Small Alphabet

- What will happen if the items set for the compression is small?
- Example of 77.49MB of bitmaps:

	n	Huffman (MB)	Arithmetic (MB)	increase (%)
each bit	2	77.49	9.64	703
k-blocks	256	13.70	7.70	78

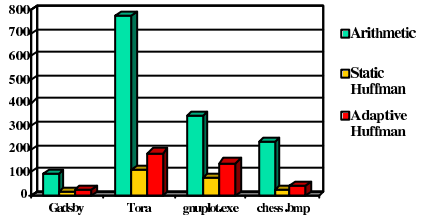
Arithmetic evaluation

- Average codeword length in bits in Huffman and Arithmetic:

	Huffman	Arithmetic	Huffman cost (%)
English	4.1854	4.1603	0.6
Finnish	4.0448	4.0134	0.8
French	4.0003	4.0376	0.9
German	4.1475	4.1129	0.8
Hebrew	4.2851	4.2490	0.8
Italian	4.0000	3.9725	0.7
Portuguese	4.0100	3.9864	0.6
Spanish	4.0469	4.0230	0.6
Russian	4.4704	4.4425	0.6
English-2	7.4446	7.4158	0.4
Hebrew-2	8.0370	8.0085	0.4

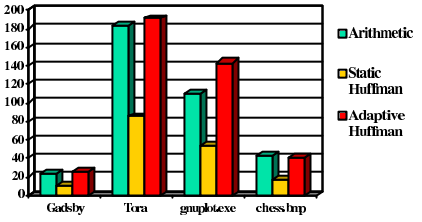
Time considerations

- Decoding time in seconds:



Time considerations

- Encoding time in seconds:



An example

- The items are:
  - a–10%
  - i–20%
  - r–30%
  - y–40%
- A Huffman tree for these item:
  - y - 0
  - r - 10
  - a - 110
  - i – 111
- The string which should be encoded is: "ii".

Quasi Arithmetic Coding

- A combination of Huffman coding and Arithmetic coding.
- Algorithm:
  - Compress data by Huffman (Or other prefix codes).
  - Compress the bits by Arithmetic coding.

An example (Cont.)

- The bit string for "ii" is: 111111.
- $[0,1) \rightarrow [8/19,1) \rightarrow [240/361,1) \rightarrow$
  $\rightarrow [5528/6859,1) \rightarrow [115680/130321,1) \rightarrow$
  $\rightarrow [2315048/2476099,1)$
  $\rightarrow [45274320/47045881,1)$

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An example (Cont.)

- ones - 0.3*1+0.1*2+0.2*3=1.1
- zeros - 0.4*1+0.3*1+0.1*1=0.8
- Ratio is 8:11
- Arithmetic code for bits:
  - $[0,8/19)$
  - $[8/19,1)$

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Advantages of Quasi Arithmetic

- Just two intervals (for 1 and 0) so can be implemented by tables instead of floating point divisions.
- These table can be implemented even in hardware.

$\Rightarrow$ Faster than Arithmetic coding but has the same efficiency.

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An example (Cont.)

- The shortest binary fraction in this interval is 0.11111 which is just 5 bits.
- Huffman represents "i" by 3 bits which is too much since $3=-\log_2 0.125$ while the probability of "i" is 0.2.
- 0.2*0.2=0.04 so the content of data in this string is $-\log_2 0.04$ which is circa 4.643856 so 5 bits are the optimal compression of this string.

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Theorem - Proof

- The derivative of the entropy is:

$$H'=[-\sum_{i=0}^{1} P_i \log_2 P_i]'=[-P\log_2 P-(1-P)\log_2(1-P)]'=$$

$$=[\frac{-P\ln P}{\ln 2}-\frac{(1-P)\ln(1-P)}{\ln 2}]'=\frac{-P}{P\ln 2}+\frac{-\ln P}{\ln 2}-\frac{(1-P)(-1)}{\ln 2(1-P)}-\frac{-\ln(1-P)}{\ln 2}=$$

$$=\frac{-1}{\ln 2}-\log_2 P+\frac{1}{\ln 2}+\log_2(1-P)=\log_2(\frac{1-P}{P})$$

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Theorem

- The second step of the Quasi Arithmetic Coding will give the worst improvement when the 1s and the 0s have exactly the same probability - 0.5
- In other words: The entropy of a binary string will be maximal, if the 1s and the 0s have exactly the same probability - 0.5
- When the ratio between the 1s and the 0s grows, the entropy will be diminished.

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Theorem - Proof (Cont.)

- The second derivative is:

$$[\log_2(1-P)-\log_2 P]'=[\frac{\ln(1-P)}{\ln 2}-\frac{\ln P}{\ln 2}]'=-\frac{1}{(1-P)\ln 2}-\frac{1}{P\ln 2}$$

- When P=1/2:

$$-\frac{2}{\ln 2}-\frac{2}{\ln 2}<0$$

- So 1/2 is a maximum point.

Theorem - Proof (Cont.)

$$\log_2(\frac{1-P}{P})=0$$

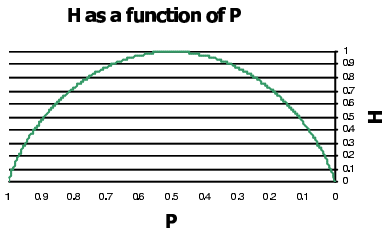
$$\frac{1-P}{P}=1$$

$$1-P=P$$

$$P=\frac{1}{2}$$

- So there is an extremum point when P=1/2.

Conclusion in a graph



Theorem - conclusion

- When P=1 or P=0:

$$\lim_{x\to 0} H=\lim_{x\to 0}-x\cdot\log_2 x-1\cdot\log_2 1=$$

$$\lim_{x\to 0}-\log_2(x^x)=-\log_2 1=0$$

- When P=1/2:

$$H=-\frac{1}{2}\log_2\frac{1}{2}-\frac{1}{2}\cdot\log_2\frac{1}{2}=1$$

Error in Arithmetic coding

- One bit in the compressed file was changed from "1" to "0":

In the beginning God created the heaven and the ea eithen e
tnos hheaheedh rd ehtesthhfo nh dtohrnh we
telw .e as io
oht o dwf esttfde evsieoeGehee mn n to ito en,nnte
denntoe asipa c., meeha odftdetredd gnt nGp og rmoa oei
ndf r pnfwh eias peedd kaeoweeihd.gomhi G ro m sarc t
eeloorAhiytgawGtlmG r
ddis.i tswot suea df.i. at
i vsha nieI,yawedd D cn eAotc ndeenogidihhese
n etdoleated oiAhettS tDeot Aee retd gctdehneds hheda tdai
sdwL toheAc ohia nrv,iaicna:dde iettweah dea Gmg k.d,ai
kafe rew:ehtd .truit h.ihnd enif e.dwgt hrihrm atn
hSetbhhdd hdtongehie

Wrong probabilities

- Excess of Huffman over Arithmetic in percents when assuming the rows to compress the columns

	English	Gadsby	German	Finnish	French
English	0.6	1.4	-2.1	-5.1	0.4
Gadsby	-2.4	1.0	-2.9	-0.8	-3.3
German	-1.8	-3.6	0.9	-5.2	-0.1
Finnish	2.4	1.2	2.7	0.8	2.9
French	0.8	2.2	-4.6	-11.3	0.9