\|(u_0)_x\|_{L^\infty} \leq \|K(u_0)_x\|_{L^\infty} + |u_0|_{W^{1,\infty}} \qquad \forall\varepsilon\in(0,1] \quad .$$

Therefore, for $\varepsilon \in (0,1]$,

$$\|K^\varepsilon(u^\varepsilon(\cdot,t))_x\|_{L^\infty(\mathbb{R})} \leq C \qquad \forall t \geq 0 \ , \tag{3.10}$$

where C is independent of ε and is given by

$$C = 2 \max_{|u|\leq\|u_0\|_{L^\infty}} |f(u)| + \|K(u_0)_x\|_{L^\infty} + |u_0|_{W^{1,\infty}} \quad .$$

Inequality (3.10) implies that

$$\sup_{\phi\in\Phi}\left|\int_{\mathbb{R}} K^\varepsilon(u^\varepsilon)\phi_x dx\right| \le C \qquad \forall t\ge 0 \quad , \tag{3.11}$$

where

$$\Phi = \{\phi \in C_0^\infty(\mathbb{R}) : \ \|\phi\|_{L^1} = 1\} \quad .$$

However, since $u^\varepsilon(\cdot,t)$ converges in $L^1_{loc}(\mathbb{R}_x)$, when $\varepsilon \downarrow 0$, to $u(\cdot,t)$ for all $t>0$, [11], it follows that

$$\left|\int_{\mathbb{R}} K^\varepsilon(u^\varepsilon)\phi_x dx - \int_{\mathbb{R}} K(u)\phi_x dx\right| \le \left|\int_{\mathbb{R}} (K(u^\varepsilon)-K(u))\phi_x dx\right| + \left|\int_{\mathbb{R}} \varepsilon u^\varepsilon \phi_x dx\right| \le \tag{3.12}$$

$$\le \|\phi_x\|_{L^\infty}\cdot \max_{|u|\le\|u_0\|_{L^\infty}} |Q(u)|\cdot\|u^\varepsilon(\cdot,t)-u(\cdot,t)\|_{L^1(Supp\phi)} + \varepsilon\|u_0\|_{L^\infty}\|\phi_x\|_{L^1} \underset{\varepsilon\to 0}{\longrightarrow} 0 \quad .$$

Hence, by (3.11) and (3.12),

$$\sup_{\phi\in\Phi}\left|\int K(u)\phi_x dx\right| \le C \qquad \forall t\ge 0 \quad ,$$

and, therefore, the derivative $K(u)_x$ exists in the sense of distributions and is uniformly bounded in $\mathbb{R}\times\mathbb{R}^+$. $\square$

Example. Consider the porous media equation, (1.4), subject to a compactly supported initial data, $u(x,0)=u_0(x)$. Assume that

$$\int_{\mathbb{R}} u_0(x)dx = 0 \quad \text{and} \quad P := -\int_{\mathbb{R}} x u_0(x) dx > 0 \quad .$$

In [7] it is shown that

$$t^{\frac{1}{m}}\|u(\cdot,t)-z(\cdot,t)\|_{L^\infty} \underset{t\to\infty}{\longrightarrow} 0 \quad ,$$

where $z(x,t)$ is the solution of (1.4) which takes a dipole as initial data, $z(x,0)=\delta'(x)$. This solution, which was published by Barenblatt and Zeldovich [2], is given by

$$z(x,t) = -dt^{-\frac{1}{m}}|\xi|^{\frac{1}{m}} sgn(\xi)\cdot(C-q|\xi|^{\frac{m+1}{m}})_+^{\frac{1}{m-1}}, \qquad (\cdot)_+ = \max(\cdot,0) \ ,$$

where $\xi = xt^{-\frac{1}{2m}}$ and d, C, q are some constants which depend on m and P.

Equation (1.4) degenerates, for this dipole solution, at $x=0$ (where z changes its sign) and at the tips of the compact support, $x = x_\pm(t) = \pm(C/q)^{\frac{m}{m+1}} t^{\frac{1}{2m}}$. Along $x=0$, $z(x,t)$ is Hölder continuous with exponent $\frac{1}{m}$. This demonstrates the sharpness of our estimate that $K(z)_x = (|z|^{m-1}z)_x$ is bounded. Note that, on the other hand, along the interfaces $x_\pm(t)$ the solution is Hölder continuous with exponent $\min\{\frac{1}{m-1},1\}$. Hence, in the neighborhood of these interfaces, where the solution is one-signed, our estimate from §2 holds, namely, $Q(z)_x = (m|z|^{m-1})_x$ is locally bounded.

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