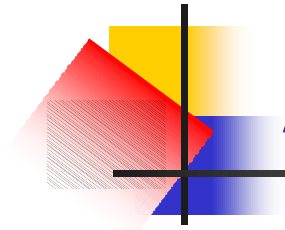
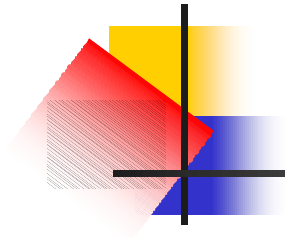


Arithmetic evaluation

- Average codeword length in bits in Huffman and Arithmetic:

	Huffman	Arithmetic	Huffman cost (%)
English	4.1854	4.1603	0.6
Finnish	4.0448	4.0134	0.8
French	4.0003	4.0376	0.9
German	4.1475	4.1129	0.8
Hebrew	4.2851	4.2490	0.8
Italian	4.0000	3.9725	0.7
Portuguese	4.0100	3.9864	0.6
Spanish	4.0469	4.0230	0.6
Russian	4.4704	4.4425	0.6
English-2	7.4446	7.4158	0.4
Hebrew-2	8.0370	8.0085	0.4

© Klein S. T. and Wiseman Y.



Small Alphabet

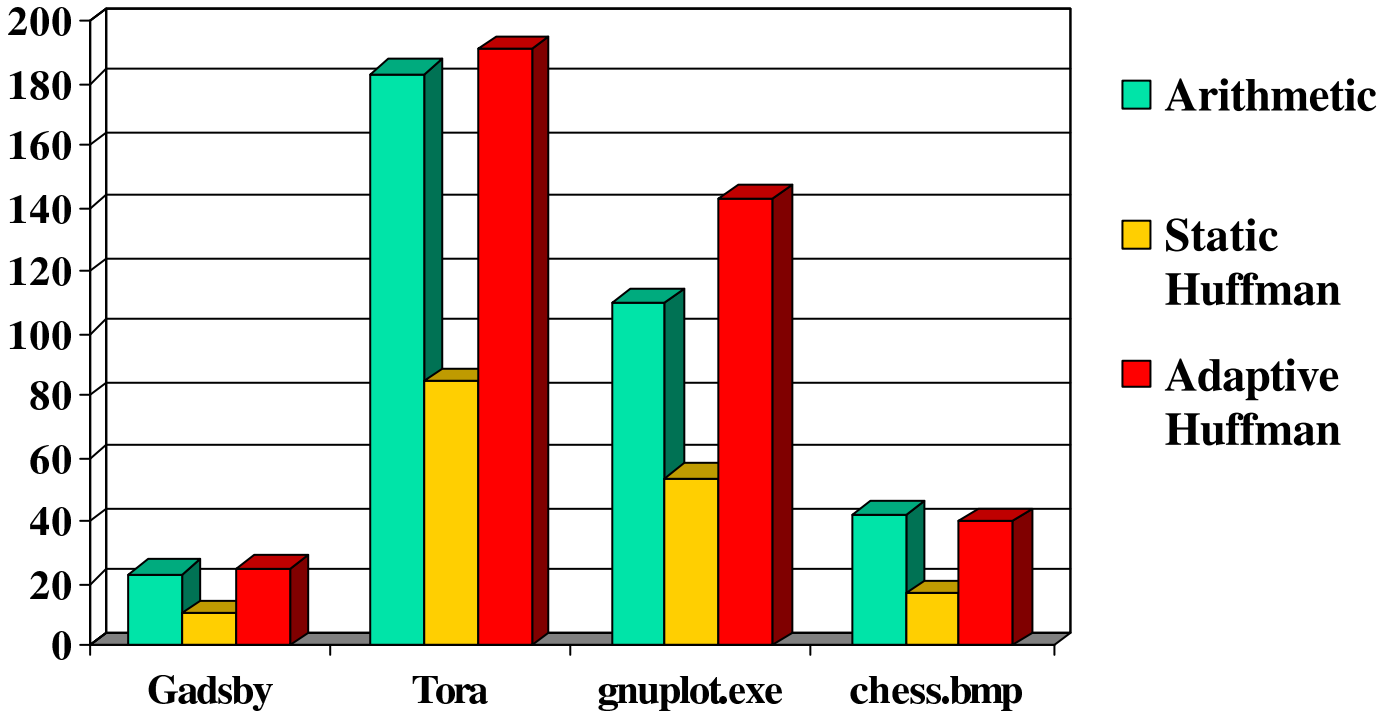
- What will happen if the items set for the compression is small?
- Example of 77.49MB of bitmaps:

	n	Huffman (MB)	Arithmetic (MB)	increase (%)
each bit	2	77.49	9.64	703
k-blocks	256	13.70	7.70	78

© Klein S. T. and Wiseman Y.

Time considerations

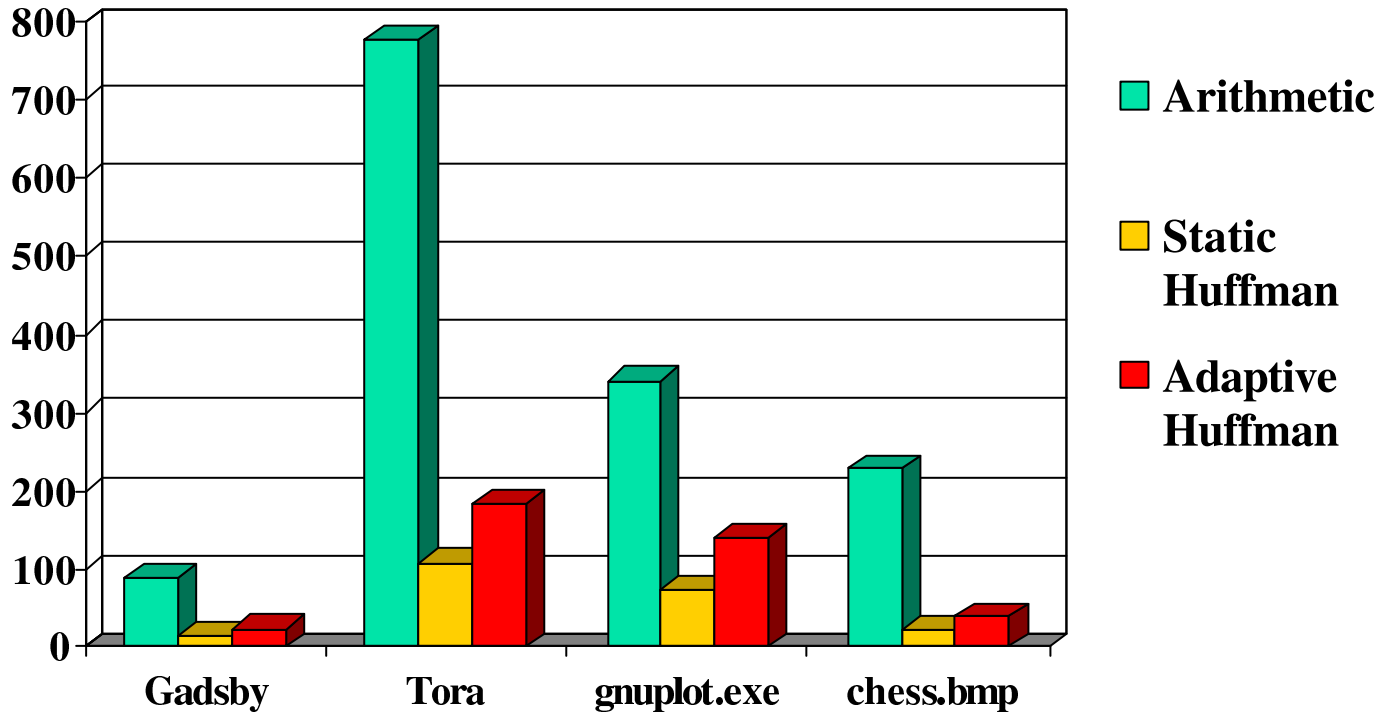
- Encoding time in seconds:



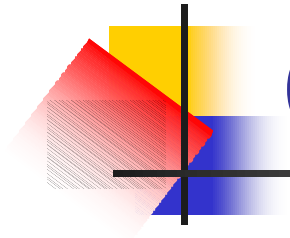
© Klein S. T. and Wiseman Y.

Time considerations

- Decoding time in seconds:



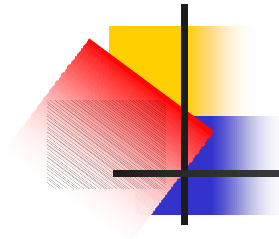
© Klein S. T. and Wiseman Y.



Quasi Arithmetic Coding

- A combination of Huffman coding and Arithmetic coding.
- Algorithm:
  - Compress data by Huffman (Or other prefix codes).
  - Compress the bits by Arithmetic coding.

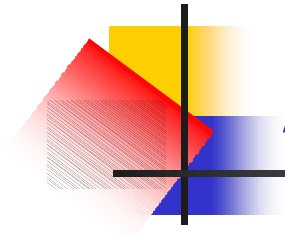
© Klein S. T. and Wiseman Y.



An example

- The items are:
  - **a-10%**
  - **i-20%**
  - **r-30%**
  - **y-40%**
- A Huffman tree for these item:
  - y - 0
  - r - 10
  - a - 110
  - i – 111
- The string which should be encoded is: "ii".

© Klein S. T. and Wiseman Y.



An example (Cont.)

- ones - 0.3*1+0.1*2+0.2*3=1.1
- zeros - 0.4*1+0.3*1+0.1*1=0.8
- Ratio is 8:11
- Arithmetic code for bits:
  - $[0, 8/19)$
  - $[8/19, 1)$

© Klein S. T. and Wiseman Y.

An example (Cont.)

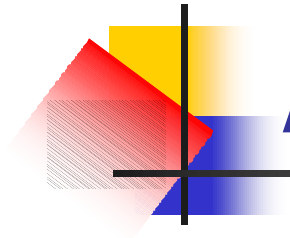
- The bit string for "ii" is: 111111.
- [0,1) $\rightarrow$ $[{}^{8}/_{19},1)$ $\rightarrow$ $[{}^{240}/_{361},1)$ $\rightarrow$

  $\rightarrow$ $[{}^{5528}/_{6859},1)$ $\rightarrow$ $[{}^{115680}/_{130321},1)$ $\rightarrow$

  $\rightarrow$ $[{}^{2315048}/_{2476099},1)$

  $\rightarrow$ $[{}^{45274320}/_{47045881},1)$

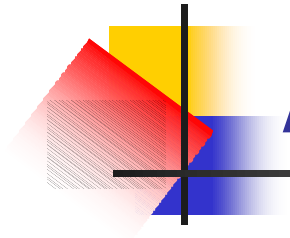
© Klein S. T. and Wiseman Y.



An example (Cont.)

- The shortest binary fraction in this interval is 0.11111 which is just 5 bits.
- Huffman represents "i" by 3 bits which is too much since $3=-\log_2 0.125$ while the probability of "i" is 0.2.
- $0.2*0.2=0.04$ so the content of data in this string is $-\log_2 0.04$ which is circa 4.643856 so 5 bits are the optimal compression of this string.

© Klein S. T. and Wiseman Y.

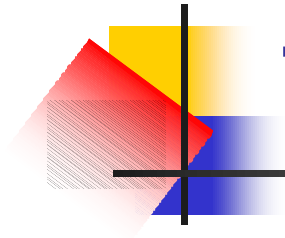


Advantages of Quasi Arithmetic

- Just two intervals (for 1 and 0) so can be implemented by tables instead of floating point divisions.
- These table can be implemented even in hardware.

$\Rightarrow$ Faster than Arithmetic coding but has the same efficiency.

© Klein S. T. and Wiseman Y.



Theorem

- The second step of the Quasi Arithmetic Coding will give the worst improvement when the 1s and the 0s have exactly the same probability - 0.5
- In other words: The entropy of a binary string will be maximal, if the 1s and the 0s have exactly the same probability - 0.5
- When the ratio between the 1s and the 0s grows, the entropy will be diminished.

© Klein S. T. and Wiseman Y.

Theorem - Proof

- The derivative of the entropy is:

$$H' = [-\sum_{i=0}^{1} P_i \log_2 P_i]' = [-P\log_2 P - (1-P)\log_2(1-P)]' =$$

$$= [\frac{-P\ln P}{\ln 2} - \frac{(1-P)\ln(1-P)}{\ln 2}]' = \frac{-P}{P\ln 2} + \frac{-\ln P}{\ln 2} - \frac{(1-P)(-1)}{\ln 2(1-P)} - \frac{-\ln(1-P)}{\ln 2} =$$

$$= \frac{-1}{\ln 2} - \log_2 P + \frac{1}{\ln 2} + \log_2(1-P) = \log_2(\frac{1-P}{P})$$

© Klein S. T. and Wiseman Y.

Theorem - Proof (Cont.)

$$\log_2\left(\frac{1-P}{P}\right) = 0$$

$$\frac{1-P}{P} = 1$$

$$1 - P = P$$

$$P = \frac{1}{2}$$

- So there is an extremum point when P=1/2.

© Klein S. T. and Wiseman Y.

Theorem - Proof (Cont.)

- The second derivative is:

$$[\log_2(1-P)-\log_2 P]'=[\frac{\ln(1-P)}{\ln 2}-\frac{\ln P}{\ln 2}]'=-\frac{1}{(1-P)\ln 2}-\frac{1}{P\ln 2}$$

- When P=1/2:

$$-\frac{2}{\ln 2}-\frac{2}{\ln 2}<0$$

- So 1/2 is a maximum point.

© Klein S. T. and Wiseman Y.

Theorem - conclusion

- When P=1 or P=0:

$$\lim_{x\to 0} H = \lim_{x\to 0} -x\cdot\log_2 x - 1\cdot\log_2 1 =$$

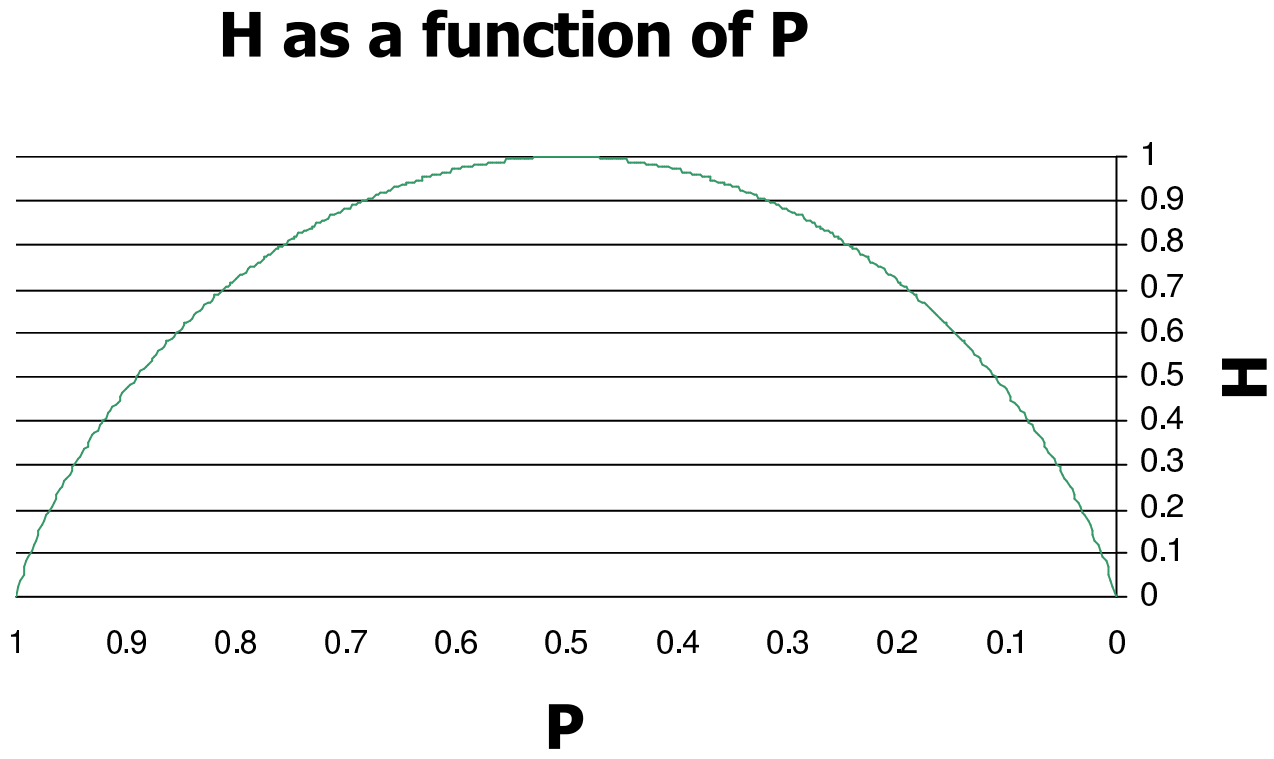
$$\lim_{x\to 0} -\log_2(x^x) = -\log_2 1 = 0$$

- When P=1/2:

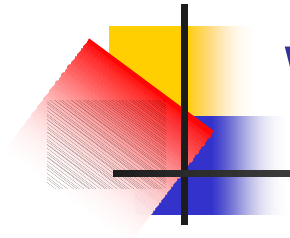
$$H = -\frac{1}{2}\log_2\frac{1}{2} - \frac{1}{2}\cdot\log_2\frac{1}{2} = 1$$

© Klein S. T. and Wiseman Y.

Conclusion in a graph



© Klein S. T. and Wiseman Y.

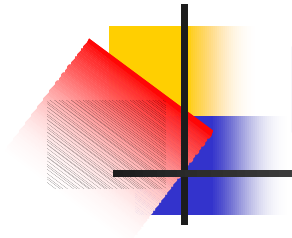


Wrong probabilities

- Excess of Huffman over Arithmetic in percents when assuming the rows to compress the columns

	English	Gadsby	German	Finnish	French
English	0.6	1.4	-2.1	-5.1	0.4
Gadsby	-2.4	1.0	-2.9	-0.8	-3.3
German	-1.8	-3.6	0.9	-5.2	-0.1
Finnish	2.4	1.2	2.7	0.8	2.9
French	0.8	2.2	-4.6	-11.3	0.9

© Klein S. T. and Wiseman Y.



Error in Arithmetic coding

- One bit in the compressed file was changed from "1" to "0":
- ***In the beginning God created the heaven and the ea*** eithen e
  tnos hheaheedh rd ehtesthhfo nh dtohrnh we
  telw .e as io
  oht o dwf esttfde evsieoeGehee mn n to ito en,nnte denntoe asipa c., meeha odftdetredd gnt nGp og rmoa oei ndf r pnfwh eias peedd kaeoweeihd.gomhi G ro m sarc t eeioorAhiytgawGtLmG r
  ddis.i tswot suea df.i. at
  i vsha nieI,yawedd D cn eAotc ndeenogidihhese
  n etdoleated oiAhettS tDeot Aee retd gctdehneds hheda tdai sdwL toheAc ohia nrv,iaicna:dde iettweah dea Gmg k.d,ai kafe rew:ehtd .truit h.ihnd enif e.dwgt hrihrm atn hSetbhhdd hdtongehie

© Klein S. T. and Wiseman Y.