HOMOGENIZATION WITH MULTIPLE SCALES†

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Abstract. We study the homogenization of oscillatory solutions of partial differential equations with a multiple number of small scales. We consider a variety of problems – nonlinear convection-diffusion equations with oscillatory initial and forcing data, the Carleman model for the discrete Boltzman equations, and two-dimensional linear transport equations with oscillatory coefficients. In these problems, the initial values, force terms or coefficients are oscillatory functions with a multiple number of small scales – $f(x, \frac{x}{\varepsilon_1}, ..., \frac{x}{\varepsilon_n})$. The essential question in this context is what is the weak limit of such functions when $\varepsilon_i \downarrow 0$ and what is the corresponding convergence rate. It is shown that the weak limit equals the average of $f(x, \cdot)$ over an affine submanifold of the torus T^n; the submanifold and its dimension are determined by the limit ratios between the scales, $\alpha_i = \lim \frac{\varepsilon_1}{\varepsilon_i}$, their linear dependence over the integers and also, unexpectedly, by *the rate* in which the ratios $\frac{\varepsilon_1}{\varepsilon_i}$ tend to their limit α_i. These results and the accompanying convergence rate estimates are then used in deriving the homogenized equations in each of the abovementioned problems.

To Judith with love.

Contents

1 Introduction

The theory of *homogenization* aims at understanding how the behavior in the microscopic level, in a given physical model, affects the behavior in the macroscopic level. In many models, this problem translates into studying the effects of high-frequency oscillations upon solutions of partial differential equations. In the simplest setting, we are given a problem with two natural length scales – a macroscopic scale of order 1 and a microscopic scale of order ε which measures the period of the oscillations. These oscillations may be introduced into the problem through the coefficients of the equation or through the data. The solution of such problems is usually complicated and hard to compute numerically. In homogenization, we look for the limiting behavior when $\varepsilon \downarrow 0$. The idea is that this limit process will 'average out' the small scale effects and the resulting *homogenized* limit solution will be of a simpler structure.

In several applications, the behavior in the microscopic level is more complex and involves a multiple number of small scales – $\varepsilon_1, ..., \varepsilon_n$. The typical form of the oscillatory functions is then

$$f_\varepsilon(x) = f(x, \frac{x}{\varepsilon_1}, ..., \frac{x}{\varepsilon_n}) \ , \tag{1.1}$$

where $f(x, y)$ is a function of a real variable x and a periodic variable on the n-dimensional unit torus, $y = (y_1, ..., y_n) \in T^n = [0, 1)^n$. Hence, it is only natural that two of the more essential questions in this context are:

- *Question 1:* What is the weak limit of $f_\varepsilon(x)$ when $\varepsilon_i \downarrow 0$?
- *Question 2:* What is the corresponding rate of convergence?

In [14] we studied oscillatory solutions to convection-diffusion equations which are subject to initial and forcing data with modulated one-scale oscillations, i.e., functions of the form (1.1) with $n = 1$. As a first step, we addressed the above two questions; the answer to them in the simple one-scale case is as follows [14, Lemma 2.1]:

Lemma 1.1 *Assume that $f = f(x, y) \in BV(\Omega \times T^1)$ and let $f_\varepsilon(x) := f(x, \frac{x}{\varepsilon})$. Then $f_\varepsilon(x) \rightharpoonup \bar{f}(x) = \int_{T^1} f(x, y)dy$ and*

$$\|f_\varepsilon(x) - \bar{f}(x)\|_{W^{-1,\infty}(\Omega)} \leq C\varepsilon \quad \textit{where} \quad C \sim \|f\|_{L^1(T^1;BV(\Omega))} \ . \tag{1.2}$$

Here, and henceforth, $\Omega = [a, b]$ – a bounded interval in $\mathbb{R}_x$, $\|\cdot\|_{W^{-1,\infty}(\Omega)}$ stands for the $W^{-1,\infty}$-norm in Ω, $\|g(x)\|_{W^{-1,\infty}(\Omega)} = \| \int_a^x g(\xi)d\xi\|_{L^\infty(\Omega)}$, and $BV(\Omega \times T^n)$ is the space of functions $f = f(x, y)$, $x \in \Omega$, $y \in T^n$, which are of bounded variation.

In this paper we are concerned with the multiple scale case, i.e., we study the homogenization of problems which depend on more than one small scale, $n \geq 2$. In all of these problems – linear or nonlinear, one- or two-dimensional, scalar equations or systems – *Questions 1* and *2* play a significant role. Hence, the first part of this paper revolves around these questions.

In our discussion, we assume that $f = f(x, y) \in BV(\Omega \times T^n)$ and view all scales as continuous functions of a common parameter, $\varepsilon_i = \varepsilon_i(\varepsilon) > 0$, such that $\varepsilon_i \downarrow 0$

when $\varepsilon \downarrow 0$. By taking the wave lengthes of the oscillations go to zero, $f_\varepsilon(x)$ tends in a weak sense to a limit $\bar{f}(x)$ which takes the form of an *average* of $f(x,\cdot)$ with respect to some measure on the torus T^n; however, unlike in the one-scale case, it is not clear beforehand what is that measure and what is the corresponding rate of convergence.

In the two-scale case, $n=2$, if the ratio between the scales remains fixed, $\frac{\varepsilon_1}{\varepsilon_2}=\alpha$, *Question 1* is analogous to classical questions in ergodic theory or the theory of numerical integration: if α is irrational, the weak limit is the average of $f(x,\cdot)$ over the entire torus T^2,

$$\bar{f}(x) = \int_{T^2} f(x,y)dy \ ; \tag{1.3}$$

if α is a nonzero rational number, $\frac{m}{n}$, the weak limit is the average of $f(x,\cdot)$ over the projection of the straight line $Span_{\mathbb{R}}\{(n,m)\}$ on T^2,

$$\bar{f}(x) = \int_{T^1} f(x,ny_1,my_1)dy_1 \ . \tag{1.4}$$

Here, however, we consider the more general situation where the ratio between the scales only *tends* to a limit, $\frac{\varepsilon_1}{\varepsilon_2}\to\alpha$. The answer, or better yet, the array of answers which we reveal here is surprising and is of interest both in the theoretical level and in the practical level, as we demonstrate later. T. Hou dealt with that situation in [7]; in his analysis he assumed that $r := \frac{\varepsilon_1}{\varepsilon_2}-\alpha$ tends to zero *faster* than ε_1 and ε_2. This assumption, however, turns out to be equivalent to assuming that the ratio between the scales is fixed (Lemma 2.1). Although he observed that the average in (1.4) is no longer the weak limit when $|r| \geq \mathcal{O}(\varepsilon_1,\varepsilon_2)$, he did not pursue the study in that direction.

In §2, §2.1–§2.3, we complete the task and unveil the entire picture in the two-scale case. If α is zero or irrational, we prove that the weak limit is as in (1.3), *regardless* of the rate in which r vanishes (Theorems 2.1 and 2.4). If, however, α is a nonzero rational number, the weak limit depends on the value of α and, in addition, on the *rate* in which α is approached by $\frac{\varepsilon_1}{\varepsilon_2}$, namely – the order of magnitude of r. In Theorem 2.2 we show that (1.4) holds only when $|r| \ll \mathcal{O}(\varepsilon_1,\varepsilon_2)$. If $|r| = \mathcal{O}(\varepsilon_1,\varepsilon_2)$, $\bar{f}(x)$ takes a similar form of an f-average over an affine curve in T^2; that curve, which may depend on x, is parallel to the linear curve along which the integral in (1.4) is taken. Finally, if $|r| \gg \mathcal{O}(\varepsilon_1,\varepsilon_2)$, the weak limit switches unexpectedly from a one-dimensional average to the two-dimensional average in (1.3), Theorem 2.3.

Regarding *Question 2* about the convergence rate: in the cases where α is rational, our convergence proofs are accompanied by sharp convergence rate estimates; this question is far more complicated when α is irrational and we address it in §2.4 by adopting ideas from number theory and the theory of quasi-Monte Carlo methods; the necessary terms and results from these theories are reviewed briefly in Appendix A.

To conclude §2, we provide in §2.5 convincing visual illustrations of our weak convergence results.

In §3 we extend our discussion to the case of a multiple number of scales. We show that, like in the two-scale case, the weak limit of $f_\varepsilon(x)$ is an average of $f(x,\cdot)$ over an affine submanifold of T^n. The manifold and its dimension are determined by the limit ratios between the scales (in particular, on their linear dependence over the integers) and on the rates in which these limit ratios are approached. One of the key points in this context is the introduction of an equivalence relation $\sim$ on the set of scales $\mathcal{S} = \{\varepsilon_i\}_{1\le i\le n}$. This relation enables us to reduce the problem of homogenization of f with respect to $\mathcal{S}$ to a problem of homogenization of another function with respect to the smaller set of scales $\mathcal{S}/\sim$. In other words, with this relation we are able to detect 'redundancies' in $\mathcal{S}$ and to eliminate redundant scales.

In §4 we apply our weak convergence analysis to a variety of homogenization problems. In §4.1 we apply our results to homogenization of nonlinear convection-diffusion equations. We consider initial value problems for nonlinear equations of mixed hyperbolic-parabolic type, where the initial and forcing data are oscillatory. The homogenized limit solution satisfies the same equation with the corresponding averaged initial and forcing data. Moreover, if the solution operator of the equation is compact, the oscillatory solution tends to its homogenized limit in a strong sense, for all positive time after an initial layer. In this context, we provide a most illuminating example which demonstrates why our refined weak convergence analysis is important not only theoretically, but also for practical applications where the value of the small scales is fixed and there is no limiting process involved.

In §4.2 we briefly discuss an application to homogenization of discrete Boltzman equations. One of the simplest models for these equations is the Carleman model where the density functions satisfy a 2×2 semilinear hyperbolic system. We consider these equations subject to initial data with modulated two-scale oscillations. Combining the techniques presented in [7] and our weak convergence results of §2, we obtain pointwise error estimates for the oscillatory solutions of these equations.

§4.3 is devoted to homogenization of two-dimensional linear flows with oscillatory velocity fields. Here, the oscillations are introduced through the coefficients of the equation. The weak limit of the solution depends on the *rotation number* which is associated with the flow and on the correlation between the two small scales: this limit solution is either a solution of a linear transport equation with constant coefficients (in which case the convergence of the oscillatory solutions is pointwise) or an average of solutions of parameter-dependent linear transport equations with possibly variable coefficients (in which case the convergence is in the weak $W^{-1,\infty}$-sense).

Finally, in §4.4 we demonstrate the connection between our weak convergence analysis and the classical problem of studying the motion of an harmonic oscillator in several dimensions; in addition, we use our analysis to study the motion of quasi-harmonic oscillators, where the frequency of the oscillations is not constant.

2 $W^{-1,\infty}$-Convergence Analysis with Two Small Scales

Throughout this section, $y = (y_1, y_2) \in T^2$, $f = f(x, y) \in BV(\Omega \times T^2)$ and $f_\varepsilon(x) = f(x, \frac{x}{\varepsilon_1}, \frac{x}{\varepsilon_2})$. We define

$$\alpha = \lim_{\varepsilon\to 0} \frac{\varepsilon_1}{\varepsilon_2} \quad \text{and} \quad r = \frac{\varepsilon_1}{\varepsilon_2} - \alpha \ . \tag{2.1}$$

With this, it is convenient to identify the common parameter ε with ε_2 and then

$$\varepsilon_1 = \alpha\varepsilon + \delta \quad \text{where} \quad \delta = r\varepsilon = o(\varepsilon) \ . \tag{2.2}$$

Our convergence rate estimates will be given in terms of ε and r.

We separate the discussion into the following three cases:

2.1 Case 1: Zero limit

Here we deal with the case $\alpha = 0$. This is the simplest case since the two small scales are of different orders of magnitude and, therefore, they do not interact. Hence, the limit process can be separated into two successive limits. Here and henceforth $Lip(y_i)$ (or $Lip(x)$) denotes the class of functions $f \in BV(\Omega \times T^n)$ which are uniformly Lipschitz continuous with respect to y_i (respectively, x) in $\Omega \times T^n$.

Theorem 2.1 *Assume that* $\frac{\varepsilon_1}{\varepsilon_2} \to 0$ *and that* $f \in Lip(x) \cap Lip(y_2)$. *Then*

$$f_\varepsilon(x) \rightharpoonup \bar{f}(x) = \int_{T^2} f(x,y)dy \ , \tag{2.3}$$

and

$$\|f_\varepsilon(x) - \bar{f}(x)\|_{W^{-1,\infty}(\Omega)} \leq Const \cdot \left(\varepsilon_2 + \frac{\varepsilon_1}{\varepsilon_2}\right) = Const \cdot (\varepsilon + r) \ . \tag{2.4}$$

Proof. Defining

$$g(x,y_1) = f(x,y_1,\frac{x}{\varepsilon_2}) \ , \qquad \bar{g}(x) = \int_{T^1} g(x,y_1)dy_1 \ ,$$

and

$$h(x,y_2) = \int_{T^1} f(x,y_1,y_2)dy_1 \ , \qquad \bar{h}(x) = \int_{T^1} h(x,y_2)dy_2 \ ,$$

the difference in (2.4) may be decomposed as follows:

$$\|f_\varepsilon(x) - \bar{f}(x)\|_{W^{-1,\infty}(\Omega)} \leq \|g(x,\frac{x}{\varepsilon_1}) - \bar{g}(x)\|_{W^{-1,\infty}(\Omega)} + \|h(x,\frac{x}{\varepsilon_2}) - \bar{h}(x)\|_{W^{-1,\infty}(\Omega)} \ . \tag{2.5}$$

Using Lemma 1.1 for the two terms on the right hand side of (2.5), we get that

$$\|f_\varepsilon(x) - \bar{f}(x)\|_{W^{-1,\infty}(\Omega)} \leq Const \cdot \left(\|g\|_{L^1(T^1;BV(\Omega))} \cdot \varepsilon_1 + \|h\|_{L^1(T^1;BV(\Omega))} \cdot \varepsilon_2\right) \ . \tag{2.6}$$

Finally, since the assumed regularity of f implies that

$$\|g\|_{L^1(T^1;BV(\Omega))} \leq Const \cdot \varepsilon_2^{-1} \ , \tag{2.7}$$

(2.4) follows from (2.6) and (2.7). □

The next two subsections will be devoted to the case where the limit ratio α is nonzero, i.e., the two small scales are of the same order of magnitude. Here, the following observation is most important:

Lemma 2.1 *Assume that*

$$\exists c \in \mathbb{R} \quad \textit{such that} \quad \frac{r}{\varepsilon} \to c \; . \tag{2.8}$$

Then

$$\| f_\varepsilon(x) - g(x, \frac{x}{\alpha\varepsilon}, \frac{x}{\varepsilon}) \|_{W^{-1,\infty}(\Omega)} \le Const \cdot \left(\frac{r^2}{\varepsilon} + \left| \frac{r}{\varepsilon} - c \right| \right) \tag{2.9}$$

where

$$g(x, y_1, y_2) = f(x, y_1 - \frac{cx}{\alpha^2}, y_2) \; . \tag{2.10}$$

Remark. In case f is not of bounded variation, but it is in the Sobolev space $W^{\theta,1}_{loc}$, $\theta < 1$, we have, instead of (2.9),

$$\| f_\varepsilon(x) - g(x, \frac{x}{\alpha\varepsilon}, \frac{x}{\varepsilon}) \|_{W^{-1,\infty}(\Omega)} \le Const \cdot \left(\frac{r^2}{\varepsilon} + \left| \frac{r}{\varepsilon} - c \right| \right)^\theta \; .$$

This lemma (which is a special case of Proposition 3.2 and, therefore, is not proved here) implies that when α is approached by the ratio $\frac{\varepsilon_1}{\varepsilon_2}$ sufficiently fast (namely, $|r| \le \mathcal{O}(\varepsilon)$) the weak limit of $f_\varepsilon(x)$ equals that of $g(x, \frac{x}{\alpha\varepsilon}, \frac{x}{\varepsilon})$ – a similar function whose two small scales, $\alpha\varepsilon$ and ε, are of a fixed ratio. Hence, we shall refer to the case where (2.8) holds as *the almost fixed ratio case*; the case where $|r| > \mathcal{O}(\varepsilon)$, i.e.,

$$\frac{|r|}{\varepsilon} \to \infty \; , \tag{2.11}$$

is *the genuinely variable ratio case.* We note that the assumption made in [7] was of an almost fixed ratio with $c = 0$; under this assumption, $g = f$ and, therefore, we can simply replace $\varepsilon_1(\varepsilon)$ with $\alpha\varepsilon$ and pass to the weak limit.

2.2 Case 2: A nonzero rational limit

Here we deal with the case where the limit ratio α is a nonzero rational,

$$\frac{\varepsilon_1}{\varepsilon_2} \to \alpha = \frac{m}{n} \in \mathbb{Q}^* = \mathbb{Q} \setminus \{0\} \; . \tag{2.12}$$

In the almost fixed ratio case, Lemma 2.1 implies the following:

Theorem 2.2 *Assume (2.12) holds and that the ratio is almost fixed, (2.8). Then if f is locally of bounded variation,*

$$f_\varepsilon(x) \rightharpoonup \bar{f}(x) = \int_{T^1} f(x, ny_1 - \frac{cx}{\alpha^2}, my_1) dy_1 \; , \tag{2.13}$$

and

$$\| f_\varepsilon(x) - \bar{f}(x) \|_{W^{-1,\infty}(\Omega)} \le Const \cdot \left(\varepsilon + \left| \frac{r}{\varepsilon} - c \right| \right) \; . \tag{2.14}$$

Proof. In view of Lemma 2.1, the weak limit of $f_\varepsilon(x)$ equals that of $g(x, \frac{x}{\alpha\varepsilon}, \frac{x}{\varepsilon})$ where g is defined in (2.10). Defining $\tilde\varepsilon = m\varepsilon$ and $\tilde g(x, y_1) = g(x, ny_1, my_1)$, we get that

$$g(x, \frac{x}{\alpha\varepsilon}, \frac{x}{\varepsilon}) = \tilde g(x, \frac{x}{\tilde\varepsilon}) \ .$$

By Lemma 1.1,

$$\|\tilde g(x, \frac{x}{\tilde\varepsilon}) - \int_{T^1} \tilde g(x, y_1)dy_1\|_{W^{-1,\infty}(\Omega)} \le Const \cdot \tilde\varepsilon \ \text{ where } Const \sim \|\tilde g\|_{L^1(T^1;BV(\Omega))} \ . \tag{2.15}$$

But the definitions of g and $\tilde g$ imply that $\int_{T^1} \tilde g(x, y_1)dy_1$ equals the weak limit $\bar f(x)$ in (2.13); this proves (2.13). The error estimate in (2.14) follows from the error estimates in (2.9) and (2.15). □

Remark. The order of magnitude of the constant in error estimate (2.14) depends linearly on the order of magnitude of m and n; this can be seen by noting that the constants in (2.9) and (2.15) are independent of m and n while in (2.15) $\tilde\varepsilon = m\varepsilon$.

The situation is completely different when the ratio is genuinely variable:

Theorem 2.3 *Assume (2.12) holds and that the ratio is genuinely variable, (2.11). Then if $f \in Lip(x) \cap Lip(y_2)$,*

$$f_\varepsilon(x) \rightharpoonup \bar f(x) = \int_{T^2} f(x, y)dy \ , \tag{2.16}$$

and

$$\|f_\varepsilon(x) - \bar f(x)\|_{W^{-1,\infty}(\Omega)} \le Const \cdot \left(\frac{\varepsilon}{|r|} + |r|\right) \ . \tag{2.17}$$

Proof. For the sake of simplicity, let us assume first that $\alpha = 1$. Then, by (2.2) $f_\varepsilon(x) = f(x, \frac{x}{\varepsilon+\delta}, \frac{x}{\varepsilon})$ where, by (2.11), $\frac{\varepsilon^2}{\delta} \to 0$. We now consider the function $g(x, y_1, y_2) = f(x, y_1, y_1 + y_2)$ and observe that

$$f(x, \frac{x}{\varepsilon+\delta}, \frac{x}{\varepsilon}) = g(x, \frac{x}{\eta_1}, \frac{x}{\eta_2}) \quad \text{where} \quad \eta_1 = \varepsilon + \delta \ , \ \eta_2 = \frac{\varepsilon^2 + \varepsilon\delta}{\delta} \ . \tag{2.18}$$

Since $\varepsilon \gg |\delta| \gg \varepsilon^2$, we have that

$$\eta_1 \sim \varepsilon \to 0 \quad \text{and} \quad \eta_2 \sim \frac{\varepsilon^2}{\delta} \to 0 \ . \tag{2.19}$$

Moreover,

$$\frac{\eta_1}{\eta_2} \sim \frac{\delta}{\varepsilon} \to 0 \ . \tag{2.20}$$

Hence, in light of (2.19)–(2.20), we may apply Theorem 2.1 to g and conclude that

$$\|g(x, \frac{x}{\eta_1}, \frac{x}{\eta_2}) - \bar g(x)\|_{W^{-1,\infty}(\Omega)} \le Const \cdot \left(\frac{\varepsilon^2}{|\delta|} + \frac{|\delta|}{\varepsilon}\right) \ , \tag{2.21}$$

where

$$\bar{g}(x) = \int_0^1 \int_0^1 g(x, y_1, y_2) dy_1 dy_2 \tag{2.22}$$

(note that g is as regular as f with respect to x and y_2 and, therefore, satisfies the assumption of Theorem 2.1). Since the definition of g and the 1-periodicity of f imply that $\bar{g}(x) = \bar{f}(x) = \int_{T^2} f(x, y)dy$, (2.16)–(2.17) follow from (2.18) and (2.21).

The general case may be reduced to the case $\alpha = 1$ by introducing the notations $\tilde{\varepsilon} = m\varepsilon$, $\tilde{\delta} = nr\varepsilon$ and rewriting our function as

$$f_\varepsilon(x) = \tilde{f}(x, \frac{x}{\tilde{\varepsilon} + \tilde{\delta}}, \frac{x}{\tilde{\varepsilon}}) \ , \tag{2.23}$$

where $\tilde{f}(x, y_1, y_2) = f(x, ny_1, my_2)$. Applying the above analysis to $\tilde{f}$ and observing that $\int_{T^2} \tilde{f}(x, y)dy = \int_{T^2} f(x, y)dy$, we arrive at (2.16) and (2.17). $\square$

Later, we provide several examples to illustrate the results of this section. However, we would like to give here one example, taken from [7, Remark 3.1]. As mentioned earlier, T. Hou concentrated on studying the almost fixed ratio case (2.8) with $c = 0$; namely – he assumed that $|r| \ll \varepsilon$. Under this assumption, the weak limit when $\alpha = \frac{m}{n}$ is as in (1.4). He observed, however, that the weak limit is different when this assumption does not hold. As an example, he considered the function $f(y_1, y_2) = \cos(2\pi y_1)\sin(2\pi y_2)$, with $\varepsilon_1 = \varepsilon + \varepsilon^2$. Here, $\alpha = 1$, $r = \varepsilon$ and, consequently, $c = 1$. A direct computation showed that

$$\int_0^{\frac{1}{2}} f(\frac{x}{\varepsilon + \varepsilon^2}, \frac{x}{\varepsilon})dx = \frac{1}{2\pi} + \mathcal{O}(\varepsilon) \tag{2.24}$$

in disagreement with the weak limit predicted by (1.4) – $\int_0^1 \int_0^1 f(y_1, y_2)dy_1 dy_2 = 0$.

Theorem 2.2 provides the answers for that: the weak limit in this case is, according to (2.13),

$$\bar{f}(x) = \int_0^1 f(y_1 - x, y_1)dy_1 = \frac{1}{2}\sin(2\pi x) \ ,$$

and the $W^{-1,\infty}$ convergence rate estimate is, in view of (2.14), $\mathcal{O}(\varepsilon)$. Indeed,

$$\int_0^{\frac{1}{2}} \bar{f}(x)dx = \frac{1}{2\pi}$$

and therefore, by (2.24),

$$\int_0^{\frac{1}{2}} \left(f(\frac{x}{\varepsilon + \varepsilon^2}, \frac{x}{\varepsilon}) - \bar{f}(x) \right) dx = \mathcal{O}(\varepsilon)$$

in agreement with Theorem 2.2.

The integral in (2.13) is taken along a closed spiral curve in T^2. The larger are m and n – the longer is the curve. Let α be an irrational number and let $\{\frac{m_k}{n_k}\}_{k \in \mathbb{N}}$ be a sequence of rational numbers which converges to α as $k \to \infty$. Let $\mathcal{L}_k = \{(n_k y_1, m_k y_1) : \ y_1 \in T^1\}$ be a typical curve in T^2 associated with $\frac{m_k}{n_k}$ by (2.13). Then, since $m_k, n_k \to \infty$, the length of $\mathcal{L}_k$ tends to infinity as $\frac{m_k}{n_k} \to \alpha$

and the "limit-curve", so to speak, covers the entire torus T^2. Hence, it is natural to expect that when $\frac{\varepsilon_1}{\varepsilon_2} \to \alpha$, α irrational, the corresponding weak limit of $f_\varepsilon(x)$ will take the form of a two-dimensional integral over T^2, like in (2.16), rather than a line integral as in (2.13). This is the subject of our discussion in the following subsection.

2.3 Case 3: An irrational limit

Here, $\frac{\varepsilon_1}{\varepsilon_2} \to \alpha \in \mathbb{R} \setminus \mathbb{Q}$. We start with the following straightforward lemmas:

Lemma 2.2 *Assume that $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, $|\delta| \ll \varepsilon \downarrow 0$ and consider the functions*

$$E_{m,n}(y_1, y_2) = e^{2\pi i(my_1+ny_2)} \ , \quad m, n \in \mathbb{Z} \ , \ y_1, y_2 \in T^1 \ . \tag{2.25}$$

Then there exists a constant $C > 0$, such that for every fixed $(m, n) \neq (0, 0)$,

$$\|E_{m,n}(\frac{x}{\alpha\varepsilon + \delta}, \frac{x}{\varepsilon})\|_{W^{-1,\infty}(\Omega)} \leq C \cdot |\eta| \quad where \quad \eta = \frac{\alpha\varepsilon^2 + \varepsilon\delta}{(m + n\alpha)\varepsilon + n\delta} \to 0 \ . \tag{2.26}$$

Proof. Denoting $E(x) = e^{2\pi ix}$, $E_{m,n}(\frac{x}{\alpha\varepsilon+\delta}, \frac{x}{\varepsilon}) = E(\frac{x}{\eta})$ with η as in (2.26). Since the irrationality of α implies that $m + n\alpha \neq 0$, we conclude that $\eta \sim \varepsilon \to 0$. Hence, applying Lemma 1.1 to $E(\frac{x}{\eta})$, which has a zero average, we obtain (2.26). □

Lemma 2.3 *Let $g \in BV(\Omega)$ and $f \in W^{-1,\infty}(\Omega)$. Then $g{\cdot}f \in W^{-1,\infty}(\Omega)$ and*

$$\|gf\|_{W^{-1,\infty}(\Omega)} \leq (\|g\|_{L^\infty(\Omega)} + \|g\|_{BV(\Omega)}) \cdot \|f\|_{W^{-1,\infty}(\Omega)} \ . \tag{2.27}$$

Proof. Let $F(x)$ denote the primitive of $f(x)$, $F(x) = \int_a^x f(\xi)d\xi$. Then

$$\int_a^x gf = g(x)F(x) - \int_a^x g'F \ . \tag{2.28}$$

Taking the supremum in absolute value over Ω on both sides of (2.28) we arrive at (2.27). □

We may now proceed to prove the main theorem of this subsection:

Theorem 2.4 *Assume that $\frac{\varepsilon_1}{\varepsilon_2} \to \alpha \in \mathbb{R} \setminus \mathbb{Q}$ and that $f \in L^\infty(\Omega, H^s(T^2))$, $s > 1$. Then*

$$\|f_\varepsilon(x) - \bar{f}(x)\|_{W^{-1,\infty}(\Omega)} \leq \gamma(\varepsilon) \underset{\varepsilon \to 0}{\longrightarrow} 0 \ \ , \ where \ \bar{f}(x) = \int_{T^2} f(x, y)dy \ . \tag{2.29}$$

Proof. Using the notations (2.1)–(2.2), we shall show that for any $\mu > 0$

$$\|f(x, \frac{x}{\alpha\varepsilon+\delta}, \frac{x}{\varepsilon}) - \bar{f}(x)\|_{W^{-1,\infty}(\Omega)} \le \mu \tag{2.30}$$

for sufficiently small ε.

Let f_N denote the Nth order Fourier approximation of f,

$$f_N(x,y) = f_N(x,y_1,y_2) = \sum_{-N\le m,n\le N} \hat{f}_{m,n}(x) E_{m,n}(y_1,y_2) \ , \tag{2.31}$$

where $\hat{f}_{m,n}(x)$ are the corresponding Fourier coefficients and $E_{m,n}$ are as in (2.25). Then, for any value of r, $1 < r < s$, it holds that

$$\|f(x,\cdot) - f_N(x,\cdot)\|_{H^r(T^2)} \le Const \cdot \frac{\|f(x,\cdot)\|_{H^s(T^2)}}{N^{s-r}} \qquad \forall x \in \Omega \tag{2.32}$$

(consult [13]). Hence, since the L^∞-norm in $\mathbb{R}^2$ is dominated by the H^r-norm for $r > 1$, we conclude that

$$\|f - f_N\|_{L^\infty(\Omega\times T^2)} = \sup_{x\in\Omega} \|f(x,\cdot) - f_N(x,\cdot)\|_{L^\infty(T^2)} \le Const \cdot \frac{\|f\|_{L^\infty(\Omega,H^s(T^2))}}{N^{s-r}} \underset{N\to\infty}{\longrightarrow} 0 \ . \tag{2.33}$$

We may now proceed to prove (2.30). By (2.33), there exists N such that

$$\|f(x,y) - f_N(x,y)\|_{L^\infty(\Omega\times T^2)} \le \frac{\mu}{2|\Omega|} \ . \tag{2.34}$$

Therefore, for this value of N,

$$\|f(x, \frac{x}{\alpha\varepsilon+\delta}, \frac{x}{\varepsilon}) - \bar{f}(x)\|_{W^{-1,\infty}(\Omega)} \le \frac{\mu}{2} + \|f_N(x, \frac{x}{\alpha\varepsilon+\delta}, \frac{x}{\varepsilon}) - \bar{f}(x)\|_{W^{-1,\infty}(\Omega)} \ . \tag{2.35}$$

Since $\bar{f}(x) = \hat{f}_{0,0}(x)$, we may upper bound the second term on the right of (2.35), using Lemma 2.3, as follows:

$$\|f_N(x, \frac{x}{\alpha\varepsilon+\delta}, \frac{x}{\varepsilon}) - \bar{f}(x)\|_{W^{-1,\infty}(\Omega)} \le \sum \|\hat{f}_{m,n}(x)\| \cdot \|E_{m,n}(\frac{x}{\alpha\varepsilon+\delta}, \frac{x}{\varepsilon})\|_{W^{-1,\infty}(\Omega)} \ , \tag{2.36}$$

where the sum is taken over $-N \le m,n \le N$, $(m,n) \ne (0,0)$ and $\|\cdot\| := \|\cdot\|_{L^\infty(\Omega)} + \|\cdot\|_{BV(\Omega)}$. Since, by Lemma 2.2, each of the terms on the right of (2.36) tends to zero when $\varepsilon \downarrow 0$,

$$\|f_N(x, \frac{x}{\alpha\varepsilon+\delta}, \frac{x}{\varepsilon}) - \bar{f}(x)\|_{W^{-1,\infty}(\Omega)} \le \frac{\mu}{2} \ , \tag{2.37}$$

for sufficiently small ε. Therefore, (2.30) follows from (2.35) and (2.37). □

<u>*Remarks.*</u>

1. The assumption $f \in L^\infty(\Omega, H^s(T^2))$, $s > 1$, could have been replaced by the weaker assumption that $\|f(x,y) - f_N(x,y)\|_{L^\infty(\Omega\times T^2)} \to 0$ as $N \to \infty$.

2. Theorem 2.4 lacks convergence rate estimates. The derivation of such estimates is the subject of the next subsection.

2.4 The case of an irrational limit – convergence rate estimates

It turns out that the distinction between the almost fixed ratio case, (2.8), and the genuinely variable ratio case, (2.11), is of great significance here as well. In the first case, where we may take the two scales as proportional to each other, Lemma 2.1, we can borrow results from the theory of quasi-Monte Carlo methods. In the second case, however, that is impossible, whence different methods should be applied.

The analysis presented here involves some terminology and results from number theory and the theory of quasi-Monte Carlo integration methods. The reader is referred to §5 where a brief review of these terms and results is provided.

We first handle the almost fixed ratio case and we start with the simpler situation where the ratio between the two scales remains fixed. The following lemma is a modification of [7, Lemma 2.2]:

Lemma 2.4 *Assume that f is differentiable with respect to x and that $\frac{\varepsilon_1}{\varepsilon_2} = \alpha \in \mathbb{R} \setminus \mathbb{Q}$. Then (2.29) holds with*
(1) $\gamma(\varepsilon) = \mathcal{O}(\varepsilon|\log \varepsilon|)$ if α is proper;
(2) $\gamma(\varepsilon) = \mathcal{O}(\varepsilon)$ if α is of type η and $f(x, y_1, \cdot)$ is in class $\mathcal{E}^k$ for $k > \eta$.

Remark. It is not surprising that an error estimate in this case involves both the smoothness of the function f and the type of α. Indeed, if we review the proof of Theorem 2.4, we see that the smoothness is required for the rate of decay of the Fourier coefficients, $\hat{f}_{m,n}(x)$, while the type of α appears when bounding the exponents $E_{m,n}$ in the lower modes of the error, (2.36). The bound on these exponents is provided by Lemma 2.2 and equals

$$\|E_{m,n}(\frac{x}{\alpha\varepsilon}, \frac{x}{\varepsilon})\|_{W^{-1,\infty}(\Omega)} \leq C \cdot \frac{\alpha\varepsilon}{m + n\alpha} \ ;$$

this bound depends on the lower bound for $|m + n\alpha|$ when $-N \leq m, n \leq N$, $(m, n) \neq (0, 0)$, and the latter is determined by the type of α.

Proof. By normalizing f, we may assume that $\bar{f}(x) \equiv 0$. For the sake of conveniency, we shift the x-domain, Ω, so that $\Omega = [0, b]$. We therefore have to show that for any $x_0 \in [0, b]$,

$$\left| \int_0^{x_0} f(x, \frac{x}{\alpha\varepsilon}, \frac{x}{\varepsilon}) dx \right| \leq \gamma(\varepsilon) \ , \tag{2.38}$$

where $\gamma(\varepsilon) = \mathcal{O}(\varepsilon|\log \varepsilon|)$ in case (1) and $\gamma(\varepsilon) = \mathcal{O}(\varepsilon)$ in case (2).

We first prove the assertion for functions f which do not depend on their first variable, $f(x, y_1, y_2) = f(y_1, y_2)$. By a change of variable in (2.38) we get that

$$\left| \int_0^{x_0} f(\frac{x}{\alpha\varepsilon}, \frac{x}{\varepsilon}) dx \right| \leq b \cdot \left| \frac{1}{M} \int_0^M f(y_1, \alpha y_1) dy_1 \right| \ , \tag{2.39}$$

where $M = \frac{x_0}{\alpha\varepsilon}$. We need to show that the right hand side of (2.39) tends to zero as $M \to \infty$. It suffices to show that only for integer values of M. Using the 1-periodicity of f with respect to y_1, we get that

$$\frac{1}{M}\int_0^M f(y_1,\alpha y_1)dy_1 = \frac{1}{M}\sum_{n=0}^{M-1}\int_0^1 f(y_1,\alpha y_1+\alpha n)dy_1 = \frac{1}{M}\sum_{n=0}^{M-1} F(n\alpha)\ , \quad (2.40)$$

where $F(z) = \int_0^1 f(y_1,\alpha y_1 + z)dy_1$. Since F is 1-periodic and has zero average, $\int_0^1 F(z)dz = \int_0^1\int_0^1 f(y_1,y_2)dy_1dy_2 = 0$, we may apply Propositions 5.2 and 5.3, combined with (2.39)–(2.40), in order to arrive at (2.38) with the desired value of $\gamma(\varepsilon)$ in each of the two cases, (1) and (2).

Next, we deal with functions which do depend on x, $f = f(x,y_1,y_2)$. Using the identity

$$f(x,\frac{x}{\alpha\varepsilon},\frac{x}{\varepsilon}) = \frac{d}{dx}\int_0^x f(x,\frac{s}{\alpha\varepsilon},\frac{s}{\varepsilon})ds - \int_0^x f_x(x,\frac{s}{\alpha\varepsilon},\frac{s}{\varepsilon})ds\ ,$$

we get that

$$\int_0^{x_0} f(x,\frac{x}{\alpha\varepsilon},\frac{x}{\varepsilon})dx = \int_0^{x_0} f(x_0,\frac{s}{\alpha\varepsilon},\frac{s}{\varepsilon})ds - \int_0^{x_0}\int_0^x f_x(x,\frac{s}{\alpha\varepsilon},\frac{s}{\varepsilon})dsdx\ . \quad (2.41)$$

Since the function $f_x(x,y_1,y_2)$ is 1-periodic with respect to y_1,y_2 and has a zero average, $\int_{T^2} f_x(x,y)dy = \frac{d}{dx}\bar{f}(x) = \frac{d}{dx}0 = 0$, we may apply our previous arguments to the integrals with respect to ds in (2.41) and thus conclude the proof. □

Lemma 2.4, combined with Lemma 2.1, imply the following:

Theorem 2.5 *Assume that $\frac{\varepsilon_1}{\varepsilon_2} \to \alpha \in \mathbb{R}\setminus\mathbb{Q}$ and that the ratio is almost fixed, (2.8). Then if f is differentiable with respect to x, (2.29) holds with*
(1) $\gamma(\varepsilon) = Const\cdot(\varepsilon|\log\varepsilon| + |\frac{r}{\varepsilon} - c|)$ if α is proper;
(2) $\gamma(\varepsilon) = Const\cdot(\varepsilon + |\frac{r}{\varepsilon} - c|)$ if α is of type η and $f(x,y_1,\cdot)$ is in class $\mathcal{E}^k$ for $k > \eta$.

Next, we are concerned with the genuinely variable case, (2.11). This more intricate case calls for a different approach. Our two convergence rate results here are as follows:

Theorem 2.6 *Assume that $\frac{\varepsilon_1}{\varepsilon_2} \to \alpha \in \mathbb{R}\setminus\mathbb{Q}$ and that the ratio is genuinely variable, (2.11). Then if $f \in Lip(x)\cap Lip(y_1)$, there exists a subsequence of the scaling parameter, $\varepsilon = \varepsilon(k)$, $k \in \mathbb{N}$, such that for the corresponding subsequence of functions,*

$$\|f_\varepsilon(x) - \bar{f}(x)\|_{W^{-1,\infty}(\Omega)} \le Const\cdot\left(\frac{\varepsilon}{|r|} + \sqrt{|r|}\right) \quad , \quad \bar{f}(x) = \int_{T^2} f(x,y)dy\ . \quad (2.42)$$

Proof. By Proposition 5.4, there exists an infinite sequence of pairs of nonzero integers, (m_k,n_k), such that

$$|n_k\alpha - m_k| \le \frac{1}{n_k}\ . \quad (2.43)$$

For a given pair in this sequence, (m_k, n_k), we define $\varepsilon(k)$ to be the first value of $\varepsilon \downarrow 0$ for which

$$\frac{|\delta(k)|}{\varepsilon(k)} = |r(k)| = \frac{2}{n_k^2} \ , \tag{2.44}$$

where $\delta(k)$ and $r(k)$ are the values of δ and r, (2.1)–(2.2), which correspond to $\varepsilon = \varepsilon(k)$. We shall show that the corresponding subsequence of functions, $f_\varepsilon(x) = f_{\varepsilon(k)}(x)$, satisfies (2.42). For the sake of conveniency, we use henceforth the notations $m, n, \varepsilon, \delta$ instead of $m_k, n_k, \varepsilon(k), \delta(k)$.

We introduce the function $g(x, y_1, y_2) = f(x, y_2 + ny_1, my_1)$. Our assumptions on f imply that g is in $Lip(x)$ and it is also in $Lip(y_2)$ with a Lipschitz constant that equals the one of $f(x, \cdot, y_2)$. Moreover, one may easily verify that $\bar{g}(x) = \int_{T^2} g(x, y)dy = \bar{f}(x)$. With this, we have that

$$f(x, \frac{x}{\alpha\varepsilon + \delta}, \frac{x}{\varepsilon}) = g(x, \frac{x}{\eta_1}, \frac{x}{\eta_2}) \tag{2.45}$$

where

$$\eta_1 = m\varepsilon \quad \text{and} \quad \eta_2 = \frac{m\varepsilon(\alpha\varepsilon + \delta)}{(m - n\alpha)\varepsilon - n\delta} \ . \tag{2.46}$$

By our choice of m, n, ε and δ, we conclude, using (2.43)–(2.44), that

$$|(m - n\alpha)\varepsilon - n\delta| \le \frac{\varepsilon}{n} + n|\delta| = \frac{3\varepsilon}{n} \ ,$$

and

$$|(m - n\alpha)\varepsilon - n\delta| = n|\delta| - |m - n\alpha|\varepsilon \ge n|\delta| - \frac{\varepsilon}{n} = \frac{\varepsilon}{n} \ .$$

The above two estimates, together with (2.46) and (2.44), imply that

$$\left|\frac{\eta_1}{\eta_2}\right| = \left|\frac{(m - n\alpha)\varepsilon - n\delta}{\alpha\varepsilon + \delta}\right| \le \frac{Const}{n} = Const \cdot \sqrt{\frac{|\delta|}{\varepsilon}} \ , \tag{2.47}$$

and

$$|\eta_2| = m\varepsilon \cdot \left|\frac{\alpha\varepsilon + \delta}{(m - n\alpha)\varepsilon - n\delta}\right| \le Const \cdot n^2\varepsilon = Const \cdot \frac{\varepsilon^2}{|\delta|} \ . \tag{2.48}$$

Hence, by Theorem 2.1,

$$\|f(x, \frac{x}{\alpha\varepsilon + \delta}, \frac{x}{\varepsilon}) - \bar{f}(x)\|_{W^{-1,\infty}(\Omega)} = \|g(x, \frac{x}{\eta_1}, \frac{x}{\eta_2}) - \bar{g}(x)\|_{W^{-1,\infty}(\Omega)} \le$$

$$Const \cdot \left(|\eta_2| + \left|\frac{\eta_1}{\eta_2}\right|\right) \le Const \cdot \left(\frac{\varepsilon^2}{|\delta|} + \sqrt{\frac{|\delta|}{\varepsilon}}\right) = Const \cdot \left(\frac{\varepsilon}{|r|} + \sqrt{|r|}\right) \ .$$

$\square$

Theorem 2.7 *Assume that* $r = \frac{\varepsilon_1}{\varepsilon_2} - \alpha \to 0$, *where* α *is an irrational number of type* $\eta = \eta(\alpha)$. *Furthermore, assume that*

$$\frac{\varepsilon}{|r|} \leq Const \cdot |r|^p \qquad \textit{for some } p > \frac{\eta - 1}{\eta + 1} \ , \tag{2.49}$$

and that $f \in Lip(x) \cap Lip(y_1)$. *Then for every fixed* $0 < q < \frac{1}{\eta}$ *the following error estimate holds,*

$$\|f_\varepsilon(x) - \bar{f}(x)\|_{W^{-1,\infty}(\Omega)} \leq Const_q \cdot \left(|r|^{\frac{q}{1+q}} + |r|^{p - \frac{1-q}{1+q}} \right) \quad , \quad \bar{f}(x) = \int_{T^2} f(x,y)dy \ . \tag{2.50}$$

<u>*Remarks.*</u>

1. Assumption (2.49) is a restriction of the genuinely variable ratio assumption (2.11).

2. The restriction on p in (2.49) guarantees that the second error term on the right of (2.50) is vanishing, for q sufficiently close to $\frac{1}{\eta}$.

3. When $\eta \uparrow \infty$, the error bound on the right of (2.50) approaches $\mathcal{O}(|r|^0)$. However, $\bar{f}(x)$ in (2.50) is still the weak limit even when α is of an infinite type, in view of Theorem 2.4.

4. If α is algebraic, its type is $\eta = 1$ [12]. In that case, (2.49) may hold with any $p > 0$ and the convergence rate estimate in (2.50) is 'roughly' $\mathcal{O}(|r|^{\frac{1}{2}} + |r|^p)$, modulo spurious factors of $|r|^{-\nu/2}$ and $|r|^{-\nu}$ where $\nu = \frac{1-q}{1+q} > 0$ can be made as small as we wish by choosing q close to 1.

Proof. Letting $g(x, y_1, y_2)$ and η_1, η_2 be the same as in the proof of Theorem 2.6, we get that

$$\|f_\varepsilon(x) - \bar{f}(x)\|_{W^{-1,\infty}(\Omega)} = \|g(x, \frac{x}{\eta_1}, \frac{x}{\eta_2}) - \bar{g}(x)\|_{W^{-1,\infty}(\Omega)} \leq Const \cdot \left(|\eta_2| + \left| \frac{\eta_1}{\eta_2} \right| \right) \ . \tag{2.51}$$

We fix $0 < q < \frac{1}{\eta}$. Then, by Definition 5.1, there exists a constant $c = c(\alpha, q)$ such that $|n\alpha - m| \geq cn^{-1/q}$ for all $\frac{m}{n} \in \mathbb{Q}$. This implies that

$$|n\alpha - m| \leq \mu \quad \Longrightarrow \quad n \geq d\mu^{-q} \qquad \forall \mu > 0 \qquad (d = c^q) \ . \tag{2.52}$$

Hence, in view of (2.52) and Proposition 5.4, for every $\mu > 0$ there exists $\frac{m}{n} \in \mathbb{Q}$ such that

$$|n\alpha - m| \leq \mu \quad \text{and} \quad d\mu^{-q} \leq n \leq \mu^{-1} \ . \tag{2.53}$$

Now, we set

$$\mu = \left(\frac{d}{2} \cdot \frac{|\delta|}{\varepsilon} \right)^{\frac{1}{1+q}} \ . \tag{2.54}$$

Multiplying the second inequality in (2.53) by the factor $\frac{2}{d}\mu^{1+q}\varepsilon$, which, in view of (2.54), equals $|\delta|$, we get

$$|n\alpha - m|\varepsilon \leq \mu\varepsilon \quad \text{and} \quad 2\mu\varepsilon \leq n|\delta| \leq \frac{2}{d}\mu^q\varepsilon \ . \tag{2.55}$$

Therefore, by (2.53)–(2.55), we conclude that η_1 and η_2, given in (2.46), satisfy

$$\left|\frac{\eta_1}{\eta_2}\right| = \left|\frac{(m-n\alpha)\varepsilon - n\delta}{\alpha\varepsilon+\delta}\right| \le Const \cdot (\mu+\mu^q) \le Const \cdot \mu^q = Const \cdot |r|^{\frac{q}{1+q}} \quad (2.56)$$

(recall that $\frac{\delta}{\varepsilon} = r$) and

$$|\eta_2| = m\varepsilon \cdot \left|\frac{\alpha\varepsilon+\delta}{n\delta-(m-n\alpha)\varepsilon}\right| \le Const \cdot \frac{m\varepsilon^2}{2\mu\varepsilon-\mu\varepsilon} \le Const \cdot \frac{\varepsilon}{\mu^2} \le Const \cdot |r|^{p-\frac{1-q}{1+q}} \quad (2.57)$$

(in the last inequality we used (2.49)). Hence, by (2.51), (2.56) and (2.57), we arrive at the conclusion that for every $0 < q < \frac{1}{\eta}$ there exists a constant, $Const_q$, for which error estimate (2.50) holds. □

2.5 Graphical demonstrations

Here we provide visual illustrations of the results of our analysis. These convincing graphs not only confirm the analysis but even reveal some other interesting phenomena in the behavior of the oscillatory function when the scales tend to zero.

Let

$$f(y_1,y_2) = \cos(2\pi y_1)\cos(2\pi y_2) \quad \text{and} \quad f_\varepsilon(x) = f(\frac{x}{\varepsilon_1}, \frac{x}{\varepsilon_2}) \ . \quad (2.58)$$

Denoting $\varepsilon = \varepsilon_2$, we consider five cases:

1. $\varepsilon_1 = \varepsilon^2$;
2. $\varepsilon_1 = \varepsilon$;
3. $\varepsilon_1 = \varepsilon + \varepsilon^2$;
4. $\varepsilon_1 = \varepsilon + \varepsilon^{1.5}$;
5. $\varepsilon_1 = \pi\varepsilon$.

In case 1 the weak limit is, according to Theorem 2.1,

$$\int_0^1\int_0^1 f(y_1,y_2)dy_1dy_2 = 0 \ . \quad (2.59)$$

Theorem 2.2 implies that the weak limit in case 2 is

$$\int_0^1 f(y_1,y_1)dy_1 = \frac{1}{2} \ , \quad (2.60)$$

while in case 3 (where $r = \frac{\varepsilon_1}{\varepsilon_2} - 1 = \varepsilon$) it is

$$\int_0^1 f(y_1 - x, y_1)dy_1 = \frac{1}{2}\cos(2\pi x) \ . \quad (2.61)$$

Finally, the weak limit in cases 4 and 5 is as in (2.59), as implied by Theorems 2.3 and 2.4, respectively.

Figures 1–5 (given in Appendix B) depict $f_\varepsilon(x)$ (in solid line) and the corresponding weak limits (in dashed line) in each of the above 5 cases, for the following two values of ε:

a. $\varepsilon = 0.0408$;
b. $\varepsilon = 0.00273$.

3 $W^{-1,\infty}$-Convergence Analysis with Multiple Scales

Here we study weak limits with respect to multiple number of scales and generalize the results of §2.

As a first step, we define an equivalence relation, $\sim$, on the set of scales, $\mathcal{S} = \{\varepsilon_i\}_{1\leq i\leq n}$. Later, we use this relation in order to reduce the problem of homogenization of f with respect to $\mathcal{S}$ to a problem of homogenization of another function with respect to the smaller set of scales, $\mathcal{S}/\sim$.

Definition 3.1 *The two scales $\varepsilon_i(\varepsilon), \varepsilon_j(\varepsilon) \in \mathcal{S}$ are said to be equivalent, $\varepsilon_i \sim \varepsilon_j$, if there exist $\alpha \in \mathbb{Q}^*$ and $c \in \mathbb{R}$, such that*

$$\frac{\frac{\varepsilon_i}{\varepsilon_j} - \alpha}{\varepsilon_j} \underset{\varepsilon\to 0^+}{\longrightarrow} c \ . \tag{3.1}$$

Remark. In the two-scale case, $n = 2$, we saw that the weak limit takes the form of a *one-dimensional* average of $f(x,\cdot)$ if $\varepsilon_1 \sim \varepsilon_2$ (Theorem 2.2) while otherwise it is the *two-dimensional* average of $f(x,\cdot)$ over T^2 (Theorems 2.1, 2.3 and 2.4).

Proposition 3.1 *The relation $\sim$ is an equivalence.*

Proof. The relation is clearly reflexive (with $\alpha = 1$ and $c = 0$). Next, we prove that it is symmetric. Let $\varepsilon_i, \varepsilon_j$ satisfy (3.1); then

$$\frac{\varepsilon_i}{\varepsilon_j} = (c+r)\varepsilon_j + \alpha \quad \text{where} \ \ r \underset{\varepsilon\to 0^+}{\longrightarrow} 0 \ . \tag{3.2}$$

Hence, using (3.2) and (3.1) we get that

$$\frac{\frac{\varepsilon_j}{\varepsilon_i} - \frac{1}{\alpha}}{\varepsilon_i} = \frac{\frac{\varepsilon_i}{\varepsilon_j} - \frac{1}{\alpha}\cdot\left(\frac{\varepsilon_i}{\varepsilon_j}\right)^2}{\varepsilon_j}\cdot\left(\frac{\varepsilon_j}{\varepsilon_i}\right)^3 = \left[\frac{\frac{\varepsilon_i}{\varepsilon_j} - \alpha}{\varepsilon_j} - 2(c+r) - \frac{(c+r)^2\varepsilon_j}{\alpha}\right]\cdot\left(\frac{\varepsilon_j}{\varepsilon_i}\right)^3 \longrightarrow -\frac{c}{\alpha^3} \ .$$

This proves that $\varepsilon_j \sim \varepsilon_i$ and, thus, the symmetry of the relation. Finally, assume that

$$\frac{\frac{\varepsilon_i}{\varepsilon_j} - \alpha}{\varepsilon_j} \underset{\varepsilon\to 0^+}{\longrightarrow} c \quad \text{and} \quad \frac{\frac{\varepsilon_j}{\varepsilon_k} - \beta}{\varepsilon_k} \underset{\varepsilon\to 0^+}{\longrightarrow} d \ .$$

Then

$$\frac{\frac{\varepsilon_i}{\varepsilon_k} - \alpha\beta}{\varepsilon_k} = \frac{\frac{\varepsilon_i}{\varepsilon_j} - \alpha}{\varepsilon_j}\cdot\left(\frac{\varepsilon_j}{\varepsilon_k}\right)^2 + \frac{\frac{\varepsilon_j}{\varepsilon_k} - \beta}{\varepsilon_k}\cdot\alpha \underset{\varepsilon\to 0^+}{\longrightarrow} c\beta^2 + d\alpha \ .$$

Hence, the relation is also transitive and, consequently, an equivalence. □

Proposition 3.2 *Let $\mathcal{C} = \{\varepsilon_1, ..., \varepsilon_k\}$, $k \geq 1$, be an equivalence class and assume that*

$$\frac{\varepsilon_i}{\varepsilon_k} \to \alpha_{i,k} = \frac{n_k}{n_i} \qquad 1 \leq i \leq k \ , \tag{3.3}$$

and

$$\frac{r_{i,k}}{\varepsilon_k} \to c_{i,k} \quad \text{where} \ \ r_{i,k} = \frac{\varepsilon_i}{\varepsilon_k} - \alpha_{i,k} \qquad 1 \leq i \leq k \ . \tag{3.4}$$

For $f(x,y) \in BV(\Omega \times T^k)$, define $g(x, y_1)$ as follows:

$$g(x, y_1) = f(x, z_1, ..., z_k) \quad \text{where} \quad z_i = n_i y_1 - \frac{c_{i,k} x}{\alpha_{i,k}^2} \quad 1 \leq i \leq k \ . \tag{3.5}$$

Then

$$\|f(x, \frac{x}{\varepsilon_1}, ..., \frac{x}{\varepsilon_k}) - g(x, \frac{x}{n_k \varepsilon_k})\|_{W^{-1,\infty}(\Omega)} \leq Const \cdot \sum_{i=1}^{k-1} \left(\frac{r_{i,k}^2}{\varepsilon_k} + \left| \frac{r_{i,k}}{\varepsilon_k} - c_{i,k} \right| \right) \ . \tag{3.6}$$

Proof. We start by estimating the difference in the corresponding arguments in the two functions on the left hand side of (3.6). Using the definition of g, (3.5), the difference in the ith argument is

$$E_i = \frac{x}{\varepsilon_i} - \left(\frac{x}{\alpha_{i,k} \varepsilon_k} - \frac{c_{i,k} x}{\alpha_{i,k}^2} \right) \ .$$

Note that since $\alpha_{k,k} = 1$ and $c_{k,k} = 0$, we have that $E_k = 0$. Simple algebraic manipulations yield that

$$E_i = \frac{x}{\alpha_{i,k}^2} \cdot \left[\frac{r_{i,k}^2}{\varepsilon_k (r_{i,k} + \alpha_{i,k})} - \left(\frac{r_{i,k}}{\varepsilon_k} - c_{i,k} \right) \right] \ .$$

Hence,

$$|E_i| \leq Const \cdot \left(\frac{r_{i,k}^2}{\varepsilon_k} + \left| \frac{r_{i,k}}{\varepsilon_k} - c_{i,k} \right| \right) \ . \tag{3.7}$$

Since f is of bounded variation, error estimate (3.6) follows from (3.7). □

Proposition 3.2 enables us to unify equivalent scales and, thus, reduce the number of different scales when taking the weak limit: assuming that $\sim$ defines m equivalence classes in $\mathcal{S} = \{\varepsilon_i\}_{1 \leq i \leq n}$, i.e., $\mathcal{S}/\sim = \{\mathcal{C}_\ell\}_{1 \leq \ell \leq m}$, where

$$\mathcal{C}_\ell = \{\varepsilon_{\ell,1}, ..., \varepsilon_{\ell,k_\ell}\} \quad \text{and} \quad \frac{\varepsilon_{\ell,i}}{\varepsilon_{\ell,k_\ell}} \to \frac{n_{\ell,k_\ell}}{n_{\ell,i}} \quad 1 \leq i \leq k_\ell \ ,$$

we may define a new set of scales, $\tilde{\varepsilon}_\ell = n_{\ell,k_\ell} \varepsilon_{\ell,k_\ell}$, $1 \leq \ell \leq m$, and a function $g(x,y) \in BV(\Omega \times T^m)$ such that

$$\|f(x, \frac{x}{\varepsilon_1}, ..., \frac{x}{\varepsilon_n}) - g(x, \frac{x}{\tilde{\varepsilon}_1}, ..., \frac{x}{\tilde{\varepsilon}_m})\|_{L^\infty(\Omega)} \leq \gamma(\varepsilon_1, ..., \varepsilon_n) \to 0 \ . \tag{3.8}$$

The exact expressions for the new function g and the bound $\gamma(\varepsilon_1, ..., \varepsilon_n)$ may be obtained by applying Proposition 3.2 to each of the equivalence classes, separately.

Hence, the problem of finding the weak limit of $f(x, \frac{x}{\varepsilon_1}, ..., \frac{x}{\varepsilon_n})$ reduces to that of finding the weak limit of $g(x, \frac{x}{\tilde{\varepsilon}_1}, ..., \frac{x}{\tilde{\varepsilon}_m})$.

In view of the above, we assume that all scales are mutually non-equivalent. In the reminder of this section we shall also make the assumption that all scales are proportional, i.e.,

$$\frac{\varepsilon_1}{\varepsilon_i} = \alpha_i > 0 \qquad 1 \le i \le n \; . \tag{3.9}$$

This assumption is made since our goal is to indicate the phenomenon of weak convergence to averages of the function on affine submanifolds of T^n; to this end, it suffices to concentrate on the simple case (3.9). After the complete study of the two-scale case in §2, it would be bothersome to repeat the entire analysis in the multiple scale setting as well. Indeed, assumption (3.9) may be avoided by separating scales of different order of magnitude along the lines of §2.1, and handling the case where the ratio between scales only *tends* to a limit by applying similar methods to those used in §2.2 and §2.3.

Therefore, we aim at finding the $W^{-1,\infty}$-weak limit of $f_\varepsilon(x) = f(x, \frac{\alpha_1 x}{\varepsilon}, ..., \frac{\alpha_n x}{\varepsilon})$. Note that since we assumed that no two scales are equivalent, α_i are irrational for $2 \le i \le n$. We now set $a = (\alpha_1, ..., \alpha_n) \in \mathbb{R}^n$ and define:

Definition 3.2 *Let $a = (\alpha_1, ..., \alpha_n)$ be a vector in $\mathbb{R}^n$. Then $\mathcal{M}(a)$ denotes the $\mathbb{Z}$-module of vectors in $\mathbb{Z}^n$ which are orthogonal to a, i.e.,*

$$\mathcal{M}(a) = \{(m_1, ..., m_n) \in \mathbb{Z}^n \;\; : \;\; \sum_i m_i \alpha_i = 0\} \; . \tag{3.10}$$

If $\mathcal{M}(a) = 0$, $\{\alpha_i\}_{1 \le i \le n}$ are said to be linearly independent.

Let $\mathcal{M}_{\mathbb{R}}(a)$ denote the $\mathbb{R}$-subspace of $\mathbb{R}^n$ spanned by the vectors of $\mathcal{M}(a)$, and $\mathcal{M}_{\mathbb{R}}(a)^\perp$ be its orthogonal complement in $\mathbb{R}^n$. Since $\mathcal{M}_{\mathbb{R}}(a)$ has a basis of vectors in $\mathbb{Z}^n$, so does $\mathcal{M}_{\mathbb{R}}(a)^\perp$. Hence, denoting $k = \dim \mathcal{M}_{\mathbb{R}}(a)^\perp$, there exist $v_1, ..., v_k \in \mathbb{Z}^n$ such that

$$\mathcal{M}_{\mathbb{R}}(a)^\perp = \{\sum_{j=1}^k z_j v_j \;\; : \;\; z_j \in \mathbb{R}\} \; . \tag{3.11}$$

Our statement is as follows:

Theorem 3.1 *Under the above assumptions, if $f \in L^\infty(\Omega, H^s(T^n))$, $s > \frac{n}{2}$, then*

$$f_\varepsilon(x) \rightharpoonup \bar{f}(x) = \int_{T^k} f(x, \sum_{j=1}^k z_j v_j) dz \quad \textit{in } W^{-1,\infty}(\Omega) \; . \tag{3.12}$$

Remark. $\bar{f}(x)$ in (3.12) is just the average of $f(x, \cdot)$ over $P_n(\mathcal{M}_{\mathbb{R}}(a)^\perp)$ – the projection of $\mathcal{M}_{\mathbb{R}}(a)^\perp$ on T^n.

Proof of Theorem 3.1.

Step 1. Let $\hat{f}_m(x)$, $m = (m_1, ..., m_n)$ being a multi-index, denote the Fourier coefficients of $f(x, \cdot)$, i.e.,

$$f(x,y) = \sum_{m\in\mathbb{Z}^n} \hat{f}_m(x)E_m(y) \quad \text{where } E_m(y) = E(m\cdot y) \quad \text{and } E(t) = e^{2\pi it} \ . \tag{3.13}$$

Then we claim that $\bar{f}(x)$, given in (3.12), may be written as follows:

$$\bar{f}(x) = \int_{T^k} f(x, \sum_{j=1}^k z_j v_j)dz = \sum_{m\in\mathcal{M}(a)} \hat{f}_m(x) \ . \tag{3.14}$$

Indeed, by (3.13),

$$f(x, \sum_{j=1}^k z_j v_j) = \sum_{m\in\mathbb{Z}^n} \hat{f}_m(x)E_m(\sum_{j=1}^k z_j v_j) = \sum_{m\in\mathcal{M}(a)} \hat{f}_m(x)E(\sum_{j=1}^k z_j(m\cdot v_j)) + \tag{3.15}$$

$$\sum_{m\notin\mathcal{M}(a)} \hat{f}_m(x)E(\sum_{j=1}^k z_j(m\cdot v_j)) = S_1(x,z) + S_2(x,z) \ .$$

Since $m \cdot v_j = 0$ for all $m \in \mathcal{M}(a)$, we get that

$$S_1(x,z) = \sum_{m\in\mathcal{M}(a)} \hat{f}_m(x) \ ,$$

and, consequently,

$$\int_{T^k} S_1(x,z)dy = \sum_{m\in\mathcal{M}(a)} \hat{f}_m(x) \ . \tag{3.16}$$

On the other hand, for every $m \notin \mathcal{M}(a)$ there exists at least one j, $1 \le j \le k$, for which $m \cdot v_j \neq 0$. Hence,

$$\int_{T^k} S_2(x,z)dz = \sum_{m\notin\mathcal{M}(a)} \hat{f}_m(x) \prod_{j=1}^k \int_{T^1} E(z_j(m\cdot v_j))dz_j = 0 \ . \tag{3.17}$$

Equality (3.14) now follows from (3.15)–(3.17).

Step 2. Let f_N be the Nth order Fourier approximation of f,

$$f_N(x,y) = \sum_{|m|\le N} \hat{f}_m(x)E_m(y) \qquad |m| = \max_{1\le i\le n} |m_i| \ . \tag{3.18}$$

Since $f \in L^\infty(\Omega, H^s(T^n))$, $s > \frac{n}{2}$, we get that

$$\|f(x,y) - f_N(x,y)\|_{L^\infty(\Omega\times T^n)} \underset{N\to\infty}{\longrightarrow} 0 \tag{3.19}$$

(for more details, see the proof of Theorem 2.4 and consult [13]). This implies that the sum on the right of (3.14) converges uniformly to $\bar{f}(x)$: denoting

$$\bar{f}_N(x) = \int_{T^k} f_N(x, \sum_{j=1}^k z_j v_j)dz = \sum_{m\in\mathcal{M}(a), |m|\le N} \hat{f}_m(x) \ , \tag{3.20}$$

(the proof of the last equality is similar to that of (3.14)), we get that

$$\|\bar{f}(x)-\bar{f}_N(x)\|_{L^\infty(\Omega)} \le \int_{T^k} \|f(x,\sum_{j=1}^{k} z_j v_j)-f_N(x,\sum_{j=1}^{k} z_j v_j)\|_{L^\infty(\Omega\times T^k)} dz \le \tag{3.21}$$

$$\|f(x,y)-f_N(x,y)\|_{L^\infty(\Omega\times T^n)} \underset{N\to\infty}{\longrightarrow} 0 \ .$$

Step 3. Let $\mu > 0$ be an arbitrary small positive number. Then, by (3.19) and (3.21), there exists $N > 0$, such that

$$\|f(x,y)-f_N(x,y)\|_{L^\infty(\Omega\times T^n)} + \|\bar{f}(x)-\bar{f}_N(x)\|_{L^\infty(\Omega)} \le \frac{\mu}{2|\Omega|} \ . \tag{3.22}$$

For this value of N,

$$\|f_\varepsilon(x)-\bar{f}(x)\|_{W^{-1,\infty}(\Omega)} \le \|f(x,\frac{\alpha_1 x}{\varepsilon},...,\frac{\alpha_n x}{\varepsilon})-f_N(x,\frac{\alpha_1 x}{\varepsilon},...,\frac{\alpha_n x}{\varepsilon})\|_{W^{-1,\infty}(\Omega)}+ \tag{3.23}$$

$$\|f_N(x,\frac{\alpha_1 x}{\varepsilon},...,\frac{\alpha_n x}{\varepsilon})-\bar{f}_N(x)\|_{W^{-1,\infty}(\Omega)}+\|\bar{f}_N(x)-\bar{f}(x)\|_{W^{-1,\infty}(\Omega)} \ .$$

The sum of the first and last terms on the right of (3.23) does not exceed $\frac{\mu}{2}$, in view of (3.22). Hence, it remains only to show that, by choosing ε sufficiently small, the second term on the right of (3.23) becomes also less than $\frac{\mu}{2}$. To this end, we observe that by (3.18) and (3.20),

$$f_N(x,\frac{\alpha_1 x}{\varepsilon},...,\frac{\alpha_n x}{\varepsilon})-\bar{f}_N(x) =$$

$$\sum_{|m|\le N} \hat{f}_m(x) E\left(\frac{x}{\varepsilon}(m\cdot a)\right) - \sum_{m\in\mathcal{M}(a),|m|\le N} \hat{f}_m(x) = \sum_{m\notin\mathcal{M}(a),|m|\le N} \hat{f}_m(x) E\left(\frac{x}{\varepsilon}(m\cdot a)\right) \ .$$

Since $m\cdot a \ne 0$ for all $m \notin \mathcal{M}(a)$, each of the terms in the last sum tends in $W^{-1,\infty}$ to zero when $\varepsilon \downarrow 0$, in view of Lemma 1.1. Hence, for sufficiently small ε,

$$\|f_N(x,\frac{\alpha_1 x}{\varepsilon},...,\frac{\alpha_n x}{\varepsilon})-\bar{f}_N(x)\|_{W^{-1,\infty}(\Omega)} \le \frac{\mu}{2} \ . \tag{3.24}$$

That concludes the proof. $\square$

Corollary 3.1 *If $\{\alpha_i\}_{1\le i\le n}$ are linearly independent,*

$$f_\varepsilon(x) \rightharpoonup \bar{f}(x) = \int_{T^n} f(x,y)dy \quad \text{in } W^{-1,\infty}(\Omega) \ .$$

Example. Consider the function $f_\varepsilon(x)=f(x,\frac{\alpha_1 x}{\varepsilon},\frac{\alpha_2 x}{\varepsilon},\frac{\alpha_3 x}{\varepsilon})$ where

$$f(x,y_1,y_2,y_3)=\cos(2\pi y_1)\cos(2\pi y_2)\cos(4\pi(x+y_3)) \ ,$$

$\alpha_1 = 1$, $\alpha_2 = \frac{1}{1-\sqrt{2}}$ and $\alpha_3 = \frac{1}{\sqrt{2}}$. Here, $\mathcal{M}(a) = \{(m,m,2m): \ m\in\mathbb{Z}\}$. Hence, $\mathcal{M}_{\mathbb{R}}(a) = \{(t,t,2t): \ t\in\mathbb{R}\}$ and

$$\mathcal{M}_{\mathbb{R}}(a)^\perp = \{z_1 v_1 + z_2 v_2 : \ z_1,z_2\in\mathbb{R} \ , \ v_1 = (2,0,-1) \ , \ v_2 = (0,2,-1)\} \ .$$

Therefore,

$$\bar{f}(x) = \int_0^1 \int_0^1 f(x, 2z_1, 2z_2, -z_1 - z_2) dz_1 dz_2 = \frac{1}{4} \cos(4\pi x) \ .$$

The graphs of $f_\varepsilon(x)$ (solid line) and $\bar{f}(x)$ (dashed line) are given in **Figure 6** in Appendix B. As in §2.5, part **a** corresponds to the value $\varepsilon = 0.0408$ while part **b** corresponds to $\varepsilon = 0.00273$.

4 Applications

In this section we apply our weak convergence results to various homogenization problems. In §4.1 we describe an application to homogenization of nonlinear convection-diffusion equations; in §4.2 we apply our analysis in the context of discrete Boltzman equations; in §4.3 we consider linear transport equations with oscillatory vector fields and in §4.4 we study the motion of an n-dimensional harmonic oscillator and a 2-dimensional quasi-harmonic oscillator.

4.1 Homogenization of nonlinear convection-diffusion equations

Here, we combine our analysis with the homogenization theory of [14] (see there for more details).

Assume that $u_0(x, y)$ and $h(x, y, t)$ (t is a parameter) are functions in $BV(\Omega \times T^n)$ which are constant for $x \notin \Omega$, and let

$$u_0^\varepsilon(x) = u_0(x, \frac{x}{\varepsilon_1}, ..., \frac{x}{\varepsilon_n}) \quad , \quad h^\varepsilon(x, t) = h(x, \frac{x}{\varepsilon_1}, ..., \frac{x}{\varepsilon_n}, t) \ , \tag{4.1}$$

where $\varepsilon_i = \varepsilon_i(\varepsilon) > 0$ vanish when $\varepsilon \downarrow 0$. Let $\bar{u}_0(x)$ and $\bar{h}(x, t)$ denote, respectively, the $W^{-1,\infty}$-weak limits of $u_0^\varepsilon(x)$ and $h^\varepsilon(x, t)$. Consider now the convection-diffusion problem

$$u_t^\varepsilon = K(u^\varepsilon, u_x^\varepsilon)_x + h^\varepsilon(x, t) \ , \qquad u^\varepsilon(x, 0) = u_0^\varepsilon(x), \tag{4.2}$$

with modulated initial and forcing data, $u_0^\varepsilon(x)$ and $h^\varepsilon(x, t)$, as given in (4.1). Here, $K = K(u, p)$ is a non-decreasing function in p and $u^\varepsilon(x, t)$ is the unique entropy solution of the problem, namely, that which corresponds to $K^\delta(u, p) = K(u, p) + \delta p$, $\delta \downarrow 0$. Then, according to [14, Theorem 2.3], $u^\varepsilon(\cdot, t)$, $t \geq 0$, tends weakly in $W^{-1,\infty}$ to $u(\cdot, t)$, the entropy solution of the homogenized problem,

$$u_t = K(u, u_x)_x + \bar{h}(x, t) \ , \qquad u(x, 0) = \bar{u}_0(x) \ . \tag{4.3}$$

Moreover, if the equation is $W^{s,r}$-regular (in the sense that its solution operator maps bounded sets in L^∞ into bounded sets in the Sobolev space $W^{s,r}_{loc}$, $s > 0$, $1 \leq r \leq \infty$), this type of weak convergence may be translated in positive times into a strong one; namely,

$$\|u^\varepsilon(\cdot, t) - u(\cdot, t)\|_{L^p(\Omega)} \underset{\varepsilon \to 0}{\longrightarrow} 0 \qquad \forall t > 0 \ , \tag{4.4}$$

for some values of $p \in [1, \infty]$, consult [14, Theorem 3.1].

As examples, we mention convex hyperbolic conservation laws, $K(u,p) = -f(u)$, $f'' \geq Const > 0$, which posses $W^{1,1}$-regularity and the subquadratic porous media equation, $K(u,p) = mu^{m-1}p$, $1 \leq m \leq 2$, $u \geq 0$, which possesses $W^{2,1}$-regularity [14, Propositions 4.1 & 5.1].

To illustrate this, let us consider the following initial value problem for the hyperbolic Burgers' equation

$$u_t^\varepsilon + \frac{1}{2}\left((u^\varepsilon)^2\right)_x = f(\frac{x}{\varepsilon}, \frac{x}{\varepsilon}) \quad ; \quad u^\varepsilon(x,0) = f(\frac{x}{\varepsilon+\varepsilon^2}, \frac{x}{\varepsilon}) \ , \tag{4.5}$$

where $f(\cdot,\cdot)$ is given in (2.58). The weak limits of the forcing term and of the initial value are given, respectively, in (2.60) and (2.61). Hence, according to (4.3), the entropy solution of (4.5), $u^\varepsilon(\cdot,t)$, tends weakly in $W^{-1,\infty}$ to $u(\cdot,t)$, the entropy solution of the homogenized problem,

$$u_t + \frac{1}{2}\left(u^2\right)_x = \frac{1}{2} \quad ; \quad u(x,0) = \frac{1}{2}\cos(2\pi x) \ . \tag{4.6}$$

Moreover, apart from an initial layer of width $\mathcal{O}(\varepsilon)$, $u^\varepsilon(\cdot,t)$ converges strongly to $u(\cdot,t)$. In **Figure 7** we plot $u^\varepsilon(\cdot,t)$, with $\varepsilon = 0.0408$, versus $u(\cdot,t)$ for four values of t in the initial layer (u^ε is described by the solid line and u by the dashed one). We see how the oscillations diminish in time and that they no longer exist at $t = 0.04 \approx \varepsilon$.

In applications, the small scales ε_i are fixed. This raises the natural question whether our convergence analysis, which involves the *limits* of ratios between the scales as well as the *rates* in which these limits are approached, has any practical significance? The next example provides an affirmative answer to this question. We consider the following initial value problem for the homogeneous Burgers' equation,

$$u_t^\varepsilon + \frac{1}{2}\left((u^\varepsilon)^2\right)_x = 0 \quad ; \quad u^\varepsilon(x,0) = f(\frac{x}{\varepsilon_1}, \frac{x}{\varepsilon_2}) \ , \tag{4.7}$$

where $f(\cdot,\cdot)$ is given in (2.58) and the values of the small scales are

$$\varepsilon_1 = 0.11 \quad , \quad \varepsilon_2 = \varepsilon = 0.10 \ . \tag{4.8}$$

What is the "correct" homogenization of this problem? In other words, what is the proper functional relation between the scales, $\varepsilon_1 = \varepsilon_1(\varepsilon)$, in which we should embed the fixed pair $(\varepsilon_1, \varepsilon_2)$ given in (4.8). There are two reasonable embeddings:

($E1$) $\varepsilon_1 = \frac{11}{10} \cdot \varepsilon$;
($E2$) $\varepsilon_1 = \varepsilon + \varepsilon^2$.

If we assumed ($E1$) then, by Theorem 2.2, the initial value in (4.7) should be approximated by its following average,

$$u^\varepsilon(x,0) \approx \int_0^1 f(10y_1, 11y_1)dy_1 = 0 \qquad \text{in } W^{-1,\infty}$$

(here $m = 11, n = 10$ and $r = 0$). Hence, the oscillatory solution at $t > 0$, $u^\varepsilon(\cdot, t)$, should be close, in a strong sense, to $u_1(\cdot, t) \equiv 0$ – the corresponding homogenized solution.

If, on the other hand, we assumed ($E2$) then, by the same theorem, $u^\varepsilon(x, 0)$ should be approximated as follows,

$$u^\varepsilon(x,0) \approx \int_0^1 f(y_1 - x, y_1)dy_1 = \frac{1}{2}\cos(2\pi x) \qquad \text{in } W^{-1,\infty}$$

(here $m = n = 1$ and $r = \varepsilon$). Hence, $u^\varepsilon(\cdot, t)$ would be close to $u_2(\cdot, t)$ – the entropy solution of the homogeneous Burgers' equation with initial value $u_2(x, 0) = \frac{1}{2}\cos(2\pi x)$.

In **Figure 8** we plot $u^\varepsilon(\cdot, t)$ (solid) versus $u_1(\cdot, t)$ (dash-dot) and $u_2(\cdot, t)$ (dashed) for four values of t. The pictures clearly show that u_2 is the correct homogenized solution. The reason for that lies in error estimate (2.14) which measures the $W^{-1,\infty}$-error in approximating the oscillatory initial value $u^\varepsilon(x, 0)$. The order of magnitude of the constant in that error estimate depends linearly on $\mathcal{O}(m, n)$ (see the remark after Theorem 2.2). Hence, while in ($E1$) $m, n \sim \frac{1}{\varepsilon}$ and, consequently, the error bound is $\mathcal{O}(1)$, in ($E2$) $m = n = 1$ and, consequently, the error bound is $\mathcal{O}(\varepsilon) \sim 0.1$. Therefore, our analysis enables us to predict which of two possible homogenization settings is preferable.

The numerical results shown in **Figures 7–8** were obtained by the Nessyahu-Tadmor scheme, [9].

4.2 Homogenization of discrete Boltzmann equations

Consider the Carleman equations, which serve as a simple model for the nonlinear discrete Boltzmann equations [5],

$$u_t + u_x + u^2 - v^2 = 0 \ , \qquad v_t - v_x + v^2 - u^2 = 0 \ . \tag{4.9}$$

Assume that the initial values are given by

$$u(x,0) = u_0(x, \frac{x}{\varepsilon_1}, \frac{x}{\varepsilon_2}) \ , \qquad v(x,0) = v_0(x, \frac{x}{\varepsilon_1}, \frac{x}{\varepsilon_2}) \ , \tag{4.10}$$

where $u_0, v_0 \in BV(\mathbb{R} \times T^2)$. We are looking for the homogenized equations for (4.9)+(4.10) which describe the weak limits of u and v when $\varepsilon_1, \varepsilon_2 \downarrow 0$. Combining the techniques presented in [7] and our results from §2, we infer the following:

Theorem 4.1 *Assume that u_0 and v_0 are smooth and non-negative and let u and v be the solution of (4.9)+(4.10). Assume that $\frac{\varepsilon_1}{\varepsilon_2} \to \alpha$ and denote $r = \frac{\varepsilon_1}{\varepsilon_2} - \alpha$. Let $U = U(x, y_1, y_2, t)$ and $V = V(x, y_1, y_2, t)$ be the solution of the homogenized equations*

$$U_t + U_x + U^2 - \overline{V^2} = 0 \quad , \quad V_t - V_x + V^2 - \overline{U^2} = 0 \ , \tag{4.11}$$

$$U(x, y_1, y_2, 0) = u_0(x, y_1, y_2) \quad , \quad V(x, y_1, y_2, 0) = v_0(x, y_1, y_2) \ , \tag{4.12}$$

where

$$\overline{U^2} = \int_0^1 \int_0^1 U^2(x, y_1, y_2, t) dy_1 dy_2 \quad , \quad \overline{V^2} = \int_0^1 \int_0^1 V^2(x, y_1, y_2, t) dy_1 dy_2$$

if

- *Case 1: $\alpha = 0$,*
- *Case 2: α is irrational, or*
- *Case 3: α is rational and $\frac{|r|}{\varepsilon_2} \to \infty$,*

while

$$\overline{U^2} = \int_0^1 U^2\left(x, ny_1 - \frac{c(x-t)}{\alpha^2}, my_1, t\right) dy_1 \quad , \quad \overline{V^2} = \int_0^1 V^2\left(x, ny_1 - \frac{c(x+t)}{\alpha^2}, my_1, t\right) dy_1$$

if

- *Case 4: $\alpha = \frac{m}{n}$ and $\frac{r}{\varepsilon_2} \to c$.*

Then the following pointwise error estimate holds for $\varepsilon_1, \varepsilon_2 \downarrow 0$,

$$\left| u(x,t) - U\left(x, \frac{x-t}{\varepsilon_1}, \frac{x-t}{\varepsilon_2}, t\right)\right| + \left| v(x,t) - V\left(x, \frac{x+t}{\varepsilon_1}, \frac{x+t}{\varepsilon_2}, t\right)\right| \le \gamma(\varepsilon_1, \varepsilon_2) \to 0 \ ,$$

where $\gamma(\varepsilon_1, \varepsilon_2)$ is as given in §2 in each of the four Cases 1–4.

Theorem 4.1 generalizes [7, Theorem 3.1] by including the cases in which the ratio $\frac{\varepsilon_1}{\varepsilon_2}$ does not tend to its limit faster than $\mathcal{O}(\varepsilon_1, \varepsilon_2)$ and improves it as well by providing the pointwise convergence rate $\gamma(\varepsilon_1, \varepsilon_2)$.

Similar results can be proved along the same lines for more complicated models, such as the Broadwell model [4] or the general model studied in [15, 16].

4.3 Homogenization of linear flows with oscillatory velocity fields

Linear transport equations with oscillatory velocity fields appear in several physical problems, e.g., miscible displacement problems in the oil reservoir simulation [1]. These equations take the form

$$u_t + \vec{a}\left(\frac{\vec{x}}{\varepsilon}\right) \cdot \frac{\partial u}{\partial \vec{x}} = 0 \quad , \quad u(\vec{x}, 0) = u_0(\vec{x}) \ , \tag{4.13}$$

where $\vec{x} \in \mathbb{R}^d$, $t \in \mathbb{R}^+$, $\vec{a}$ is a periodic vector field with no stagnation points on the torus T^d and u_0 is Lipschitz continuous. If the small scale ε depends on the dimension, problem (4.13) becomes a multiple scale problem, e.g., when $d = 2$,

$$u_t + a_1\left(\frac{x_1}{\varepsilon_1}, \frac{x_2}{\varepsilon_2}\right) \frac{\partial u}{\partial x_1} + a_2\left(\frac{x_1}{\varepsilon_1}, \frac{x_2}{\varepsilon_2}\right) \frac{\partial u}{\partial x_2} = 0 \quad , \quad u(x_1, x_2, 0) = u_0(x_1, x_2) \ . \tag{4.14}$$

We concentrate here on shear vector fields, i.e., vector fields with a constant direction,

$$\begin{pmatrix} a_1(y) \\ a_2(y) \end{pmatrix} = a(y) \cdot \begin{pmatrix} 1 \\ \gamma \end{pmatrix} \qquad \forall y \in T^2 \ ; \tag{4.15}$$

here, $a(y)$ is a scalar non-vanishing function and γ is a constant, called *the rotation number*. Hence, we wish to find the limit solution of

$$u_t + a\left(\frac{x_1}{\varepsilon_1}, \frac{x_2}{\varepsilon_2}\right)\frac{\partial u}{\partial x_1} + \gamma a\left(\frac{x_1}{\varepsilon_1}, \frac{x_2}{\varepsilon_2}\right)\frac{\partial u}{\partial x_2} = 0 \quad , \quad u(x_1, x_2, 0) = u_0(x_1, x_2) \ , \tag{4.16}$$

when $\varepsilon_1, \varepsilon_2 \downarrow 0$. In view of a theorem due to Kolmogorov [8], if the flow in (4.14) is measure preserving in the sense that there exists a smooth measure density on the torus, $\mu > 0$, such that $\nabla \cdot (\mu \vec{a}) = 0$, we can always reduce (4.14) to a shear flow problem, (4.16), through a diffeomorphism on T^2.

Equation (4.16) may be solved explicitly, using the method of characteristics. Let us define the function

$$A_\eta(x) = \int_0^x \frac{dz}{a(\frac{z}{\varepsilon_1}, \frac{\gamma z + \eta}{\varepsilon_2})} \ . \tag{4.17}$$

Since a is non-vanishing, $A_\eta(x)$ is well-defined, monotonic and, therefore, invertible. For each point (x_1, x_2, t) we set

$$\eta = x_2 - \gamma x_1 \quad , \quad x_1^0 = A_\eta^{-1}(-t + A_\eta(x_1)) \ \text{ and } \ x_2^0 = \gamma x_1^0 + \eta \ . \tag{4.18}$$

With these notations, the solution of (4.16) is given by

$$u(x_1, x_2, t) = u_0(x_1^0, x_2^0) \ . \tag{4.19}$$

The limit of u when $\varepsilon_1, \varepsilon_2 \downarrow 0$ depends on $\lim_{\varepsilon_1, \varepsilon_2 \to 0} A_\eta(x_1)$ and the latter is determined by the correlation between the two small scales – ε_1 and ε_2/γ. Using our usual notations, $\alpha = \lim \gamma \frac{\varepsilon_1}{\varepsilon_2}$ and $r = \gamma \frac{\varepsilon_1}{\varepsilon_2} - \alpha$, we get the following:

If $\alpha = 0$, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ or $\alpha \in \mathbb{Q}$ but $|r| \gg \mathcal{O}(\varepsilon_1, \varepsilon_2)$, we get that $A_\eta(x)$ converges pointwise to x/a^*, *independently* of the value of η, where a^* stands for the harmonic average of $a(\cdot)$ over T^2, $1/a^* = \int_{T^2} dy/a(y)$. This implies that the pointwise limit of the solution in (4.18)–(4.19) is

$$u(x_1, x_2, t) \to w(x_1, x_2, t) = u_0(x_1 - a^* t, x_2 - \gamma a^* t) \ . \tag{4.20}$$

Hence, the limit solution in this case, w, satisfies a homogenized equation which is a linear transport equation with constant coefficients,

$$w_t + a^* \frac{\partial w}{\partial x_1} + \gamma a^* \frac{\partial w}{\partial x_2} = 0 \quad , \quad w(x_1, x_2, 0) = u_0(x_1, x_2) \ . \tag{4.21}$$

If, on the other hand, $\alpha = \frac{m}{n} \in \mathbb{Q}$ and $\gamma \frac{r}{\varepsilon_2} \to c$, we get, in view of (4.17) and Theorem 2.2, that

$$A_\eta(x) = \int_0^x \frac{dz}{a(\frac{z}{\varepsilon_1}, \frac{\gamma z + \eta}{\varepsilon_2})} = \int_0^x \frac{dz}{a^*_{\eta/\varepsilon_2}(z)} + \mathcal{O}\left(\varepsilon_2 + \left|\gamma \frac{r}{\varepsilon_2} - c\right|\right)$$

where

$$\frac{1}{a_\eta^*(z)} = \int_{T^1} \frac{dy_1}{a(ny_1 - \frac{cz}{\alpha^2}, my_1 + \eta)} \ .$$

Hence, we conclude by (4.18)–(4.19) that

$$|u(x_1, x_2, t) - v(x_1, x_2, t)| \le Const \cdot \left(\varepsilon_2 + \left| \gamma \frac{r}{\varepsilon_2} - c \right| \right) \ ,$$

where v is the solution of

$$v_t + a^*_{(x_2 - \gamma x_1)/\varepsilon_2}(x_1) \frac{\partial v}{\partial x_1} + \gamma a^*_{(x_2 - \gamma x_1)/\varepsilon_2}(x_1) \frac{\partial v}{\partial x_2} = 0 \ \ , \ \ v(x_1, x_2, 0) = u_0(x_1, x_2) \ .$$

Finally, since $a_\eta^*(\cdot)$ is 1-periodic in η, we get by Lemma 1.1 that

$$u(x_1, x_2, t) \rightharpoonup \bar{w}(x_1, x_2, t) = \int_{T^1} w(x_1, x_2, t; \eta) d\eta \qquad \text{in } W^{-1,\infty} \tag{4.22}$$

where $w(x_1, x_2, t; \eta)$ is the solution of the parameter-dependent equation

$$w_t + a_\eta^*(x_1) \frac{\partial w}{\partial x_1} + \gamma a_\eta^*(x_1) \frac{\partial w}{\partial x_2} = 0 \ \ , \ \ w(x_1, x_2, 0; \eta) = u_0(x_1, x_2) \quad ; \quad \eta \in T^1 \ . \tag{4.23}$$

Hence, in this case we have not one homogenized equation, like (4.21), but a continuum of such, (4.23), and the average of their solutions is the homogenized weak limit of u, (4.22).

4.4 Harmonic and quasi-harmonic oscillators

For a fixed $a = (\alpha_1, ..., \alpha_n) \in \mathbb{R}^n$, $n \ge 2$, let $G(a) = \{g^{ta}\}_{t \in \mathbb{R}}$ denote the group of the following continuous one-to-one mappings of the torus T^n onto itself:

$$\Phi \in T^n \mapsto g^{ta}(\Phi) = P_n(\Phi + ta) \ , \tag{4.24}$$

P_n being the projection operator of $\mathbb{R}^n$ onto T^n. Let $\mathcal{L}_\Phi(a)$ be the $G(a)$-orbit in T^n which passes at $t = 0$ through the point Φ, i.e., $\mathcal{L}_\Phi(a) = G(a)\Phi = \{g^{ta}(\Phi) \ : \ t \in \mathbb{R}\}$. Then Theorem 3.1 implies the following:

Theorem 4.2 *The orbit $\mathcal{L}_\Phi(a)$ is dense in the T^n-submanifold $\Sigma_\Phi(a) = P_n(\Phi + \mathcal{M}_\mathbb{R}(a)^\perp)$, where $\mathcal{M}_\mathbb{R}(a)^\perp$ is given in (3.11).*

Proof. Since $ta \in \mathcal{M}_\mathbb{R}(a)^\perp$ for all $t \in \mathbb{R}$, it follows that $\mathcal{L}_\Phi(a) \subset \Sigma_\Phi(a)$. Next, we prove that $\mathcal{L}_\Phi(a)$ is dense in $\Sigma_\Phi(a)$. Assume, by contradiction, that there is an open set of a positive measure in $\Sigma_\Phi(a)$, S, such that $S \cap \mathcal{L}_\Phi(a) = \emptyset$. Let $f(y)$ be a smooth function on T^n such that $f(y) > 0$ for $y \in S$ and $f(y) = 0$ for $y \in \Sigma_\Phi(a) \setminus S$. By our assumption, the function $f_\varepsilon(x) = f(\Phi + \frac{x}{\varepsilon} a)$ is identically zero in $\mathbb{R}_x$ for all values of $\varepsilon > 0$. However, by Theorem 3.1, $f_\varepsilon(x)$ tends weakly to the average of f over $\Sigma_\Phi(a)$, which is positive. This establishes the contradiction and the proof is therefore complete. $\square$

Corollary 4.1 *$\mathcal{L}_\Phi(a)$ is dense in T^n iff α_i are linearly independent over $\mathbb{Z}$ and it is a closed curve in T^n iff all α_i are rationally proportional, i.e., $\alpha_i = r_i\alpha_1$, where $r_i \in \mathbb{Q}^*$, $1 \le i \le n$.*

Remark. The case $n = 2$ is of special interest. Here, the orbits are dense in T^2 iff the ratio $\frac{\alpha_1}{\alpha_2}$ is irrational. This well-known result is a consequence of Poincare's Recurrence Theorem (consult [2, §16]).

Now, we use the above in order to study the motion of the n-dimensional harmonic oscillator, $\ddot{x}_i(t) = -\omega_i^2 x_i(t)$, $1 \le i \le n$. The general solution is given by

$$x_i(t) = A_i \cos(\omega_i t + \phi_i) \qquad 1 \le i \le n \ . \tag{4.25}$$

Hence, the orbit of the oscillator,

$$X = \{x(t) = (x_1(t), ..., x_n(t)) \ : \ t \in \mathbb{R}\} \ , \tag{4.26}$$

is confined to the box $B^n = \prod_{i=1}^n [-A_i, A_i]$ (such orbits are called *Lissajous figures* [2, §5]). Let F_i denote the one-to-one mapping of $[-A_i, A_i]$ onto $[0, 1]$, $F_i(x) = \frac{1}{\pi}\cos^{-1}\left(\frac{x}{A_i}\right)$, and F_i^{-1} denote its inverse, $F_i^{-1}(y) = A_i\cos(\pi y)$. Furthermore, we denote by F the tensor product of F_i which maps B^n onto $[0, 1]^n$ and, similarly, let F^{-1} denote the tensor product of F_i^{-1} which maps $[0, 1]^n$ onto B^n. Letting $y(t) = F(x(t))$ and Y be the corresponding orbit in T^n, i.e., $Y = \{y(t) \ : \ t \in \mathbb{R}\}$, we get, using our previous notations, that

$$y(t) = P_n(\Phi + ta) \quad \text{where} \quad \Phi = \frac{1}{\pi}(\phi_1, ..., \phi_n) \ , \ a = \frac{1}{\pi}(\omega_1, ..., \omega_n) \ , \tag{4.27}$$

and, hence, $Y = \mathcal{L}_\Phi(a)$. Since, by Theorem 4.2, Y is dense in the T^n-submanifold $\Sigma_\Phi(a)$, we conclude the following:

Theorem 4.3 *The orbit of the n-dimensional harmonic oscillator (4.25)–(4.26) is dense in the B^n-submanifold $F^{-1}(\Sigma_\Phi(a))$, where Φ and a are given in (4.27).*

Remark. We do not claim that the result of Theorem 4.3 is new; however, we failed to find a reference for this result in the multi-dimensional case, $n > 2$.

Example. Consider the 3-dimensional oscillator

$$x(t) = (\cos(\pi t), \cos(\pi t - \frac{\pi}{2}), \cos(t)) \ .$$

Here, $a = (1, 1, \frac{1}{\pi})$ and, therefore, $\mathcal{M}_\mathbb{R}(a)^\perp = \{(t_1, t_1, t_2) \ : \ t_1, t_2 \in \mathbb{R}\}$. Since $\Phi = (0, -\frac{1}{2}, 0)$ in this case, the orbit is dense in the following submanifold of $B^3 = [-1, 1]^3$,

$$F^{-1}(\Sigma_\Phi(a)) = \{(\cos(\pi t_1), \cos(\pi t_1 - \frac{\pi}{2}), \cos(\pi t_2)) \ : \ t_1, t_2 \in T^1\} \ ,$$

which is the cylindrical manifold $\{x \in B^3 \ : \ x_1^2 + x_2^2 = 1\}$.

Next, we would like to study the motion of a quasi-harmonic oscillator,

$$x_i(t) = A_i \cos(f_i(t)) \quad , \quad f_i(t) \underset{t\to\infty}{\longrightarrow} \infty \qquad 1 \le i \le n \ . \tag{4.28}$$

The harmonic oscillator, (4.25), is a special case of (4.28) where the frequencies are constant in time, namely, $f_i(t)$ are all linear. We concentrate on the 2-dimensional case and examine the orbits of such oscillators,

$$X = \{x(t) = (x_1(t), x_2(t)) \ : \ t \in \mathbb{R}\} \ . \tag{4.29}$$

Let us assume that $\lim_{t\to\infty} \frac{f_1(t)}{f_2(t)} = \alpha$. Then, by §2, we conclude (along the lines of the proof of Theorem 4.2) that when α is zero or irrational, X is dense in $B^2 = [-A_1, A_1] \times [-A_2, A_2]$. The case of a nonzero rational limit, $\alpha = \frac{m}{n} \neq 0$, is more interesting. In this case,

$$f_1(t) = \frac{m}{n} \cdot f_2(t) + g(t) \quad \text{where} \quad \frac{g(t)}{f_2(t)} \underset{t\to\infty}{\longrightarrow} 0 \ . \tag{4.30}$$

We consider three subcases:

1. $\lim_{t\to\infty} |g(t)| = \infty$. Here, Theorem 2.3 may be applied in order to conclude that X is dense in B^2. See **Figure 9-a** where the orbit of the quasi-harmonic oscillator with $f_1(t) = \pi \cdot (t + 0.6)$ and $f_2(t) = 2\pi t + t^{0.2}$ is depicted for $0 \le t \le 20$.

2. $\lim_{t\to\infty} g(t) = g_\infty$. In this case, the orbit $\Xi(m, n; g_\infty) = \{(\xi_1(t), \xi_2(t)) : \ t \in \mathbb{R}\}$ of the harmonic oscillator

$$\xi_1(t) = \cos(mt + g_\infty) \quad , \quad \xi_2(t) = \cos(nt) \ ,$$

serves as an attractor for X when $t \to \infty$. **Figure 9-b** depicts the orbit which corresponds to $f_1(t) = \pi \cdot (t + 0.6)$ and $f_2(t) = 2\pi t + (1 + t)^{-1}$ whose attractor is $\Xi(1, 2; 0.6\pi)$, **Figure 9-c**.

3. $g_1 \le g(t) \le g_2$. Here,

$$X \subset \cup_{\gamma\in[g_1,g_2]} \Xi(m, n; \gamma) \ , \tag{4.31}$$

where $\Xi(m, n; \gamma)$ is, as before, the orbit of the harmonic oscillator

$$\xi_1(t) = \cos(mt + \gamma) \quad , \quad \xi_2(t) = \cos(nt) \ .$$

We note in passing that if $|g_2 - g_1| \ge \frac{1}{n}$, the union in (4.31) covers the entire box B^2. This union of orbits is very apparent in **Figure 9-d** which corresponds to $f_1(t) = \pi \cdot (t + 0.6)$ and $f_2(t) = 2\pi t + 0.1 \cdot \sin(t)$.

5 Appendix A

Here we provide a brief review of some results from the theory of quasi-Monte Carlo numerical integration methods and related topics in number theory, [10, 11].

Let $f(y)$ be a function in $BV[0,1]$ and $\{x_n\}$ be a sequence of points in $[0,1]$. Then, a quasi-Monte Carlo approximation for the integral of f is given by

$$\int_0^1 f(y)dy \approx \frac{1}{N}\sum_{n=1}^{N} f(x_n) \ . \tag{5.1}$$

In order for the approximation to converge, the sequence of points must be "well-distributed" in the interval of integration. The *discrepancy* of the sequence, defined as

$$D_N(x_1, x_2, ..., x_N) = \sup_{0\le r\le 1}\left|\frac{\#\{x_i: \ 1\le i\le N, \ x_i\in[0,r]\}}{N} - r\right| \ , \tag{5.2}$$

is a mean to quantify how well the sequence is distributed. With this definition, the following error estimate holds [10, Theorem 2.9]:

$$\left|\int_0^1 f(y)dy - \frac{1}{N}\sum_{n=1}^{N} f(x_n)\right| \le D_N(x_1, x_2, ..., x_N)\cdot \|f\|_{BV[0,1]} \ . \tag{5.3}$$

Hence, in order to obtain an error estimate for the quasi-Monte Carlo method, one must upper-bound the discrepancy of the corresponding sequence of points. Since it was proved that for any sequence

$$D_N \ge 0.06\cdot N^{-1}\log N \qquad \text{for infinitely many } N \ ,$$

we cannot hope for an error estimate better than $\mathcal{O}(N^{-1}\log N)$, unless we assume more on the smoothness of f.

We are interested here in sequences of the form $x_n = P_1(n\alpha)$, where α is an irrational number and P_1 denotes, as before, the projection of $\mathbb{R}$ onto T^1 (namely, $P_1(x)$ is the fractional part of x). For such sequences we may apply the ergodic theorem of equi-partition modulo 1 (Bohl-Serpinskii-Weyl) [3], which implies that

$$\left|\int_0^1 f(y)dy - \frac{1}{N}\sum_{n=1}^{N} f(x_n)\right| \underset{N\to\infty}{\longrightarrow} 0 \ . \tag{5.4}$$

Convergence rate estimates are available in some special cases. We cite below two of the more important results in this direction.

Proposition 5.1 *If α is a proper irrational number (defined below) then $D_N = \mathcal{O}(N^{-1}\log N)$, where D_N is the discrepancy of the sequence $x_n = P_1(n\alpha)$.*

An irrational number, α, is called *proper* if the partial quotients a_i in its (unique) continued fraction expansion,

$$\alpha = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \dots}} \quad , \quad a_i \in \mathbb{Z} \ , \ a_i \ge 1 \ \ \forall i \ge 1 \ ,$$

are such that $\sum_{i=1}^{m} a_i = \mathcal{O}(m)$.

In view of (5.3) and Proposition 5.1 we conclude:

Proposition 5.2 *If $f \in BV[0,1]$ and $x_n = P_1(n\alpha)$, α being a proper irrational number, then*

$$\left| \int_0^1 f(y)dy - \frac{1}{N} \sum_{n=1}^{N} f(x_n) \right| \leq Const \cdot N^{-1} \log N \ . \tag{5.5}$$

By further assumptions on the smoothness of f, we may obtain an $\mathcal{O}(N^{-1})$-error estimate. To this end we define the following:

Definition 5.1 *Let α be an irrational number and let $S = S_\alpha$ be defined as*

$$S = \{\sigma \ : \ \exists c = c(\alpha, \sigma) \ \textit{such that} \ \operatorname{dist}(\alpha n, \mathbb{Z}) \geq \frac{c}{n^\sigma} \quad \forall n \in \mathbb{N}\} \ .$$

Then if $S \neq \emptyset$, α is said to be of type η, where $\eta =$inf S.

Definition 5.2 *Let f be a 1-periodic function and assume that $|\hat{f}_n| \leq \mathcal{O}(|n|^{-k})$ for all $n \neq 0$, where $\hat{f}_n$ are the Fourier coefficients of f, $f(y) = \sum_{n\in\mathbb{Z}} \hat{f}_n e^{2\pi i n y}$. Then f is said to be of class $\mathcal{E}^k$.*

Proposition 5.3 *Let f be a 1-periodic function of class $\mathcal{E}^k$ and α be an irrational number of type $\eta < k$. Then the following error estimate holds:*

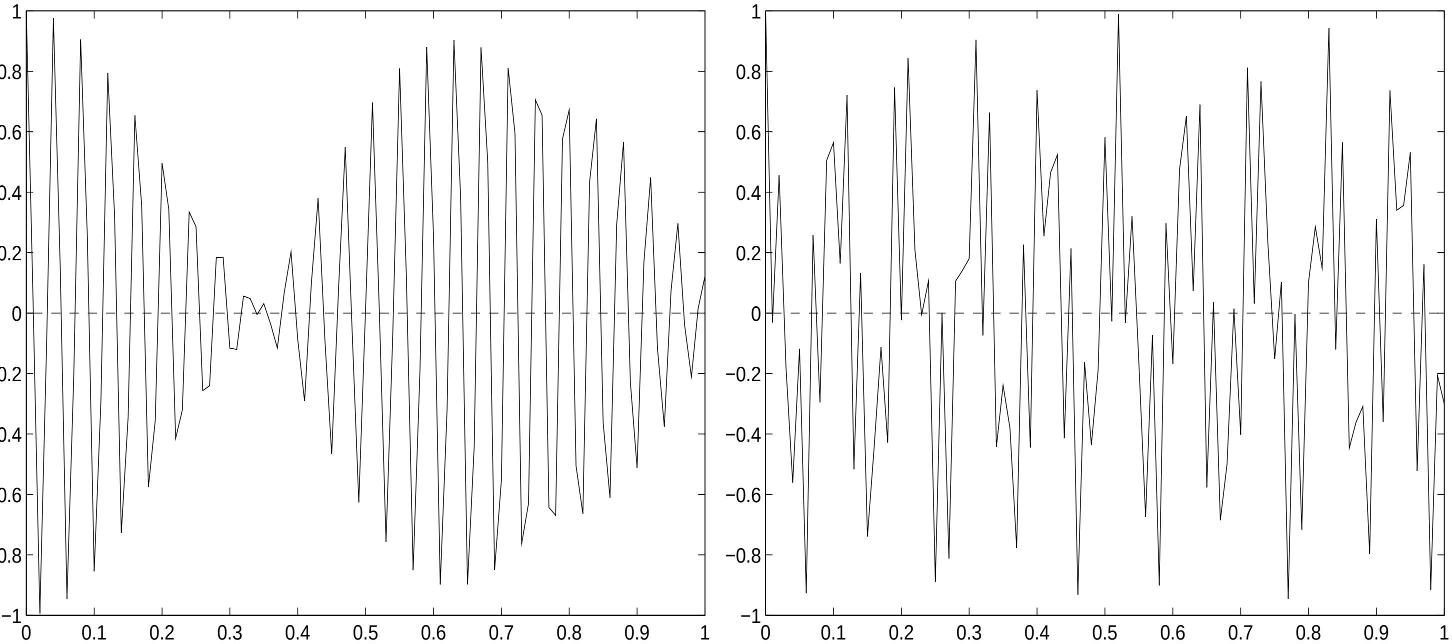
$$\left| \int_0^1 f(y)dy - \frac{1}{N} \sum_{n=1}^{N} f(n\alpha) \right| \leq Const \cdot N^{-1} \ .$$

We conclude this brief review with some remarks on the type of irrational numbers. The type of α, $\eta(\alpha)$, indicates how well can α be approximated by rational numbers. The greater $\eta(\alpha)$ is – the better are the rational approximations of α. Whenever the type is defined, it is greater than or equal to 1, as implied by Proposition 5.4 below. Moreover, the type of algebraic numbers is 1 (an immediate consequence of a theorem by Roth [12]).

Proposition 5.4 *[6, Theorem 185] For any $\alpha \in \mathbb{R}$, there exist infinitely many $\frac{m}{n} \in \mathbb{Q}$, such that $|\alpha n - m| \leq n^{-1}$.*

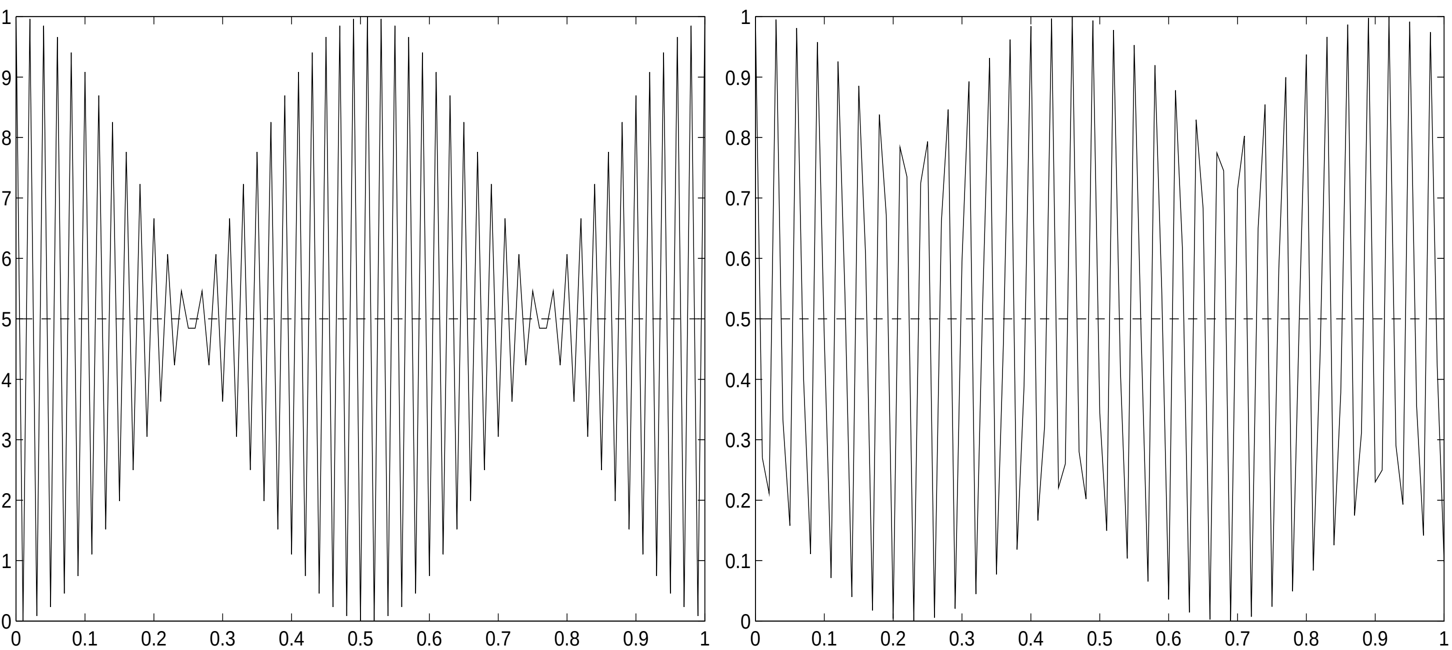
6 Appendix B

The following pages contain **Figures 1–9** to which we referred in the previous sections.



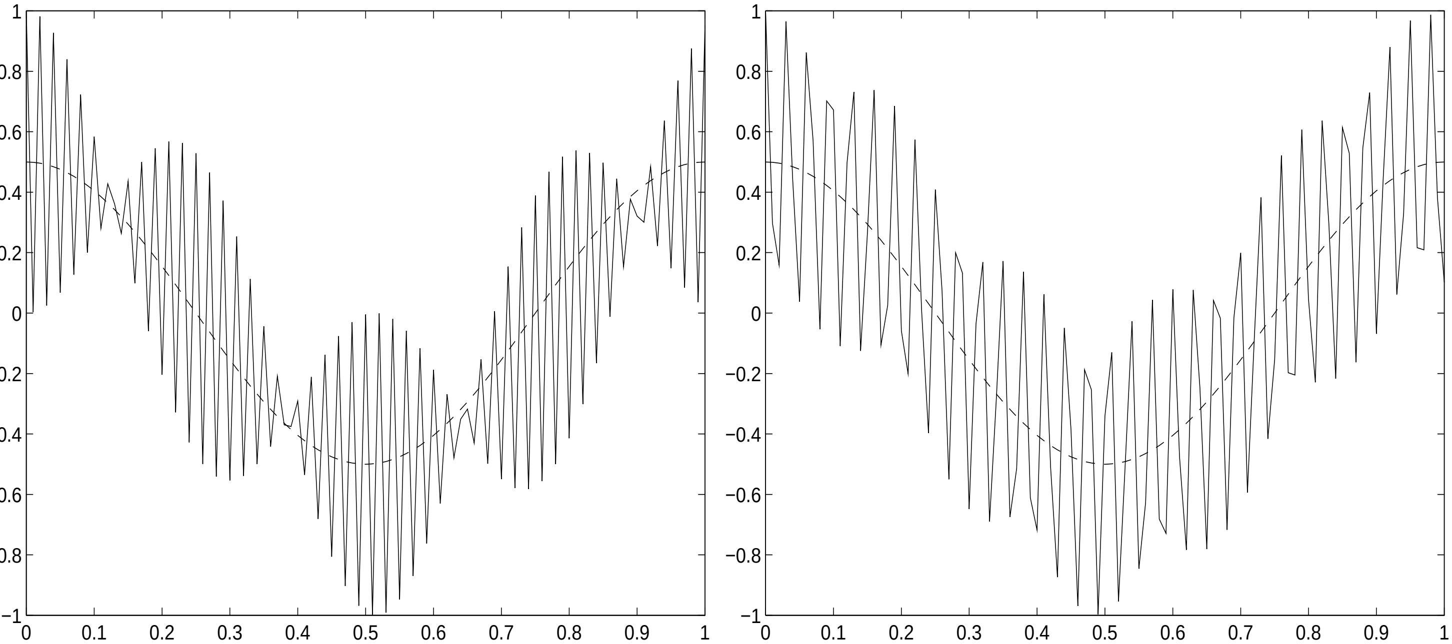
a **b**

Figure 1



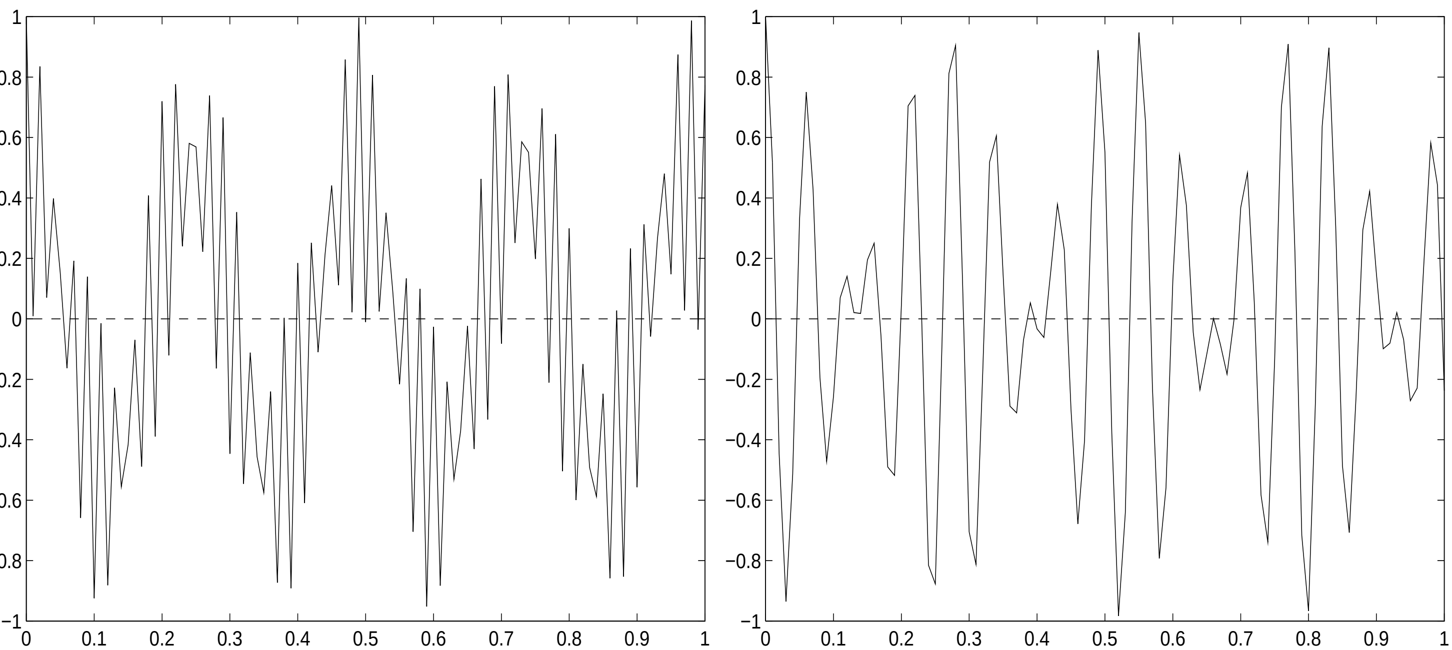
a **b**

Figure 2



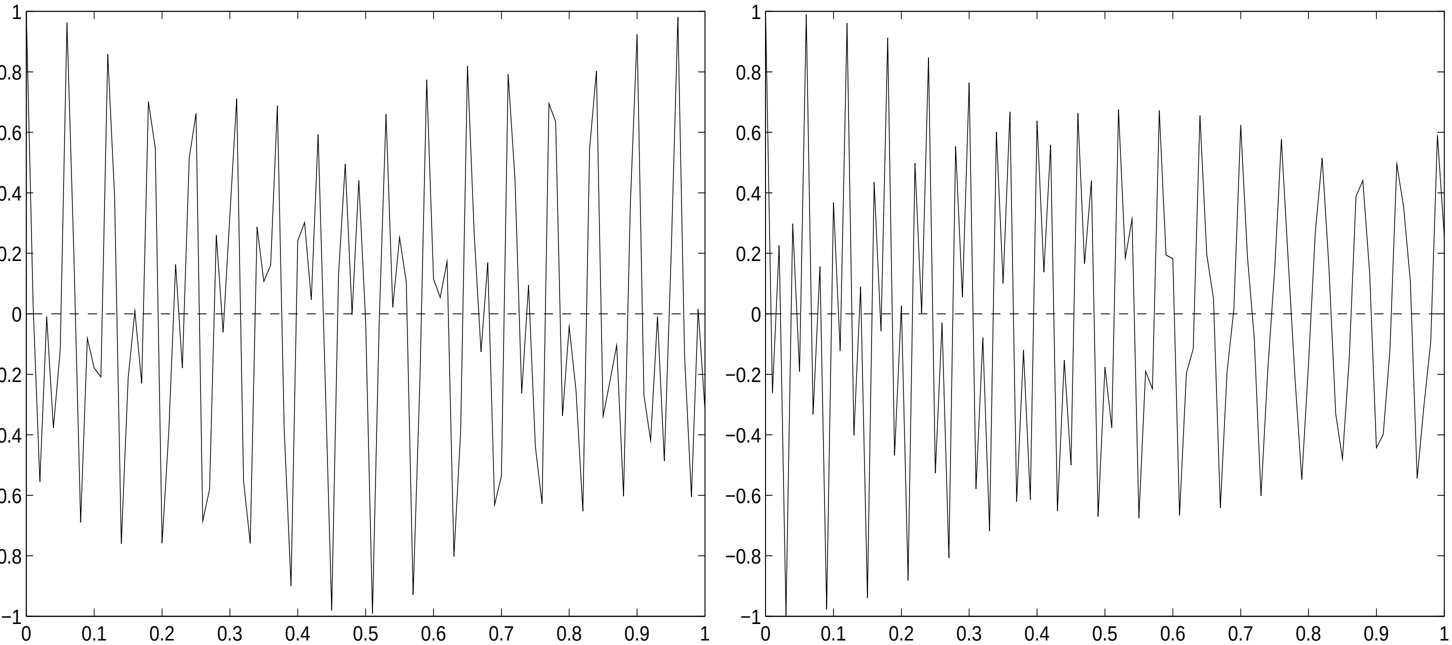
a **b**

Figure 3



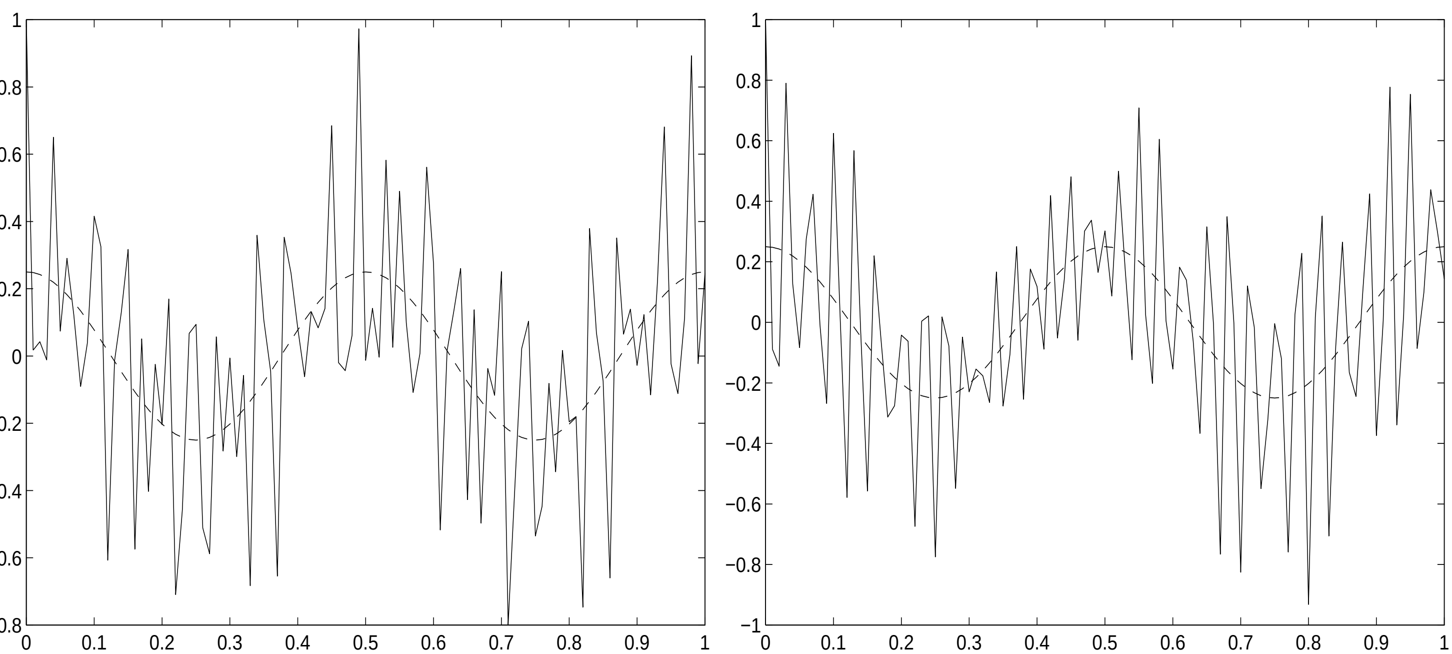
a **b**

Figure 4



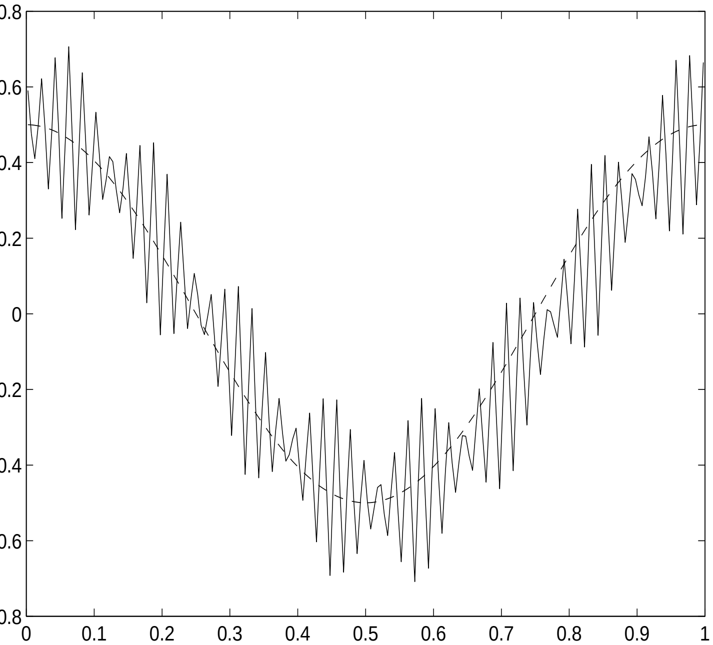
a **b**

Figure 5

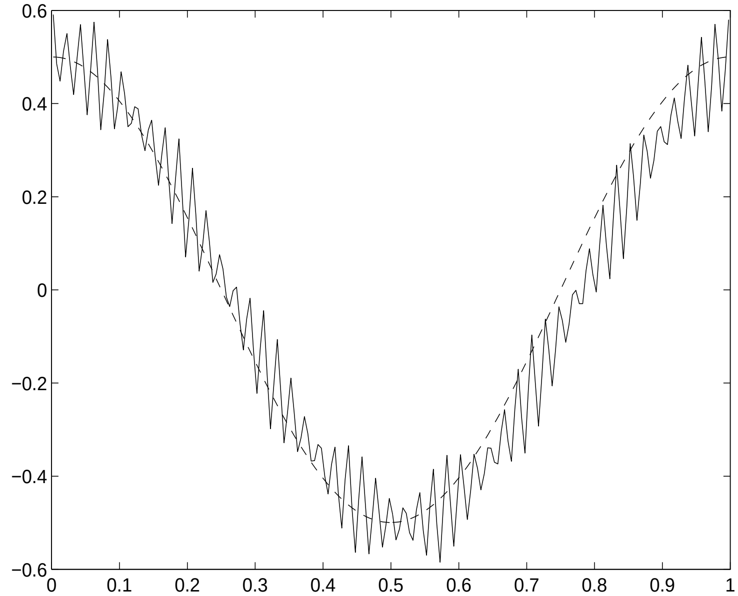


a **b**

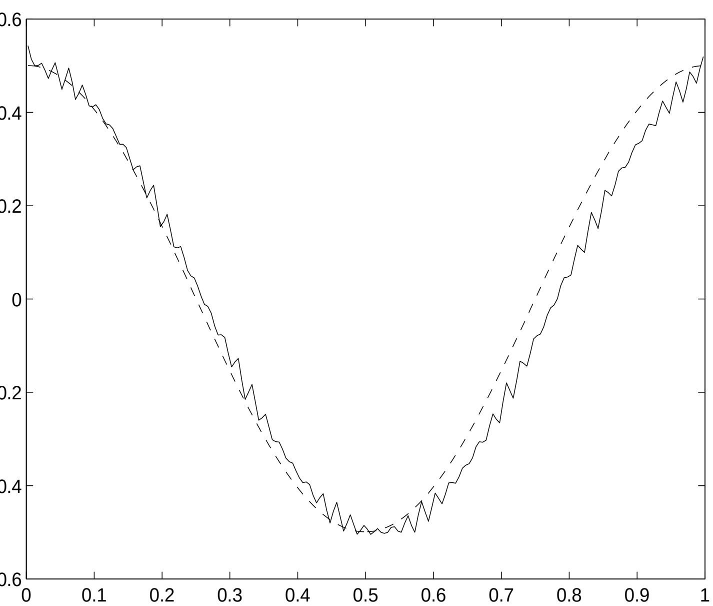
Figure 6



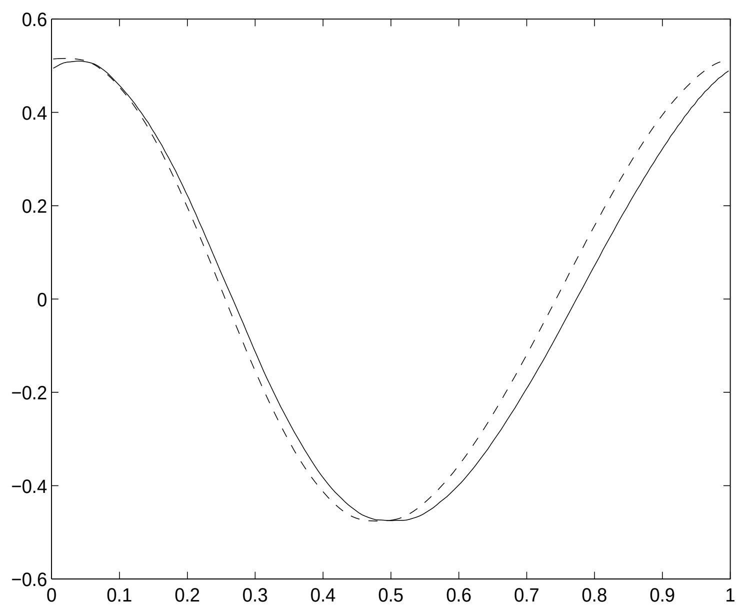
$t = 3.75 * 10^{-4}$



$t = 7.5 * 10^{-3}$

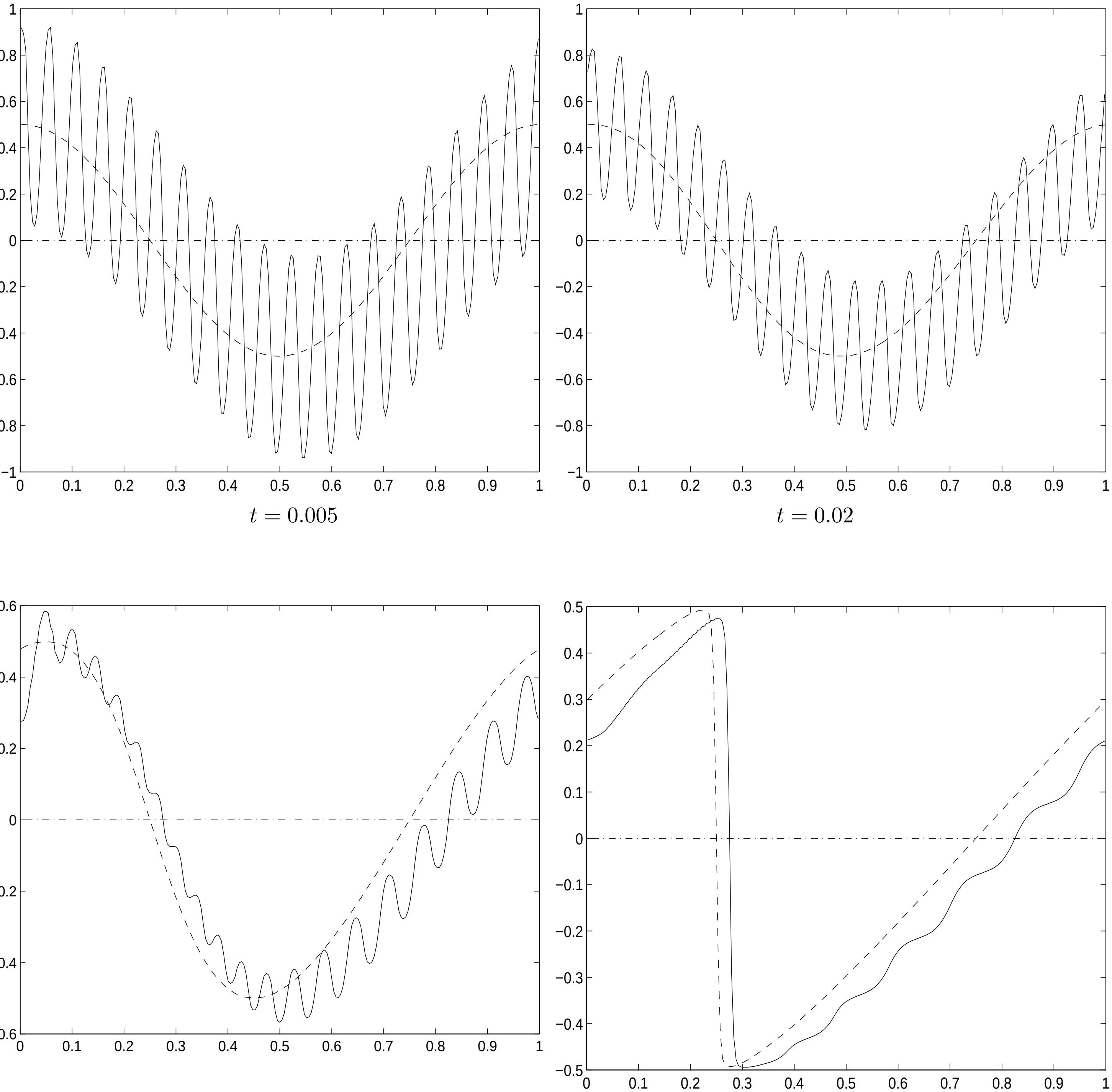


$t = 1.5 * 10^{-3}$



$t = 0.04$

Figure 7



$t = 0.005$ $t = 0.02$

$t = 0.1$ $t = 0.5$

Figure 8

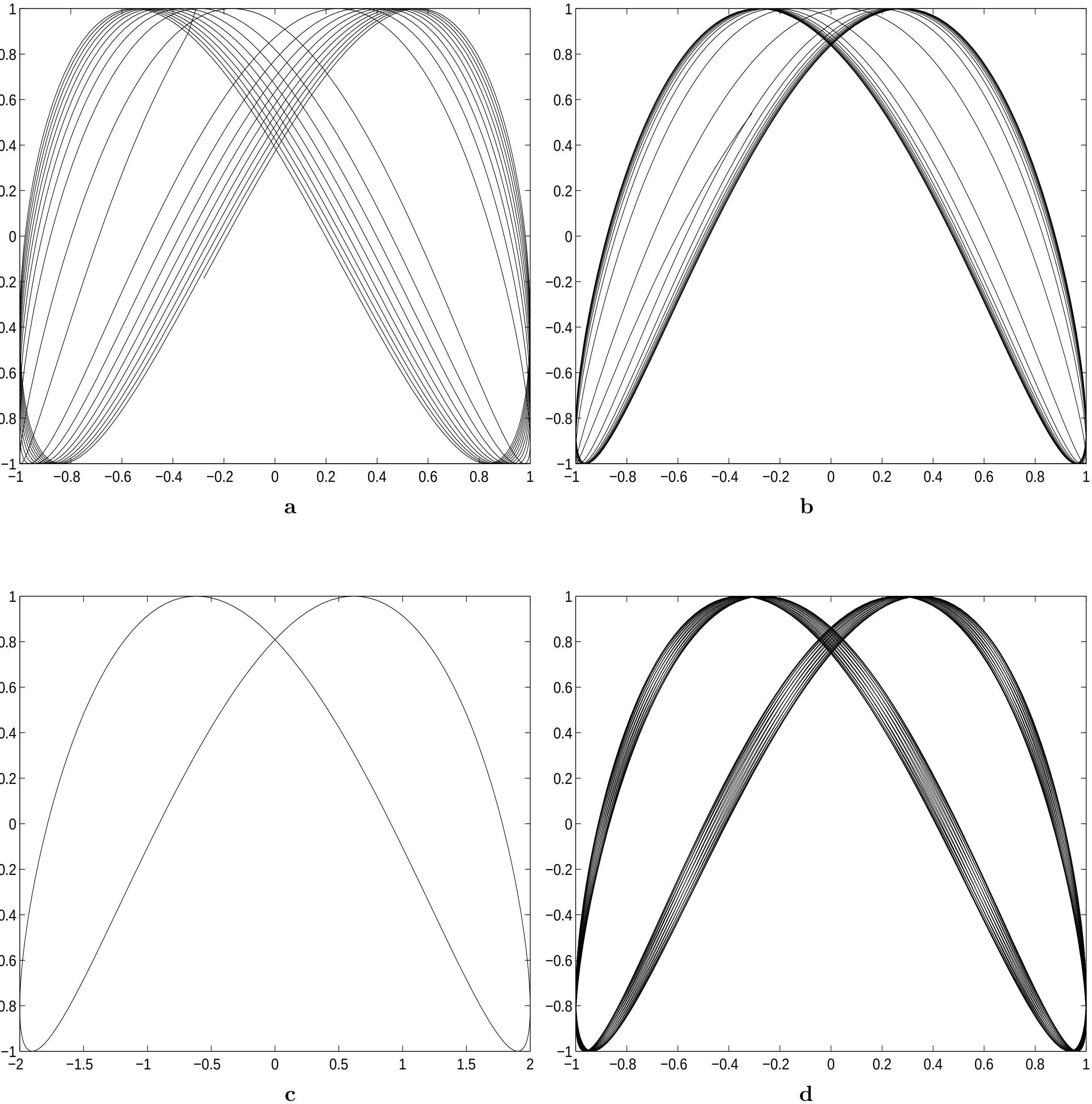


Figure 9

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