CONVERGENCE RATE OF APPROXIMATE SOLUTIONS TO CONSERVATION LAWS WITH INITIAL RAREFACTIONS *

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Abstract. We address the question of local convergence rate of conservative Lip^+-stable approximations, $u^\varepsilon(x,t)$, to the entropy solution, $u(x,t)$, of a genuinely nonlinear conservation law. This question has been answered in the case of rarefaction free, i.e. Lip^+-bounded, initial data. It has been shown that, by post-processing u^ε, pointwise values of u and its derivatives may be recovered with an error as close to $O(\varepsilon)$ as desired, where ε measures, in $W^{-1,1}$, the truncation error of the approximate solution u^ε.

In this paper we extend the previous results by including Lip^+-unbounded initial data. Specifically, we show that for arbitrary $L_\infty \cap BV$ initial data, u and its derivatives may be recovered with an almost optimal, modulo a spurious log factor, error of $O(\varepsilon|\ln\varepsilon|)$. Our analysis relies on obtaining new Lip^+-stability estimates for the speed, $a(u^\varepsilon)$, rather than for u^ε itself. This enables us to establish an $O(\varepsilon|\ln\varepsilon|)$ convergence rate in $W^{-1,1}$, which in turn, implies the above mentioned local convergence rate.

We demonstrate our analysis for four types of approximate solutions: viscous parabolic regularizations, pseudo-viscosity approximations, the regularized Chapman-Enskog expansion and spectral-viscosity methods. Our approach does not depend on the geometry of the characteristics of the solution and, therefore, applies equally to finite-difference approximations of the conservation law.

Key words. Conservation laws, Lip^+-stability, $W^{-1,1}$-consistency, error estimates, parabolic regularizations, spectral viscosity methods

1. Intoduction.

We study the convergence rate of approximate solutions of the single convex conservation law

$$\frac{\partial}{\partial t}[u(x,t)] + \frac{\partial}{\partial x}[f(u(x,t))] = 0 \quad , \quad t>0 \quad , \quad f'' \geq \alpha > 0 \quad , \tag{1.1}$$

with compactly supported (or periodical) initial condition

$$u(x,t=0) = u_0(x) \quad , \quad u_0 \in L_\infty \cap BV \quad . \tag{1.2}$$

Our main focus in this paper is the extension of previous convergence results by allowing possibly Lip^+-unbounded initial conditions,

$$\|u_0(x)\|_{Lip^+} \leq \infty \ , \tag{1.3}$$

where, $\|\cdot\|_{Lip^+}$ denotes the usual Lip^+-semi-norm

$$\|w(x)\|_{Lip^+} \equiv \operatorname*{ess\,sup}_{x\neq y}\left(\frac{w(x)-w(y)}{x-y}\right)^+ \qquad , \qquad (\cdot)^+ \equiv \max(\cdot,0) \quad .$$

It is well known that the solution of (1.1) is not uniquely determined by the initial condition (1.2) in the class of weak solutions. The unique physically relevant weak solution is the one which may be realized as a small viscosity solution of the parabolic regularization

$$\frac{\partial}{\partial t}\left[u^\varepsilon(x,t)\right] + \frac{\partial}{\partial x}\left[f(u^\varepsilon(x,t))\right] = \varepsilon\frac{\partial^2}{\partial x^2}\left[Q(u^\varepsilon(x,t))\right] \quad , \quad Q' \geq 0 \quad , \quad \varepsilon \downarrow 0 \ . \tag{1.4}$$

We recall that these admissible, so-called entropy solutions, are characterized by their Lip^+-stability [18]:

$$\|a(u(\cdot,t))\|_{Lip^+} \leq \frac{\|a(u(\cdot,0))\|_{Lip^+}}{1+t\|a(u(\cdot,0))\|_{Lip^+}} \quad , \quad a(\cdot) = f'(\cdot) \ . \tag{1.5}$$

We therefore seek the convergence rate of conservative approximations to (1.1),

$$\int_x u^\varepsilon(x,t)dx = \int_x u_0(x)dx \quad , \quad t \geq 0 \ ,$$

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which mimic this one sided Lipschitz stability of the exact entropy solution. This leads to

Definition 1.1. *A family $\{u^\varepsilon(x,t)\}_{\varepsilon>0}$ of approximate solutions of the conservation law (1.1) is strongly Lip^+-stable if*

$$\|a(u^\varepsilon(\cdot,t))\|_{Lip^+} \le \frac{\|a(u^\varepsilon(\cdot,0))\|_{Lip^+}}{1+t\|a(u^\varepsilon(\cdot,0))\|_{Lip^+}} \quad , \quad \varepsilon>0\ . \tag{1.6}$$

Our first convergence rate result is the content of the following theorem:

Theorem 1.2. *Let $\{u^\varepsilon(x,t)\}_{\varepsilon>0}$ be a family of conservative and strongly Lip^+-stable approximations to the entropy solution of (1.1)–(1.2), $u(x,t)$. Then,*

(i) *If $\|u_0\|_{Lip^+}<\infty$, the following error estimate holds (K_1 and K_2 denote constants which depend on T):*

$$\|u^\varepsilon(\cdot,T)-u(\cdot,T)\|_{W^{-1,1}} \le \tag{1.7a}$$

$$K_1\|u^\varepsilon(\cdot,0)-u(\cdot,0)\|_{W^{-1,1}} + K_2\|u^\varepsilon_t+f(u^\varepsilon)_x\|_{L_\infty([0,T],W^{-1,1}(\Re_x))} \quad ;$$

(ii) *If $\|u_0\|_{Lip^+}=\infty$ and the approximate solutions are also L_1-stable, the following error estimate holds:*

$$\|u^\varepsilon(\cdot,T)-u(\cdot,T)\|_{W^{-1,1}} \le O\left(\frac{1}{\varepsilon}\right)\|u^\varepsilon(\cdot,0)-u(\cdot,0)\|_{W^{-1,1}}+ \tag{1.7b}$$

$$+O(\varepsilon)\|u^\varepsilon(\cdot,0)\|_{BV}+O(\varepsilon)\|u(\cdot,0)\|_{BV}+O\left(|\ln\varepsilon|\right)\|u^\varepsilon_t+f(u^\varepsilon)_x\|_{L_\infty([0,T],W^{-1,1}(\Re_x))} \quad .$$

Remarks.

1. An approximate solution operator, $S^\varepsilon(t)$, is considered L_1-stable, if for any two initial conditions, u_0 and v_0,

$$\|S^\varepsilon(t)u_0-S^\varepsilon(t)v_0\|_{L_1(\Re_x)} \le \mathrm{Const}_t\|S^\varepsilon(0)u_0-S^\varepsilon(0)v_0\|_{L_1(\Re_x)} \quad , \quad t>0\ . \tag{1.8}$$

2. The norm $\|w(x,t)\|_{W^{-1,1}}$ is defined, when $\int_\Re w(x,t)dx=0$, as follows:

$$\|w(x,t)\|_{W^{-1,1}} = \|w(x,t)\|_{W^{-1,1}(\Re_x)} \equiv \left\|\int_{-\infty}^x w(\xi,t)d\xi\right\|_{L_1(\Re_x)} \quad .$$

3. The use of stability with respect to the Lip^+-semi-norm in order to establish uniqueness for the Cauchy problem (1.1)–(1.2), goes back to Oleinik [12] (see also Theorem 1.8 later on). Stability, in a similar sense, with respect to that semi-norm, was also used in [2] in order to obtain the total variation boundedness and entropy consistency of some finite difference approximations to (1.1) and, consequently, their convergence to the entropy solution. However, this analysis lacks convergence rate estimates.

The first to have used Lip^+-stability in order to *quantify* the convergence rate, was Tadmor [18]. He used the Lip^+-stability of both the entropy solution and its parabolic regularization, (1.4), in order to quantify the convergence rate of the regularization. The same ideas were also used in [10, 11] in the context of finite difference approximations. These works employed the Lip^+-stability of the approximation itself, $u^\varepsilon(x,t)$, namely, an estimate of the sort

$$\|u^\varepsilon(\cdot,t)\|_{Lip^+} \le \frac{\|u^\varepsilon(\cdot,0)\|_{Lip^+}}{1+\beta t\|u^\varepsilon(\cdot,0)\|_{Lip^+}} \quad , \quad 0\le\beta\le\alpha\ , \tag{1.9}$$

in order to obtain convergence rate in the case of Lip^+-bounded initial data. In fact, in that case, our first $W^{-1,1}$-error estimate, (1.7a), holds even if the family of approximate solutions is merely Lip^+-bounded,

$$\|u^\varepsilon(\cdot,t)\|_{Lip^+} \le \mathrm{Const}_t \quad , \quad \varepsilon>0\ , \tag{1.10}$$

and does not satisfy the strong Lip^+-stability requirement (1.6).

However, estimates such as (1.9) or (1.10) are not sufficient in the case of Lip^+-unbounded initial data and a stronger Lip^+-stability, (1.6), of $a(u^\varepsilon(x,t))$ is required.

As a counter-example we mention the Roe scheme (consult [1]): When $\|u_0\|_{Lip^+} < \infty$ this scheme remains Lip^+-bounded, (1.10), and converges to the exact entropy solution. However, it is not strongly Lip^+-stable and, therefore, it fails to converge to the entropy solution in case of Lip^+-unbounded initial data (as demonstrated by the steady state solution obtained by this scheme for $u_0(x) = \mathrm{sgn}(x)$).

The strong Lip^+-stability, (1.6), is indeed one of the main ingredients in establishing convergence rate estimates when initial rarefactions are present. Unfortunately, many well-known approximations of (1.1) fail to satisfy this restricted condition. However, these approximations are still Lip^+-stable in a weaker sense than that of Definition 1.1. This weaker Lip^+-stability proves sufficient in order to establish the same convergence rates as in Theorem 1.2.

Definition 1.3. *Let* $\{u^\varepsilon(x,t)\}_{\varepsilon>0}$ *be a family of approximate solutions of (1.1) and let*

$$W^\varepsilon(t) \equiv \|a(u^\varepsilon(\cdot,t))\|_{Lip^+} \ .$$

Then this family is ε*-weakly* Lip^+*-stable if there exists a constant* M *such that whenever*

$$W^\varepsilon(0) \le \frac{M}{\varepsilon} \ ,$$

the following estimates hold for every $T > 0$*:*

$$e^{\int_0^T W^\varepsilon(t)dt} \le O\left(\frac{1}{\varepsilon}\right) \ ; \tag{1.11}$$

$$\int_0^T e^{\int_t^T W^\varepsilon(\tau)d\tau} dt \le O\left(|\ln\varepsilon|\right) \ . \tag{1.12}$$

Remarks.

1. Any strongly Lip^+-stable family of approximate solutions is also ε-weakly Lip^+-stable (for any value of the constant M).
2. We henceforth refer by Lip^+-stability to either weak or strong Lip^+-stability. This notion of Lip^+-stability is stronger than (1.9), in view of the monotonicity of $a(\cdot)$.

The following theorem asserts that the convergence rate estimates, given in Theorem 1.2 for strongly Lip^+-stable approximations, hold also for ε-weakly Lip^+-stable ones.

Theorem 1.4. *Let* $\{u^\varepsilon(x,t)\}_{\varepsilon>0}$ *be a family of conservative and* Lip^+*-stable approximations to the entropy solution of (1.1),* $u(x,t)$*. Then,*

(i) *If* $\|u_0\|_{Lip^+} < \infty$*, error estimate (1.7a) holds;*
(ii) *If* $\|u_0\|_{Lip^+} = \infty$ *and the approximate solutions are also* L_1*-stable, error estimate (1.7b) holds.*

In order to have convergence, the stability of the family of approximate solutions is not sufficient. The second crucial ingredient is consistency.

Definition 1.5. *The family* $\{u^\varepsilon(x,t)\}_{\varepsilon>0}$ *of approximate solutions is* $W^{-1,1}$*-consistent with (1.1)–(1.2) if*

$$\|u^\varepsilon(\cdot,0) - u_0(\cdot)\|_{W^{-1,1}} \le \begin{cases} \mathrm{Const}\cdot\varepsilon & \text{if } \ \|u_0\|_{Lip^+} < \infty \\ \mathrm{Const}\cdot\varepsilon^2|\ln\varepsilon| & \text{if } \ \|u_0\|_{Lip^+} = \infty \end{cases} \tag{1.13}$$

and

$$\|u_t^\varepsilon + f(u^\varepsilon)_x\|_{L_\infty([0,T],W^{-1,1}(\Re_x))} \le \mathrm{Const}_T\cdot\varepsilon \ . \tag{1.14}$$

In view of Theorem 1.4 and Definition 1.5, we may now conclude the following convergence rate estimates.

COROLLARY 1.6. *($W^{-1,1}$-Error Estimates). If the family $\{u^\varepsilon(x,t)\}_{\varepsilon>0}$ of approximate solutions is conservative, $W^{-1,1}$-consistent with (1.1)–(1.2), L_1-stable and Lip^+-stable, then for every $T>0$ there exists a constant C_T such that*

$$\|u^\varepsilon(\cdot,T)-u(\cdot,T)\|_{W^{-1,1}} \le C_T\cdot\tilde{\varepsilon} \quad , \tag{1.15a}$$

where

$$\tilde{\varepsilon}=\left\{\begin{array}{ll}\varepsilon & \text{if } \ \|u_0\|_{Lip^+}<\infty \\ \varepsilon|\ln\varepsilon| & \text{if } \ \|u_0\|_{Lip^+}=\infty\end{array}\right. \quad . \tag{1.15b}$$

Remarks.

1. Error estimate (1.15) suggests that whenever initial rarefactions are present, the convergence rate in $W^{-1,1}$ is nearly $O(\varepsilon)$. The $|\ln\varepsilon|$ term, which somewhat slows the rate of convergence, is a consequence of the initial rarefaction (as we show later on).
2. Error estimate (1.15) relates to that of Harabetian in [3]. He has shown an $O(\varepsilon|\ln\varepsilon|)$ convergence rate in L_1 for the viscous parabolic regularizations, (1.4), when the exact entropy solution amounts to a pure rarefaction wave.

The $W^{-1,1}$ error estimate (1.15) may be translated, along the lines of [18, 10], into various global, as well as local, error estimates which we summarize as follows:

COROLLARY 1.7. *(Global and Local Error Estimates). Let $\{u^\varepsilon(x,t)\}_{\varepsilon>0}$ be a family of conservative, $W^{-1,1}$-consistent, L_1-stable and Lip^+-stable approximate solutions of the conservation law (1.1)–(1.2). Then the following error estimates hold ($\tilde{\varepsilon}$ is as in (1.15b)):*

$$\|u^\varepsilon(\cdot,T)-u(\cdot,T)\|_{W^{s,p}} \le C_T\cdot\tilde{\varepsilon}^{\frac{1-sp}{2p}} \quad , \quad -1\le s\le\frac{1}{p} \quad , \quad 1\le p\le\infty \quad ; \tag{E1}$$

$$|(u^\varepsilon(\cdot,T)*\phi_\delta)(x)-u(x,T)| \le \mathrm{Const}_{x,T}\cdot\tilde{\varepsilon}^{\frac{p}{p+2}} \quad , \quad \delta\sim\tilde{\varepsilon}^{\frac{1}{p+2}} \quad , \tag{E2}$$

where

$$\mathrm{Const}_{x,T}=\mathrm{Const}_T\cdot\left(1+\frac{1}{p!}\cdot\left\|\frac{\partial^p}{\partial x^p}u(\cdot,T)\right\|_{L_\infty(x-\delta,x+\delta)}\right)$$

and $\phi_\delta(x)=\frac{1}{\delta}\phi\left(\frac{x}{\delta}\right)$ is any unit mass $C_0^1(-1,1)$-mollifier, satisfying

$$\int_{-1}^{1}x^k\phi(x)dx=0 \quad \text{for } k=1,2,...,p-1 \quad ;$$

$$|u^\varepsilon(x,T)-u(x,T)| \le \mathrm{Const}_{x,T}\cdot\sqrt[3]{\tilde{\varepsilon}} \quad , \tag{E3}$$

where

$$\mathrm{Const}_{x,T}=\mathrm{Const}_T\cdot\left(1+\|u_x(\cdot,T)\|_{L_\infty(x-\sqrt[3]{\tilde{\varepsilon}},x+\sqrt[3]{\tilde{\varepsilon}})}\right) \quad .$$

Remark. A similar treatment enables the recovery of the derivatives of $u(x,t)$ as well, consult [18, §4].

We would like to point out two straightforward consequences of Theorem 1.2, interesting for their own sake. The first is a simple proof of the uniqueness of Lip^+-stable solutions to (1.1)–(1.2), Theorem 1.8, and the second is the $W^{-1,1}$-stability of entropy solutions of (1.1), Theorem 1.9.

THEOREM 1.8. *Weak solutions of the convex conservation law (1.1) which are Lip^+-stable, (1.5), are uniquely determined by their initial value.*

THEOREM 1.9. *Let u and v denote two entropy solutions of the conservation law (1.1), subject to the $L_\infty \cap BV$ initial data u_0 and v_0, respectively. Then*

$$\|v(\cdot,t) - u(\cdot,t)\|_{W^{-1,1}} \leq \mathrm{Const}_t \cdot \|v_0 - u_0\|^{\eta}_{W^{-1,1}} \ , \tag{1.16}$$

where $\eta = 1$ if u_0 and v_0 are Lip^+-bounded and $\eta = \frac{1}{2}$ otherwise.

This paper is organized as follows:

After §2 in which we prove our main results, Theorems 1.2–1.9, the rest of the paper is devoted to applications to various types of approximations.

In §3 we deal with the family of viscous parabolic regularizations, (1.4). We prove that these approximations are L_1-contractive, $W^{-1,1}$-consistent and Lip^+-stable, in order to conclude that they converge to the exact entropy solution and satisfy the convergence rate estimates (E1)–(E3). We further show that if the viscosity coefficient satisfies

$$\left(\frac{Q'}{a'}\right)'' \leq 0 \ , \tag{1.17}$$

then the resulting approximation is even strongly Lip^+-stable. The most natural choice (already presented by Von-Neumann, Lax and Wendroff, [16]) of a monotone regularization coefficient, $Q(u)$, which satisfies (1.17) is $Q(\cdot) = a(\cdot)$. Hence, we refer to regularizations which satisfy condition (1.17) as "speed-like".

In §4 we apply our analysis to pseudo-viscosity approximations. These approximations are parabolic regularizations with a gradient dependent viscosity,

$$u^\varepsilon_t + f(u^\varepsilon)_x = \varepsilon Q(u^\varepsilon, p^\varepsilon)_x \quad , \quad p^\varepsilon := u^\varepsilon_x \quad , \quad \varepsilon \frac{\partial Q}{\partial p^\varepsilon} \downarrow 0 \ .$$

Such approximations, with $Q = Q(p^\varepsilon)$, were introduced by von Neumann and Richtmeyer in [9] and discussed later in [7]. We derive conditions on the pseudo-viscosity coefficient, Q, under which the resulting approximation is Lip^+-stable and $W^{-1,1}$-consistent and, consequently, satisfies error estimates (E1)–(E3).

In §5 we discuss the regularized Chapman-Enskog expansion for hydrodynamics (consult [14, 17]). We focus our attention on Burgers' equation and demonstrate our analysis in this case.

Finally, in §6, we show how the Spectral Viscosity (SV) method (consult [8, 19, 20]) fits into our framework as well. In the course of the analysis performed there, we introduce an extension argument which removes the need for an a-priori L_∞-bound. This argument may also be used for other approximate solutions of (1.1) for which an a-priori L_∞-bound is not known in advance.

We close the Introduction by referring to the applicability of our framework to finite difference schemes, $\{v^{\Delta x}\}_{\Delta x > 0}$. It is shown in [10, 11] that finite difference schemes in viscosity form are conservative, BV-bounded and $W^{-1,1}$-consistent with (1.1)–(1.2). Hence, so that our convergence rate results will apply to these schemes, all that remains to show is that they satisfy our strict notion of Lip^+-stability, (1.6) or (1.11)–(1.12). However, the best Lip^+-stability estimates which have been established for finite difference schemes are of the form (1.9). Since we have not been able, so far, to sharpen those estimates, we do not include a treatment of these approximations in the present paper.

2. Proof of main results. We begin this section by proving our basic convergence rate estimates, as stated in Theorems 1.2 and 1.4 in the Introduction. Since Theorem 1.2 deals with strongly Lip^+-stable approximations, which are, as noted before, weakly Lip^+-stable as well, it suffices to prove Theorem 1.4.

Proof (of Theorem 1.4). We deal with conservative approximations to (1.1) which take the following form

$$\frac{\partial}{\partial t}[u^\varepsilon(x,t)] + \frac{\partial}{\partial x}[f(u^\varepsilon(x,t))] = r^\varepsilon(x,t) \quad , \quad t>0 \quad , \quad \varepsilon \downarrow 0 \ , \tag{2.1}$$

where $r^\varepsilon(x,t)$ is the truncation error of the approximation, and we need to estimate, in $W^{-1,1}$, the error

$$e^\varepsilon(x,t) \equiv u^\varepsilon(x,t) - u(x,t) \ .$$

Step 1. We first assume that both the exact entropy solution, $u(x,t)$, and its approximation, $u^\varepsilon(x,t)$, have a Lip^+-bounded initial data, i.e.,

$$L_0^+ = \max\{\|a(u(\cdot,0))\|_{Lip^+}, \|a(u^\varepsilon(\cdot,0))\|_{Lip^+}\} < \infty \ . \tag{2.2}$$

Subtracting (1.1) from (2.1) we arrive at the equation which governs the error $e^\varepsilon(x,t)$,

$$\frac{\partial}{\partial t}[e^\varepsilon(x,t)] + \frac{\partial}{\partial x}[\bar{a}^\varepsilon(x,t)e^\varepsilon(x,t)] = r^\varepsilon(x,t) \quad , \quad t>0 \ , \tag{2.3}$$

where

$$\bar{a}^\varepsilon(x,t) = \int_0^1 a\left(\xi u^\varepsilon(x,t) + (1-\xi)u(x,t)\right) d\xi \ .$$

Note that the monotonicity of $a(\cdot)$ implies that

$$\min\{a(u), a(u^\varepsilon)\} \le \bar{a}^\varepsilon(x,t) \le \max\{a(u), a(u^\varepsilon)\} \ . \tag{2.4}$$

Integration of (2.3) with respect to x yields

$$\frac{\partial}{\partial t}[E^\varepsilon(x,t)] + \bar{a}^\varepsilon(x,t)\frac{\partial}{\partial x}[E^\varepsilon(x,t)] = R^\varepsilon(x,t) \quad , \quad t>0 \ , \tag{2.5}$$

where

$$E^\varepsilon(x,t) = \int_{-\infty}^x e^\varepsilon(\xi,t)d\xi \quad , \quad R^\varepsilon(x,t) = \int_{-\infty}^x r^\varepsilon(\xi,t)d\xi \ .$$

Integration of (2.5) over $\Re$ against $\mathrm{sgn}(E^\varepsilon)$ and rearranging, yield that

$$\frac{d}{dt}\|E^\varepsilon(\cdot,t)\|_{L_1} \le \int_x \bar{a}^\varepsilon(x,t)\left(-\frac{\partial}{\partial x}|E^\varepsilon(x,t)|\right)dx + \|R^\varepsilon(\cdot,t)\|_{L_1} \ . \tag{2.6}$$

The main effort henceforth is concentrated on upper bounding the integral on the right hand side of (2.6). To this end we suggest to divide the real line into intervals,

$$\Re = \cup_n I_n(t) \quad , \quad I_n(t) = [x_n(t), x_{n+1}(t)) \ ,$$

in such a way that neither $\mathrm{sgn}(e^\varepsilon)$ nor $\mathrm{sgn}(E^\varepsilon)$ change within the interior of these intervals (the implicit assumption of piecewise smoothness of the solution, as in [5], may be removed by considering a further vanishing parabolic regularization which is omitted). We use this division to define the following function:

$$\hat{a}^\varepsilon(x,t) = \begin{cases} a(u(x,t)) & \text{if } x \in I_n(t) \text{ and } E^\varepsilon(x,t) \ge 0\Big|_{I_n(t)} \\ & \\ a(u^\varepsilon(x,t)) & \text{if } x \in I_n(t) \text{ and } E^\varepsilon(x,t) \le 0\Big|_{I_n(t)} \end{cases} \ . \tag{2.7}$$

We now claim (and prove later on) that

$$\int_x \bar{a}^\varepsilon(x,t)\left(-\frac{\partial}{\partial x}|E^\varepsilon(x,t)|\right)dx \le \int_x \hat{a}^\varepsilon(x,t)\left(-\frac{\partial}{\partial x}|E^\varepsilon(x,t)|\right)dx \quad . \tag{2.8}$$

Integration by parts of the right hand side of (2.8) yields

$$\int_x \bar{a}^\varepsilon(x,t)\left(-\frac{\partial}{\partial x}|E^\varepsilon(x,t)|\right)dx \le \int_x \frac{\partial}{\partial x}[\hat{a}^\varepsilon(x,t)]\cdot|E^\varepsilon(x,t)|dx \quad . \tag{2.9}$$

The following inequality (whose proof is postponed) provides us an upper bound for the integral on the right hand side of (2.9):

$$\int_x \frac{\partial}{\partial x}[\hat{a}^\varepsilon(x,t)]\cdot|E^\varepsilon(x,t)|dx \le L^\varepsilon(t)\|E^\varepsilon(\cdot,t)\|_{L_1} \quad , \tag{2.10a}$$

where

$$L^\varepsilon(t) = \max\left\{\frac{L_0^+}{1+tL_0^+}\ ,\ W^\varepsilon(t) = \|a(u^\varepsilon(\cdot,t))\|_{Lip^+}\right\} . \tag{2.10b}$$

Inserting (2.9) and (2.10a) into (2.6), we arrive at the inequality

$$\frac{d}{dt}\|e^\varepsilon(\cdot,t)\|_{W^{-1,1}} \le L^\varepsilon(t)\|e^\varepsilon(\cdot,t)\|_{W^{-1,1}} + \|r^\varepsilon(\cdot,t)\|_{W^{-1,1}} \ , \tag{2.11}$$

which implies that

$$\|e^\varepsilon(\cdot,T)\|_{W^{-1,1}} \le e^{\int_0^T L^\varepsilon(t)dt}\cdot\|e^\varepsilon(\cdot,0)\|_{W^{-1,1}} + \int_0^T e^{\int_t^T L^\varepsilon(\tau)d\tau}\|r^\varepsilon(\cdot,t)\|_{W^{-1,1}}dt \ . \tag{2.12}$$

Since, by the definition of L_0^+ in (2.2), $W^\varepsilon(0) \le L_0^+$, we conclude, in view of Lip^+-stability (see Definition 1.3, that

$$e^{\int_0^T W^\varepsilon(t)dt} \le \mathrm{Const}_1 \quad , \quad \mathrm{Const}_1 \sim \frac{L_0^+}{M} \quad , \tag{2.13}$$

and

$$\int_0^T e^{\int_t^T W^\varepsilon(\tau)d\tau}dt \le \mathrm{Const}_2 \quad , \quad \mathrm{Const}_2 \sim |\ln M - \ln L_0^+| \quad . \tag{2.14}$$

Using (2.13), (2.14) and (2.10b) in (2.12), proves the desired error estimate (1.7a).

Finally, in order to conclude Step 1, we return to justify (2.8) and (2.10):

First, we prove (2.8) by showing that the inequality holds in each interval $I_n(t)$, i.e,

$$\int_{I_n(t)} \bar{a}^\varepsilon(x,t)\left(-\frac{\partial}{\partial x}|E^\varepsilon(x,t)|\right)dx \le \int_{I_n(t)} \hat{a}^\varepsilon(x,t)\left(-\frac{\partial}{\partial x}|E^\varepsilon(x,t)|\right)dx \quad . \tag{2.15}$$

Suppose that $E^\varepsilon(\cdot,t) \ge 0$ in $I_n(t)$. Then by definition (2.7),

$$\hat{a}^\varepsilon(x,t) = a(u(x,t)) \quad \forall x \in I_n(t) \ . \tag{2.16}$$

There are two possibilities to consider. If $e^\varepsilon(x,t) \ge 0$ in $I_n(t)$ then by (2.4)

$$\bar{a}^\varepsilon(x,t) \ge a(u(x,t)) \quad , \quad -\frac{\partial}{\partial x}|E^\varepsilon(x,t)| = -\mathrm{sgn}(E^\varepsilon(x,t))\cdot e^\varepsilon(x,t) \le 0 \quad \forall x \in I_n(t) \ . \tag{2.17}$$

Therefore, (2.15) follows in this case by (2.16) and (2.17). If, on the other hand, $e^\varepsilon(x,t) \leq 0$ in $I_n(t)$, then

$$\bar{a}^\varepsilon(x,t) \leq a(u(x,t)) \quad , \quad -\frac{\partial}{\partial x}|E^\varepsilon(x,t)| \geq 0 \qquad \forall x \in I_n(t) \tag{2.18}$$

and (2.15) follows in this case as well. The case $E^\varepsilon(\cdot,t) \mid_{I_n(t)} \leq 0$ is treated similarly. This concludes the proof of (2.8).

Next, we prove inequality (2.10). In view of definitions (2.7) and (2.10b), we conclude, using the Lip^+-stability of the exact solution,

$$\|a(u(\cdot,t))\|_{Lip^+} \leq \frac{L_0^+}{1+tL_0^+} \quad ,$$

that $\frac{\partial}{\partial x}[\hat{a}^\varepsilon(x,t)]$ satisfies the following inequality in the sense of distributions:

$$\frac{\partial}{\partial x}[\hat{a}^\varepsilon(x,t)] \leq L^\varepsilon(t) + \sum [\hat{a}^\varepsilon(x_n(t)+0,t) - \hat{a}^\varepsilon(x_n(t)-0,t)]\delta(x-x_n(t)) \quad , \tag{2.19}$$

the sum being taken over all division points $x_n(t)$ where $\hat{a}^\varepsilon(\cdot,t)$ experiences a jump discontinuity, namely where $\text{sgn}(E^\varepsilon(\cdot,t))$ changes. But, $E^\varepsilon(\cdot,t)$ – being a continuous primitive function – vanishes at these points. Hence, integration of (2.19) against $|E^\varepsilon(x,t)|$ proves (2.10a) and completes Step 1.

Step 2. Now we turn to the case of initial rarefactions and prove error estimate (1.7b). To this end we introduce the function $\psi_\delta(\cdot) = \frac{1}{\delta}\psi(\frac{\cdot}{\delta})$, $\delta > 0$, which is the dilated mollifier of

$$\psi(x) = \begin{cases} 1 & |x| \leq \frac{1}{2} \\ 0 & |x| > \frac{1}{2} \end{cases} \quad . \tag{2.20}$$

Clearly

$$\|\psi_\delta * w - w\|_{L_1} \leq O(\delta)\|w\|_{BV} \ , \tag{2.21}$$

and

$$\|\psi_\delta * w\|_{Lip^+} \leq O\left(\frac{1}{\delta}\right) \qquad \delta \downarrow 0 \ . \tag{2.22}$$

With this in mind we return to the conservation law (1.1) and its approximate solution (2.1) and define a new pair of solutions, u_δ and u_δ^ε, corresponding to the mollified initial data:

$$\frac{\partial}{\partial t}[u_\delta(x,t)] + \frac{\partial}{\partial x}[f(u_\delta(x,t))] = 0 \quad , \quad u_\delta(\cdot,0) = \psi_\delta * u(\cdot,0) \ ; \tag{2.23}$$

$$\frac{\partial}{\partial t}[u_\delta^\varepsilon(x,t)] + \frac{\partial}{\partial x}[f(u_\delta^\varepsilon(x,t)] = r_\delta^\varepsilon(x,t) \quad , \quad u_\delta^\varepsilon(\cdot,0) = \psi_\delta * u^\varepsilon(\cdot,0) \ . \tag{2.24}$$

We are now able to estimate the $W^{-1,1}$-error in (1.7b) by decomposing it as follows:

$$\|u^\varepsilon(\cdot,T) - u(\cdot,T)\|_{W^{-1,1}} \leq \tag{2.25}$$

$$\|u^\varepsilon(\cdot,T) - u_\delta^\varepsilon(\cdot,T)\|_{W^{-1,1}} + \|u_\delta^\varepsilon(\cdot,T) - u_\delta(\cdot,T)\|_{W^{-1,1}} + \|u_\delta(\cdot,T) - u(\cdot,T)\|_{W^{-1,1}} \ .$$

Since for compactly supported functions, $\|w\|_{W^{-1,1}} \leq |\text{supp}\{w\}| \cdot \|w\|_{L_1}$, we may bound the first term on the right hand side of (2.25), using (1.8), (2.24) and (2.21), as follows (Ω_T denotes the compact support[1] at $t = T$):

$$\|u^\varepsilon(\cdot,T) - u_\delta^\varepsilon(\cdot,T)\|_{W^{-1,1}} \leq |\Omega_T| \cdot \|u^\varepsilon(\cdot,T) - u_\delta^\varepsilon(\cdot,T)\|_{L_1} \leq \tag{2.26}$$

[1]Note that in case $u^\varepsilon(\cdot,T)$ is not compactly supported, the exponential decay which characterizes the tail of various viscosity-like approximations will suffice for our estimates.

$$\leq |\Omega_T| \cdot C_T \|u^\varepsilon(\cdot,0) - u^\varepsilon_\delta(\cdot,0)\|_{L_1} \leq |\Omega_T| \cdot C_T \cdot O(\delta)\|u^\varepsilon(\cdot,0)\|_{BV} = O(\delta)\|u^\varepsilon(\cdot,0)\|_{BV} \ .$$

Similarly, the last term on the right hand side of (2.25), may be bounded by

$$\|u_\delta(\cdot,T) - u(\cdot,T)\|_{W^{-1,1}} \leq O(\delta)\|u(\cdot,0)\|_{BV} \ . \tag{2.27}$$

Hence, it remains only to deal with the term $\|u^\varepsilon_\delta(\cdot,T) - u_\delta(\cdot,T)\|_{W^{-1,1}}$. This requires δ to be appropriately chosen so that

$$W^\varepsilon_\delta(0) \leq \frac{M}{\varepsilon} \quad , \quad W^\varepsilon_\delta(t) = \|a(u^\varepsilon_\delta(\cdot,t))\|_{Lip^+} \tag{2.28}$$

and, consequently, the Lip^+-stability estimates x (1.11)–(1.12) hold. If D denotes the largest positive jump in $u^\varepsilon(\cdot,0)$ then the choice $\delta = 2D\max[a'(u^\varepsilon(\cdot,0))]\varepsilon/M$ will do for (2.28). By doing so, we may conclude the ε-weak Lip^+-stability estimates, (1.11)–(1.12), for $W^\varepsilon_\delta(t)$:

$$e^{\int_0^T W^\varepsilon_\delta(t)dt} \leq O\left(\frac{1}{\varepsilon}\right) \quad ; \quad \int_0^T e^{\int_t^T W^\varepsilon_\delta(\tau)d\tau} dt \leq O(|\ln\varepsilon|) \quad .$$

These estimates, together with error estimate (2.12) for $e^\varepsilon_\delta = u^\varepsilon_\delta - u_\delta$, imply that

$$\|u^\varepsilon_\delta(\cdot,T) - u_\delta(\cdot,T)\|_{W^{-1,1}} \leq \tag{2.29}$$

$$O\left(\frac{1}{\varepsilon}\right)\|u^\varepsilon_\delta(\cdot,0) - u_\delta(\cdot,0)\|_{W^{-1,1}} + O(|\ln\varepsilon|)\|r^\varepsilon_\delta\|_{L_\infty([0,T],W^{-1,1}(\Re_x))} \ .$$

Since $\|\psi_\delta * w\|_{W^{-1,1}} \leq \|w\|_{W^{-1,1}}$, estimate (2.29) implies that

$$\|u^\varepsilon_\delta(\cdot,T) - u_\delta(\cdot,T)\|_{W^{-1,1}} \leq \tag{2.30}$$

$$O\left(\frac{1}{\varepsilon}\right)\|u^\varepsilon(\cdot,0) - u(\cdot,0)\|_{W^{-1,1}} + O(|\ln\varepsilon|) \cdot \|r^\varepsilon\|_{L_\infty([0,T],W^{-1,1}(\Re_x))} \ .$$

Therefore, since $\delta = O(\varepsilon)$, (1.7b) follows from (2.25), (2.26), (2.27) and (2.30) and the proof is thus concluded. ◻

Remark. Note that if the approximate solution smoothens the initial data so that

$$\|u^\varepsilon(\cdot,0)\|_{Lip^+} \leq O\left(\frac{1}{\varepsilon}\right) \ ,$$

e.g. – the SV-method, there is no need to mollify the initial data of the approximation, as we did in (2.24). Hence, in this case, the error term (2.26) does not exist and, therefore, error estimate (1.7b) holds even if the approximate solution is not L_1-stable.

We close this section with the proof of Theorems 1.8 and 1.9.

Proof (of Theorem 1.8). Let u be the entropy solution of (1.1)–(1.2) and v be another weak solution of (1.1)–(1.2) which is also Lip^+-stable x in the sense of (1.5). Setting $u^\varepsilon = v$, $\varepsilon > 0$, we have

$$u^\varepsilon(\cdot,0) - u(\cdot,0) = 0 \quad \text{and} \quad u^\varepsilon_t + f(u^\varepsilon)_x = 0 \qquad \forall\varepsilon > 0 \ .$$

Hence, error estimate (1.7b) implies that

$$\|v(\cdot,T) - u(\cdot,T)\|_{W^{-1,1}} = \|u^\varepsilon(\cdot,T) - u(\cdot,T)\|_{W^{-1,1}} \leq O(\varepsilon)\|u_0\|_{BV} \qquad \forall\varepsilon > 0 \ .$$

Letting $\varepsilon \downarrow 0$, we conclude that $u = v$. ◻

Proof (of Theorem 1.9). We set $u^\varepsilon = v$ for all $\varepsilon > 0$ and use error estimates (1.7a) and (1.7b), given in Theorem (1.1). Since u^ε is an exact entropy solution of (1.1), the truncation error term on the right hand side of both estimates vanishes.

In case that both u_0 and v_0 are Lip^+-bounded, estimate (1.7a) holds and (1.16) follows with $\mathrm{Const}_t = K_1$ and $\eta = 1$.

If either of the initial conditions is Lip^+-unbounded, estimate (1.7b) holds and we conclude that

$$\|v(\cdot,t) - u(\cdot,t)\|_{W^{-1,1}} \leq O\left(\frac{1}{\varepsilon}\right) \|v_0 - u_0\|_{W^{-1,1}} + O(\varepsilon)\Big(\|v_0\|_{BV} + \|u_0\|_{BV}\Big)$$

for all $\varepsilon > 0$. Taking $\varepsilon = \|v_0 - u_0\|^{\frac{1}{2}}_{W^{-1,1}}$, proves (1.16) with $\eta = \frac{1}{2}$. $\square$

3. Viscous parabolic regularizations. We consider here viscous parabolic regularizations to (1.1) of the form (1.4). These regularizations are:

- Conservative;
- L_∞-bounded, $\|u^\varepsilon(\cdot,t)\|_{L_\infty} \leq \|u_0\|_{L_\infty}$;
- L_1-contractive and, therefore, thanks to translation invariance, BV-bounded (see Theorem 4.1, later on, for a proof of L_1-contraction in a more general setting);
- $W^{-1,1}$-consistent in the sense of Definition 1.5, since $u^\varepsilon(\cdot,0) = u_0(\cdot)$ and

$$\|u^\varepsilon_t + f(u^\varepsilon)_x\|_{W^{-1,1}} = \|\varepsilon Q(u^\varepsilon)_x\|_{L_1} \leq \varepsilon \cdot \max_{|u| \leq \|u_0\|_{L_\infty}} |Q'(u)| \cdot \|u^\varepsilon(\cdot,t)\|_{BV} \leq O(\varepsilon) \ ;$$

- Lip^+-stable (Theorem 3.1).

In view of the above, error estimates (E1)–(E3), x given in Corollary 1.7, apply to this family of approximate solutions.

We are therefore left only with the task of proving Lip^+-stability; this is done in the following theorem and lemma.

THEOREM 3.1. *The (possibly degenerate) parabolic regularization of (1.1),*

$$\frac{\partial}{\partial t}[u^\varepsilon(x,t)] + \frac{\partial}{\partial x}[f(u^\varepsilon(x,t))] = \varepsilon \frac{\partial^2}{\partial x^2}[Q(u^\varepsilon(x,t))] \quad , \quad Q' \geq 0 \quad , \quad \varepsilon \downarrow 0 \ , \tag{3.1}$$

is strongly Lip^+-stable if

$$\left(\frac{Q'}{a'}\right)'' \leq 0 \ , \tag{3.2}$$

and ε-weakly Lip^+-stable otherwise.

Proof. Let us first assume that Q' is strictly positive so that the solution u^ε is smooth. Multiplying (3.1) by $a'(u^\varepsilon(x,t))$ we get

$$\frac{\partial}{\partial t}[a(u^\varepsilon)] + a(u^\varepsilon)\frac{\partial}{\partial x}[a(u^\varepsilon)] = \varepsilon a'(u^\varepsilon)\frac{\partial^2}{\partial x^2}[Q(u^\varepsilon)] \ . \tag{3.3}$$

By denoting

$$w^\varepsilon = w^\varepsilon(x,t) = \frac{\partial a(u^\varepsilon)}{\partial x} = a'(u^\varepsilon)\frac{\partial u^\varepsilon}{\partial x} \ , \tag{3.4}$$

the right hand side of (3.3) may be rewritten as follows:

$$\varepsilon a'(u^\varepsilon)\frac{\partial^2}{\partial x^2}[Q(u^\varepsilon)] = \varepsilon \left[Q'(u^\varepsilon)\frac{\partial w^\varepsilon}{\partial x} + \left(\frac{Q'(u^\varepsilon)}{a'(u^\varepsilon)}\right)' (w^\varepsilon)^2 \right] \ . \tag{3.5}$$

Differentiation of (3.3) with respect to x and using identity (3.5) yields

$$\frac{\partial w^\varepsilon}{\partial t} + (w^\varepsilon)^2 + a(u^\varepsilon)\frac{\partial w^\varepsilon}{\partial x} = \tag{3.6}$$

$$\varepsilon\left[Q'(u^\varepsilon)\frac{\partial^2 w^\varepsilon}{\partial x^2} + \frac{Q''(u^\varepsilon)}{a'(u^\varepsilon)}w^\varepsilon\frac{\partial w^\varepsilon}{\partial x} + 2\left(\frac{Q'(u^\varepsilon)}{a'(u^\varepsilon)}\right)' w^\varepsilon\frac{\partial w^\varepsilon}{\partial x} + \left(\frac{Q'(u^\varepsilon)}{a'(u^\varepsilon)}\right)''\frac{(w^\varepsilon)^3}{a'(u^\varepsilon)}\right] .$$

Since u^ε is smooth and compactly supported, $w^\varepsilon(\cdot,t)$ attains its maximal value, say in $x = x(t)$, and

$$w^\varepsilon(x(t),t) \geq 0 \quad , \quad \frac{\partial w^\varepsilon}{\partial x}(x(t),t) = 0 \quad , \quad \frac{\partial^2 w^\varepsilon}{\partial x^2}(x(t),t) \leq 0 \ . \tag{3.7}$$

Hence, denoting

$$W^\varepsilon(t) = w^\varepsilon(x(t),t) = \|a(u^\varepsilon(\cdot,t))\|_{Lip^+} \ , \tag{3.8}$$

we conclude by (3.6), (3.7) and the positivity of a' and Q', that

$$\frac{dW^\varepsilon}{dt} + (W^\varepsilon)^2 \leq \varepsilon K (W^\varepsilon)^3 \ , \tag{3.9}$$

where

$$K \equiv \frac{1}{\alpha}\max_{|u|\leq\|u_0\|_{L_\infty}}\left[\left(\frac{Q'(u^\varepsilon)}{a'(u^\varepsilon)}\right)''\right]^+ \ . \tag{3.10}$$

In view of Lemma 3.2 below, inequality (3.9) implies ε-weak Lip^+-stability. In case that condition (3.2) holds, $K = 0$ and inequality (3.9) amounts to Ricatti's inequality

$$\frac{dW^\varepsilon}{dt} + (W^\varepsilon)^2 \leq 0 \ ,$$

which implies strong Lip^+-stability.

If $Q' \geq 0$, equation (3.1) is degenerate and, therefore, admits non-smooth solutions. This case may be treated, as in [21], by introducing a further regularization. We replace $Q(\cdot)$ by the strictly monotone regularization term $Q_\delta(\cdot) = Q(\cdot) + \delta a(\cdot)$. Note that with this choice of Q_δ, the value of K, (3.10), does not change. Hence, the corresponding solution, u^ε_δ, satisfies inequality (3.9) and by letting $\delta \downarrow 0$, we obtain the same inequality for the limit solution. ☐

Remark. The most common choice of a regularization coefficient is $Q(u) = u$. For this special choice of $Q(u)$, the speed-like condition (3.2) reads $\left(\frac{1}{a'}\right)'' \leq 0$, consult [6].

Lemma 3.2. *Let $y^\varepsilon(t)$ denote the solution of*

$$\frac{dy^\varepsilon}{dt} + (y^\varepsilon)^2 = \varepsilon K (y^\varepsilon)^3 \quad , \quad K > 0 \quad , \quad t > 0 \ , \tag{3.11}$$

where

$$y^\varepsilon(t=0) = \frac{c^\varepsilon}{\varepsilon K} \tag{3.12}$$

and c^ε satisfies

$$0 < \underline{c} \leq c^\varepsilon \leq \overline{c} < 1 \quad , \quad \varepsilon \downarrow 0 \ . \tag{3.13}$$

Then, for any $T > 0$,

$$e^{\int_0^T y^\varepsilon(t)dt} \leq O\left(\frac{1}{\varepsilon}\right) \tag{3.14}$$

and

$$\int_0^T e^{\int_t^T y^\varepsilon(\tau)d\tau} dt \leq O(|\ln \varepsilon|) \ . \tag{3.15}$$

The proof of this Lemma is postponed to the Appendix. Note that Lemma 3.2, together with (3.8) and inequality (3.9), show that the approximate solutions $u^\varepsilon(x,t)$ are ε-weakly Lip^+-stable with any constant $M < 1/K$ (see Definition 1.3).

4. Pseudo-viscosity approximations. One of the methods for the approximation of phenomena governed by hyperbolic conservation laws is considering parabolic regularizations with a gradient dependent viscosity. These so-called pseudo-viscosity approximations take the form

$$u_t^\varepsilon + f(u^\varepsilon)_x = \varepsilon Q(u^\varepsilon, p^\varepsilon)_x \quad , \quad p^\varepsilon := u_x^\varepsilon \quad , \quad \varepsilon \downarrow 0 \ , \tag{4.1}$$

$$u^\varepsilon(x,0) = u_0(x) \ , \tag{4.2}$$

where

$$\frac{\partial Q}{\partial p^\varepsilon} \geq 0 \ . \tag{4.3}$$

Note that this class of parabolic regularizations is wider than the class of viscous parabolic approximations, (3.1).

First, we note that these conservative approximations satisfy the maximum principle and, therefore, the solution remains uniformly bounded by $\|u_0\|_{L_\infty}$.

Next, the following theorem (whose proof is postponed to the Appendix) asserts that the solution operator of the pseudo-viscosity approximation is L_1-contractive. Therefore, thanks to translation invariance, the solution u^ε remains BV-bounded.

THEOREM 4.1. *(L_1-Contraction). Let u^ε and v^ε be two solutions of (4.1), (4.3). Then*

$$\|u^\varepsilon(\cdot,t) - v^\varepsilon(\cdot,t)\|_{L_1} \leq \|u^\varepsilon(\cdot,0) - v^\varepsilon(\cdot,0)\|_{L_1} \quad , \quad t > 0 \ . \tag{4.4}$$

Finally, we address the question of Lip^+-stability. We show that under suitable assumptions on the pseudo-viscosity coefficient, $Q(u,p)$, the solution of (4.1) is weakly Lip^+-stable.

THEOREM 4.2. *(Lip^+-Stability). Let Ω denote the domain in $\Re^2$,*

$$\Omega = [\inf\ u_0 \ , \ \sup\ u_0] \times [0, \infty) \ .$$

Assume that the following hold for all $(u,p) \in \Omega$ (M_1 and M_2 denote some constants):

$$|Q_p(u,p)| \ , \ |Q_{up}(u,p)| \leq M_1 \ ; \tag{A1}$$

$$Q_{uu}(u,p) \leq M_2 \cdot p \quad ; \tag{A2}$$

$$(A3) \qquad Q_{pp}(u,p) \leq 0 \ .$$

Then the solution of (4.1)–(4.3) is ε-weakly Lip^+-stable.

Proof. We first deal with the uniformly parabolic case, $Q_p \geq \delta > 0$. Let us denote

$$w^\varepsilon(x,t) = \frac{\partial}{\partial x}[a(u^\varepsilon(x,t))] \quad , \quad W^\varepsilon(t) = \max_x w^\varepsilon(x,t) = \|a(u^\varepsilon(\cdot,t))\|_{Lip^+} \ .$$

In view of Lemma 3.2, it suffices to show that there exists a constant $K > 0$, such that

$$\frac{d}{dt}W^\varepsilon(t) + (W^\varepsilon(t))^2 \leq \varepsilon K (W^\varepsilon(t))^3 \quad , \quad t > 0 \ . \tag{4.5}$$

Multiplying (4.1) by $a'(u^\varepsilon)$ and differentiating with respect to x, we find that $w = w^\varepsilon(x,t)$ satisfies

$$w_t + w^2 + a w_x = \varepsilon \cdot \left[Q_{uu}\frac{w^2}{a'} + 2Q_{up}\frac{w}{a'}(w_x + A'w^2) + Q_{pp}\frac{(w_x + A'w^2)^2}{a'} + \right.$$

$$\left. + \ Q_u w_x + Q_p \cdot \left(w_{xx} + 2A'ww_x + A''\frac{w^3}{a'} \right) \right] \quad ,$$

where $a = a(u^\varepsilon)$ and $A = A(u^\varepsilon) = 1/a'(u^\varepsilon)$.

Let $(x(t),t)$ be a positive local maximum of w. Then $w > 0$ in that point and, since $a' \geq \alpha > 0$, (1.1), also $p^\varepsilon = u^\varepsilon_x > 0$ there. Furthermore, $w_x = 0$ and $w_{xx} \leq 0$ in that point. Therefore, in view of (4.3) and assumptions (A1)–(A3), the above inequality implies that

$$w_t + w^2 \leq \varepsilon K w^3$$

in $(x(t),t)$, for some constant K which depends on M_1, M_2, α and the uniform bounds on A' and A''. Therefore, (4.5) holds and that concludes the proof for the non-degenerate case.

In the degenerate case, we replace $Q(u,p)$ by $Q_\delta(u,p) = Q(u,p) + \delta p$ so that the resulting pseudo-viscous approximation will be uniformly parabolic, $\partial Q_\delta/\partial p \geq \delta > 0$, and admit a smooth solution, u^ε_δ. Note that Q_δ, $\delta \downarrow 0$, still satisfies conditions (A1)–(A3) with constants, say, $M_1 + 1$ and M_2. Therefore, inequality (4.5), with K independent of δ, holds for u^ε_δ, $\delta \downarrow 0$, and consequently it holds for u^ε as well. ∎

Remark. Theorem 4.2 implies, in particular, the (ε-weak) Lip^+-stability of viscous parabolic regularizations, (3.1), stated earlier in Theorem 3.1. These regularizations are identified by viscosity coefficients of the form

$$Q(u,p) = q(u) \cdot p \quad , \quad q(u) \geq 0 \ . \tag{4.6}$$

Such coefficients satisfy assumptions (A1)–(A3), provided that $q(\cdot)$ is sufficiently smooth.

We therefore conclude, in light of Theorems 4.1 and 4.2, that Theorem 1.4 applies to approximation (4.1) under assumptions (4.3) and (A1)–(A3). Hence, if in addition, approximation (4.1) is $W^{-1,1}$-consistent with (1.1), i.e.,

$$\|u^\varepsilon_t + f(u^\varepsilon)_x\|_{W^{-1,1}(\Re_x)} \leq O(\varepsilon) \ ,$$

or simply,

$$\|Q\left(u^\varepsilon, u^\varepsilon_x\right)\|_{L_1(\Re_x)} \leq \text{Const} \ , \tag{4.7}$$

Corollary 1.7 may be applied and error estimates (E1)–(E3) hold. We propose below a condition on $Q(u,p)$ which guarantees $W^{-1,1}$-consistency, (4.7).

PROPOSITION 4.3. *If there exists a constant $C > 0$, such that*

$$|Q(u,p)| \leq C|p| \qquad \forall (u,p) \in [\inf\, u_0\ ,\ \sup\, u_0] \times \Re\ , \tag{4.8}$$

then equation (4.1) is $W^{-1,1}$-consistent with (1.1).

Proof. Condition (4.8) implies that

$$\|Q\left(u^\varepsilon, u_x^\varepsilon\right)\|_{L_1(\Re_x)} \leq C\|u_x^\varepsilon\|_{L_1} = C\|u^\varepsilon\|_{BV} \leq C\|u_0\|_{BV}\ .$$

Therefore, (4.7) holds and the proof is concluded. □

An example of a family of pseudo-viscosity coefficients which satisfy all the above requirements, i.e., (4.3), (A1)–(A3) and (4.8), is the following:

$$Q(u,p) = Q^{q(u),\beta}(u,p) = q(u)\left[(1+|p|)^\beta - 1\right]\operatorname{sgn}(p) \quad , \quad q(u) \geq 0 \quad , \quad 0 < \beta \leq 1\ . \tag{4.9}$$

Note that by letting β go to zero we obtain $Q \equiv 0$, which corresponds to the hyperbolic conservation law, while the other extreme case, $\beta = 1$, coincides with the standard viscous parabolic coefficient, (4.6).

A special class of pseudo-viscosity approximations, (4.1), where $Q = Q(p)$,

$$u_t^\varepsilon + f(u^\varepsilon)_x = \varepsilon Q(p^\varepsilon)_x \quad , \quad Q' \geq 0 \quad , \quad \varepsilon \downarrow 0\ , \tag{4.10}$$

was introduced by von Neumann and Richtmeyer in [9]. In [7] it is shown, by means of compensated compactness, that under further assumptions on the pseudo-viscosity coefficient, there exists a subsequence of weak solutions of (4.10), subject to the initial data (4.2), which converges in L^p_{loc} to the corresponding entropy solution of (1.1), provided that $u_0 \in W^{2,\infty}$.

One of the additional restrictions assumed on Q in [7] is that it acts only on shock-waves and does not smear out rarefactions. Namely,

$$Q'(p) = 0 \quad \forall p \geq 0 \quad \text{and} \quad Q'(p) > 0 \quad \forall p < 0\ . \tag{4.11}$$

Note that restriction (4.11) guarantees Lip^+-stability, since conditions (4.3) and (A1)–(A3) are clearly satisfied in this case.

An example of a family of such pseudo-viscosity coefficients which lead to $W^{-1,1}$-consistent approximations (in view of Proposition 4.3) is

$$Q^\beta(p) = \left[Q^{1,\beta}(u,p)\right]^- = 1 - (1-p^-)^\beta \quad , \quad 0 < \beta \leq 1\ , \tag{4.12}$$

$Q^{q(u),\beta}(u,p)$ being defined in (4.9). The choice which corresponds to $\beta = 1$, $Q^1(p) = p^-$, activates the regular parabolic regularization only on shock-waves and leaves rarefactions untouched.

5. The regularized Chapman-Enskog expansion. In this section we discuss the regularized Chapman-Enskog expansion for hydrodynamics, proposed by Rosenau [14]. This so-called R-C-E approximation is studied in [17], where it is shown that it shares many of the properties of the viscosity approximation, e.g. existence of traveling waves, monotonicity, L_1-contraction and Lip^+-stability.

Let us briefly recall the main results of [17]. The R-C-E approximation is presented in the form

$$u_t^\varepsilon + f(u^\varepsilon)_x = \varepsilon\left[Q_{m\varepsilon} * u_x^\varepsilon\right]_x \quad , \quad \varepsilon \downarrow 0\ , \tag{5.1}$$

$$u^\varepsilon(\cdot,0) = u_0(\cdot)\ , \tag{5.2}$$

with the choice of the unit-mass viscosity kernel

$$Q(x) = \frac{1}{2}e^{-|x|} \quad , \quad Q_{m\varepsilon}(x) = \frac{1}{m\varepsilon}Q\left(\frac{x}{m\varepsilon}\right)\ . \tag{5.3}$$

This is a pseudo-local dissipative approximation of the conservation law, where the viscosity coefficient is being activated by means of convolution rather than multiplication (compare (5.1) to (3.1)).

When $m \to 0$, $Q_{m\varepsilon}$ tends to the Dirac measure and the R-C-E approximation, (5.1), turns into the viscous parabolic approximation

$$u_t^\varepsilon + f(u^\varepsilon)_x = \varepsilon u_{xx}^\varepsilon \ .$$

Equation (5.1) may be rewritten in the equivalent form

$$u_t^\varepsilon + f(u^\varepsilon)_x = -\frac{1}{m^2\varepsilon}\left[u^\varepsilon - Q_{m\varepsilon} * u^\varepsilon\right] \quad . \tag{5.4}$$

The solution of (5.4) remains as smooth as its initial data ([17, Theorem 2.1]) and, therefore, if the initial data are discontinuous, weak solutions must be admitted. Since such solutions are not uniquely determined by the initial data, (5.4) is augmented with a Kružkov-like [4] entropy condition ([17, (4.1)]),

$$\frac{\partial}{\partial t}|u^\varepsilon - c| + \frac{\partial}{\partial x}\{\mathrm{sgn}(u^\varepsilon - c)[f(u^\varepsilon) - f(c)]\} \leq \tag{5.5}$$

$$-\frac{1}{m^2\varepsilon}\{|u^\varepsilon - c| - \mathrm{sgn}(u^\varepsilon - c)[Q_\varepsilon * (u^\varepsilon - c)]\} \ ,$$

for all $c \in \Re$. In particular, by substituting $c = +\sup|u^\varepsilon|$ or $c = -\sup|u^\varepsilon|$, we obtain from (5.5) that u^ε is, respectively, a supersolution or a subsolution of (5.4) and therefore a weak solution. Hence, u^ε is considered an entropy solution of (5.4) if it satisfies inequality (5.5) in the sense of distributions for all $c \in \Re$.

The above inequality, (5.5), implies L_1-contraction,

$$\|u^\varepsilon(\cdot,t) - v^\varepsilon(\cdot,t)\|_{L_1} \ \leq \ \|u^\varepsilon(\cdot,0) - v^\varepsilon(\cdot,0)\|_{L_1} \ ,$$

and hence BV-boundedness,

$$\|u^\varepsilon(\cdot,t)\|_{BV} \ \leq \ \|u_0\|_{BV} \ .$$

Since, by (5.1),

$$\|u_t^\varepsilon + f(u^\varepsilon)_x\|_{W^{-1,1}} \leq \varepsilon\|Q_{m\varepsilon} * u_x^\varepsilon\|_{L_1} \leq \varepsilon\|Q_{m\varepsilon}\|_{L_1}\|u_0\|_{BV} \leq O(\varepsilon) \quad ,$$

we also have $W^{-1,1}$-consistency.

Finally, we deal with the question of Lip^+-stability. Adding the smoothing viscosity term $\delta u_{xx}^{\varepsilon,\delta}$ to (5.4) and differentiating with respect to x, we get that $w \equiv u_x^{\varepsilon,\delta}$ satisfies

$$w_t + a'(u^{\varepsilon,\delta}) \cdot w^2 + a(u^{\varepsilon,\delta}) \cdot w_x = -\frac{1}{m^2\varepsilon}\left[w - Q_{m\varepsilon} * w\right] + \delta w_{xx} \ .$$

Letting $\delta \downarrow 0$, we get that $W(t) \equiv \max_x w(x,t)$ is governed by the Ricatti differential inequality

$$W'(t) + \alpha W^2(t) \leq 0 \ . \tag{5.6}$$

Restricting our attention to Burgers' equation, $a(u) = u$, the R-C-E approximation turns to be strongly Lip^+-stable, in virtue of (5.6).

Therefore, we conclude, in view of Theorem (1.1), that the R-C-E approximation converges to the entropy solution of Burgers' equation and error estimates (E1)–(E3) hold. This extends, for Burgers' equation, the convergence rate result of ([17, Corollary 5.2]) which was restricted to $u_0 \in C^1$.

6. The spectral viscosity method. The method of Spectral Viscosity (SV) is used for the approximate solution of (1.1) in the 2π-periodic case. The family of approximate solutions, $\{u_N(x,t)\}$, constructed by this method, consists of trigonometric polynomials, $u_N(x,t) = \sum_{k=-N}^{N} \hat{u}_k(t)e^{ikx}$, which approximate the spectral projection of the exact entropy solution, $P_N u$.

This method takes the following conservative form (consult [20]):

$$\frac{\partial}{\partial t}u_N(x,t) + \frac{\partial}{\partial x}P_N(f(u_N(x,t))) = \varepsilon_N \frac{\partial}{\partial x}Q_N(x,t) * \frac{\partial}{\partial x}u_N(x,t) \ , \tag{6.1}$$

$$u_N(\cdot,0) = P_N u_0(\cdot) \ . \tag{6.2}$$

The right hand side of (6.1) consists of a vanishing viscosity amplitude of size $\varepsilon_N \downarrow 0$ and a viscosity kernel, $Q_N(x,t) = \sum_{|k|=m_N}^{N} \hat{Q}_k(t)e^{ikx}$, activated only on high wave numbers, $|k| \geq m_N >> 1$. As in [20] we deal with real viscosity kernels with increasing Fourier coefficients, $\hat{Q}_k \equiv \hat{Q}_{|k|}$, which satisfy

$$1 - \left(\frac{m_N}{|k|}\right)^{2q} \leq \hat{Q}_k(t) \leq 1 \quad , \quad |k| \geq m_N \quad , \quad q = \text{Const} > 1.5 \ , \tag{6.3}$$

and the spectral viscosity parameters, ε_N and m_N, behave asymptotically as

$$\varepsilon_N \sim \frac{1}{N^\theta \log N} \quad , \quad m_N \sim N^{\frac{\theta}{2q}} \quad , \quad 0 < \theta < 1 \ . \tag{6.4}$$

The use of the projection P_N on the initial data is problematic since even if u_0 has a bounded variation, $\|P_N u_0\|_{BV}$ may grow as much as $O(\log N)$. This may be avoided by taking, for instance, the spectrally accurate de la Vallee Poussin projection,

$$u_N(x,0) = VP_N u_0 \equiv \sum_{k=-N}^{N} \sigma_k \hat{u}_{0k} e^{ikx} \quad , \quad \sigma_k = \begin{cases} 1 & |k| \leq \frac{N}{2} \\ 2 - \frac{2k}{N} & |k| > \frac{N}{2} \end{cases} \ , \tag{6.5}$$

which satisfies

$$\|u_N(\cdot,0)\|_{BV} = \|VP_N u_0\|_{BV} \leq 3\|u_0\|_{BV} \ .$$

This, according to the total-variation boundedness of the SV method (consult [20, Corollary 2.3]), implies that

$$\|u_N(\cdot,t)\|_{BV} \leq \text{Const}_\text{T} \quad , \quad t \in [0,T] \ . \tag{6.6}$$

Hence, we hereafter assume (6.5). At the end of this section we will deal with the case described in (6.2) of employing the regular spectral projection on the initial datat.

The SV method smoothens the initial data by smearing its discontinuities: Since $u_0(x) = \sum_{k=-\infty}^{\infty} \hat{u}_{0k}e^{ikx} \in BV$, it follows that $\hat{u}_{0k} = O\left(\frac{1}{k}\right)$. Hence

$$\left|\frac{\partial}{\partial x}[u_N(\cdot,0)]\right| = \left|\sum_{k=-N}^{N} ik\sigma_k \hat{u}_{0k} e^{ikx}\right| \leq \sum_{k=-N}^{N} |k| \cdot |\hat{u}_{0k}| \leq O(N) \ ,$$

and therefore

$$\|u_N(\cdot,0)\|_{Lip^+} \leq O(N) < \infty \ . \tag{6.7}$$

We now turn to deal with the Lip^+-stability of this approximation. To this end we rewrite (6.1), as in [20, (2.4)], in the following form,

$$\frac{\partial}{\partial t}u_N(x,t) + \frac{\partial}{\partial x}f(u_N(x,t)) = \tag{6.8a}$$

$$= \varepsilon_N \frac{\partial^2}{\partial x^2} u_N(x,t) - \varepsilon_N \frac{\partial}{\partial x} R_N(x,t) * \frac{\partial}{\partial x} u_N(x,t) + E_N \ ,$$

where $E_N = \frac{\partial}{\partial x}(I - P_N) f(u_N)$ is a spectrally small error term and

$$R_N(x,t) = \sum_{k=-N}^{N} \hat{R}_k(t) e^{ikx} \quad , \quad \hat{R}_k(t) = \begin{cases} 1 & |k| < m_N \\ 1 - \hat{Q}_k(t) & |k| \geq m_N \end{cases} \ . \tag{6.8b}$$

Multiplying (6.8a) by $a'(u_N)$ and differentiating with respect to x yields for $w = \frac{\partial}{\partial x} a(u_N)$:

$$w_t + a(u_N) w_x + w^2 = \varepsilon_N \left[w_{xx} + 2A'(u_N) w w_x + A''(u_N) \frac{w^3}{a'(u_N)} \right] - \tag{6.9}$$

$$-\varepsilon_N \left[a''(u_N) A(u_N) \left(\frac{\partial}{\partial x} R_N * \frac{\partial}{\partial x} u_N \right) w + a'(u_N) \left(\frac{\partial^2}{\partial x^2} R_N * \frac{\partial}{\partial x} u_N \right) \right] +$$

$$+ a''(u_N) A(u_N) w E_N + a'(u_N) \frac{\partial}{\partial x} E_N \ .$$

Here, as in §4, $A(\cdot) = 1/a'(\cdot)$. As before, we find that $W(t) = \max_x w(x,t)$ is governed by

$$\frac{d}{dt} W(t) + (W(t))^2 \leq \varepsilon_N K (W(t))^3 + \beta_N W(t) + \gamma_N \ , \tag{6.10}$$

where

$$K \equiv \max_{|u| \leq \|u_N\|_{L_\infty}} \left(\frac{A''(u)}{a'(u)} \right)^+ \ , \tag{6.11}$$

$$\beta_N = M_1 \cdot \left[\varepsilon_N \| \frac{\partial}{\partial x} R_N * \frac{\partial}{\partial x} u_N \|_{L_\infty} + \|E_N\|_{L_\infty} \right] \ ; \ M_1 \equiv \max_{|u| \leq \|u_N\|_{L_\infty}} |a''(u) A(u)| \tag{6.12}$$

and

$$\gamma_N = M_2 \cdot \left[\varepsilon_N \| \frac{\partial^2}{\partial x^2} R_N * \frac{\partial}{\partial x} u_N \|_{L_\infty} + \| \frac{\partial}{\partial x} E_N \|_{L_\infty} \right] \ ; \ M_2 \equiv \max_{|u| \leq \|u_N\|_{L_\infty}} a'(u) \ . \tag{6.13}$$

We now use estimates, obtained in [20], in order to estimate β_N and γ_N. First, we recall that [20, Lemma 3.1] supplies us with a uniform bound for the spatial derivatives of R_N:

$$\| \frac{\partial^s}{\partial x^s} R_N(\cdot,t) \|_{L_\infty} \leq \text{Const} \cdot m_N^{s+1} \log N \quad , \quad 0 \leq s \leq 2q-1 \ . \tag{6.14}$$

Using (6.14) with $s = 1, 2$ and the BV-boundedness (6.6), we conclude that

$$\| \frac{\partial}{\partial x} R_N * \frac{\partial}{\partial x} u_N \|_{L_\infty} \leq \| \frac{\partial}{\partial x} R_N \|_{L_\infty} \cdot \|u_N\|_{BV} \leq \text{Const} \cdot m_N^2 \log N \tag{6.15}$$

and

$$\| \frac{\partial^2}{\partial x^2} R_N * \frac{\partial}{\partial x} u_N \|_{L_\infty} \leq \| \frac{\partial^2}{\partial x^2} R_N \|_{L_\infty} \cdot \|u_N\|_{BV} \leq \text{Const} \cdot m_N^3 \log N \ . \tag{6.16}$$

Since $\|E_N\|_{L_\infty}$ and $\|\frac{\partial}{\partial x} E_N\|_{L_\infty}$ are spectrally small, hence negligible, we conclude by (6.12), (6.13), (6.15), (6.16), (6.4) and (6.3) that

$$\beta_N \sim N^{\theta(\frac{1}{q}-1)} \downarrow 0 \quad , \quad \gamma_N \sim N^{\theta(\frac{3}{2q}-1)} \downarrow 0 \ . \tag{6.17}$$

We may now state and prove the following weak Lip^+-stability result:

THEOREM 6.1. *Consider the SV method (6.1), (6.3)–(6.5), approximating the conservation law (1.1)–(1.2). Assume that $a = f'$ satisfies $\left(\frac{1}{a'}\right)'' \leq 0$. Then the approximate solutions are ε-weakly Lip^+-stable, with $\varepsilon = \frac{1}{N}$.*

Proof. Our assumption on $a(\cdot)$ implies that K, given in (6.11), equals zero. Hence, (6.10) reads in this case:

$$\frac{d}{dt}W(t) \leq -(W(t))^2 + \beta_N W(t) + \gamma_N \ . \tag{6.18}$$

Solving (6.18) we get that

$$W(t) \leq w_+ + \frac{w_+ - w_-}{\eta e^{(w_+ - w_-)t} - 1} \ , \tag{6.19}$$

where

$$w_\pm = \frac{\beta_N \pm \sqrt{\beta_N^2 + 4\gamma_N}}{2} \quad , \quad \eta = \frac{W(t=0) - w_-}{W(t=0) - w_+} \quad . \tag{6.20}$$

Note that $w_\pm$ and η depend on N. Furthermore, by (6.20), (6.17) and (6.3) it follows that

$$w_\pm = O\left(\sqrt{\gamma}_N\right) \sim N^{\theta(\frac{3}{4q} - \frac{1}{2})} \underset{N\to\infty}{\longrightarrow} 0 \quad . \tag{6.21}$$

Also, since by (6.7)

$$W(t=0) \sim N \ , \tag{6.22}$$

we conclude by (6.20) and (6.21) that

$$\eta \underset{N\to\infty}{\longrightarrow} 1 \quad . \tag{6.23}$$

We claim that the weak Lip^+-stability conditions, (1.11)–(1.12), hold here with $\varepsilon = \frac{1}{N}$. Namely,

$$e^{\int_0^T W(t)dt} \leq O(N) \tag{6.24}$$

and

$$\int_0^T e^{\int_t^T W(\tau)d\tau} dt \leq O(\log N) \quad . \tag{6.25}$$

In order to prove these two estimates we integrate (6.19) and find that

$$\int_t^T W(\tau)d\tau \leq w_-(T-t) + \log\left[\frac{\eta e^{(w_+ - w_-)T} - 1}{\eta e^{(w_+ - w_-)t} - 1}\right] \quad . \tag{6.26}$$

Hence

$$\exp\left[\int_0^T W(\tau)d\tau\right] \leq e^{w_- T} \cdot \frac{\eta e^{(w_+ - w_-)T} - 1}{\eta - 1} = e^{w_+ T} + e^{w_- T} \cdot \frac{e^{(w_+ - w_-)T} - 1}{\eta - 1} \quad .$$

But since

$$\eta - 1 = \frac{w_+ - w_-}{W(t=0) - w_+} \quad , \tag{6.27}$$

we conclude that

$$\exp\left[\int_0^T W(\tau)d\tau\right] \le e^{w_+T} + e^{w_-T}\cdot\frac{e^{(w_+-w_-)T}-1}{w_+-w_-}\left(W(t=0)-w_+\right)$$

and (6.24) follows by using (6.21) and (6.22).

As for (6.25), inequality (6.26) implies (note that $w_- \le 0$):

$$\int_0^T \exp\left[\int_t^T W(\tau)d\tau\right]dt \le \tag{6.28}$$

$$\le -T\left(\eta e^{(w_+-w_-)T}-1\right) + \frac{\eta e^{(w_+-w_-)T}-1}{w_+-w_-}\log\left(\frac{\eta e^{(w_+-w_-)T}-1}{\eta-1}\right) .$$

First, we observe that (6.21) and (6.23) imply that

$$\eta e^{(w_+-w_-)T}-1 \underset{N\to\infty}{\longrightarrow} 0 . \tag{6.29}$$

Now, in order to estimate the second term on the right hand side of (6.28) we deal with each of its two multiplicands. Using (6.27), (6.21) and (6.22) we find that

$$\frac{\eta e^{(w_+-w_-)T}-1}{w_+-w_-} = \frac{e^{(w_+-w_-)T}}{W(t=0)-w_+} + \frac{e^{(w_+-w_-)T}-1}{w_+-w_-} \underset{N\to\infty}{\longrightarrow} 0+T=T . \tag{6.30}$$

Furthermore, by (6.27), (6.21) and (6.22),

$$\frac{\eta e^{(w_+-w_-)T}-1}{\eta-1} = e^{(w_+-w_-)T} + \frac{e^{(w_+-w_-)T}-1}{w_+-w_-}\cdot\left(W(t=0)-w_+\right) \sim N . \tag{6.31}$$

Hence, (6.28)–(6.31) prove (6.25) and the proof is thus concluded. □

Corollary 6.2. *Consider the SV method (6.1), (6.3)–(6.5), approximating the conservation law (1.1)–(1.2). Then*

$$\|u_N(\cdot,T)-u(\cdot,T)\|_{W^{-1,1}} \le \begin{cases} C_T\varepsilon & \text{if } \|u_0\|_{Lip^+}<\infty \\ C_T\varepsilon|\ln\varepsilon| & \text{if } \|u_0\|_{Lip^+}=\infty \text{ and } \left(\frac{1}{a'}\right)''\le 0 \end{cases} , \tag{6.32}$$

with $\varepsilon = N^{-\theta}$.

Proof. The case of *Lip*$^+$-bounded initial data is straightforward and we, therefore, concentrate on the case that $\|u_0\|_{Lip'}=\infty$ and $(\frac{1}{a'})''\le 0$. Since, by Theorem 6.1, we have $\frac{1}{N}$-weak *Lip*$^+$-stability in that case, and since $\frac{1}{N}<\varepsilon=N^{-\theta}$, u_N are also ε-weakly *Lip*$^+$-stable. Hence, it remains to show ε-$W^{-1,1}$-consistency.
$W^{-1,1}$-consistency with (1.1),

$$\left\|\frac{\partial}{\partial t}u_N + \frac{\partial}{\partial x}f(u_N)\right\|_{L_\infty([0,T],W^{-1,1}(\Re_x))} \le K_T N^{-\theta} ,$$

has already been shown in [20, (3.9b)]. As for $W^{-1,1}$-consistency with the initial condition, we claim that

$$\|u_N(\cdot,0)-u(\cdot,0)\|_{W^{-1,1}} = \|VP_NU_0-U_0\|_{L_1} \le K_0N^{-2}\log N , \tag{6.33a}$$

where

$$U_0(x) = \int_{-\pi}^{x} u_0(\xi)d\xi . \tag{6.33b}$$

In order to prove (6.33), we recall that (consult [13, (2.12), (2.14), (2.15)])

$$\|P_N U_0 - U_0\|_{L_1} \leq \text{Const} \cdot \log N \cdot N^{-m} \|U_0^{(m)}\|_{L_1} \quad , \quad m \geq 0 \ . \tag{6.34}$$

Taking $m = 2$ in (6.34) we find that the initial error allowed by $W^{-1,1}$-consistency, is exhausted in this case:

$$\|P_N U_0 - U_0\|_{L_1} \leq \text{Const} \cdot N^{-2} \log N \|u_0\|_{BV} \ . \tag{6.35}$$

We leave the reader to verify that

$$\|V P_N U_0 - P_N U_0\|_{L_1} \leq \text{Const} \cdot N^{-2} \log N \ . \tag{6.36}$$

Hence, (6.33) follows from (6.35), (6.36) and the proof is completed. □

We have thus far restricted our attention to the case $\left(\frac{1}{a'}\right)'' \leq 0$. In the general case, the cubic term on the right hand side of (6.10) does not vanish. Still, one can prove (along the lines of the proof of Lemma 3.2) weak Lip^+-stability of order $\varepsilon_N = N^{-\theta} \log N$, provided that

$$W(t=0) \leq \frac{\bar{c}}{\varepsilon_N K}$$

for some $\bar{c} < 1$. Alas, this condition does not hold in our case (consult (6.4) and (6.22)). We, therefore, suggest to overcome this problem by considering a speed-like SV method,

$$\frac{\partial}{\partial t} u_N(x,t) + \frac{\partial}{\partial x} P_N f(u_N(x,t)) = \varepsilon_N \frac{\partial}{\partial x} Q_N(x,t) * \frac{\partial}{\partial x} a(u_N(x,t)) \tag{6.37}$$

with (6.3)–(6.5) as before. This method, still conservative, differs from the regular SV method, (6.1), only in the spectral viscosity term on the right hand side, where u_N was replaced by $a(u_N)$.

Remark (on an a-priori L_∞ bound).

The question of uniform L_∞-boundedness of this modified SV method may be tackled along the lines of [15]. However, we suggest here a simple argument which enables us to circumvent that question:

Since the initial data are always assumed bounded, (1.2), the exact entropy solution of (1.1)–(1.2) will not be affected if we change the flux f outside the interval $I_0 \equiv [\min u_0, \max u_0]$. Therefore, we choose to smoothly extend f from I_0 to $\Re$, so that f, $a = f'$,a',a'', etc. remain uniformly bouded on $\Re$. By doing so we may conclude that $f^{(i)}(u_N)$, and by convexity, $A^{(i)}(u_N)$ as well, $i \geq 0$, are all uniformly bounded even if u_N is not. Since our estimates depend only on $\|f^{(i)}(u_N)\|_{L_\infty}$ and $\|A^{(i)}(u_N)\|_{L_\infty}$ and never on the L_∞-bound of u_N itself, this argument is sufficient for our needs and no a-priori L_∞-bound is required.

We would like to comment that L_∞-boundedness proofs for approximate solutions of (1.1)–(1.2) may be sometimes tedious (as in our present case). Hence, it is sometimes customary to assume an a-priori L_∞-bound, based, for instance, on numerical evidence. The above, to the best of our knowledge, innovative extension argument, may be applied to such approximations as well, so that assumptions, not fully justified, may be avoided.

The convergence rate estimates for this modified SV method are given in the following theorem.

THEOREM 6.3. *Consider the modified SV method (6.37), (6.3)–(6.5), approximating the conservation law (1.1)–(1.2). Then u_N converges to the exact entropy solution $u(x,t)$, as $N \to \infty$, and for every $T > 0$ there exists a constant C_T such that*

$$\|u_N(\cdot,T) - u(\cdot,T)\|_{W^{-1,1}} \leq C_T \cdot \tilde{\varepsilon} \quad , \tag{6.38a}$$

where

$$\tilde{\varepsilon} = \begin{cases} \varepsilon & \text{if } \ \|u_0\|_{Lip^+} < \infty \\ \varepsilon |\ln \varepsilon| & \text{if } \ \|u_0\|_{Lip^+} = \infty \end{cases} \qquad \text{and} \quad \varepsilon = N^{-\theta} \ . \tag{6.38b}$$

Proof. We first note that (6.37) is still L_1-stable (consult the proof of [20, Lemma 2.2]) and hence (6.6) still holds. Therefore, (6.37) describes a family of conservative, L_1-stable and BV-bounded approximate solutions of (1.1)–(1.2).

Next, we address the question of weak Lip^+-stability. We rewrite (6.37) as

$$\frac{\partial}{\partial t}u_N + \frac{\partial}{\partial x}f(u_N) = \varepsilon_N \frac{\partial^2}{\partial x^2}a(u_N) - \varepsilon_N \frac{\partial}{\partial x}R_N * \frac{\partial}{\partial x}a(u_N) + E_N \ ,$$

where R_N and E_N are as in (6.8). Multiplying by $a'(u_N)$ and differentiating with respect to x, we find that $w = a(u_N)_x$ satisfies (compare to (6.9)):

$$w_t + a(u_N)w_x + w^2 = \varepsilon_N \left[a'(u_N)w_{xx} + a''(u_N)A(u_N)ww_x\right] -$$

$$-\varepsilon_N \left[a''(u_N)A(u_N)\left(\frac{\partial}{\partial x}R_N * \frac{\partial}{\partial x}a(u_N)\right)w + a'(u_N)\left(\frac{\partial^2}{\partial x^2}R_N * \frac{\partial}{\partial x}a(u_N)\right)\right] +$$

$$+a''(u_N)A(u_N)wE_N + a'(u_N)\frac{\partial}{\partial x}E_N \ .$$

We conclude that $W(t) = \max_x w(x,t)$ satisfies

$$\frac{d}{dt}W(t) + (W(t))^2 \leq \beta_N W(t) + \gamma_N \quad ,$$

where β_N and γ_N are not the same as in (6.12), (6.13) but still satisfy (6.17) (since $\|a(u_N)\|_{BV}$ remains uniformly bounded). This, according to the proof of Theorem 6.1, implies the $\frac{1}{N}$-weak Lip^+-stability of (6.37).

Hence, by Theorem 1.2, error estimates (1.7) hold with $\varepsilon = \frac{1}{N}$. Since it is easy to verify that our modified SV method is also $W^{-1,1}$-consistent of order $N^{-\theta}$, error estimates (6.38) follow. ☐

Before concluding this section we consider the case of P_N projecting the initial data, (6.2). We recall that the resulting approximation, u_N, may not be bounded in BV and in fact $\|u_N\|_{BV}$ may grow as much as $O(\log N)$. We note that this slightly changes our convergence rate results, stated in Corollary 6.2 and Theorem 6.3, so that (6.32) and (6.38) hold with $\varepsilon = N^{-\theta}\log N$, rather than $\varepsilon = N^{-\theta}$.

The first effect of replacing VP_N by P_N is that estimate (6.17) changes to

$$\beta_N \sim N^{\theta(\frac{1}{q}-1)}\log N \downarrow 0 \quad , \quad \gamma_N \sim N^{\theta(\frac{3}{2q}-1)}\log N \downarrow 0 \ ,$$

(consult (6.12), (6.13), (6.15) and (6.16)). This, however, does not change the final result of ε-weak-Lip^+-stability with $\varepsilon = \frac{1}{N}$. Hence, by Theorem 1.2, error estimates (1.7) still hold with $\varepsilon = \frac{1}{N}$. In view of (6.35) it remains only to consider the $W^{-1,1}$-consistency of $u_N(x,t)$ with (1.1). Ignoring the spectrally small discretization error $E_N = \frac{\partial}{\partial x}(I - P_N)f(u_N)$, we obtain from (6.1) (the proof for (6.37) is similar) that

$$\|\frac{\partial}{\partial t}u_N(\cdot,t) + \frac{\partial}{\partial x}f(u_N(\cdot,t))\|_{W^{-1,1}} \leq \varepsilon_N \|Q_N(\cdot,t) * \frac{\partial}{\partial x}u_N(\cdot,t)\|_{L_1} \ .$$

Using (6.4), Young inequality and the fact that $\|Q_N(\cdot,t)\|_{L_1}$ does not exceed $O(\log N)$ (consult [20, (3.9b)]), we get

$$\|\frac{\partial}{\partial t}u_N(\cdot,t) + \frac{\partial}{\partial x}f(u_N(\cdot,t))\|_{W^{-1,1}} \leq \varepsilon_N \|Q_N(\cdot,t)\|_{L_1}\|u_N(\cdot,t)\|_{BV} \leq \tag{6.39}$$

$$\leq \text{Const} \cdot \frac{1}{N^\theta \log N} (\log N)^2 = O(N^{-\theta} \log N) \ .$$

Hence, the order of $W^{-1,1}$-consistency reduced from $O(N^{-\theta})$ to $O(N^{-\theta} \log N)$. Therefore, (1.7), (6.35) and (6.39) imply an $O(N^{-\theta} \log N)$ convergence rate in $W^{-1,1}$.

Appendix A. Appendix.

Proof (of Lemma 3.2). By rescaling ε we may assume that $K = 1$. Since $y^\varepsilon(t)$ is the solution of a perturbated Ricatti's equation, (3.11), we denote by $y(t)$ the solution of the regular Ricatti's equation,

$$\text{(A.1a)} \qquad \frac{dy}{dt} + y^2 = 0 \ ,$$

subject to the same initial condition,

$$\text{(A.1b)} \qquad y(0) = y^\varepsilon(0) = \frac{c^\varepsilon}{\varepsilon} \quad , \quad 0 < \underline{c} \leq c^\varepsilon \leq \overline{c} < 1 \ .$$

The solution of (A.1) is

$$\text{(A.2)} \qquad y(t) = \left(t + \frac{1}{y^\varepsilon(0)} \right)^{-1} \quad ,$$

while the solution of (3.11)–(3.12) is given implicitly by

$$\text{(A.3)} \qquad y^\varepsilon(t) = \left(t + D^\varepsilon + \varepsilon \ln \left(\frac{y^\varepsilon}{\frac{1}{\varepsilon} - y^\varepsilon} \right) \right)^{-1} \quad ,$$

with

$$\text{(A.4)} \qquad D^\varepsilon = \frac{1}{y^\varepsilon(0)} - \varepsilon \ln \left(\frac{y^\varepsilon(0)}{\frac{1}{\varepsilon} - y^\varepsilon(0)} \right) \quad .$$

First, we note that (3.12) and (3.13) imply that $y^\varepsilon(t)$ is monotonically decreasing. Hence

$$\text{(A.5)} \qquad y^\varepsilon(t) \leq y^\varepsilon(0) \qquad \forall t \geq 0 \ .$$

Furthermore, since by (3.11) and (A.1a)

$$\text{(A.6)} \qquad y^\varepsilon(t) \geq y(t) \qquad \forall t \geq 0 \ ,$$

it follows, using (A.2) and monotonicity, (A.5), that

$$\text{(A.7)} \qquad y^\varepsilon(t) \geq y^\varepsilon(T) \geq y(T) = \left(T + \frac{1}{y^\varepsilon(0)} \right)^{-1} \qquad \forall t \in [0, T] \ .$$

With the upper and lower bounds on $y^\varepsilon(t)$, (A.5) and (A.7), we may estimate the terms in (A.3) and (A.4). We start with the last term in the brackets in (A.3). Using (A.5) and (A.1b) it may be upper-bounded as follows, for all $t \geq 0$:

$$\text{(A.8)} \qquad \varepsilon \ln \left(\frac{y^\varepsilon}{\frac{1}{\varepsilon} - y^\varepsilon} \right) = -\varepsilon \ln \left(\frac{1}{\varepsilon y^\varepsilon} - 1 \right) \leq -\varepsilon \ln \left(\frac{1}{\varepsilon y^\varepsilon(0)} - 1 \right) \leq -\varepsilon \ln \left(\frac{1}{\overline{c}} - 1 \right) = O(\varepsilon) \quad .$$

On the other hand, using (A.7) together with (A.1b), we get a lower bound for this term:

$$\text{(A.9)} \qquad \varepsilon \ln \left(\frac{y^\varepsilon}{\frac{1}{\varepsilon} - y^\varepsilon} \right) \geq -\varepsilon \ln \left(\frac{T + \frac{1}{y^\varepsilon(0)}}{\varepsilon} - 1 \right) = O\left(\varepsilon |\ln \varepsilon|\right) \quad , \quad 0 \leq t \leq T \ .$$

Next, we estimate the constant D^ε, given in (A.4). Using (A.1b), (A.8) and (A.9) we obtain the following bounds:

$$D^\varepsilon \leq \frac{\varepsilon}{\underline{c}} + \varepsilon \ln \left(\frac{T + \frac{1}{y^\varepsilon(0)}}{\varepsilon} - 1 \right) = O\left(\varepsilon |\ln \varepsilon|\right) \ ; \tag{A.10}$$

$$D^\varepsilon \geq \frac{\varepsilon}{\bar{c}} + \varepsilon \ln \left(\frac{1}{\bar{c}} - 1 \right) = O(\varepsilon) \ . \tag{A.11}$$

Hence we conclude by (A.3) and (A.8)–(A.11) that

$$y^\varepsilon(t) = \Big[t + O(\varepsilon) + O\left(\varepsilon |\ln \varepsilon|\right) \Big]^{-1} , \qquad 0 \leq t \leq T \ . \tag{A.12}$$

With (A.12), estimates (3.14) and (3.15) may easily be verified. Indeed,

$$e^{\int_0^T y^\varepsilon(t) dt} = \left| \frac{T + O(\varepsilon) + O\left(\varepsilon |\ln \varepsilon|\right)}{O(\varepsilon) + O\left(\varepsilon |\ln \varepsilon|\right)} \right| \leq O\left(\frac{1}{\varepsilon}\right)$$

and

$$\int_0^T e^{\int_t^T y^\varepsilon(\tau) d\tau} dt = \int_0^T \frac{T + O(\varepsilon) + O\left(\varepsilon |\ln \varepsilon|\right)}{t + O(\varepsilon) + O\left(\varepsilon |\ln \varepsilon|\right)} dt \leq O\left(|\ln \varepsilon|\right) \ ,$$

and the proof is thus completed. $\square$

Proof (of Theorem 4.1). Let $u^\varepsilon(x,t)$ and $v^\varepsilon(x,t)$ be two solutions of (4.1). We assume that the regularization (4.1) is uniformly parabolic, $Q_p \geq \delta > 0$, hence u^ε and v^ε are smooth. L_1-contraction for the degenerate case, $Q_p \geq 0$, easily follows by adding the term δp to the pseudo-viscosity coefficient $Q(u,p)$ and letting $\delta \downarrow 0$.

As in [5], we divide the real line into intervals, $\Re = \cup_n I_n(t), \ I_n(t) = [x_n(t), x_{n+1}(t))$, so that

$$(-1)^n [u^\varepsilon(\cdot,t) - v^\varepsilon(\cdot,t)] \Big|_{I_n(t)} \geq 0 \tag{A.13}$$

and consequently

$$u^\varepsilon(x_n(t),t) = v^\varepsilon(x_n(t),t) \ . \tag{A.14}$$

Using (A.13) and (A.14) we conclude that

$$\frac{d}{dt} \|u^\varepsilon(\cdot,t) - v^\varepsilon(\cdot,t)\|_{L_1(\Re)} = \tag{A.15}$$

$$= \frac{d}{dt} \sum_n (-1)^n \int_{x_n(t)}^{x_{n+1}(t)} [u^\varepsilon(x,t) - v^\varepsilon(x,t)] dx = \sum_n (-1)^n \int_{x_n(t)}^{x_{n+1}(t)} [u_t^\varepsilon(x,t) - v_t^\varepsilon(x,t)] dx \ .$$

Using (4.1) and carrying out the integral on the right hand side of (A.15) we find that

$$\frac{d}{dt} \|u^\varepsilon(\cdot,t) - v^\varepsilon(\cdot,t)\|_{L_1(\Re)} = \tag{A.16}$$

$$= \sum_n (-1)^n \Big[- f(u^\varepsilon) + f(v^\varepsilon) \Big]_{x_n(t)}^{x_{n+1}(t)} + \varepsilon \sum_n (-1)^n \Big[Q(u^\varepsilon, u_x^\varepsilon) - Q(v^\varepsilon, v_x^\varepsilon) \Big]_{x_n(t)}^{x_{n+1}(t)} \ .$$

The first term on the right hand side of (A.16) vanishes in view of (A.14). Equality (A.14) also implies that the second term may be written as

$$\varepsilon \sum_n \left[Q_p(u^\varepsilon, w^\varepsilon) \cdot \frac{\partial}{\partial x} \Big[(-1)^n (u^\varepsilon(x,t) - v^\varepsilon(x,t)) \Big] \right]_{x_n(t)}^{x_{n+1}(t)} , \tag{A.17}$$

where w^ε is a mid-value between u_x^ε and v_x^ε . Since (A.13) implies that

$$\frac{\partial}{\partial x} \Big[(-1)^n (u^\varepsilon(x,t) - v^\varepsilon(x,t)) \Big] \Big|_{x=x_{n+1}(t)} \leq 0$$

and

$$\frac{\partial}{\partial x} \Big[(-1)^n (u^\varepsilon(x,t) - v^\varepsilon(x,t)) \Big] \Big|_{x=x_n(t)} \geq 0$$

and since $Q_p > 0$, we conclude that (A.17) is non-positive. Therefore, (A.16) implies that

$$\frac{d}{dt} \| u^\varepsilon(\cdot, t) - v^\varepsilon(\cdot, t) \|_{L_1(\Re)} \leq 0 ,$$

namely, the solution operator of (4.1) is L_1-contractive. □

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