ON THE PIECEWISE SMOOTHNESS OF ENTROPY SOLUTIONS TO SCALAR CONSERVATION LAWS

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ABSTRACT

The behavior and structure of entropy solutions of scalar convex conservation laws are studied. It is well known that such entropy solutions consist of at most countable number of C^1-smooth regions. We obtain new upper bounds on the higher order derivatives of the entropy solution in any one of its C^1-smoothness regions. These bounds enable us to measure the *high order* piecewise smoothness of the entropy solution. To this end we introduce an appropriate new C^n-semi norm – localized to the smooth part of the entropy solution, and we show that the entropy solution is stable with respect to this norm. We also address the question regarding the *number* of C^1-smoothness pieces; we show that if the initial speed has a finite number of decreasing inflection points then it bounds the number of future shock discontinuities.

Loosely speaking this says that in the case of such generic initial data the entropy solution consists of a finite number of smooth pieces, each of which is as smooth as the data permits. It is this type of *piecewise* smoothness which is assumed — sometime implicitly — in many finite-dimensional computations for such discontinuous problems.

1. INTRODUCTION

We study here the behavior and structure of entropy solutions of the single hyperbolic conservation law

$$(1.1a) \qquad u_t + f(u)_x = 0 \quad , \quad -\infty < x < \infty \quad , \quad t > 0 \quad ,$$

subject to the smooth initial condition

$$(1.1b) \qquad u(x,0) = u_0(x) \quad , \quad -\infty < x < \infty \quad ,$$

where the flux f is strictly convex

$$(1.2) \qquad f'' \geq \alpha > 0 \quad .$$

The structure of such solutions has been determined by Oleinik [8,9,10] and Lax [6]; more refined information was obtained by Dafermos [2]. The entropy solutions are continuous except on the union of an at most countable set of Lipschitz continuous shock curves. The complement of the shock set is open, [2], and from each point (x,t) in this open set one can trace a straight characteristic backward in time to $t = 0$, where the initial condition is given. Since the slope of this characteristic equals $a(u(x,t)) = f'(u(x,t))$, the entropy solution is given by the implicit relation

$$(1.3) \qquad u(x,t) = u_0(x - a(u(x,t))t) \quad .$$

The Implicit Function Theorem implies that if $a, u_0 \in C^N$, $N \geq 1$, then $u \in C^N$ in its region of continuity, since in that region

$$1 + a'(u)u_0'(x - a(u)t)t > 0 \qquad \forall t \geq 0,$$

consult [2, Theorem 5.1].

In this paper we *quantify* the regularity of the entropy solution using sharp upper bounds for its high order spatial derivatives in its region of C^1-smoothness, and we determine the size of the complement set of that region, namely — the set of shock discontinuities.

In §2 we examine the behavior of $|\partial_x^n u| \equiv |\partial^n u/\partial x^n|,\;\; 2 \le n \le N$. The behavior of the first derivative, u_x, in the region where it is non-negative, has been thoroughly studied and shown to be $O(t^{-1})$ e.g., [1,4,8,12]. We derive sharp estimates for the higher order derivatives and show (Theorem 2.1) that their behavior depends on the sign of u_x: There exist constants, $\mathrm{Const_n}$, which depend solely on initial condition, u_0, such that the following holds.

- Along characteristics where u_x is positive we have

  $$|\partial_x^n u| \le \mathrm{Const_n}(u_x)^n \qquad ,$$

  and therefore — since u_x decays like $O(t^{-1})$ along those curves, the higher order derivatives decay at a rate which increases with n;

- Along characteristics where u_x is negative we have

  $$|\partial_x^n u| \le \mathrm{Const_n}|u_x|^{2n-1} \qquad ,$$

  and therefore — since the solution breaks in a finite time, t_c, along these characteristics, $|\partial_x^n u|$ tends to infinity as $t \to t_c$ at a rate which increases with n;

- Finally, along characteristics where $u_x = 0$ we have $|\partial_x^n u| \le \mathrm{Const_n} t^{n-2}$, $n > 1$.

These estimates on the spatial high order derivatives can be converted into an appropriate stability estimate on the *piecewise* regularity of the entropy solution. This is carried out in §3 in terms of a suitable C^n semi-norm which is localized to the C^1-smoothness part of the entropy solution. Theorem 3.1 shows that the solution operator of the convex conservation law (1.1) is stable with respect to that semi-norm. In this context we refer to DeVore & Lucier, [3], for a different type of high order regularity result which manifests itself in terms of an high-order spatial Besov stability estimate.

Finally, for the sake of completeness we discuss in §4 the complement of the C^1-smoothness part of the entropy solution, that is, we determine the size of the set of shocks. Theorem 4.1 asserts that this set is equivalent to the set of negative minima of $a(u_0)'$. Thus Theorem 4.1 complements Schaeffer's regularity theorem [11], by realizing the first category set of infinitely smooth initial conditions , $\{u_0\}$, which evolve into entropy solutions with infinitely many shock discontinuities.

In summary we conclude that if $a(u_0)$ has a finite number of decreasing inflection points, then only a finite number of shocks will occur. Hence, if $a, u_0 \in C^N$ and $a(u_0)$ has a finite number of decreasing inflection points, then the corresponding entropy solution consists of finite number of pieces, each of which is C^N-smooth; moreover, the regularity of these pieces is bounded by the initial regularity. It is this type of *piecewise* regularity of the entropy solution which is assumed — sometime implicitly — in many finite-dimensional computations.

2. HIGH ORDER REGULARITY ESTIMATES

We consider solutions of the single convex conservation law (1.1) where

$$u_0(x) \in C^N(\Re) \bigcap W^{N,\infty}(\Re) \quad , \quad N \geq 2 \tag{2.1}$$

and

$$a := f' \in C^N[\inf u_0, \sup u_0] \quad . \tag{2.2}$$

The behavior of the solution's first spatial derivative has been thoroughly studied (see [1,4,8,12]): Whenever it is non-negative it decays like $O(t^{-1})$, while elsewhere it decreases unboundedly, and becomes infinite in a finite time on the shock curves. We examine here the behavior of the higher order spatial derivatives $\partial_x^n u = \partial^n u / \partial x^n$, $2 \leq n \leq N$, the existence of which is guaranteed by (2.1-2) everywhere apart from the singular set of shock curves.

Since the solution u is smooth in the open complement of the set of shocks, we may multiply equation (1.1a) by $a'(u)$ to find out that $v := a(u)$ satisfies Burgers' equation in that region,

$$v_t + v v_x = 0 \quad . \tag{2.3}$$

We now differentiate (2.3) $n \le N$ times with respect to x to obtain the equation which governs the evolution of $w^n := \partial_x^n v$ in the smooth region:

$$w_t^n + \partial_x^n(vv_x) = 0 \quad .$$

Leibnitz rule gives

$$w_t^n + \sum_{k=0}^{n} \binom{n}{k} (\partial_x^k v)(\partial_x^{n-k} v_x) = 0 \quad ,$$

or equivalently ,

$$w_t^n + v w_x^n = -n w^1 w^n - \sum_{k=2}^{n} \binom{n}{k} w^k w^{n-k+1} \qquad 1 \le n \le N \quad . \tag{2.4}$$

Observe that all the spatial derivatives of v are governed by a first order quasi-linear equation (2.4) with the same principal part as the governing equation for v itself in (2.3) , hence having the same characteristic geometry. However – unlike equation (2.3) which tells us that v remains constant along characteristics, the non-vanishing right hand side of (2.4) implies that w^n changes along the characteristics. Let the value of w^n along a characteristic $x(t)$ denoted by $w^n(t) = w^n(x(t), t)$, then (2.4) implies that

$$\frac{dw^n(t)}{dt} = -n w^1(t) w^n(t) - \sum_{k=2}^{n} \binom{n}{k} w^k(t) w^{n-k+1}(t) \qquad 1 \le n \le N \quad . \tag{2.5}$$

We start by examining the first derivative $w^1 = v_x = a(u)_x$. Since it proves to play a significant role in our analysis we denote it, for convenience, by w. Equation (2.5) reduces in that case, $n = 1$, to the well known Riccati equation

$$\frac{dw}{dt} = -(w)^2 \tag{2.6}$$

whose solution is:

$$w(t) = \frac{w(0)}{1 + w(0)t} \quad . \tag{2.7}$$

We see that if $w(0) > 0$, $w(t)$ remains positive and decays to zero like $O(t^{-1})$; if $w(0) = 0$ then $w(t) = 0$ for all $t > 0$ while if $w(0) < 0$, $w(t)$ remains negative and decreases until it becomes infinite.

We now use (2.5) and (2.7) in order to estimate $w^n(t)$, arriving at the following.

Proposition 2.1. *For every* $2 \le n \le N$ *and* $t \ge 0$ *there holds*

$$|w^n(t)| \le C_n(1 + w(0)t)^{-n-1} \qquad \text{if } w(0) > 0 \ ; \tag{2.8a}$$

$$|w^n(t)| \le D_n(1 + w(0)t)^{-2n+1} \qquad \text{if } w(0) < 0 \ ; \tag{2.8b}$$

$$w^n(t) = w^n(0) + P_{n-2}(t) \qquad \text{if } w(0) = 0 \ . \tag{2.8c}$$

Here the constants C_n *and* D_n *are given recursively by*

$$C_n = |w^n(0)| + \frac{1}{w(0)} \sum_{k=2}^{n-1} \binom{n}{k} C_k C_{n-k+1} \qquad 2 \le n \le N \tag{2.9a}$$

$$D_n = |w^n(0)| + \frac{1}{|w(0)|(n-2)} \sum_{k=2}^{n-1} \binom{n}{k} D_k D_{n-k+1} \qquad 2 \le n \le N \tag{2.9b}$$

and $P_{n-2}(t)$ *is a polynomial of degree* $n-2$ *which vanishes for* $t = 0$.

Remarks.

1. Throughout this section we shall use the notations C, C_n, D_n etc. to denote constants which do not depend on t, and P_n to denote polynomials of degree n. Note that these notations can stand for different constants or polynomials in different occurrences.

2. Equality (2.7) allows us to rewrite (2.8a-b) as

$$|w^n(t)| \le \begin{cases} \tilde{C}_n w(t)^{n+1} & w(0) > 0 \\ \\ \tilde{D}_n |w(t)|^{2n-1} & w(0) < 0 \end{cases} \qquad t \ge 0 \ , \tag{2.10}$$

where the constants $\tilde{C}_n$ and $\tilde{D}_n$,

$$\tilde{C}_n = \frac{C_n}{w(0)^{n+1}} \quad , \quad \tilde{D}_n = \frac{D_n}{|w(0)|^{2n-1}} \tag{2.11}$$

depend solely on the initial condition.

Proof. Equation (2.5) may be written for $n \geq 2$ as follows:

$$\frac{dw^n}{dt} = -(n+1)w(t)w^n(t) + q_n(t) \quad , \tag{2.12a}$$

$$q_n(t) := -\sum_{k=2}^{n-1} \binom{n}{k} w^k(t) w^{n-k+1}(t) \quad . \tag{2.12b}$$

Using (2.7), the solution of (2.12a) is

$$w^n(t) = (1+w(0)t)^{-n-1}\left[w^n(0) + \int_0^t (1+w(0)\tau)^{n+1} q_n(\tau)d\tau\right] \quad . \tag{2.13}$$

We prove (2.8) by induction. The case $n = 2$ is immediate since $q_2 = 0$ and therefore, by (2.13),

$$w^2(t) = (1+w(0)t)^{-3} w^2(0) \quad . \tag{2.14}$$

Hence (2.8) is proved for $n = 2$ with $C_2 = D_2 = |w^2(0)|$ (in agreement with (2.9)) and $P_0(t) \equiv 0$.

We turn now to the proof of (2.8) for $2 < n \leq N$, assuming it holds for all $2 \leq k < n$. The proof is separated for three cases according to the sign of $w(0)$.

If $w(0) > 0$ then by (2.12b) and induction we get that

$$|q_n(t)| \leq \sum_{k=2}^{n-1} \binom{n}{k} |w^k(t)||w^{n-k+1}(t)| \leq \tag{2.15}$$

$$\leq \sum_{k=2}^{n-1} \binom{n}{k} C_k (1+w(0)t)^{-k-1} C_{n-k+1}(1+w(0)t)^{-n+k-2} =$$

$$= \sum_{k=2}^{n-1} \binom{n}{k} C_k C_{n-k+1}(1+w(0)t)^{-n-3} \quad .$$

Therefore, by (2.13) and (2.15)

$$|w^n(t)| \leq \tag{2.16}$$

$$(1+w(0)t)^{-n-1}\left[|w^n(0)| + \sum_{k=2}^{n-1} \binom{n}{k} C_k C_{n-k+1} \int_0^t (1+w(0)\tau)^{-2} d\tau\right]$$

Evaluating the integral in (2.16) proves (2.8a) and (2.9a).

Similarly, if $w(0) < 0$ then

$$
(2.17) \qquad |q_n(t)| \le \sum_{k=2}^{n-1} \binom{n}{k} |w^k(t)||w^{n-k+1}(t)| \le
$$

$$
\le \sum_{k=2}^{n-1} \binom{n}{k} C_k (1+w(0)t)^{-2k+1} C_{n-k+1} (1+w(0)t)^{-2n+2k-1} =
$$

$$
= \sum_{k=2}^{n-1} \binom{n}{k} C_k C_{n-k+1} (1+w(0)t)^{-2n} \quad .
$$

Hence, by (2.13) and (2.17),

$$
(2.18) \qquad |w^n(t)| \le
$$

$$
(1+w(0)t)^{-n-1} \left[|w^n(0)| + \sum_{k=2}^{n-1} \binom{n}{k} C_k C_{n-k+1} \int_0^t (1+w(0)\tau)^{-n+1} d\tau \right]
$$

and (2.8b), (2.9b) follow by evaluating the integral in (2.18).

Finally, if $w(0) = 0$, (2.13) implies that

$$
(2.19a) \qquad w^n(t) = w^n(0) + \int_0^t q_n(\tau) d\tau \quad .
$$

But, by (2.12b) and the induction assumption

$$
(2.19b) \quad q_n(t) = -\sum_{k=2}^{n-1} \binom{n}{k} (w^k(0) + P_{k-2}(t))(w^{n-k+1}(0) + P_{n-k-1}(t)) = P_{n-3}(t)
$$

Therefore, $\int_0^t q_n(\tau)d\tau$ is a polynomial of degree $n-2$ which vanishes for $t = 0$, hence (2.8c) is proved, and that concludes the proof. □

Example. The estimates offered by Proposition 2.1 are sharp, as demonstrated by Burgers' equation, $u_t + uu_x = 0$, subject to initial condition

$$
u(x,0) = u_0(x) = \begin{cases} \frac{x^2-1}{2} & -1 < x < 1 \\ 0 & elsewhere \end{cases} \quad .
$$

Its solution along characteristics $x(t)$ for which $-1 < x(0) < 1$ is given by

$$
u(x(t),t) = \frac{1 + x(t)t - \sqrt{1 + 2x(t)t + t^2}}{t^2} \quad .
$$

Therefore, for $n \geq 2$ we get that

$$(2.20) \qquad w^n(t) = \frac{\partial^n u(x(t),t)}{\partial x^n} = (-1)^n C_n (1 + 2x(t)t + t^2)^{-n+\frac{1}{2}} t^{n-2} \quad ,$$

where

$$C_n = \prod_{k=1}^{n-1} (2k-1) \quad .$$

Let $x(t)$ be the characteristic which starts at $x_0 \in (-1,1)$. Its speed is $u_0(x_0) = \frac{1}{2}(x_0^2 - 1)$ and therefore

$$(2.21) \qquad x(t) = x_0 + \frac{1}{2}(x_0^2 - 1)t \quad .$$

For that characteristic $w(0) = u_0'(x_0) = x_0$ and therefore , by (2.21) :

$$(2.22) \qquad (1 + 2x(t)t + t^2)^{\frac{1}{2}} = 1 + x_0 t = 1 + w(0)t \quad .$$

Using (2.22) in (2.20) gives :

$$(2.23) \qquad w^n(t) = (-1)^n C_n (1 + w(0)t)^{-2n+1} t^{n-2} \quad .$$

If $x_0 > 0$ then $w(0) > 0$ and therefore, for $t \gg w(0)^{-1}$,

$$|w^n(t)| = C_n (1 + w(0)t)^{-2n+1} t^{n-2} \approx$$

$$C_n (1 + w(0)t)^{-2n+1} \left(\frac{1 + w(0)t}{w(0)} \right)^{n-2} == \frac{C_n}{w(0)^{n-2}} (1 + w(0)t)^{-n-1} \; .$$

If $x_0 < 0$ then $w(0) < 0$ and the characteristic will not exist beyond the critical time $t_c = 1/|w(0)|$. Therefore, by (2.23), when $t \to t_c$

$$|w^n(t)| = C_n (1 + w(0)t)^{-2n+1} t^{n-2} \approx \frac{C_n}{|w(0)|^{n-2}} (1 + w(0)t)^{-2n+1} \quad .$$

If $x_0 = 0$ then $w(0) = 0$ and therefore $w^n(t) = (-1)^n C_n t^{n-2}$. Since $w^2(0) = 1$ and $w^n(0) = 0$ for $n > 2$, (2.8c) is met with $P_0(t) \equiv 0$ and $P_{n-2}(t) = (-1)^n C_n t^{n-2}$ for $n > 2$. □

After establishing estimates for $w^n = \partial_x^n a(u)$ we are ready to translate them into analogous estimates for $\partial_x^n u$. For that matter we observe that w^n has the form (successive chain rule)

$$(2.24a) \qquad w^n = \partial_x^n a(u) = a'(u)\partial_x^n u + \sum_i K_i a^{(m_i)}(u) \prod_{j=1}^{m_i} \partial_x^{r_j^i} u \quad ,$$

where K_i are positive integer coefficients and

$$(2.24b) \qquad m_i \geq 2 \quad ; \quad 1 \leq r_j^i \leq n+1-m_i \quad ; \quad \sum_{j=1}^{m_i} r_j^i = n \quad .$$

We denote

$$(2.25) \qquad M := \max_{2\leq n\leq N} ||a^{(n)}(u)||_{L_\infty} = \max_{2\leq n\leq N} ||a^{(n)}(u_0)||_{L_\infty} \quad .$$

With (2.24) and (2.25) we get, using (1.2), that for $n \leq N$

$$(2.26) \qquad |\partial_x^n u| \leq \frac{1}{\alpha}\left(|w^n| + \sum_i K_i M \prod_{j=1}^{m_i} |\partial_x^{r_j^i} u|\right) \quad .$$

Note that for Burgers' equation $\alpha = 1$ and $M = 0$ and (2.26) holds with an equality.

If we now denote $\partial_x^n u(t) := \partial_x^n u(x(t), t)$, where $x(t)$ is a characteristic curve, we may state the analogous of Proposition 2.1.

Theorem 2.1. *For every $1 \leq n \leq N$ and $t \geq 0$ there holds*

$$(2.27a) \qquad |\partial_x^n u(t)| \leq C_n(1+w(0)t)^{-n} \qquad \text{if } \partial_x u(t) > 0 \ ;$$

$$(2.27b) \qquad |\partial_x^n u(t)| \leq D_n(1+w(0)t)^{-2n+1} \qquad \text{if } \partial_x u(t) < 0 \ ;$$

$$(2.27c) \qquad \partial_x^n u(t) = \partial_x^n u(0) + P_{n-2}(t) \qquad \text{if } \partial_x u(t) = 0 \ .$$

Here C_n and D_n are constants which depend on the initial condition and $P_{n-2}(t)$ is a polynomial of degree $n-2$ which vanishes for $t = 0$.

Proof. Since u remains constant along its characteristics, (2.7) implies that

$$(2.28) \qquad \partial_x u(t) = \frac{\partial_x u(0)}{1 + w(0)t} \quad .$$

Hence, (2.27) holds for $n = 1$ with $C_1 = D_1 = |\partial_x u(0)|$ and $P_{-1}(t) \equiv 0$. (2.28) and (1.2) imply that $\partial_x u(t), \partial_x u(0)$ and $w(0)$ have the same sign.

As for $n \geq 2$, we proceed by induction.

If $\partial_x u(t) > 0$, (2.26) and (2.8a), together with the induction assumption, imply that

$$(2.29) \quad |\partial_x^n u| \le \frac{1}{\alpha}\left(C_n(1+w(0)t)^{-n-1} + \sum_i K_i M \prod_{j=1}^{m_i} C_{r_j^i}(1+w(0)t)^{-r_j^i}\right) \quad .$$

But, by (2.24b),

$$(2.30) \qquad \sum_i K_i M \prod_{j=1}^{m_i} C_{r_j^i}(1+w(0)t)^{-r_j^i} =$$

$$= \sum_i C_i(1+w(0)t)^{-n} = C_n(1+w(0)t)^{-n} \quad .$$

Hence (2.27a) follows from (2.29) and (2.30).

Similarly, if $\partial_x u(t) < 0$ then (2.26), (2.8b) and induction imply that

$$(2.31) \qquad |\partial_x^n u| \le$$

$$\frac{1}{\alpha}\left(C_n(1+w(0)t)^{-2n+1} + \sum_i K_i M \prod_{j=1}^{m_i} C_{r_j^i}(1+w(0)t)^{-2r_j^i+1}\right) \quad .$$

Using (2.24b) we get that

$$\sum_i K_i M \prod_{j=1}^{m_i} C_{r_j^i}(1+w(0)t)^{-2r_j^i+1} = \sum_i C_i(1+w(0)t)^{-2n+m_i} \quad .$$

But $m_i \ge 2$ and therefore the first term on the right hand side of (2.31) is the dominant one as t tends to the critical time, $t_c = 1/|w(0)|$, hence (2.27b) follows.

As for the case $\partial_x u(t) = 0$, since u remains constant along $x(t)$, (2.24a) implies that

$$w^n(t) - w^n(0) =$$

$$a'(u)(\partial_x^n u(t) - \partial_x^n u(0)) + \sum_i K_i a^{(m_i)}(u)\left[\prod_{j=1}^{m_i} \partial_x^{r_j^i} u(t) - \prod_{j=1}^{m_i} \partial_x^{r_j^i} u(0)\right] \quad .$$

Using (2.8c) we therefore conclude that

$$\partial_x^n u(t) =$$

$$\partial_x^n u(0) + \frac{1}{a'(u)}\left(P_{n-2}(t) - \sum_i K_i a^{(m_i)}(u)\left[\prod_{j=1}^{m_i} \partial_x^{r_j^i} u(t) - \prod_{j=1}^{m_i} \partial_x^{r_j^i} u(0)\right]\right) \quad .$$

But since by induction the term in the brackets is a polynomial of degree $n-2$ and it vanishes at $t = 0$, (2.27c) is proved and we are done. $\square$

Remarks.

1. We call attention that (2.27a) is slightly different from (2.8a). This difference in the exponent is the reason why (2.27a) holds for $n \geq 1$ while (2.8a) holds only for $n \geq 2$.

2. Equality (2.28) allows us to rewrite (2.27a-b) in the form announced in the Introduction:

$$|\partial_x^n u(t)| \leq \begin{cases} \tilde{C}_n(\partial_x u(t))^n & \partial_x u(t) > 0 \\ \tilde{D}_n|\partial_x u(t)|^{2n-1} & \partial_x u(t) < 0 \end{cases} \qquad t \geq 0 \quad , \tag{2.32}$$

with constants

$$\tilde{C}_n = \frac{C_n}{(\partial_x u(0))^n} \quad , \quad \tilde{D}_n = \frac{D_n}{|\partial_x u(0)|^{2n-1}} \quad , \tag{2.33}$$

which depend solely on the initial condition.

3. The large time behavior of the second spatial derivative in (planar) rarefaction waves has been studied before by Xin in [13]. Xin considered the scalar viscous conservation law

$$u_t + f(u)_x = \varepsilon u_{xx}$$

subject to the C^2-smooth and bounded initial condition, u_0, satisfying

$$u_0' > 0 \tag{2.34}$$

and

$$|u_0''| \leq k_0 u_0' \quad , \quad 0 \leq k_0 = \mathrm{Const} \quad . \tag{2.35}$$

He showed that in that case there exists a positive constant K such that

$$|u_{xx}(x,t)| \leq K u_x(x,t) \qquad \forall (x,t) \in \Re \times \Re^+ \ . \tag{2.36}$$

This estimate can be recovered for the inviscid hyperbolic conservation law (1.1) from our analysis. Let us denote

$$L^+ \equiv \max_{x,t} u_x(x,t) = \max_x u_0'(x) \quad . \tag{2.37}$$

By (2.7) and (2.14) we get that

$$w^2(t) = w(t)\frac{w^2(0)}{w(0)(1+w(0)t)^2} \quad . \tag{2.38}$$

Therefore, since by (2.34) and (1.2) $w(t) = a'(u)u_x > 0$, (2.38) implies that

$$(2.39) \qquad |w^2(t)| \leq \frac{|w^2(0)|}{w(0)} w(t) \quad .$$

As $w(0)$ and $w^2(0)$ are given by (consult (2.24a))

$$(2.40) \qquad w(0) = a'(u_0)u_0' \quad , \quad w^2(0) = a''(u_0)(u_0')^2 + a'(u_0)u_0'' \quad ,$$

we get from (2.39) that

$$|w^2(t)| \leq \left[\frac{|u_0''|}{u_0'} + \frac{|a''(u_0)|u_0'}{a'(u_0)} \right] w(t) \quad .$$

Using (1.2), (2.25), (2.35) and (2.37) we conclude that

$$(2.41) \qquad |w^2(t)| \leq K_1 w(t) \quad , \quad K_1 \equiv \left(k_0 + \frac{ML^+}{\alpha} \right) \quad .$$

Thus, $v = a(u)$ satisfies inequality (2.36) since, by definition, $w(t) = v_x(x(t), t)$ and $w^2(t) = v_{xx}(x(t), t)$. The desired inequality for u easily follows from (1.2), (2.25), (2.37) and (2.41):

$$\frac{|u_{xx}|}{u_x} = \frac{|w^2 - a'' u_x^2|}{w} \leq \frac{|w^2|}{w} + \frac{|a''|}{a'} u_x \leq K \equiv K_1 + \frac{ML^+}{\alpha}$$

Note that (2.36) holds even if condition (2.34) is replaced by $u_0' \geq 0$, since along characteristics where $u_x = 0$, u_{xx} remains constant (by (2.27c)) which must be zero in view of restriction (2.35).

Theorem 2.1 tells us the behavior of the high order derivatives of the entropy solution along its characteristics, depending on the sign of the first derivative there: if the first derivative is positive, then according to (2.27a) the higher derivatives decay in time; if it is negative – the higher derivatives tend, in absolute value, to infinity as the characteristic approaches the shock curve, (2.27b); and along characteristics where the first derivative is zero, the higher order derivatives experience a polynomial growth rate indicated in (2.27c). Furthermore, the rate of decay or growth increases with the order of the derivative.

3. HIGH ORDER PIECEWISE STABILITY ESTIMATES

The estimates obtained in §2, consult (2.10-11) and (2.32-33), show how the smoothness of the entropy solution depends on the *distance* from the set of shock discontinuities, where this distance is measured by the size of $\partial_x u(t)$. These estimates involve, apart from $\partial_x u(t)$, also the value of the first derivative of the initial condition, $\partial_x u(0)$.

We now turn to upper bound the higher order derivatives in regularity regions solely in terms of the local value of the first derivative, thus extending the special case of an estimate for the second spatial derivative of planar rarefaction waves in (2.36). Moreover, our bound will indicate the dependence of the high order regularity on the *distance* from the singular set of shocks. The distance from the singular set is measured by a lower bound of the first derivative. To quantify this dependence we define for every $L \leq 0$ the following semi-norm:

$$\|g(x)\|_{C_L^n} \equiv \sup_{x \in D_{g,L}} \left| \frac{d^n g}{dx^n} \right| \quad , \quad D_{g,L} \equiv \{x : \frac{dg}{dx}(x) \geq L\} \quad . \tag{3.1}$$

This is a localized version of the regular C^n (or $W^{n,\infty}$) semi-norm which may be obtained from $\|\cdot\|_{C_L^n}$ by letting $L \to -\infty$.

We show that the solution operator of (1.1) is stable with respect to this semi-norm. As before, we deal first with the "Burgerized" equation, (2.3), in the unknown $v = a(u)$.

Proposition 3.1. *For every $2 \leq n \leq N$ and $L < 0$ there holds*

$$\|v(\cdot, t)\|_{C_L^n} \leq e^{(n+1)|L|t} \|v(\cdot, 0)\|_{C_L^n} + P_{n-2}(|L|^{-1}) e^{3(n-1)|L|t} \quad , \tag{3.2}$$

where the coefficients of P_{n-2} depend on $\{\|v(\cdot, 0)\|_{C_L^k}\}_{2 \leq k < n}$.

Proof. We recall equation (2.12a) which governs the evolution of $w^n(t)$ along a characteristic $(x(t), t)$. Let (x, t) be located on a characteristic $x = x(t)$ and assume that $x(t) \in D_{v(\cdot,t),L}$, i.e.,

$$v_x(x(t), t) = w(t) \geq L \quad . \tag{3.3}$$

Since by (2.7) $w(t)$ can only decrease along a characteristic, (3.3) implies that

$$w(\tau) \geq L \quad ; \quad 0 \leq \tau \leq t \quad , \tag{3.4a}$$

or equivalently,

$$(3.4b) \qquad x(\tau) \in D_{v(\cdot,\tau),L} \qquad ; \qquad 0 \le \tau \le t \quad .$$

The solution of (2.12a) is

$$(3.5) \qquad w^n(t) = e^{\int_0^t -(n+1)w(\tau)d\tau} w^n(0) + \int_0^t e^{\int_\tau^t -(n+1)w(s)ds} q_n(\tau)d\tau \quad .$$

Therefore, by (3.4a) and (3.5) we get that in $D_{v(\cdot,t),L}$

$$(3.6) \qquad |w^n(t)| \le e^{-(n+1)Lt}\left[|w^n(0)| + \int_0^t e^{(n+1)L\tau}|q_n(\tau)|d\tau\right] \quad .$$

We start by dealing with $n = 2$. Here $q_2 = 0$ and (3.6) reads

$$|w^2(t)| \le e^{3|L|t}|w^2(0)| \quad ,$$

hence (3.2) follows with $P_0(|L|^{-1}) = 0$.

We proceed by induction assuming (3.2) holds for all $2 \le k < n$,

$$(3.7) \qquad \|v(\cdot,t)\|_{C_L^k} \le e^{(k+1)|L|t}\|v(\cdot,0)\|_{C_L^k} + P_{k-2}(|L|^{-1})e^{3(k-1)|L|t} \quad ,$$

where the coefficients of $P_{k-2}(|L|^{-1})$ depend on $\{\|v(\cdot,0)\|_{C_L^m}\}_{2\le m<k}$. Clearly, since for $k \ge 2$ we have that $3(k-1) \ge k+1$, (3.7) may be rewritten as follows

$$(3.8) \qquad \|v(\cdot,t)\|_{C_L^k} \le P_{k-2}(|L|^{-1})e^{3(k-1)|L|t} \qquad , \qquad 2 \le k < n \quad ,$$

where the coefficients of $P_{k-2}(|L|^{-1})$ in (3.8) depend on $\{\|v(\cdot,0)\|_{C_L^m}\}_{2\le m\le k}$.

Using (3.6), (2.12b) and (3.8) we arrive at

$$|w^n(t)| \le e^{-(n+1)Lt}\left[|w^n(0)| + \int_0^t e^{(n+1)L\tau}\sum_{k=2}^{n-1}\binom{n}{k}|w^k(\tau)||w^{n-k+1}(\tau)|d\tau\right] \le$$

$$\le e^{-(n+1)Lt}\left[|w^n(0)| + \tilde{P}_{n-3}(|L|^{-1})\int_0^t e^{(-2n+4)L\tau}d\tau\right] \quad ,$$

where $\tilde{P}_{n-3}(\cdot)$ abbreviates

$$\tilde{P}_{n-3}(|L|^{-1}) = \sum_{k=2}^{n-1}\binom{n}{k}P_{k-2}(|L|^{-1})P_{n-k-1}(|L|^{-1})$$

which depends on $\{\|v(\cdot,0)\|_{C^k_L}\}_{2\le k<n}$. Evaluating the last integral we arrive at

$$(3.9a)\qquad |w^n(t)| \le e^{-(n+1)Lt}\left[|w^n(0)| + P_{n-2}(|L|^{-1})e^{(-2n+4)Lt} - P_{n-2}(|L|^{-1})\right]$$

where

$$(3.9b)\qquad P_{n-2}(|L|^{-1}) = \frac{1}{(-2n+4)L}\tilde{P}_{n-3}(|L|^{-1}) = \frac{1}{2(n-2)|L|}\tilde{P}_{n-3}(|L|^{-1}) \ .$$

Since $L<0$ and $n>2$, $P_{n-2}(|L|^{-1})$ is positive and therefore by (3.9a) we conclude that

$$(3.10)\qquad |w^n(t)| \le e^{-(n+1)Lt}\left[|w^n(0)| + P_{n-2}(|L|^{-1})e^{(-2n+4)Lt}\right] =$$
$$e^{(n+1)|L|t}|w^n(0)| + P_{n-2}(|L|^{-1})e^{3(n-1)|L|t}$$

which proves (3.2). □

Remark. It can be easily shown, in the same manner, that for $L=0$

$$\|v(\cdot,t)\|_{C^n_0} \le \|v(\cdot,0)\|_{C^n_0} + P_{n-2}(t) \ ,$$

where $P_{n-2}(t)$ depends on $\{\|v(\cdot,0)\|_{C^k_0}\}_{2\le k<n}$. This result is not surprising in view of (2.8a) and (2.8c).

Finally, we translate the estimates offered by Proposition 3.1 for $v\equiv a(u)$, into analogous estimates for u itself.

Theorem 3.1. (Piecewise Stability). *For every $2\le n\le N$ and $L<0$ there holds*

$$(3.11)\qquad \|u(\cdot,t)\|_{C^n_L} \le e^{(n+1)\hat{L}t}\|u(\cdot,0)\|_{C^n_L} + P_{n-2}(\hat{L}^{-1})e^{3(n-1)\hat{L}t} \ ,$$

where $\hat{L} = A|L|$, $A = \|a'(u)\|_{L_\infty}$ and the coefficients of P_{n-2} depend on $\{\|u(\cdot,0)\|_{C^k_L}\}_{2\le k<n}$.

Proof. The verification of (3.11) for $n=2$ is left to the reader and we proceed by induction. Let (x,t) be a point on the characteristic $x=x(t)$ where $x(t)\in D_{u(\cdot,t),L}$. The definition of $v=a(u)$ and (1.2) therefore imply that

$$(3.12)\qquad x(t)\in D_{v(\cdot,t),\tilde{L}} \quad , \quad \tilde{L} = a'(u(x(t),t))L \ .$$

Furthermore, by (2.7) and (2.28) we conclude that

$$x(\tau) \in D_{u(\cdot,\tau),L} \bigcap D_{v(\cdot,\tau),\tilde{L}} \qquad 0 \le \tau \le t \quad . \tag{3.13}$$

This together with (3.12) and (3.10) imply that

$$|w^n(t)| \le e^{(n+1)|\tilde{L}|t}|w^n(0)| + P_{n-2}(|\tilde{L}|^{-1})e^{3(n-1)|\tilde{L}|t} \quad . \tag{3.14}$$

Using (2.24a) we obtain

$$|w^n(0)| \le a'(u)|\partial_x^n u(0)| + C \quad , \tag{3.15}$$

where C depends on $\{|\partial_x^k u(0)|\}_{2\le k<n}$. Therefore, since $n+1 \le 3(n-1)$ we conclude from (3.14) and (3.15) that

$$|w^n(t)| \le a'(u)e^{(n+1)|\tilde{L}|t}|\partial_x^n u(0)| + P_{n-2}(|\tilde{L}|^{-1})e^{3(n-1)|\tilde{L}|t} \quad . \tag{3.16}$$

Recalling (2.24) and (2.25), the inequality (3.16) implies

$$|\partial_x^n u(t)| \le \frac{1}{a'(u)}\left(|w^n(t)| + \sum_i K_i M \prod_{j=1}^{m_i} |\partial_x^{r_j^i} u(t)|\right) \le$$

$$\le e^{(n+1)|\tilde{L}|t}|\partial_x^n u(0)| + P_{n-2}(|\tilde{L}|^{-1})e^{3(n-1)|\tilde{L}|t} + C\sum_i \prod_{j=1}^{m_i} |\partial_x^{r_j^i} u(t)| \quad .$$

By induction we may conclude, as we did in the proof of Proposition 3.1, that

$$|\partial_x^n u(t)| \le e^{(n+1)|\tilde{L}|t}|\partial_x^n u(0)| + P_{n-2}(|\tilde{L}|^{-1})e^{3(n-1)|\tilde{L}|t} \quad , \tag{3.17}$$

and taking the supremum over $x(t) \in D_{u(\cdot,t),L}$ in (3.17) we arrive at (3.11). □

Remarks.

1. In the case of Burgers' equation $A = 1$ and therefore (3.11) reduces in that case to the stability estimate (3.2).
2. The analogous of (3.11) for $L = 0$ is

$$\|u(\cdot,t)\|_{C_0^n} \le \|u(\cdot,0)\|_{C_0^n} + P_{n-2}(t) \ , \tag{3.18}$$

where $P_{n-2}(t)$ depends on $\{\|u(\cdot,0)\|_{C_0^k}\}_{2\le k<n}$.

4. ON THE SIZE OF THE SET OF SHOCK DISCONTINUITIES

We show in this section that generically, the set of shocks is finite, and identify the initial conditions for which an infinite number of shock curves is generated.

The first result concerning the size of the shock set was Oleinik's. She has shown [8,9,10] that the shock set is countable at the most. Her result, however, still allows a very complicated structure such as an everywhere dense shock set.

Two proceeding results have simplified the picture : Dafermos [2] has shown that in case that both the (convex) flux and the initial condition are infinitely smooth the solution is C^∞ a.e. apart from the shock set which must be closed. Thus, the shock set cannot be everywhere dense but shocks may still accumulate.

Schaeffer [11] has proved that, generically, the shock set is finite when the initial condition is infinitely smooth. He has shown that if $f \in C^\infty$ satisfies (1.2), there exists a subset, Ω, of the first category in Schwartz space, $S(\Re)$, such that if $u_0 \in S(\Re) \setminus \Omega$ then u is $C^\infty(\Re \times (0,\infty) \setminus \Gamma)$ where Γ is a finite set of smooth shock curves. He furthermore gives an example of such an initial condition $u_0 \in \Omega$ which evolves , according to the Burgers' equation, to an almost everywhere C^∞ function with infinitely many shock curves in a bounded region. However, we are left unable to check whether a given initial condition is in Ω or not.

It seems to be a part of the folklore [5,7] that if u_0 has a finite number of inflection points, then the corresponding entropy solution of Burgers' equation experiences a finite number of shock discontinuities. In the general case the function whose inflection points are to be examined is $a(u_0)$.

Theorem 4.1. *Let u be the entropy solution of the convex hyperbolic conservation law* (1.1a), (1.2), *subject to the bounded and piecewise C^1 initial condition, u_0, satisfying*

$$\lim_{|x|\to\infty} a(u_0)' = 0 \quad . \tag{4.1}$$

Then the number of disjoint shock curves equals to the number of negative minima of $a(u_0)'$.

Remarks.

1. Since u_0 is assumed to be only piecewise C^1 it may be discontinuous and therefore will not have a classical derivative. Therefore, we refer by $a(u_0)'$ to the generalized derivative of $a(u_0)$. Hence, in decreasing discontinuities of u_0 , $a(u_0)'$ has a negative (infinite) minimum.

2. If $a(u_0)'$ has a continuum of negative minimal points , namely, $a(u_0)$ linearly decreases along some interval, it is considered as one minimum.

3. Shocks which occur as a consequence of an interaction of two (or more) other shocks, are not counted. We consider only "original" shocks. Obviously, the number of original shocks dominates the number of simultaneous shocks in every $t \geq 0$.

Corollary 4.1. *If $a(u_0)$ has a finite number of decreasing inflection points, then the set of shock discontinuities is finite.*

Theorem 4.1 implies that the set of functions $u_0 \in S(\Re)$ for which $a(u_0)'$ has infinitely many negative minima, is the set $\Omega \subset S(\Re)$ of the first category that Schaeffer refers to in [11].

Proof. Denote the set of disjoint (original) shock curves by $S = \{X_i(t)\}_{i \in I}$ and the set of points where $a(u_0)'$ has a negative minimum by $M = \{x_j\}_{j \in J}$. We will establish an equivalence between these two sets to prove our statement.

For every $X_i(t) \in S$ let t_i^0 denote its creation time ($t_i^0 \geq 0$) and t_i^∞ its termination time ($t_i^0 < t_i^\infty \leq \infty$). t_i^∞ is finite if $X_i(t)$ collides with another shock, and infinite otherwise.

According to Lax entropy condition

$$a(u(X_i(t_i)-, t_i)) > a(u(X_i(t_i)+, t_i)) \qquad , \qquad t_i^0 < t_i < t_i^\infty \quad . \tag{4.2}$$

We choose one value of t_i in that time interval,

$$t_i \in (t_i^0, t_i^\infty) \tag{4.3}$$

and denote by x_i^- and x_i^+ the two initial points of the characteristics which impinge upon the shock $X_i(t)$ from both sides at $t = t_i$. (4.2) implies that

$$a(u_0(x_i^-)) > a(u_0(x_i^+)) \qquad , \qquad x_i^- < x_i^+ \quad . \tag{4.4}$$

A consequence of (4.4) is that $a(u_0)'$ must become negative somewhere along the interval $[x_i^-, x_i^+]$. Let x_i denote the point in that interval where $a(u_0)'$ achieves its minimal value. The shock's creation time is determined by this minimum ,

$$t_i^0 = -\frac{1}{a(u_0)'(x_i)} \quad . \tag{4.5}$$

On the other hand

$$-\frac{1}{a(u_0)'(x_i^{\pm})} \geq t_i > t_i^0 \quad , \tag{4.6}$$

since otherwise , the characteristics which start at $(x_i^{\pm}, 0)$ would not have lasted until $t = t_i$. (4.5) and (4.6) imply that

$$a(u_0)'(x_i) < a(u_0)'(x_i^{\pm}) \tag{4.7}$$

and therefore x_i is a negative local minimum of $a(u_0)'$, i.e., $x_i \in M$.

We have thus shown that to each $X_i(t) \in S$ corresponds a $x_i \in M$. This correspondence is one-to-one since if $X_i(t)$ and $X_j(t)$ are two disjoint shocks then our choice of t_i , (4.3), implies that

$$[x_i^-, x_i^+] \bigcap [x_j^-, x_j^+] = \emptyset$$

and therefore $x_i \neq x_j$.

Now we show an one-to-one correspondence from M to S to conclude the equivalence of the two sets. Let $x_1, x_2 \in M$ and let ξ be the point where $a(u_0)'$ reaches its maximal value in the interval $[x_1, x_2]$. Let $x_i(t)$ be the characteristic which starts at $(x_i, 0)$, $i = 1, 2$, and $\xi(t)$ be the one which starts at $(\xi, 0)$. The solution along $x_i(t)$ becomes discontinuous at time

$$t_i = -\frac{1}{a(u_0(x_i))'} \quad , \qquad i = 1, 2 \quad . \tag{4.8}$$

Therefore, each of the points $(x_i(t_i), t_i)$, $i = 1, 2$, is on a shock. By Lagrange mean value theorem and since $a(u_0)'$ has local minima in x_i we conclude that

$$\frac{a(u_0(x_i)) - a(u_0(\xi))}{x_i - \xi} > a(u_0(x_i))' \quad , \qquad i = 1, 2 \quad .$$

or, by (4.8),

$$-\frac{x_i - \xi}{a(u_0(x_i)) - a(u_0(\xi))} > t_i \qquad , \qquad i = 1,2 \quad . \tag{4.9}$$

Since the left hand side of (4.9) indicates the time when $x_i(t)$ and $\xi(t)$ were to meet, we conclude that $x_1(t_1) < \xi(t_1)$ and $x_2(t_2) > \xi(t_2)$. Therefore , the points $(x_i(t_i), t_i)$, $i = 1, 2$, lay on two different shocks, the first is on the left side of $\xi(t)$ and the second is from its right side.

Finally, we note that if $a(u_0)'$ has a continuum of negative minimal points, i.e., if $a(u_0)$ is linearly decreasing on some interval, $[x_1, x_2]$, this minimum creates only one shock since the characteristics from that interval will all meet at

$$t = -\frac{1}{a(u_0)'\Big|_{[x_1,x_2]}}$$

to start that shock. □

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