THE CONVERGENCE RATE OF GODUNOV TYPE SCHEMES *

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Abstract. Godunov type schemes form a special class of transport projection methods for the approximate solution of nonlinear hyperbolic conservation laws. We study the convergence rate of such schemes in the context of scalar conservation laws. We show how the question of consistency for Godunov type schemes can be answered solely in terms of the behavior of the associated projection operator. Namely, we prove that Lip'-consistent projections guarantee the Lip'-convergence of the corresponding Godunov scheme, provided that the latter is Lip^+-stable. This Lip'-error estimate is then translated into the standard $W^{s,p}$ global error estimates ($-1 \leq s \leq \frac{1}{p}$, $1 \leq p \leq \infty$) and finally to a local L^∞_{loc} convergence rate estimate. We apply these convergence rate estimates to a variety of scalar Godunov type schemes on a uniform grid as well as variable mesh size ones.

Key words. Conservation laws, Lip^+-stability, Lip'-consistency, error estimates, Godunov type schemes

1. Introduction. In this paper we study the convergence rate of Godunov type variable mesh approximations to the solution of the scalar convex conservation law

$$u_t + f(u)_x = 0 \quad , \quad t > 0 \quad , \quad f'' \geq \alpha > 0 \quad , \tag{1.1}$$

subject to the compactly supported, Lip^+-bounded initial condition

$$u(x, t = 0) = u_0(x) \quad , \qquad \|u_0(x)\|_{Lip^+} < \infty \quad . \tag{1.2}$$

Here, $\|\cdot\|_{Lip^+}$ denotes the usual Lip^+-seminorm:

$$\|w(x)\|_{Lip^+} \equiv \operatorname*{ess\,sup}_{x \neq y} \left(\frac{w(x) - w(y)}{x - y} \right)^+ \qquad , \qquad (\cdot)^+ \equiv \max(\cdot, 0) \quad . \tag{1.3}$$

Godunov type schemes form a special class of transport projection methods for the approximate solution of nonlinear hyperbolic conservation laws. This class of schemes takes the following form:

$$v^{\Delta x}(\cdot, t) = \begin{cases} E(t - t^{n-1}) v^{\Delta x}(\cdot, t^{n-1}) & t^{n-1} < t < t^n \\ P(\{I_j^n\}) v^{\Delta x}(\cdot, t^n - 0) & t = t^n = n\Delta t \end{cases} \qquad n \geq 1 \quad , \tag{1.4a}$$

where the initialization step is:

$$v^{\Delta x}(\cdot, t^0 = 0) = P(\{I_j^0\}) u_0(\cdot) \quad . \tag{1.4b}$$

These schemes are composed of the following four ingredients:

(i) The possibly variable size grid cells, $I_j^n \equiv [x^n_{j-\frac{1}{2}}, x^n_{j+\frac{1}{2}})$, where the grid is regular in the sense that:

$$\Delta x \equiv \Delta x_{\min} \leq |I_j^n| \leq \Delta x_{\max} \qquad ; \qquad \frac{\Delta x_{\max}}{\Delta x_{\min}} \leq \text{Const} \quad ; \tag{1.5}$$

(ii) A conservative piecewise polynomial grid projection, $P = P(\{I_j^n\})$,

$$\int_x P w(x) dx = \int_x w(x) dx \quad ; \tag{1.6}$$

(iii) The exact entropy solution operator associated with (1.1), $E = E(t)$;

(iv) The time step Δt, which is restricted by the CFL condition:

$$\lambda \max_{x,t} |f'(v^{\Delta x}(x,t))| \leq 1 \quad , \quad \lambda = \frac{\Delta t}{\Delta x} \quad . \tag{1.7}$$

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Let us recall that entropy solutions of (1.1) are Lip^+-bounded, e.g. [2, 13],

$$\|u(\cdot,t)\|_{Lip^+} \le C \quad , \quad t \ge 0 \quad .$$

We, therefore, concentrate on Lip^+-stable approximations, i.e. approximate solutions $v^{\Delta x}(x,t)$ for which

$$\|v^{\Delta x}(\cdot,t)\|_{Lip^+} \le C \quad , \quad t \ge 0 \quad . \tag{1.8}$$

We use the results of [8] (Theorem 2.1 and Corollary 2.2) which assert that Lip'-consistency and Lip^+-stability imply convergence whose rate may be quantified in terms of the Lip'-size of the truncation error. These results are summarized in the following:

THEOREM 1.1. *Let $\{v^{\Delta x}(x,t)\}_{\Delta x>0}$ be a family of conservative, Lip^+-stable approximate solutions of the conservation law (1.1), subject to the Lip^+-bounded initial condition (1.2). Assume that $v^{\Delta x}(x,t)$ is Lip'-consistent* [1] *with (1.1)-(1.2) in the sense that there exists $\varepsilon = \varepsilon(\Delta x)$ such that $\varepsilon(\Delta x) \downarrow 0$ for $\Delta x \downarrow 0$ and*

$$\|v^{\Delta x}(x,0) - u_0(x)\|_{Lip'} + \|v^{\Delta x}(x,t)_t + f(v^{\Delta x}(x,t))_x\|_{Lip'(x,[0,T])} \le O(\varepsilon) \ . \tag{1.9}$$

Then the following error estimates hold:

$$\|v^{\Delta x}(\cdot,t) - u(\cdot,t)\|_{W^{s,p}} \le O(\varepsilon^{\frac{1-sp}{2p}}) \quad -1 \le s \le \frac{1}{p} \ , 1 \le p \le \infty \quad . \tag{1.10}$$

Remarks.

1. When $(s,p) = (-1,1)$, the error estimate (1.10) turns into the Lip' error estimate:

$$\|v^{\Delta x}(\cdot,t) - u(\cdot,t)\|_{Lip'} \le O(\varepsilon) \quad . \tag{1.11}$$

2. (1.10) implies an $O(\varepsilon^{\frac{1}{3}})$ local error estimate and an $O(\varepsilon^{\frac{r}{r+2}})$ local error estimate for the post-processed grid values, away from shocks, where r is the degree of local smoothness of the exact solution (consult [8], (3.9b) and (2.26) there). In other words, (1.10) implies local kth order accuracy wherever the exact solution is infinitely smooth if $\varepsilon = O(\Delta x^k)$.

3. The parameter ε is a function of the smallest scale Δx. If $\varepsilon(\Delta x) = O(\Delta x^k)$ the corresponding scheme will be kth order accurate in Lip' in view of remark 1. Our analysis presented here is, however, limited to Lip'-first order accuracy, i.e. $\varepsilon = \Delta x$. A more delicate analysis will hopefully demonstrate (1.9) with $\varepsilon = O(\Delta x^k)$, $k > 1$, for higher-order schemes.

In view of the last remark we henceforth use the notation Δx instead of ε. Therefore, (1.11) now reads

$$\|v^{\Delta x}(\cdot,t) - u(\cdot,t)\|_{Lip'} \le O(\Delta x) \quad . \tag{1.12}$$

In §2 we deal with the Lip'-consistency and Lip^+-stability of Godunov type schemes, (1.4). We show that the question of Lip'-consistency of such schemes is reduced to estimating the Lip'-size of $P - I$, P denoting the projection operator of the scheme. As for the Lip^+-stability, since discontinuous piecewise polynomial grid functions are generically Lip^+-unbounded, we show that instead of (1.8) it suffices to prove discrete Lip^+-stability:

$$\|v^{\Delta x}(\cdot,t^n)\|_{DLip^+} \equiv \max_x \left(\frac{v^{\Delta x}(x+\Delta x,t^n) - v^{\Delta x}(x,t^n)}{\Delta x} \right)^+ \le C \quad , \quad n \ge 0 \quad . \tag{1.13}$$

[1] We let $\|w\|_{Lip'}$ denote the Lip-dual seminorm w.r.t $L^2(x)$, $L^2(x,t)$ inner-products,$(\cdot,\cdot)$ and $(\cdot,\cdot)_{x,t}$: $\|w(x)\|_{Lip'} \equiv \sup_\phi \frac{|(w-\bar{w},\phi)|}{\|\phi(x)\|_{Lip}}$, $\|w(x,t)\|_{Lip'(x,[0,T])} \equiv \sup_\phi \frac{|(w-\bar{w},\phi)_{x,t}|}{\|\phi(x,t)\|_{Lip}}$, where $\bar{w} = \frac{1}{|\mathrm{supp}(w)|}\int_{\mathrm{supp}(w)} w$, $\phi \in C_0^\infty$, and $\|\phi(x)\|_{Lip} = \mathrm{ess\,sup}_{x\neq y} \left|\frac{\phi(x)-\phi(y)}{x-y}\right|$, $\|\phi(x,t)\|_{Lip} = \mathrm{ess\,sup}_{(x,t)\neq(y,\tau)} \left|\frac{\phi(x,t)-\phi(y,\tau)}{|x-y|+|t-\tau|}\right|$.

The seminorm $\|\cdot\|_{DLip^+}$, defined in (1.13), is the discrete analogous of the Lip^+-seminorm (1.3). The infinite divided difference in (1.3) is replaced here by differences divided by the (finite) smallest scale of the underlying grid, Δx. Finally, we prove (Theorem 2.3) that discrete Lip^+-stable Godunov type schemes, for which $\|(P-I)w\|_{Lip'} \leq O(\Delta x^2)\|w\|_{BV}$, satisfy error estimate (1.10).

In §3 we demonstrate these convergence rate estimates on a variety of scalar Godunov type schemes, including variable mesh schemes and formally second order ones.

2. Statement and proof of main results. The convergence Theorem 1.1 requires to verify the Lip'-consistency and Lip^+-stability of the scheme in question. We begin by reducing the question of Lip'-consistency to the level of a mere approximation problem, namely, measuring in Lip'-seminorm the distance between the exact solution and its grid projection. Thus, our first theorem below enables us to avoid the delicate bookkeeping of error accumulation due to the dynamic transport part of the scheme.

THEOREM 2.1. *(Lip'-consistency). The Godunov type approximation (1.4) satisfies the following truncation error estimate:*

$$\|v_t^{\Delta x} + f(v^{\Delta x})_x\|_{Lip'(x,[0,T])} \leq \frac{T}{\Delta t} \max_{0<t^n\leq T} \|(P-I)v^{\Delta x}(\cdot, t^n - 0)\|_{Lip'} \tag{2.1}$$

Remark. We emphasize that this theorem applies to both fixed and variable grid schemes.

Proof. Let N denote the number of time steps in $[0,T]$, i.e.

$$T = t^N = N\Delta t \quad . \tag{2.2}$$

Then for every $\phi \in C_0^1(\Re \times [0,T])$

$$(v_t^{\Delta x} + f(v^{\Delta x})_x, \phi)_{x,t} = \sum_{n=1}^{N} \left[\int_{t^{n-1}}^{t^n} \int_x v_t^{\Delta x} \phi dx dt + \int_{t^{n-1}}^{t^n} \int_x f(v^{\Delta x})_x \phi dx dt \right] \quad .$$

Integration by parts gives that

$$(v_t^{\Delta x} + f(v^{\Delta x})_x, \phi)_{x,t} = \sum_{n=1}^{N} \left[(v^{\Delta x}, \phi)\Big|_{t^{n-1}}^{t^n} - \int_{t^{n-1}}^{t^n} \big((v^{\Delta x}, \phi_t) + (f(v^{\Delta x}), \phi_x) \big)\, dt \right] \quad . \tag{2.3}$$

But since $v^{\Delta x}$ is a weak solution in the strip $\Re \times (t^{n-1}, t^n)$, as definition (1.4a) implies, then

$$\int_{t^{n-1}}^{t^n} \big((v^{\Delta x}, \phi_t) + (f(v^{\Delta x}), \phi_x) \big)\, dt = (v^{\Delta x}, \phi)\Big|_{t^{n-1}+0}^{t^n - 0} \quad . \tag{2.4}$$

Therefore, by (2.3) and (2.4),

$$(v_t^{\Delta x} + f(v^{\Delta x})_x, \phi)_{x,t} = \sum_{n=1}^{N} \left[(v^{\Delta x}, \phi)\Big|_{t^{n-1}}^{t^n} - (v^{\Delta x}, \phi)\Big|_{t^{n-1}+0}^{t^n - 0} \right] \quad ,$$

and since, by (1.4a), $v^{\Delta x}(\cdot, t^{n-1}+0) = v^{\Delta x}(\cdot, t^{n-1})$, we have that

$$(v_t^{\Delta x} + f(v^{\Delta x})_x, \phi)_{x,t} = \sum_{n=1}^{N} (v^{\Delta x}, \phi)\Big|_{t^n - 0}^{t^n} = \sum_{n=1}^{N} ((P-I)v^{\Delta x}(\cdot, t^n - 0), \phi(\cdot, t^n)) \quad .$$

By the conservation of P, (1.6), $\overline{(P-I)v^{\Delta x}} = 0$. Therefore, using the definition of the Lip'-seminorm, together with (2.2), we get

$$|(v_t^{\Delta x} + f(v^{\Delta x})_x, \phi)_{x,t}| \leq \frac{T}{\Delta t} \max_{1\leq n\leq N} \|(P-I)v^{\Delta x}(\cdot, t^n - 0)\|_{Lip'} \|\phi(\cdot, t^n)\|_{Lip} \quad .$$

Dividing by $\|\phi(x,t)\|_{Lip}$ and taking the supremum over ϕ, we arrive at (2.1). ☐

Next, we turn to the question of Lip^+-stability. As noted in the Introduction, the Lip^+-seminorm $\|\cdot\|_{Lip^+}$, (1.3), does not suit discontinuous piecewise polynomial functions and hence we replace it by its discrete analogous – $\|\cdot\|_{DLip^+}$, defined in (1.13). To this end, we employ a compactly supported non-negative unit mass mollifier,

$$\psi_\delta(x) = \frac{1}{\delta}\psi(\frac{x}{\delta}) \quad , \quad \int_x \psi_\delta(x)dx = \int_x \psi(x)dx = 1 \quad . \tag{2.5}$$

In the following theorem we show that Lip'-consistency of order $O(\Delta x)$ remains invariant under a mollification with ψ_δ where $\delta = O(\Delta x)$.

THEOREM 2.2. *Assume $v^{\Delta x}(x,t)$ has a bounded variation and is Lip'-consistent with (1.1) of order $O(\Delta x)$,*

$$\|F^{\Delta x}(x,t)\|_{Lip'} \le O(\Delta x) \qquad , \qquad F^{\Delta x}(x,t) \equiv v_t^{\Delta x} + f(v^{\Delta x})_x \quad . \tag{2.6}$$

Then $v^{\Delta x,\delta} \equiv \psi_\delta * v^{\Delta x}$ is Lip'-consistent with (1.1) of order $O(\Delta x) + O(\delta)$.

Proof. We begin by stating the following three straightforward facts:

$$\|\psi_\delta * F\|_{Lip'} \le \|F\|_{Lip'} \quad ; \tag{2.7}$$

$$\|\psi_\delta * w - w\|_{L_1} \le O(\delta)\cdot\|w\|_{BV} \quad ; \tag{2.8}$$

$$\|w\|_{Lip'} \le \|\int^x (w-\bar{w})\|_{L_1} \qquad , \qquad \bar{w} = \frac{1}{|\mathrm{supp}(w)|}\int_{\mathrm{supp}(w)} w \quad . \tag{2.9}$$

Next, we upper bound the truncation error as follows:

$$\|v_t^{\Delta x,\delta} + f(v^{\Delta x,\delta})_x\|_{Lip'} = \|\psi_\delta * [v_t^{\Delta x} + f(v^{\Delta x})_x] - \psi_\delta * f(v^{\Delta x})_x + f(v^{\Delta x,\delta})_x\|_{Lip'} \le$$

$$= \|\psi_\delta * F^{\Delta x}\|_{Lip'} + \|\psi_\delta * f(v^{\Delta x})_x - f(v^{\Delta x,\delta})_x\|_{Lip'} \quad .$$

The first term on the right hand side is of order $O(\Delta x)$ by (2.6) and (2.7). In order to conclude our proof we shall now show that the second term is of order $O(\delta)$. Let us denote $w = \psi_\delta * f(v^{\Delta x})_x - f(v^{\Delta x,\delta})_x = [\psi_\delta * f(v^{\Delta x}) - f(v^{\Delta x,\delta})]_x$. As w is a complete derivative of a function which is constant, $f(0)$, outside the support of $v^{\Delta x}$, w is compactly supported and $\bar{w} = 0$. Therefore, by (2.9) and (2.8)

$$\|\psi_\delta * f(v^{\Delta x})_x - f(v^{\Delta x,\delta})_x\|_{Lip'} \le \|\psi_\delta * f(v^{\Delta x}) - f(\psi_\delta * v^{\Delta x})\|_{L_1} \le$$

$$\le \|\psi_\delta * f(v^{\Delta x}) - f(v^{\Delta x})\|_{L_1} + \|f(v^{\Delta x}) - f(\psi_\delta * v^{\Delta x})\|_{L_1} \le$$

$$\le \|\psi_\delta * f(v^{\Delta x}) - f(v^{\Delta x})\|_{L_1} + \|a\|_{L_\infty}\|v^{\Delta x} - \psi_\delta * v^{\Delta x}\|_{L_1} = O(\delta) \quad .$$

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Finally, we combine Theorems 2.1 and 2.2 to achieve our main convergence rate estimate for Godunov type schemes.

THEOREM 2.3. *(Convergence rate estimates). Assume that the Godunov type approximation (1.4) is discrete Lip^+-stable, (1.13), and Lip'-consistent in the sense that*

$$\|(P-I)w\|_{Lip'} \le O(\Delta x^2)\|w\|_{BV} \quad . \tag{2.10}$$

Then the following error estimates hold:

$$\|v^{\Delta x}(\cdot,t)-u(\cdot,t)\|_{W^{s,p}} \le O(\Delta x^{\frac{1-sp}{2p}}) \quad , \quad -1\le s\le \frac{1}{p} \;\; , \;\; 1\le p\le\infty \;\; . \tag{2.11}$$

Proof. Let us denote $\tilde{v}^{\Delta x}(\cdot,t)\equiv\psi_{\Delta x}*v^{\Delta x}(\cdot,t)$, where $\psi_{\Delta x}$ is the dilated mollifier of

$$\psi(x)=\begin{cases} 1 & |x|\le\frac{1}{2} \\ 0 & |x|>\frac{1}{2}\end{cases} \quad . \tag{2.12}$$

This choice of mollifier satisfies the following Lip'-error estimate (the proof of which is postponed to the Appendix):

$$\|\psi_{\Delta x}*w-w\|_{Lip'}\le O(\Delta x^2)\|w\|_{BV} \quad . \tag{2.13}$$

We show that $\tilde{v}^{\Delta x}$ satisfies Lip^+-stability (1.8) and Lip'-consistency (1.9) in order to use Theorem 1.1.

We start with the Lip^+-stability question. The definitions of the regular and discrete Lip^+-seminorms, (1.3) and (1.13), imply that $\|\tilde{v}^{\Delta x}(\cdot,t^n)\|_{Lip^+}=\|v^{\Delta x}(\cdot,t^n)\|_{DLip^+}$. As $v^{\Delta x}$ is assumed to be discrete Lip^+-stable we conclude that at each time level, t^n,

$$\|\tilde{v}^{\Delta x}(\cdot,t^n)\|_{Lip^+}=D_n\le C \quad . \tag{2.14}$$

This, together with the fact that the intermediate exact solution operator decreases the Lip^+-seminorm [2, 13], imply Lip^+-boundedness for all $t\ge0$:

$$\|\tilde{v}^{\Delta x}(\cdot,t)\|_{Lip^+}\le C \qquad \forall t\ge 0 \quad . \tag{2.15}$$

Namely, the mollified approximation $\tilde{v}^{\Delta x}$ is Lip^+-stable.

We note in passing that $v^{\Delta x}(\cdot,t)$, being compactly supported and Lip^+-bounded, has bounded variation (e.g. [2, Lemma 1]). Turning to the question of Lip'-consistency we, therefore, conclude from assumption (2.10) together with the truncation error estimate (2.1) that $v^{\Delta x}$ is Lip'-consistent with (1.1) of order $O(\Delta x)$; in view of Theorem 2.2 so is $\tilde{v}^{\Delta x}$, i.e.,

$$\|\tilde{v}^{\Delta x}_t+f(\tilde{v}^{\Delta x})_x\|\le O(\Delta x) \quad .$$

Furthermore, $\tilde{v}^{\Delta x}$ is also Lip'-consistent with the initial condition (1.2), since by (2.13), (1.4b) and (2.10),

$$\|\tilde{v}^{\Delta x}(\cdot,0)-u(\cdot,0)\|_{Lip'}\le\|\tilde{v}^{\Delta x}(\cdot,0)-v^{\Delta x}(\cdot,0)\|_{Lip'}+\|v^{\Delta x}(\cdot,0)-u_0(\cdot)\|_{Lip'}\le O(\Delta x^2) \quad .$$

Therefore, Theorem 1.1 holds; in particular (1.12) tells us that

$$\|\tilde{v}^{\Delta x}(\cdot,T)-u(\cdot,T)\|_{Lip'}\le O(\Delta x) \quad . \tag{2.16}$$

In addition, we have by (2.13),

$$\|\tilde{v}^{\Delta x}(\cdot,T)-v^{\Delta x}(\cdot,T)\|_{Lip'}\le O(\Delta x^2) \quad . \tag{2.17}$$

Combining (2.16) and (2.17) we end up with

$$\|v^{\Delta x}(\cdot,T)-u(\cdot,T)\|_{Lip'}\le O(\Delta x) \quad . \tag{2.18}$$

The Lip'-error estimate (2.18) may now be interpolated into the $W^{s,p}$-error estimates (2.11) along the lines of [8, Corollary 2.2]. ∎

3. Examples. In this section we demonstrate our results for a variety of Godunov type schemes. The Godunov scheme is a Godunov type scheme par excellence and is identified by the choice of projection $P = A$, where $A = A(I_j^n)$ is the cell averaging operator,

$$Aw(x) \equiv \frac{1}{|I_j^n|} \int_{I_j^n} w(\xi) d\xi \qquad \forall x \in I_j^n \quad . \tag{3.1}$$

We denote the cell averaged values of the approximation and their differences by:

$$v_j^n = Av^{\Delta x}(\cdot, t^n - 0)\Big|_{I_j^n} \qquad ; \qquad \Delta v_{j+\frac{1}{2}}^n = v_{j+1}^n - v_j^n \quad .$$

Using these notations we may introduce a different discrete Lip^+-seminorm (compare to definition (1.13)),

$$\|v^{\Delta x}(\cdot, t^n)\|_{lip^+} \equiv \max_j \left(\frac{\Delta v_{j+\frac{1}{2}}^n}{\Delta x} \right)^+ \quad , \tag{3.2}$$

which we refer to as the lip^+-seminorm of the cell averages. The need for this additional discrete Lip^+-seminorm will be clarified in the course of the discussion.

3.1. E-Schemes – on a fixed mesh. We begin by dealing with piecewise constant Godunov type approximations where the grid cells are fixed:

$$I_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}) \qquad ; \qquad x_{j\pm\frac{1}{2}} = (j \pm \frac{1}{2})\Delta x \quad .$$

The simplest choice of a projection in this case is $P = A$. There are two schemes which take precisely this form: The Godunov and the staggered Lax-Friedrichs (LxF) schemes (in the latter, the mesh moves in each time step, by $\frac{\Delta x}{2}$, to the right or to the left, alternately). The following straightforward consequence of Lemma A.1 (which is given in the Appendix) proves the Lip'-consistency of these schemes.

PROPOSITION 3.1. *The averaging operator, A, satisfies*

$$\|(A - I)w\|_{Lip'} \leq O(\Delta x^2)\|w\|_{BV} \quad . \tag{3.3}$$

Remark. Note that this proposition applies to variable mesh averaging operators as well as for fixed mesh ones, provided that the mesh is regular, (1.5).

Since the discrete Lip^+-seminorm, $\|\cdot\|_{DLip^+}$, and the cell averages lip^+-seminorm, $\|\cdot\|_{lip^+}$, coincide in the case of piecewise constant grid functions, the discrete Lip^+-stability condition (1.13) reads in this case:

$$\|v^{\Delta x}(\cdot, t^n)\|_{lip^+} \leq C \qquad , \qquad n \geq 0 \quad . \tag{3.4}$$

A proof of the (discrete) Lip^+-stability of Godunov and LxF schemes can be found in [3, 11]. Hence, our convergence rate estimates are easily obtained for these schemes by Theorem 2.3.

Godunov and LxF schemes are members of the family of essentially three point schemes. This family consists of schemes which admit the following viscosity form [12]:

$$v_j^{n+1} = v_j^n - \frac{\lambda}{2}[f(v_{j+1}^n) - f(v_{j-1}^n)] + \frac{1}{2}[Q_{j+\frac{1}{2}}^n \Delta v_{j+\frac{1}{2}}^n - Q_{j-\frac{1}{2}}^n \Delta v_{j-\frac{1}{2}}^n] \ . \tag{3.5}$$

The Godunov and LxF schemes are identified by the viscosity coefficients:

$$Q_{j+\frac{1}{2}}^{G,n} = \lambda \max_v \left[\frac{f(v_{j+1}^n) + f(v_j^n) - 2f(v)}{\Delta v_{j+\frac{1}{2}}^n} \right] \qquad , \qquad Q_{j+\frac{1}{2}}^{LxF,n} = 1 \quad .$$

To extend our discussion to this family of schemes, we present them in terms of a projection operator, $P = MA$. With this choice of projection we modify the cell averages by an appropriate operator M

tailored to the specific essentially three point scheme in question. In the following proposition we prove Lip'-consistency for these schemes:

PROPOSITION 3.2. *The modifying operators, M, which correspond to fixed mesh essentially three point BV schemes (3.5), satisfy*

$$\|(M-I)Av^{\Delta x}\|_{Lip'} \leq O(\Delta x^2) \ , \tag{3.6}$$

provided that the viscosity coefficients are uniformly bounded,

$$0 \leq Q^n_{j+\frac{1}{2}} \leq C \quad . \tag{3.7}$$

Proof. M is the operator which generates the grid values of the scheme, given in (3.5), from the cell averages of Godunov scheme,

$$v_j^{n+1} = MAv^{\Delta x}(\cdot, t^{n+1}-0)\Big|_{I_j} \quad .$$

On the other hand, since Godunov scheme uses the exact solver, its averaged value on I_j^{n+1} is given by

$$v_j^{G,n+1} = Av^{\Delta x}(\cdot, t^{n+1}-0)\Big|_{I_j} \quad .$$

Hence, in view of (3.5), the difference which we need to estimate in Lip' is a piecewise constant grid function,

$$w(x) \equiv (M-I)Av^{\Delta x}(x, t^{n+1}) = \sum_j w_j^{n+1}\chi_{I_j}(x) \quad , \tag{3.8a}$$

where w_j^{n+1} depends upon the difference between the viscosity coefficients,

$$w_j^{n+1} = \frac{1}{2}[(Q^n_{j+\frac{1}{2}} - Q^{G,n}_{j+\frac{1}{2}})\Delta v^n_{j+\frac{1}{2}} - (Q^n_{j-\frac{1}{2}} - Q^{G,n}_{j-\frac{1}{2}})\Delta v^n_{j-\frac{1}{2}}] \quad . \tag{3.8b}$$

Since $\bar{w} = 0$ (conservation), (2.9) shows that $\|w\|_{Lip'}$ in (3.6) is upper-bounded by the L_1-norm of the primitive function, $W(x) = \int_{-\infty}^x w(\xi)d\xi$. This primitive function is piecewise linear and is given by

$$W(x) = \sum_{i=-\infty}^{j-1} w_i^{n+1}\Delta x + (x - x_{j-\frac{1}{2}})w_j^{n+1} = \tag{3.9}$$

$$= \frac{\Delta x}{2}(Q^n_{j-\frac{1}{2}} - Q^{G,n}_{j-\frac{1}{2}})\Delta v^n_{j-\frac{1}{2}} + (x - x_{j-\frac{1}{2}})w_j^{n+1} \qquad \forall x \in I_j \quad .$$

Since by (3.7)

$$|Q^n_{j+\frac{1}{2}} - Q^{G,n}_{j+\frac{1}{2}}| \leq C \quad , \tag{3.10}$$

it follows that w_j^{n+1}, given in (3.8b), may be bounded as follows:

$$|w_j^{n+1}| \leq \frac{C}{2}\left(|\Delta v^n_{j+\frac{1}{2}}| + |\Delta v^n_{j-\frac{1}{2}}|\right) \quad . \tag{3.11}$$

Therefore, (3.9)–(3.11) imply that

$$|W(x)| \leq \frac{C}{2}|\Delta v^n_{j-\frac{1}{2}}|\Delta x + \frac{C}{2}(x - x_{j-\frac{1}{2}})\left(|\Delta v^n_{j+\frac{1}{2}}| + |\Delta v^n_{j-\frac{1}{2}}|\right) \qquad \forall x \in I_j \quad . \tag{3.12}$$

Equipped with (3.12) we conclude, by carrying out the integration, that

$$\|w(x)\|_{Lip'} \leq \|W(x)\|_{L_1} = \sum_j \int_{I_j} |W(\xi)|d\xi \leq C\Delta x^2 \sum_j |\Delta v^n_{j+\frac{1}{2}}| \leq$$

$$\leq C\Delta x^2 \|v^{\Delta x}(\cdot, t^n)\|_{BV} = O(\Delta x^2) \quad ,$$

which proves (3.6). ▯

Propositions 3.1 and 3.2 imply that essentially three point schemes with bounded viscosity coefficients, (3.7), are Lip'-consistent of (at least) order $O(\Delta x)$. Hence, all our error estimates follow for such Lip^+-stable (hence BV) schemes. Two more examples of Lip^+-stable members of this family are Roe and Engquist-Osher schemes (e.g. [1, 8]).

Remark. The Godunov and LxF schemes are the two extreme members of the well known family of E-schemes. This family consists of all essentially three point schemes, (3.5), for which $Q^{G,n}_{j+\frac{1}{2}} \leq Q^n_{j+\frac{1}{2}} \leq Q^{LxF,n}_{j+\frac{1}{2}}$. These schemes are known to be of first order resolution (consult [9]).

3.2. The Godunov Scheme – on a variable mesh. As a prototype example of using a variable grid we concentrate on Godunov's scheme. We briefly recall the variable mesh algorithm advocated in [5]. The fixed-mesh Godunov scheme is modified to a variable-mesh one, by adjusting the grid to follow the dynamics of the solution: when two neighboring grid values are connected through a shock wave, the mesh algorithm places one of the next step mesh points on the shock's path to enable its perfect resolution. The above choice of mesh points $\{x^n_{j+\frac{1}{2}}\}$ is done so that the mesh regularity condition (1.5) will not be violated.

Clearly, this variable-mesh Godunov scheme is Lip'-consistent (consult Theorem 2.1 and Proposition 3.1). The question of discrete Lip^+-stability, however, is more delicate and, therefore, we introduce a further slight modification. The above described mesh algorithm, chooses the variable mesh points $x^n_{j+\frac{1}{2}}$ so that $x^n_{j+\frac{1}{2}} \in [x_j, x_{j+1})$, where $\{x_j\}$ is an underlying fixed uniform mesh. Our modification applies when two neighboring grid values are connected through a rarefaction wave; in this case we suggest to choose the next step mesh point as the center of the fixed underlying mesh. By doing so, the evolution procedure coincides with the regular fixed mesh Godunov scheme whenever the solution is increasing. Hence, this modified algorithm describes a Lip^+-stable scheme without affecting the shock resolution of the original variable mesh scheme. Therefore, this modified scheme converges to the exact solution of (1.1) and satisfies all our error estimates.

3.3. MUSCL schemes. We now turn to MUSCL schemes which employ a piecewise linear reconstruction of the cell averages in order to increase the resolution. These schemes are Godunov type schemes with a projection of the form $P \equiv RA$, [6, 4]. The reconstruction $R = R(\{I_j\})$ acts on piecewise constant grid functions by rotating the constant value in each cell I_j around its center, $x_j = j\Delta x$:

$$RAv^{\Delta x}(x, t^n - 0) = R\left[\sum_j v^n_j \chi_{I_j}(x)\right] \equiv v^n_j + (x - x_j)s^n_j \qquad \forall x \in I_j \quad . \tag{3.13}$$

The reconstruction is identified by the choice of a limiter function $s(\cdot, \cdot)$ which defines the slopes,

$$s^n_j = s\left(\frac{\Delta v^n_{j-\frac{1}{2}}}{\Delta x}, \frac{\Delta v^n_{j+\frac{1}{2}}}{\Delta x}\right) \quad , \tag{3.14}$$

and usually constrained to satisfy

$$\min(a, b) \leq s(a, b) = s(b, a) \leq \max(a, b) \quad . \tag{3.15}$$

This choice of projection is conservative, i.e. $AP = A$.

Lip'-consistency of these schemes follows directly from Lemma A.1 and Proposition 3.1, as stated in the following proposition:

PROPOSITION 3.3. *The projection $P = RA$ satisfies*

$$\|(P - I)w\|_{Lip'} \leq O(\Delta x^2)\|w\|_{BV} \quad . \tag{3.16}$$

The verification of the discrete Lip^+-stability condition, (1.13), is rather delicate for this family of schemes. In the following proposition we show the equivalence of the discrete Lip^+-seminorm – $\|\cdot\|_{DLip^+}$ – and the lip^+-seminorm of the cell averages – $\|\cdot\|_{lip^+}$ – for a subclass of limiters.

PROPOSITION 3.4. *If the limiter $s(\cdot,\cdot)$ satisfies*

$$\text{minmod}(a,b) \leq s(a,b) \leq \max(a,b) \quad , \tag{3.17}$$

then for every function $w(x)$

$$\|RAw\|_{lip^+} \leq \|RAw\|_{DLip^+} \leq K \cdot \|RAw\|_{lip^+} \quad , \tag{3.18}$$

where $1 \leq K \leq 1.5$.

The proof of Proposition 3.4 is given in the Appendix.

Remarks.

1. The class of limiters defined in (3.17) forms a subclass of the one given in (3.15). The lower most limiter in the latter – min – is replaced here by the well known minmod limiter,

$$\text{minmod}(a,b) \equiv \frac{1}{2}[\text{sgn}(a) + \text{sgn}(b)] \cdot \min(|a|,|b|) \quad .$$

Minmod based reconstructions are often used in practice, since they yield non-oscillatory schemes, [4, 10].

2. Proposition 3.4 enables us, when dealing with Lip^+-stability of MUSCL schemes satisfying (3.17), to concentrate on the cell averaged values and check condition (3.4) rather than the intricate condition (1.13).

3. We note that condition (3.17) is indeed necessary – consult the counter example in the Appendix.

Example – The Maxmod Scheme.

The upper extreme case of (3.17) is the maxmod scheme. This scheme is shown to be Lip^+-stable in [2].

The reconstruction of this scheme, $R_{\max}$, has the unique feature that it avoids increasing discontinuities, hence it yields Lip^+-bounded piecewise linear functions, $\|R_{\max}Aw\|_{Lip^+} < \infty$. Furthermore, all three Lip^+-seminorms, the regular one – (1.3), the discrete one – (1.13) and the cell averaged values one – (3.2), are equal in this case, i.e.

$$\|R_{\max}Aw\|_{Lip^+} = \|R_{\max}Aw\|_{DLip^+} = \|R_{\max}Aw\|_{lip^+} \quad . \tag{3.19}$$

Brenier and Osher show [2] that the maxmod scheme is Lip^+ monotonically decreasing, namely

$$\|v^{\Delta x}(\cdot,t^{n+1})\|_{Lip^+} < \|v^{\Delta x}(\cdot,t^n)\|_{Lip^+} \qquad \forall n \geq 0 \quad .$$

Therefore, (1.8) (and in view of (3.19) also (1.13) and (3.4)) are met with $C = \|v^{\Delta x}(\cdot,t^0)\|_{Lip^+}$.

The maxmod scheme is, to the best of our knowledge, the only MUSCL scheme for which lip^+-stability has been established. Other reconstructions, such as the minmod, may increase the cell averages lip^+-seminorm. However, numerical experiments confirm our strong belief that MUSCL schemes based on such reconstructions are lip^+-bounded, though their lip^+-seminorm is not monotonically decreasing. Given this lip^+-stability together with our proof of Lip'-consistency, we obtain the convergence rate estimates (2.11).

3.4. MUSCL Schemes with approximate evolution. MUSCL schemes involve the exact evolution for a short time of a piecewise linear initial condition, namely, solving a generalized Riemann problem. This difficulty is intricate to carry out and, therefore, simpler alternative projections are sought. We present here two such projections being commonly used in practice.

One way of diffusing the problem of solving a generalized Riemann problem is by replacing the piecewise linear initial condition $v^{\Delta x}(\cdot,t^n) = RAv^{\Delta x}(\cdot,t^n-0)$ by $v^{\Delta x}(\cdot,t^n) = MRAv^{\Delta x}(\cdot,t^n-0)$, where the operator M decomposes the reconstructed piecewise linear profile at each time step into a piecewise constant one as follows:

$$MRAv^{\Delta x}(x,t^n-0) = \sum_j \left[v^n_{j,-}\chi_{I_{j,-}}(x) + v^n_{j,+}\chi_{I_{j,+}}(x) \right] \quad . \tag{3.20}$$

Here $v^n_{j,\pm}$ denote the values of the reconstruction in the two end points of I_j, $x_{j-\frac{1}{2}}$ and $x_{j+\frac{1}{2}}$,

$$v^n_{j,\pm} = v^n_j \pm \frac{\Delta x}{2} s^n_j$$

and $I_{j,\pm}$ denote the left and right halves of the interval I_j, i.e.,

$$I_{j,-} = [x_{j-\frac{1}{2}}, x_j) \quad , \quad I_{j,+} = [x_j, x_{j+\frac{1}{2}}) \quad .$$

By this modification, the solution of (1.1) consists of a successive sequence of non-interacting Riemann problems, provided that we half the CFL condition (1.7),

$$\lambda \max_{x,t} |f'(v^{\Delta x}(x,t))| \leq \frac{1}{2} \quad . \tag{3.21}$$

Let $W(x/t; u_L, u_R)$ denote the Riemann solver of (1.1). Then our modified schemes recast, after integration of the exact solution over a typical cell $I_j \times [t^n, t^{n+1}]$, into the final form

$$v^{n+1}_j = v^n_j - \lambda \left[f(W(0+; v^n_{j,+}, v^n_{j+1,-})) - f(W(0+; v^n_{j-1,+}, v^n_{j,-})) \right] \quad . \tag{3.22}$$

These modified schemes fit into our framework of Godunov type schemes with the projection $P = MRA$, where the piecewise constant decomposition operator, M, is given in (3.20). With this formulation in mind we observe that our modified schemes are *Lip*$'$-consistent. Indeed, the definition of M and Lemma A.1 imply that

$$\|(M - I)RAv^{\Delta x}\|_{Lip'} \leq O(\Delta x^2)\|RAv^{\Delta x}\|_{BV} \leq O(\Delta x^2)\|v^{\Delta x}\|_{BV} \quad ,$$

and, therefore, condition (2.10) is met by the modified projection $P = MRA$. Thus, the *Lip*$'$-consistency of the original MUSCL schemes is retained. Hence, these modified MUSCL schemes, if *Lip*$^+$-stable, satisfy our error estimates.

Another way to avoid the solution of the generalized Riemann problem is replacing the exact evolution operator E by an approximate one, $\tilde{E}$ (compare to (1.4a)),

$$v^{\Delta x}(\cdot, t^{n+1}) = RA\tilde{E}(t^{n+1} - t^n)v^{\Delta x}(\cdot, t^n) \quad . \tag{3.23}$$

This modification fits into our framework, (1.4), by rewriting the evolution procedure (3.23) as

$$v^{\Delta x}(\cdot, t^{n+1}) = PE(t^{n+1} - t^n)v^{\Delta x}(\cdot, t^n) \qquad , \qquad P = RMA \quad , \tag{3.24}$$

where M takes care of the differences between the averaged values of the exact and approximate evolutions.

In the following proposition we show that our convergence rate estimates are not affected by the use of an approximate evolution, provided that the local truncation error is of second order.

PROPOSITION 3.5. *If the modified MUSCL scheme (3.24) is conservative, discrete Lip$^+$-stable and the operator M, which identifies the approximate evolution $\tilde{E}$, satisfies*

$$|(MAE - AE)v^{\Delta x}| \leq O(\Delta x^2) \quad , \tag{3.25}$$

then the $W^{s,p}$ error estimates (2.11) hold.

Proof. In view of Theorem 2.3, we have only to show that for $w = Ev^{\Delta x}$,

$$\|(P - I)w\|_{Lip'} \leq O(\Delta x^2) \quad . \tag{3.26}$$

Applying the triangle inequality we may decompose this error term into three different error terms,

$$\|(P - I)w\|_{Lip'} = \|(RMA - I)w\|_{Lip'} \leq \tag{3.27}$$

$$\|(R - I)MAw\|_{Lip'} + \|(MA - A)w\|_{Lip'} + \|(A - I)w\|_{Lip'} = T_1 + T_2 + T_3 \quad .$$

Lemma A.1 implies that

$$T_1 = O(\Delta x^2) \quad . \tag{3.28}$$

As for T_2, we let $g = (MAE - AE)v^{\Delta x}$ and $G = \int^x g$. Since the scheme is conservative, (1.6), the averaged value of g over its compact support, which we denote by Ω, is zero. This implies that G is also compactly supported on Ω. Therefore, by (2.9) and (3.25):

$$T_2 = \|(MA - A)Ev^{\Delta x}\|_{Lip'} = \|g\|_{Lip'} \le \|G\|_{L_1} \le \tag{3.29}$$

$$|\Omega| \cdot \|G\|_{L_\infty} \le |\Omega| \cdot \|g\|_{L_1} \le |\Omega|^2 \cdot \|g\|_{L_\infty} \le O(\Delta x^2) \quad .$$

Finally, (3.26) follows from (3.27), (3.28), (3.29) and (3.3). ☐

Example – Non-Oscillatory Central Difference Scheme.

We consider a family of MUSCL-type non-oscillatory central differencing schemes, presented in [7]. We briefly recall the construction of these schemes and present them in our notations. The grid in use is a staggered one, namely, the cell size Δx is fixed, but the grid moves in each time step by $\frac{\Delta x}{2}$.

The exact solution of the generalized Riemann problems is averaged on the staggered grid (i.e., use $A = A(\{I_j\})$ or $A = A(\{I_{j+\frac{1}{2}}\})$ every other step). This central Lax-Friedrichs type solver may be written exactly, using (1.4), as (compare the following formulation to [7, (2.11)]):

$$v^{n+1}_{j+\frac{1}{2}} = \frac{1}{\Delta x}\left[\int_{x_j}^{x_{j+\frac{1}{2}}} v^{\Delta x}(x,t^n)dx + \int_{x_{j+\frac{1}{2}}}^{x_{j+1}} v^{\Delta x}(x,t^n)dx\right] - \tag{3.30}$$

$$-\frac{1}{\Delta x}\left[\int_{t^n}^{t^{n+1}} f(v^{\Delta x}(x_{j+1},\tau))d\tau - \int_{t^n}^{t^{n+1}} f(v^{\Delta x}(x_j,\tau))d\tau\right] \quad .$$

The time step Δt is restricted by the CFL condition (3.21) so that no interaction occurs between two neighboring Riemann problems.

The evaluation of the temporal integrals in (3.30) requires the exact solution of the generalized Riemann problems along the lines $x = x_j$. This is being avoided by using the mid-point rule,

$$\int_{t^n}^{t^{n+1}} f(v^{\Delta x}(x_j,\tau))d\tau \approx \Delta t \cdot f(v^{\Delta x}(x_j, t^n + \frac{\Delta t}{2})) \quad , \tag{3.31a}$$

where the mid-point value is linearly approximated,

$$v^{\Delta x}(x_j, t^n + \frac{\Delta t}{2}) \approx w_j^{n+\frac{1}{2}} \equiv v_j^n - \frac{\Delta t}{2} a(v_j^n) s_j^n \quad . \tag{3.31b}$$

Thus, with $v^{n+1}_{j+\frac{1}{2}}$ in (3.30) denoting the **exact** evolution averages, these approximations result in the modified averaged values, $Mv^{n+1}_{j+\frac{1}{2}}$, given by

$$Mv^{n+1}_{j+\frac{1}{2}} = \frac{1}{2}\left(v_j^n + v_{j+1}^n\right) + \frac{\Delta x}{8}\left(s_j^n - s_{j+1}^n\right) - \lambda\left(f(w_{j+1}^{n+\frac{1}{2}}) - f(w_j^{n+\frac{1}{2}})\right) \quad . \tag{3.32}$$

With this modification in mind we turn to show the Lip'-consistency of this family of schemes. To this end we show that the modifying operator M, given in (3.32), satisfies the consistency condition (3.25).

Since the Riemann problems do not interact, the solution $v^{\Delta x}(x_j,\tau)$ is smooth on the line $x_j \times [t^n, t^{n+1}]$. Hence, the mid-point rule local truncation error gives that

$$\Big|\int_{t^n}^{t^{n+1}} f(v^{\Delta x}(x_j,\tau))d\tau - \Delta t \cdot f(v^{\Delta x}(x_j, t^n + \frac{\Delta t}{2}))\Big| = O(\Delta t^3) \quad . \tag{3.33}$$

Furthermore, by Taylor expansion and (3.31b)

$$v^{\Delta x}(x_j,t^n+\frac{\Delta t}{2}) = v^{\Delta x}(x_j,t^n)+\frac{\Delta t}{2}v_t^{\Delta x}(x_j,t^n)+O(\Delta t^2) =$$

$$v^{\Delta x}(x_j,t^n)-\frac{\Delta t}{2}a(v^{\Delta x}(x_j,t^n))v_x^{\Delta x}(x_j,t^n)+O(\Delta t^2) = v_j^n-\frac{\Delta t}{2}a(v_j^n)s_j^n+O(\Delta t^2) = w_j^{n+\frac{1}{2}}+O(\Delta t^2)\ ,$$

which implies that

$$|v^{\Delta x}(x_j,t^n+\frac{\Delta t}{2})-w_j^{n+\frac{1}{2}}| = O(\Delta t^2)\quad . \tag{3.34}$$

Comparing (3.30) to (3.31) and (3.32) gives, using (3.33) and (3.34), that

$$|Mv_{j+\frac{1}{2}}^{n+1}-v_{j+\frac{1}{2}}^{n+1}| = |(MAE-AE)v^{\Delta x}(x,t^n)|\Big|_{I_{j+\frac{1}{2}}} \le O(\Delta t^2) = O(\Delta x^2)\quad .$$

Thus, according to Proposition 3.5, the above described family of schemes is Lip'-consistent. Augmented with Lip^+-stability we conclude that non-oscillatory central differencing schemes satisfy our global as well as local error estimates.

3.5. Epilogue. MUSCL schemes are viewed as second-order accurate since for C^2-smooth functions, w, $s_j = w_x(x_j)+O(\Delta x)$. However, local second order accuracy away from discontinuities has not been yet proven. We proved here a weaker result for Lip^+-stable MUSCL schemes, namely, a local first order accuracy (for the post-processed values, consult Remark 2 in the Introduction) whenever the exact solution is infinitely smooth. The error estimates given in Theorem 1.1, are the optimal ones. The problem is due to the Lip'-seminorm which proves to be appropriate for first order convergence rate only: It is easy to see that

$$\|(RA-I)w\|_{Lip'} = O(\Delta x^3)\|w\|_{BV} \tag{3.35}$$

whenever w is C^1 in the interior of the grid cells I_j. However, if w experiences a discontinuity inside a grid cell, (3.35) no longer holds and the weaker error estimate (3.16) is then sharp. Comparing the two Lip'-error estimates (3.3) and (3.16) shows that the reconstruction R does not improve the Lip'-accuracy in that case. Therefore, when shocks are present, formally second order schemes are only first order accurate in Lip'.

Motivated by this discussion we suggest to surpass this Lip'-first order accuracy barrier by moving the mesh so that no shock will occur in the interior of a grid cell. By doing so, the better error estimate (3.35) will hold, and the resulting scheme, if Lip^+-stable, will be second-order accurate in Lip' and local second order accuracy, for the post-processed grid values, will follow wherever the exact solution is infinitely smooth.

Appendix A. Appendix.

We start by proving a basic error estimate in Lip', which we used in §3.

LEMMA A.1. *Let u and v be two compactly supported Δx-grid functions. Assume there exist constants K and L, such that:*

(i) $\|u-v\|_{L_1} \le K\Delta x$;

(ii) *the distance between two successive zeroes of* $W(x) = \int_{-\infty}^{x}(u-v)$ *is* $L\Delta x$ *at the most.*

Then the following estimate holds:

$$\|u-v\|_{Lip'} \le LK\Delta x^2\quad . \tag{A.1}$$

Proof. Let z_j denote the zeroes of $W(x)$ and $L_j = [z_j, z_{j+1}]$. Then

$$\|W\|_{L_1} = \int_x\left|\int_{-\infty}^{x}u-v\right|dx = \sum_j\int_{L_j}\left|\int_{z_j}^{x}u-v\right|dx \le \sum_j\int_{L_j}\left(\int_{L_j}|u-v|\right)dx = \tag{A.2}$$

$$= \sum_j |L_j| \cdot \int_{L_j} |u-v| \le L\Delta x \cdot \sum_j \int_{L_j} |u-v| = L\Delta x \|u-v\|_{L_1} \le LK\Delta x^2 \quad .$$

Since u and v have a compact support, condition (ii) implies that $\overline{u-v} = 0$. Therefore (A.1) follows from (A.2) and (2.9). □

With this Lip' error estimate in our hands we may prove the mollification Lip' error estimate (2.13).

Proof (of (2.13)). The mollification error may be decomposed into three simpler error terms,

$$\text{(A.3)} \qquad \|\psi_{\Delta x} * w - w\|_{Lip'} \le$$

$$\|\psi_{\Delta x} * (w - Aw)\|_{Lip'} + \|\psi_{\Delta x} * (Aw) - Aw\|_{Lip'} + \|Aw - w\|_{Lip'} = T_1 + T_2 + T_3 \quad ,$$

where A, defined in (3.1), denotes here the fixed Δx-grid averaging operator. By Proposition 3.1,

$$\text{(A.4)} \qquad T_3 \le O(\Delta x^2)\|w\|_{BV} \quad .$$

Hence, in view of (2.7),

$$\text{(A.5)} \qquad T_1 \le T_3 \le O(\Delta x^2)\|w\|_{BV}.$$

As for T_2, since Aw is piecewise constant, $Aw(x) = \sum_j w_j \chi_{I_j}(x)$ (w_j being the averaged values of w in the cell I_j), $\psi_{\Delta x} * (Aw)$ is a continuous linear interpolant of Aw at $\{x_j\}$ - the centers of the fixed grid cells. It can be easily verified that the two functions, Aw and $\psi_{\Delta x} * (Aw)$, satisfy conditions $(i)-(ii)$ in Lemma A.1 with $K = \frac{1}{4}\|w\|_{BV}$ and $L = 1$. Therefore,

$$\text{(A.6)} \qquad T_2 \le O(\Delta x^2)\|w\|_{BV} \quad .$$

Error estimate (2.13) now follows from (A.3-A.6). □

We close the Appendix by proving the equivalence of the $\|\cdot\|_{DLip^+}$ and $\|\cdot\|_{lip^+}$ seminorms for the sub-class of reconstructions (3.17).

Proof (of Proposition 3.4). Recalling the definitions of the two seminorms, (1.13) and (3.2), the left inequality in (3.18) is trivial, since

$$\left.\frac{RAw(x+\Delta x) - RAw(x)}{\Delta x}\right|_{x=x_j} = \frac{\Delta w_{j+\frac{1}{2}}}{\Delta x} \quad .$$

As for the second inequality in (3.18) we observe that every x can be expressed as $x = x_j + \theta\Delta x$ for some x_j and $|\theta| \le \frac{1}{2}$ and, therefore, by (3.13) and (3.14),

$$\frac{RAw(x+\Delta x) - RAw(x)}{\Delta x} = \frac{\Delta w_{j+\frac{1}{2}}}{\Delta x} + \theta\left(s\left(\frac{\Delta w_{j+\frac{1}{2}}}{\Delta x}, \frac{\Delta w_{j+\frac{3}{2}}}{\Delta x}\right) - s\left(\frac{\Delta w_{j-\frac{1}{2}}}{\Delta x}, \frac{\Delta w_{j+\frac{1}{2}}}{\Delta x}\right)\right) \quad .$$

Hence, in order to prove (3.18) it suffices to show that

$$\frac{\Delta w_{j+\frac{1}{2}}}{\Delta x} + \theta\left(s\left(\frac{\Delta w_{j+\frac{1}{2}}}{\Delta x}, \frac{\Delta w_{j+\frac{3}{2}}}{\Delta x}\right) - s\left(\frac{\Delta w_{j-\frac{1}{2}}}{\Delta x}, \frac{\Delta w_{j+\frac{1}{2}}}{\Delta x}\right)\right) \le$$

$$\le K \cdot \max\left(\frac{\Delta w_{j-\frac{1}{2}}}{\Delta x}, \frac{\Delta w_{j+\frac{1}{2}}}{\Delta x}, \frac{\Delta w_{j+\frac{3}{2}}}{\Delta x}\right)^+ \quad ,$$

or in the more abstract form:

$$I \equiv b + \theta[s(b,c) - s(a,b)] \leq K \cdot \max(a,b,c)^+ \qquad , \qquad |\theta| \leq \frac{1}{2} \quad . \tag{A.7}$$

We note that due to the symmetry of $s(\cdot,\cdot)$ it suffices to deal with $\theta \geq 0$. Therefore, in order to upper-bound I, we have to upper-bound $s(b,c)$ and lower-bound $s(a,b)$.

First we show that if $b \leq 0$, (A.7) holds with $K = \frac{1}{2}$. Using the limitation assumption (3.17), we can summarize the upper-bounds for I as follows:

$$I \leq \begin{cases} b + \theta(c-0) \leq \theta c & a \geq 0, c \geq 0 \\ b + \theta(0-0) \leq 0 & a \geq 0, c \leq 0 \\ b + \theta(c-b) \leq \theta c & a \leq 0, c \geq 0 \\ b + \theta(0-b) \leq 0 & a \leq 0, c \leq 0 \end{cases} \tag{A.8}$$

Since $0 \leq \theta \leq \frac{1}{2}$, (A.7) follows from (A.8) with $K = \frac{1}{2}$.

Now we turn to the case $b \geq 0$. Using (3.17) we arrive at

$$I \leq \begin{cases} b + \theta(c-b) \leq (1-\theta)b + \theta c \leq c & a \geq b, c \geq b \\ b + \theta(b-b) = b & a \geq b, c \leq b \\ b + \theta(c-a^+) \leq b + \theta c \leq 1.5c & a \leq b, c \geq b \\ b + \theta(b-a^+) \leq b + \theta b \leq 1.5b & a \leq b, c \leq b \end{cases} \tag{A.9}$$

Hence, (A.7) holds with $K = 1.5$. □

Remarks.

1. If $s(\cdot,\cdot) = \max(\cdot,\cdot)$, (A.7) holds with $K = 1$, since in the last two cases of (A.9), which are the only cases where $K > 1$ may appear, $s(a,b) = b$, $s(b,c) = \max(b,c)$ and therefore

$$I = b + \theta(\max(b,c) - b) \leq \max(b,c) \leq \max(a,b,c)^+ \quad .$$

2. If $s(\cdot,\cdot) = \text{minmod}(\cdot,\cdot)$, the estimate $K \leq 1.5$ is sharp since if $b = c > 0$, $a \leq 0$ and $\theta = \frac{1}{2}$ we have

$$I = b + \theta(b-0) = 1.5b = 1.5\max(a,b,c)^+ \quad .$$

3. The equivalence (3.18) does not hold for $s(\cdot,\cdot) = \min(\cdot,\cdot)$: For example, if $b = c = 0$ and $a < 0$ then $I = -\theta a$ which violates (A.7).

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