Investment Externalities and Market Quotas: a Welfare Analysis under Uncertainty

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Abstract

We study the competitive equilibrium in a market in which production requires irreversible investment, profitability is stochastic, welfare-harming production externalities exist, and a cap on quantity is considered. We search for the optimal size of the cap and also analyze the welfare damages of a non-optimal cap. Our main results are: (i) If the externality is not severe enough then the uncertainty and irreversibility make firms invest at prices high enough to eliminate the externalities' adverse welfare effect. Thus, in that case no cap is needed and imposing a cap nonetheless causes a welfare loss as welfare-bearing units are not produced; (ii) the cap may create a "run" in which at a certain point in time the remaining allowed investment takes places at once. The run harms welfare as it ignites an immediate and massive production of new units at a price below the full social marginal cost. (iii) If the externality is sufficiently severe to justify a cap then the optimal cap is at the current quantity, implying that further investments should be immediately banned. This result hinges on the realistic assumptions that the market is already at its free-market equilibrium before the cap is set.

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1. Introduction

Quotas exist in a variety of markets and activities – from fishing to trade and all the way to immigration. An increasing attention has recently been devoted to their economic impact following President Trump's statements concerning the possibility of US withdrawing from international trade agreements, which in the past two decades have eliminated many import quotas (The Economist, 2018), and the need of tightening immigration quotas (The Economist, 2017).

The economic effect of quotas are becoming more interesting recently as the USA under President Trump is considering withdrawing from international trade agreement which in the past two decades have eliminated many import quotas. The Trump administration is also connected with tightening immigration quotas.

One of the main reasons why quotas are used is that economic intuition suggests that if production costs are not fully internalized by the producer, then a cap on quantity can increase social welfare. The cap does so by preventing production from reaching the range where output price only meets the marginal cost internalized by the producer and not the full marginal cost.

Despite the existence of caps in many markets, so far only a handful of studies have analyzed their effect on market equilibrium within a dynamic setting and under the realistic assumptions of stochastic market conditions and a need for irreversible investment prior to production.

To fill this void, in this article we study the competitive equilibrium in a market in which production requires prior irreversible investment, production cost is only partly internalized by the producer, and a cap on quantity is considered as a tool for limiting the welfare loss that the externality causes. We search for the optimal size of the cap, and its dependence on different parameters, and we analyze the welfare damages of a non-optimal cap.

Our analysis highlights the following three elements of the dynamic setting we use:

The first of these elements springs from one of the most well-known results of the literature on investment under uncertainty – that the stochastic nature of the profitability from the investment makes firms invest only when the output price is sufficiently above its marginal cost. Thus, despite the externality, the price that triggers investment may be above the entire marginal cost. Assume, for example, that only 80% of the full marginal cost falls on the producers, while due to uncertainty the producers invest only when the price is 50% higher than their marginal cost. In that case, multiplying 0.8 by 1.5 shows that a new unit will be produced only when the output price is 20% higher than the full marginal cost. Thus, in that market, the externality does not lead to a welfare loss, and no cap should be imposed. In this study we formally derive the conditions for this situation in which the uncertainty premium dominates the externality and a cap is unnecessary.

Another important element of a stochastic and dynamic equilibrium with a cap is that the cap may create a "run" in which at a certain point in time the remaining allowed investment takes places at once, and at a price below the full marginal cost. The emergence of this run as part of the competitive equilibrium in the presence of a cap was first derived by Bartolini (1993). As Bartolini (1993) shows, the run occurs because potential investors wish to avoid a situation in which while they have delayed their investments waiting for better profitability – their competitors have already exhausted the allowed quantity. The run harms welfare as it brings about an immediate and massive production of new units at a price below the full marginal quantity. The emergence of the run relies on the assumption that the right to invest in the market is given to all as a later study, Bartolini (1995), shows. That later study presented a model where investing and entering the market is not free to all, but requires a license. The licenses are issued at a size that matches the still allowed quantity, and thus the potential entrants know that their investment options will not be exhausted if the delay their execution as they wait for better profitability.

The third element of the dynamic environment that its role is highlighted by our analysis is the history of the market prior to the time when the cap is imposed. Under the assumption that production requires irreversible investment, it is a reasonable to further assume that some production capacity already exists when the cap is set, and is not easily removed. This imposes a lower bound on the possible size of the cap.

The dominant role that these three elements of the dynamic settings play in the market equilibrium leads our analysis to the following results:

- If the internalized part of the marginal cost is sufficiently large, then, in a free market, the quantity is raised only when the output price is above the full marginal cost and it is optimal to have no cap at all.

- The greater the uncertainty of the profitability in the market, the greater the "uncertainty premium" required by producers for investing in production ability, and the greater the plausibility of the case where no cap is required

- If in that case a cap is imposed then it harms welfare as it prevents the production of units that would have been offered at a price larger than the full marginal cost and thus add to welfare.

- If the internalized part of the marginal cost is sufficiently small, then, in a free market, the quantity is raised only when the output price is below the full marginal cost and limiting production via a cap contributes to welfare. The optimal cap in that case should be at the quantity in the market at the time when the cap is imposed, because the free conduct of the market prior to that point in time have already lead to an equilibrium with a price below the full marginal cost.

- If in that case a cap is imposed at a quantity above the current market quantity then it harms welfare as it allows the production of units that would be produced when the price is below the full marginal cost.

- Under both possibilities for the severity of the externality – a non-optimal cap may lead to a run at the still allowed quantity and harms welfare even further as the units added during the run are produced while the market price is below the full marginal cost. This additional damage to welfare by a non-optimal cap occurs only in the case were there is free entry to the market, and does not occur if licenses for producing the still allowed quantity are issued when the cap is imposed

The following table summarizes these results:

Case / Issue	**Externality dominates the uncertainty effect**		**Uncertainty effect dominates the externality**	
Optimal cap	At current quantity		None	
Welfare damages of non-optimal cap	Before the cap is reached, units are added with price below entire marginal cost		When the cap is reached, no more units are added, even when the price rises above the entire marginal cost	
	Free entry	Licensing	Free entry	Licensing
More welfare damages of non-optimal cap	A run of new units at a price below entire marginal cost	None	A run of new units at a price below entire marginal cost	None

Table 1: Summery of results

As mentioned before, so far only a handful of studies have analyzed how a cap affects the market equilibrium within a dynamic setting and under the realistic assumptions of stochastic market conditions and a need for irreversible investment prior to production. The earliest are (1993) and Bartolini (1995) mentioned above. In both articles Bartolini focuses on the equilibrium investment policy and the possible emergence of the run, and does not focus on welfare issues. In particular, in both articles he takes the cap size as exogenous and does not search for the optimal size of the cap. Our analysis of the market equilibrium with a cap, follows those of Bartolini (1993) and (1995), but differs in our search of the welfare-maximizing size of the cap, and our modeling of the welfare damages of a non-optimal cap.

The other studies related to the topic of the current article are several recent studies, with Moretto and Vergalli (2010) and Di Corato, Moretto and Vergalli (2013) being the most prominent and representative ones

Moretto and Vergalli (2010) is a theoretical study of how a cap on immigration in the host country affects the decisions of potential immigrants. Moretto and Vergalli view the immigration act as an irreversible investment and impose the standard investment under uncertainty analysis on it. The potential immigrant chooses to perform this irreversible costly act if its profitability, mostly based on labor market conditions at the host country, is sufficiently large. If the host country wishes to limit immigration via a cap – the Bartolini dynamic pattern emerges with a cap-attack at some point in time. In addition to applying these dynamics in the immigration context, Moretto and Vergalli show that the government can delay the cap-attack by creating uncertainty about the size of the cap.

Di Corato, Moretto and Vergalli (2013) apply the Bartolini analysis to the case of transforming forest land into agricultural land. Forest land generates welfare that the owners cannot fully charge for it, and mostly not for the utility derived from the beauty of the forested environment. Thus, the social loss of forest land is only partially internalized by the land owners when they convert their land to agriculture. This motivates a cap on the allowed amount of agricultural land. Yet, as Di Corato, Moretto and Vergalli show, the cap may create a cap-attack, which speeds-up the socially undesired land conversion. In addition to describing these dynamics, they also focus on conditions for land owners to voluntarily participate in government program meant to protect forestry, and also on the long-run average rate of investment in agricultural land.

These articles, as well as several related ones, do refer to policy-makers' welfare aims in setting the cap, and even point at how the resulting cap-attack harms welfare. Yet – they do not explicitly model welfare and therefore do not search for the welfare-maximizing size of the cap.

The article is organized as follows. Section 2 presents the basic model without any policy intervention, and portrays the equilibrium firms' policy and the resulting welfare n that case. Section 3 presents the case where a cap is imposed in the market under free entry, presents the optimal investment policy, including the possibility of a run, and also presents an analysis of the welfare and cap optimality in that case. Section 4 studies the case where a cap is imposed with licenses for potential production of the still allowed quantity. Section 5 compares the results under the two regimes for the cap – free entry and licensing. Section 6 adds some concluding results. Some of the more technical proofs were relegated to appendices.

2. **The basic model**

Within a continuous time setting, consider a market for a perfectly durable good, named A, that at each point in time its demand is given by:

(1) $$P_t = \frac{X_t}{Q_t},$$

where Q_t and P_t are, respectively, the quantity and the price of good A at time t. Demand changes stochastically over time according the swings in the process X_t. All producers face the same cost structure where supplying the quantity q_t at time t entails the instantaneous total cost to society as a whole:

(2) $$STC(q) = M \cdot q ,$$

where M is constant. Part of this cost is an externality that the producers do not incur, and the instantaneous total cost of a producer that supplies the quantity q is:

(3) $$TC(q) = \lambda \cdot M \cdot q ,$$

where $0 < \lambda < 1$.

Due to the perfect durability, the quantity Q_t is a stock and producing an additional unit is an investment that is based on the expected discounted flow of profits from that unit. This flow is a stochastic process due to stochastic nature of X_t. More specifically, X_t is the following Geometric Brownian Motion:

(4) $$dX_t = \mu \cdot X_t \cdot dt + \sigma \cdot X_t \cdot dZ_t ,$$

where μ and σ are constants which measure, respectively, the drift and the variance of X_t, and dZ_t is the increment of the standard Wiener process satisfying at each instant:

(5) $$E(dZ_t) = 0, \quad E(dZ_t)^2 = 1 .$$

By properties of the Geometric Brownian Motion, at time intervals when Q_t is unchanged, P_t is also a Geometric Brownian Motion with the same parameters as those of X_t. The interest rate, denoted r, is constant over time. Convergence of the value of owning a unit of the good requires that the expected rate of growth of X_t does not exceed the discount rate, i.e., that $r > \mu$.

There is free entry to this market with an infinite amount of potential investors. Yet, the investment, i.e. producing a new unit, commits the producer to permanently offer it and therefore to an infinite flow of the cost $\lambda \cdot M$. The discounted present value of this flow is $\frac{\lambda \cdot M}{r}$ and it can be viewed as an irreversible investment cost.

2.1 Optimal investment in the absence of a cap

Under the setup described above, the potential investor in this model is facing the same situation as the investors in Leahy (1993). In this sub-section we use Leahy's analysis to present the potential investors' optimal investment policy.

At each instant, each potential investor has to decide whether to produce and supply a new increment of good A, or not. The decision depends on the expected profitability of this investment, and therefore takes place only when X_t is sufficiently large, where $X^*(Q)$ denotes the investment threshold and presents it as a function of the current quantity in the market. A larger level of Q implies, ceteris paribus, lower profitability, so the threshold $X^*(Q)$ is an increasing function of Q.

Let $V(Q, B)$ be the value of owning such an incremental parcel of good A. The standard no-arbitrage analysis in Appendix A shows that

(6) $$V(Q, X) = Y(Q) \cdot X^{\beta} + \frac{X}{Q \cdot (r - \mu)} - \frac{\lambda \cdot M}{r}$$

where β is the positive root of the quadratic:

(7) $$\tfrac{1}{2} \cdot \sigma^2 \cdot x^2 + \left(\mu - \tfrac{1}{2} \cdot \sigma^2\right) \cdot x - r = 0 .$$

Additional boundary conditions are required for finding $Y(Q)$ and the threshold function $X^*(Q_t)$. The first one is the following *Value Matching Condition*:

(8) $$V\left[Q, X^*(Q)\right] = 0 .$$

The second one is the following *Smooth Pasting Condition*:

(9) $$V_X\left[Q, X^*(Q)\right] = 0 .$$

Applying (6) in (8) and (9) yields:

(10) $$X^*(Q) = \overline{\beta} \cdot (r - \mu) \cdot \lambda \cdot \frac{M}{r} \cdot Q ,$$

where $\overline{\beta} \equiv \frac{\beta}{\beta - 1}$. Note that $\overline{\beta} > 1$ since $\beta > 1$.

2.2 Welfare in the absence of a cap

Following the same procedure as that conducted for the value of a unit of good A, yields that given the current levels of *X* and *Q* the value of social welfare satisfies:

$$(11) \qquad W(Q, X) = C(Q) \cdot X^{\beta} + \frac{X \cdot \ln(Q)}{r - \mu} - \frac{M \cdot Q}{r},$$

where $C(Q)$ is to be determined by boundary conditions. The first such condition is the following *Value Matching Condition* at times of hitting the investment threshold:

$$(12) \qquad W_Q\left[Q, B^*(Q)\right] = 0$$

Applying (11) in (12), partially differentiating with respect to *Q*, applying (10), and rearranging terms, yields:

$$(13) \qquad C'(Q) = K \cdot \frac{1 - \overline{\beta} \cdot \lambda}{Q^{\beta}},$$

where:

$$(14) \qquad K \equiv \frac{1}{\left(\frac{M}{r}\right)^{\beta - 1} \cdot \lambda^{\beta} \cdot \overline{\beta}^{\beta} \cdot (r - \mu)^{\beta}} > 0.$$

Integrating (13) yields:

(15) $$C(Q)=\frac{K}{\beta-1}\cdot\frac{\overline{\beta}\cdot\lambda-1}{Q^{\beta-1}}+G,$$

To find the value of the integration constant, *G*, note from (10) that when $Q\to\infty$ the threshold $X^*(Q)$ goes to infinity too, and the probability of *X* hitting it goes to 0. In that case no further changes in *Q* are expected, and therefore the value of the possibility of such changes is zero. Formally put:

(16) $$\underset{Q\to\infty}{Lim}\, C(Q)=0.$$

Since $\beta > 1$, (16) and (15) imply that $G = 0$, and therefore:

(17) $$C(Q)=\frac{K}{\beta-1}\cdot\frac{\overline{\beta}\cdot\lambda-1}{Q^{\beta-1}}.$$

Note from (17) that if, and only if, λ is sufficiently small, specifically – below $\frac{1}{\overline{\beta}}$, then $C(Q)<0$. This implies that if, and only if, the market imperfection is sufficiently strong then the value of the possibility of further investments in *Q* is negative.

To have a better insight into the role that $\overline{\beta}\cdot\lambda-1$ plays, it is convenient to look at the case of $\mu = 0$, i.e., the case where the dynamics in *B* are purely stochastic, as there is no deterministic drift. In that case the optimal investment rule of investing when $X\geq X^*(Q)$ which can be presented, by applying (1) and (10) as $P\geq\overline{\beta}\cdot\lambda\cdot M$. Thus, if $\overline{\beta} \cdot \lambda - 1 > 0$, then investment takes place when $P \geq M$ and therefore increase welfare. It can be concluded then that the market imperfection lowers the price that triggers producing additional unit below the total social cost, but investors' reaction to uncertainty raises it back above the total social cost.[1]

3. The model with a cap and free entry

The possibility that $C(Q)$ may be negative could lead policy makers to limit future investments with a cap on the level of *Q*. We denote the size of the cap by $\overline{Q}$. The analysis in this case follows Bartolini (1993). Similar to the analysis conducted in sub-section 2.1 above for the case with no cap on *Q*, the analysis in this case too starts with the definition of $V(Q, X)$ as the value of owning a unit of *Q* and leads to the functional form given by (6). Then, to find $Y(Q)$ and the threshold function $X^*(Q)$, Bartolini too uses the *Value Matching Condition* (8). From here on the analysis for the case of a cap departs from that conducted in sub-section 2.1 as the other boundary condition that Bartolini uses is:

(18) $$V_Q\left[Q, X^*(Q)\right] = 0$$

Bartolini (1993) proves the existence of condition (18) in *Proposition 1* of his article. As he shows there, the condition springs from:

(19) $$V\left[Q, X^*(Q+\Delta Q)\right] = V\left[Q+\Delta Q, X^*(Q+\Delta Q)\right].$$

[1]For a similar explanation for the case where $\mu \neq 0$ it is helpful to use the manner by which Kongstead (1996) separates from one another the effects that the drift component and the uncertainty component have on the optimal investment thresholds.

The condition shows that when the quantity is Q and X passes its corresponding threshold level, then, by definition of X^* as a threshold level, Q is increased by another increment with probability 1. This probability, together with the no-arbitrage condition, equates the value function between the two states. Dividing both sides by ΔQ and taking the limit $\Delta Q \to 0$ leads to (18). Note that (8) and (18) are not optimality conditions and should hold for any $X^*(Q)$, not necessarily the optimal one, as they merely reflect the no-arbitrage condition on the value of the firm, given a certain threshold. This means that (18) holds for all levels of Q, which implies that the derivatives with respect to Q of both its sides should equal one another, i.e.:

(20) $$\frac{dV\left[Q, X^*(Q)\right]}{dQ} = 0\,.$$

Expanding (20) and applying (18) in it yield the condition:

(21) $$V_X\left[Q, X^*(Q)\right] \cdot \frac{dX^*(Q)}{dQ} = 0$$

For (21) to hold it requires either $V_X\left[Q, X^*(Q)\right] = 0$ or $\frac{dX^*(Q)}{dQ} = 0$. In the former case – the *Smooth Pasting Condition* (9) holds, and the threshold $X^*(Q)$ is given by (10), as in the case of no cap.

In the latter case, the *Smooth Pasting Condition* (9) and the resulting (10) do not hold. To further understand this case, recall that $Y(Q)$ represents how the changes in Q over time are expected to affect the value of an increment of good A. At the upper limit of Q no such changes can happen and the value of this possibility is 0. Thus:

(22) $$Y(\overline{Q}) = 0 .$$

Applying (22) and (6) and in (8) yields that when Q is at its cap, $\overline{Q}$, the investment threshold is:

(23) $$\overline{X} = (r - \mu) \cdot \lambda \cdot \frac{M}{r} \cdot \overline{Q}$$

Thus, Smooth pasting does not hold in $\overline{Q}$ and, by continuity, also not within a sufficiently close vicinity of $\overline{Q}$. This vicinity is $\left[\tilde{Q}, \overline{Q}\right]$ where $\tilde{Q}$ satisfies:

(24) $$X^*(\tilde{Q}) = \overline{X} ,$$

due to $\frac{dX^*(Q)}{dQ} = 0$. Applying (10) and (23) in (24) yields:

(25) $$\tilde{Q} = \frac{1}{\beta} \cdot \overline{Q} .$$

To summarize the resulting investment dynamics:

- As long as $Q < \tilde{Q}$, when X hits the threshold $X^*(Q)$ investment occurs. The rising Q makes $X^*(Q)$ rise too, so that X is once again below its threshold and investment stops, until the next time X hits the threshold. In *Figure 1* below this is described by the move from point E to point F.

- If $Q = \tilde{Q}$, then when X hits the threshold $X^*(Q)$ investment occurs, but in this case the threshold is not increased by the rising Q as $\frac{dX^*(Q)}{dQ} = 0$. Thus, X is still at the threshold and investment continues and Q immediately hits its cap. In *Figure 1* below this is described by the move from point G to point H.

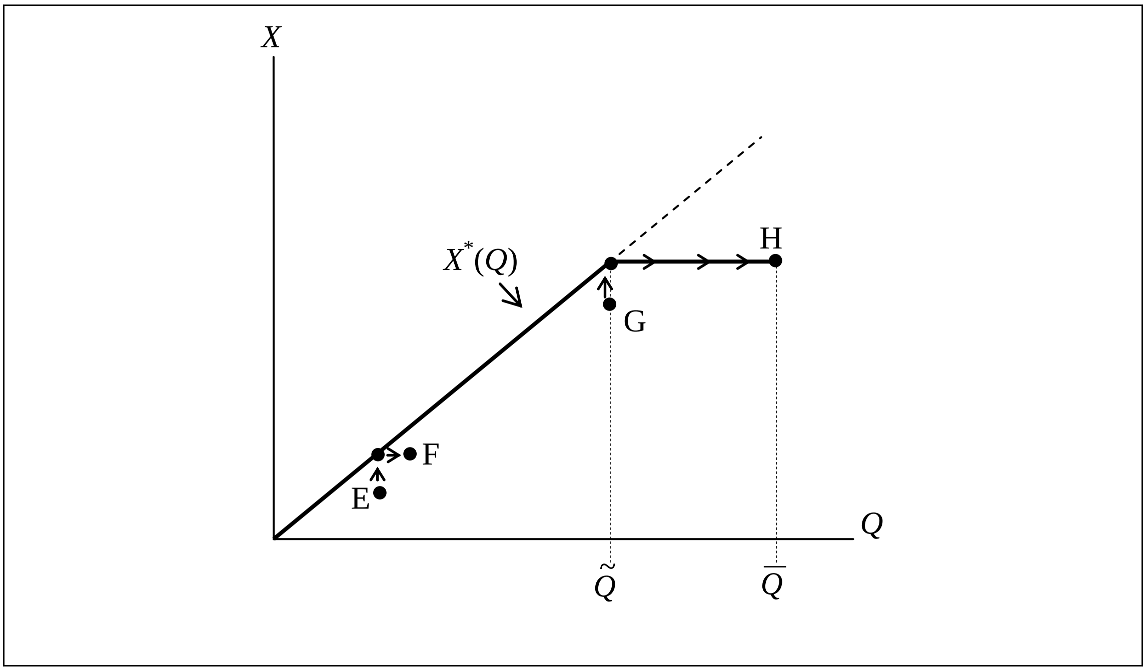

Figure 1: Investment dynamics.

3.1 Welfare with a cap on *Q*

Following the analysis in previous sections, welfare in the case of a cap is given by:

(26) $$W(Q, X, \overline{Q}) = C(Q, \overline{Q}) \cdot X^{\beta} + \frac{X \cdot \ln(Q)}{r-\mu} - \frac{M \cdot Q}{r}.$$

Notice that (26) is almost identical to the welfare function given by (15) in the case with no cap, with the main difference being that $W(Q, X, \overline{Q})$ and $C(Q, \overline{Q})$ are also functions of the size of the cap, $\overline{Q}$, and not merely functions of *Q* and *X*.

Clearly from (26) a welfare maximizing choice of $\overline{Q}$ is a choice that maximizes $C(Q, \overline{Q})$, as the other elements of $W(Q, X, \overline{Q})$ do not contain $\overline{Q}$. To find this optimal level of $\overline{Q}$, two different ranges have to be examined:

- If $\overline{Q}$ is set sufficiently close to the current level of *Q*, specifically – within the range $Q \leq \overline{Q} \leq \overline{\beta} \cdot Q$, then *Q* is above $\tilde{Q}$ and the next change in *Q* is a run that will be ignited when *X* will hit the threshold $\overline{X}$ given by (23).

- If $\overline{Q}$ is set sufficiently far from the current level of *Q*, specifically – in the range $\overline{Q} > \overline{\beta} \cdot Q$, then *Q* is below $\tilde{Q}$ and the next changes in *Q* (as long as $Q < \tilde{Q}$) are according to the incremental and gradual process described in the previous section.

3.1.1 $C(Q, \overline{Q})$ when $Q \leq \overline{Q} \leq \overline{\beta} \cdot Q$

In the first range, $Q \leq \overline{Q} \leq \overline{\beta} \cdot Q$, the function $C(Q, \overline{Q})$ is found by the condition:

(27) $$W(Q, \overline{X}, \overline{Q}) = \frac{\overline{X} \cdot \ln(\overline{Q})}{r-\mu} - \frac{M \cdot \overline{Q}}{r},$$

which states that in that range Q is above $\tilde{Q}$ and if the threshold $\overline{X}$ is hit then a run brings the quantity to the cap at once, no more changes in Q shall take place, and welfare is therefore the present value of the welfare flow based on $\overline{Q}$.

Evaluating (26) at $\overline{X}$, comparing it to (27), applying (23) and simplifying, yields that in the range $Q \le \overline{Q} \le \overline{\beta} \cdot Q$ the function $C(Q, \overline{Q})$ is given by:

$$(28) \qquad C(Q, \overline{Q}) = \overline{\beta}^{\beta} \cdot K \cdot \frac{\lambda \cdot \overline{Q} \cdot \left[\ln(\overline{Q}) - \ln(Q)\right] - (\overline{Q} - Q)}{\overline{Q}^{\beta}}.$$

Note from (28) that if the cap is set at its lowest possible level, i.e., at the current level of Q, then $C(Q, \overline{Q})$ = 0. This reflects the fact that the first term in the welfare function, $C(Q, \overline{Q}) \cdot \overline{X}^{\beta}$ shows the welfare value of future changes in Q, and no such changes will take place if the cap is imposed at the current level of Q.

Proposition 1: Within the range $Q \le \overline{Q} \le \overline{\beta} \cdot Q$, the function $C(Q, \overline{Q})$, as given by (28), is a u-shape function.

Proof: The proof of this proposition is in appendix B. □

3.1.2 $C(Q, \overline{Q})$ when $\overline{Q} > \overline{\beta} \cdot Q$

We now turn to the case where $\overline{Q}$ is set in the range $\overline{Q} > \overline{\beta} \cdot Q$. The point where the two ranges connect is at $Q = \tilde{Q} = \frac{1}{\overline{\beta}} \cdot \overline{Q}$, and applying this value of Q in (28) yields:

(29) $$C\left(\tilde{Q}, \overline{Q}\right) = \overline{\beta}^{\beta} \cdot K \cdot \frac{\ln\left(\overline{\beta}\right) \cdot \lambda - \frac{1}{\beta}}{\overline{Q}^{\beta}} \cdot \overline{Q} .$$

In the internal points of the range $\overline{Q} > \overline{\beta} \cdot Q$, the initial value of Q is below $\tilde{Q}$ and therefore the changes in Q are the incremental additions when X hits its threshold function $X^{*}(Q)$, as long as Q is still below $\tilde{Q}$. The analysis of welfare in this case is therefore similar to the one presented in section 2.2 for the case of no cap and leads once again to (15), only with $C\left(Q, \overline{Q}\right)$ at its LHS, rather than $C(Q)$.

In the case of no cap the integration constant, G, was found using the boundary condition (16) which was based on properties of $C(Q)$ when Q goes to infinity. In the case of a cap, Q is bounded at the cap and therefore (27), evaluated at $Q = \tilde{Q}$, provides us the necessary boundary condition. Specifically, evaluating $C\left(Q, \overline{Q}\right)$ at $Q = \tilde{Q} = \frac{1}{\overline{\beta}} \cdot \overline{Q}$ in (19), equating it to the LHS of (29), and simplifying, yields:

(30) $$C\left(Q, \overline{Q}\right) = \frac{K}{\beta - 1} \cdot \left[\frac{\overline{\beta} \cdot \lambda - 1}{Q^{\beta - 1}} - \frac{g(\beta) \cdot \lambda \cdot \overline{\beta}^{\beta}}{\overline{Q}^{\beta - 1}} \right].$$

where:

$$g(\beta) \equiv 1 - (\beta - 1) \cdot \ln\left(\overline{\beta}\right).$$

The following *Proposition 2* shows properties of $C\left(Q, \overline{Q}\right)$ in the range $\overline{Q} > \overline{\beta} \cdot Q$:

Proposition 2: In the range $\overline{Q} > \overline{\beta} \cdot Q$, the function $C(Q, \overline{Q})$ satisfies the following:

(a) $C_{\overline{Q}}(Q, \overline{Q}) > 0$

(b) $\underset{\overline{Q} \to \infty}{Lim}\, C(Q, \overline{Q}) > 0$ if and only if $\lambda > \dfrac{1}{\overline{\beta}}$.

Proof: Appendix D shows that $g(\beta) > 0$ within the range $\beta > 1$.[2] This, together with (30) leads immediately to (a) and (b). □

3.1.3 $C(Q, \overline{Q})$ throughout it entire definition range

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Bringing together the analyses of $C(Q, \overline{Q})$ in each of the two parts of its definition range, *Figure 3* summarizes the properties of this function. Part a of the figure refer to the case where $\lambda > \frac{1}{\overline{\beta}}$ and part b refers to the case where $\lambda < \frac{1}{\overline{\beta}}$.

Both parts of the figure show that $C(Q, \overline{Q})$ equals 0 at the left end of its definition range, then falls to negative values as $\overline{Q}$ rises, hits a minimum point within the range $Q \le \overline{Q} \le \overline{\beta} \cdot Q$ and rises with $\overline{Q}$ from then on. Part (a) of the figure shows that if $\lambda > \frac{1}{\overline{\beta}}$ then at the part in which it is rising in $\overline{Q}$ the function $C(Q, \overline{Q})$ becomes positive. Part (b) of the figure shows that if $\lambda < \frac{1}{\overline{\beta}}$ then even in the part in which it is rising, $C(Q, \overline{Q})$ remains negative throughout its definition range.

[2] $g(\beta)$ is a function of β alone, with no free parameters. Therefore, it is possible to see that $g(\beta)$>0 throughout the range β > 1 numerically, by plotting this function. Readers interested in an analytical proof will find it in Appendix D.

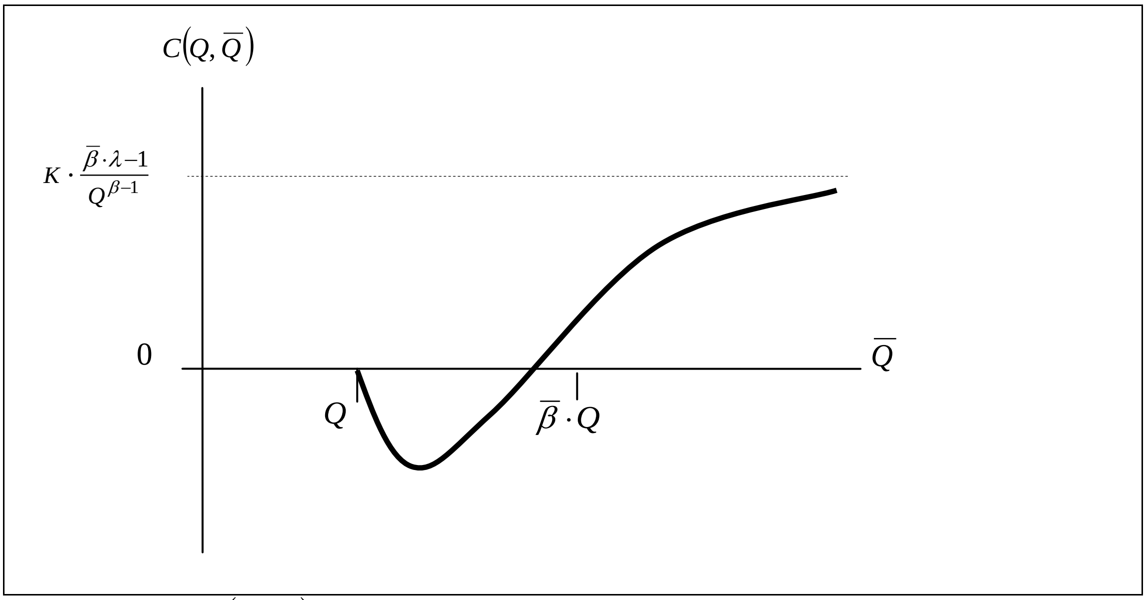

Figure 3 (a): $C(Q,\overline{Q})$ **when** $\lambda > \frac{1}{\overline{\beta}}$**.**

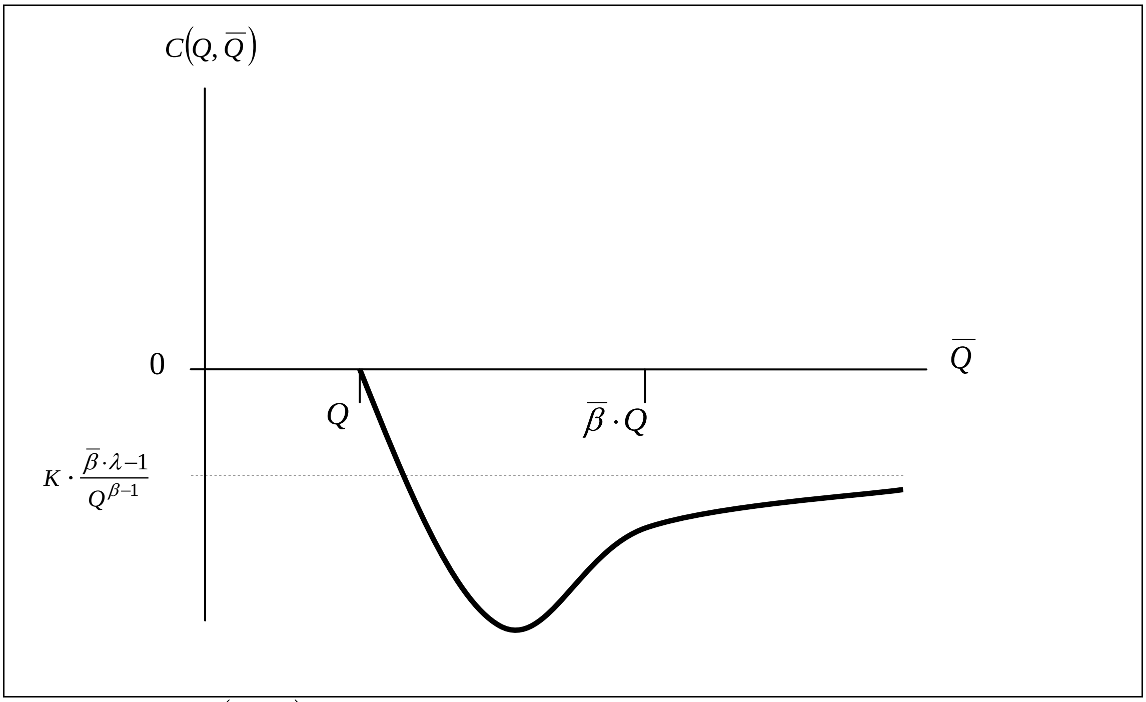

Figure 3(b): $C(Q,\overline{Q})$ **when** $\lambda < \frac{1}{\overline{\beta}}$**.**

The following proposition presents the resulting two possibilities for a welfare maximizing choice of the cap $\overline{Q}$:

Proposition 3:

- If $\lambda < \frac{1}{\beta}$ then the cap should be set at the current level of Q, which implies allowing no further changes in Q from now on.

- If $\lambda > \frac{1}{\beta}$ then it is optimal to push the cap to infinity, which actually implies to have no cap at all.

Proof: The proof is immediate from the analysis in section 3.1.3 □

4. Optimal cap with rationing

In this subsection we look at the case where entry licenses are distributed when the cap is announced. Each license is for an infinitesimally small increment of Q, and their quantity covers exactly the gap between the current quantity in the market and the cap. Bartolini (1995) has introduced this case and analyzed the firms' optimal policy and the social welfare in the resulting equilibrium. Our analysis in this section follows Bartolini (1995) and adds to it a derivation of the optimal size of the cap.

Following Bartolini (1995), we avoid the question of how the licenses were distributed, whether by auction, lottery, or any other way. We merely assume that the distribution act itself has no other implications except for providing each license owner with a right to invest at any time it wishes to.

As Bartolini (1995) shows, the competitive run of the free entry case does not emerge in the equilibrium of the licensing case. This happens because licenses owners do not have to fear that due to the cap their investment option will vanish.

Absent the run, the analysis of the firm's optimal policy under licensing is very similar to the analysis in section 2.1 for the case of no cap at all. The only difference between the cases is that under licensing the option to invest is an asset that the firm gives up when investing. Thus, alongside the function $V(Q, X)$ which shows the value of what the investment yields, we also define the function $F(Q, X)$ which shows the value of the option to make the investment at the optimal time. In the case of free entry analyzed in previous sections, this option was worthless because of free entry, which implied $F(Q, X) \equiv 0$. In the current case, a no-arbitrage analysis similar to one carried out in sub-section 2.1 yields that:

$$(31) \qquad F(Q, X) = H(Q) \cdot X^{\beta},$$

where $H(Q)$ is to be found by boundary conditions. The first one is the following *Value Matching Condition*:

$$(32) \qquad V\left[Q, X^{*}(Q)\right] = F\left[Q, X^{*}(Q)\right].$$

The second one is the following *Smooth Pasting Condition*:

$$(33) \qquad V_X\left[Q, X^{*}(Q)\right] = F_X\left[Q, X^{*}(Q)\right].$$

Note that conditions (8) and (9) from section 2.1 are specific cases, with $F(Q, X) \equiv 0$ of conditions (32) and (33).

Despite this small difference from the analysis in section 2.1, applying (6) and (31) in (32) and (33) yields the same threshold function, $X^*(Q)$, given by (10).

Alongside $X^*(Q)$, the solution of this system yields an expression for $H(Q)-Y(Q)$ from which $F(Q,X)$ is found, after $V(Q,X)$ is found using (10), (18) and (22).

As in the case of a cap under free entry, welfare is given by (26) and the analysis is similar to that from section 2.2, up until (15) is obtained, with $C(Q,\overline{Q})$ at its LHS. Then the integration constant is found using the following boundary condition:

(34) $$C(\overline{Q},\overline{Q})=0,$$

which states that when Q is at $\overline{Q}$ no more changes in Q are going to take place, and therefore $C(\overline{Q},\overline{Q})\cdot X^{\beta}$, which shows the value of such changes within the welfare function (26), should equal zero.

Applying (15) in (34), extracting the integration coefficient G, applying it in (15) and simplifying, yields:

(35) $$C(Q,\overline{Q})=\frac{K}{\beta-1}\cdot(\overline{\beta}\cdot\lambda-1)\cdot\left(\frac{1}{Q^{\beta-1}}-\frac{1}{\overline{Q}^{\beta-1}}\right).$$

From (35) it follows that $C(Q,\overline{Q})$, and therefore welfare, is a monotonic function of $\overline{Q}$ with the monotonicity type depending on the sign of $\overline{\beta}\cdot\lambda-1$. Specifically, if $\overline{\beta}\cdot\lambda>1$, then welfare rises in $\overline{Q}$ and it is optimal therefore to push $\overline{Q}$ to infinity, i.e., to have no cap at all. Otherwise, if $\overline{\beta} \cdot \lambda < 1$, then welfare decreases in $\overline{Q}$ and it is optimal therefore to set the cap at the lowest possible level which is the current level of Q. Both parts of figure 1 present $C(Q, \overline{Q})$ as a function of $\overline{Q}$.

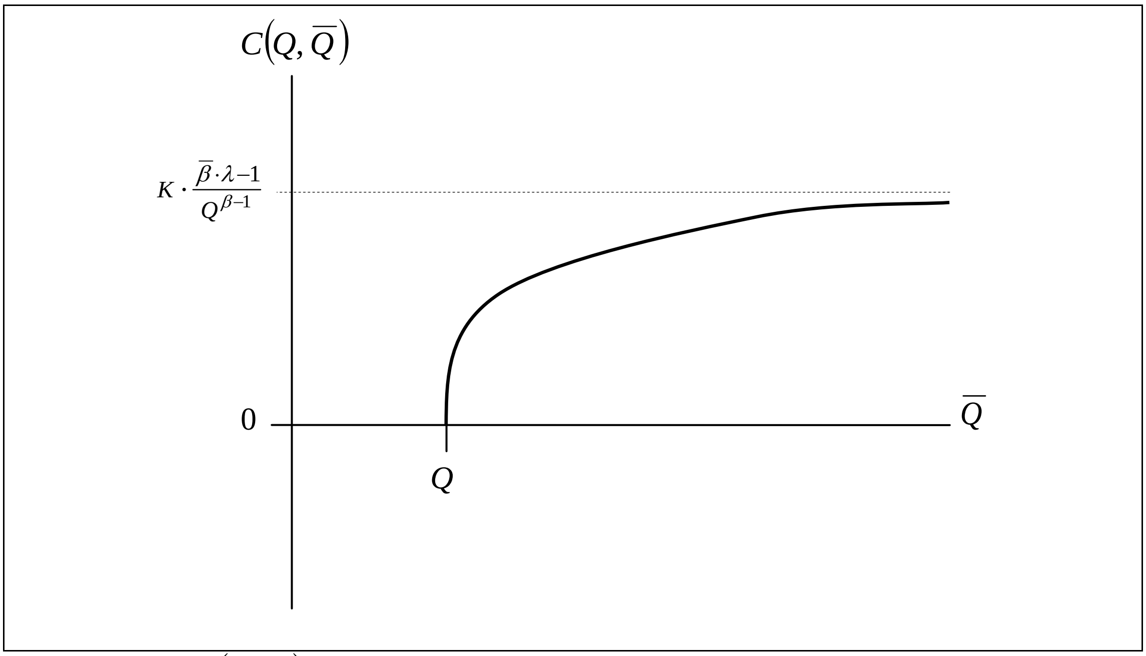

Figure 4(a): $C(Q, \overline{Q})$ **when** $\lambda > \frac{1}{\overline{\beta}}$**.**

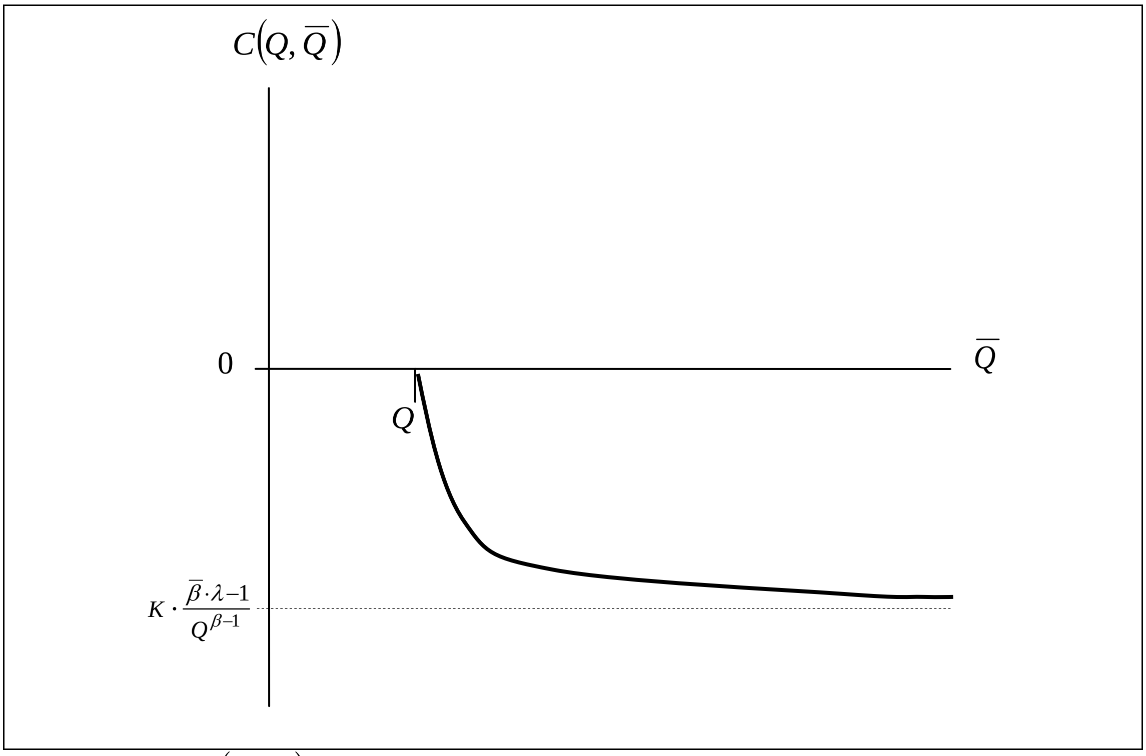

Figure 4(b): $C(Q, \overline{Q})$ **when** $\lambda < \frac{1}{\overline{\beta}}$**.**

As the figure shows, in both cases $C(Q,\overline{Q})=0$ when $\overline{Q}$ is set at the current level of Q, and in both cases $C(Q,\overline{Q})$ converges to $K\cdot\frac{\overline{\beta}\cdot\lambda-1}{Q^{\beta-1}}$ as $\overline{Q}$ goes to infinity. The difference between the two cases is about this limit being positive or negative and about $C(Q,\overline{Q})$ rising or falling towards it.

4.1 Comparing optimal cap equilibrium under free entry and under licensing

In both cases, free entry and rationing, the same result has emerged: if $\lambda\cdot\overline{\beta}<1$ then it is optimal to set the cap at the current level of Q and ban any further production; if $\lambda\cdot\overline{\beta}>1$ then it is optimal to have no cap at all.

What comparing $\lambda\cdot\overline{\beta}$ to 1 determines is whether when investment takes place the price exceeds the entire production cost (and not merely the cost the producer faces) or not. If it does, then despite the externality, production of additional units is done only when it adds to welfare and therefore should not be banned. Otherwise, if $\lambda\cdot\overline{\beta}<1$, then investments take place when the price is below the entire production cost and therefore lowers welfare and should be banned altogether.

This reasoning is not based on the nature of the cap regime and therefore under both regimes the same result about the optimal cap emerges.

The two cases differ in the welfare damages that a non-optimal cap causes. We now describe these damages and start with the case where $\lambda\cdot\overline{\beta}>1$.

When $\lambda \cdot \overline{\beta} > 1$, additional units contribute to welfare and therefore $C(Q, \overline{Q})$, which represents the welfare value of future change in Q, is positive. The cap harms welfare because it prevents the production of more of these welfare-promoting units. If this is the only harm that the cap causes to welfare, as in the rationing case, then, as figure 4a shows, raising the cap also raises $C(Q, \overline{Q})$ and welfare.

Figure 3a shows that under free entry the cap harms welfare in an additional manner – it ignites a run under which new units are added to the market at a price below the entire marginal cost and at high speed. The closer the cap to the already existing quantity, the the sooner the expected time of the run. therefore, at sufficiently small level of the cap, the possibility of producing more units harms welfare and $C(Q, \overline{Q})$ falls below 0. Only if the cap is sufficiently far from the initial quantity, then the production of units with price above the entire production cost is the dominant effect and $C(Q, \overline{Q})$ rises with the size of the cap, as in the rationing case.

The same logic applies in the case where $\lambda \cdot \overline{\beta} < 1$ too. In this case units are added to the market when the price does not exceed the entire production cost and therefore each such unit harms welfare. If this is the only harm that the cap inflicts on welfare, as in the rationing case, then, as figure 4a shows, $C(Q, \overline{Q})$ is negative and falls as the cap is raised and the production of more of these welfare-harming units is allowed.

Under free entry the cap also harms welfare with the run in brings about, in which units are added to the market at a price below the entire marginal cost. As the cap is raised the run is becoming larger but also expected later, and these competing effects creates the u-shape effect of the run which figure 3a shows.

Proposition 4: Welfare under licensing is larger than welfare under free entry for each level of $\overline{Q}$ that exceeds the current level of Q.

Proof: See Appendix C. □

5. Conclusion

In this article, we consider a market for a durable good in which producers pay only part of the marginal production cost and let the other part fall on society. We then study the opportunity of introducing a cap on the aggregate market quantity in order to limit the welfare losses associated with this externality. We consider its introduction under a scenario where firms may freely enter the market and also under a scenario where the right to enter the market is rationed by distributing licenses when the cap is announced. We then determine at which level a welfare maximizing cap should be set and find that, irrespective of the way by which the right to enter is allocated among firms, the planner should either ban further market entries or have no cap at all. The decision will merely depend on the magnitude of the externality with respect to the expected net surplus associated with a higher market quantity. If the welfare loss associated with the externality is higher than the expected net surplus of having additional units of the considered good, then market entries should be banned, otherwise, as a higher market quantity is beneficial, no cap should be imposed. This result is relevant in that it implies that the justification of a cap on the aggregate market quantity based on social welfare considerations is not plausible. This means that the introduction of caps in the reality results from the consideration of objectives other than the actual social welfare mirroring, for instance, the opportunism of political parties in the office aiming at maximizing the chances of conserving power by favoring particular parts of the society. In this respect, our model shed light on the cost for society as a whole of certain choices.

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Appendix A

In this appendix we show that equation (6) presents the general form of the function $V(Q, X)$. For that, we use the standard no-arbitrage analysis of the literature of investment under uncertainty, with Dixit 91989) and Leahy (1993) as representative examples for it. We start with the following no-arbitrage condition states that the instantaneous profit, $\frac{X}{Q} - \lambda \cdot M$, along with the expected instantaneous capital gain from a change in X, must equal the instantaneous normal return:

(A.1) $$r \cdot V(Q, X) \cdot dt = \frac{X}{Q} - \lambda \cdot M + E[dV(Q, X)].$$

By Ito's lemma:

(A.2) $$E[dV(Q, X)] = \tfrac{1}{2} \cdot \sigma^2 \cdot B^2 \cdot V_{XX}(Q, X) + \mu \cdot X \cdot V_X(Q, X)$$

Applying (A.2) in (A.1) yields:

(A.3) $$\tfrac{1}{2} \cdot \sigma^2 \cdot X^2 \cdot V_{XX}(Q, X) + \mu \cdot X \cdot V_X(Q, X) - r \cdot V(Q, X) + \frac{X}{Q} - \lambda \cdot M = 0$$

Trying a solution of the type X^b for the homogenous part of this differential equation and a linear form as a particular solution to the entire equation, yields:

(A.4) $$V(Q, X) = Z(Q) \cdot X^{\alpha} + Y(Q) \cdot X^{\beta} + \frac{X}{Q \cdot (r - \mu)} - \frac{\lambda \cdot M}{r},$$

where α and β are the roots of the quadratic:

(A.5) $$\tfrac{1}{2} \cdot \sigma^2 \cdot x^2 + \left(\mu - \tfrac{1}{2} \cdot \sigma^2\right) \cdot x - r = 0.$$

The assumption that $r > \mu$ asserts that $\beta > 1$ and $\alpha < 0$.

$\frac{X}{Q \cdot (r - \mu)} - \frac{\lambda \cdot M}{r}$ describes the expected extra value this unit generates if Q remains forever in its current level. The two other elements of the RHS of (A.4) represent therefore how the changes in Q over time are expected to affect the value of the unit.

By properties of the Geometric Brownian Motion, when X goes to 0 the probability of it ever rising to $X^*(Q_t)$, and Q consequently changing, approaches 0. Thus implies:

(A.6) $$\lim_{B \to \infty} \left[Z(Q) \cdot X^{\alpha} + Y(Q) \cdot X^{\beta} \right] = 0,$$

which leads to $Z(Q) = 0$, since $\alpha < 0$, and therefore the function given by (6).

Appendix B

In this appendix we prove *Proposition 1* which states that in the range $Q \le \overline{Q} \le \overline{\beta} \cdot Q$ the function $C(Q, \overline{Q})$, as given by (28), is a u-shape function of $\overline{Q}$.

By (28), in the range $Q \le \overline{Q} \le \overline{\beta} \cdot Q$:

(B.1) $$C_{\overline{Q}}\left(Q, \overline{Q}\right) = K \cdot \frac{f\left(\overline{Q}\right)}{\overline{Q}^{\beta+1}}$$

where:

(B.2) $$f\left(\overline{Q}\right) \equiv -(\beta - 1) \cdot \lambda \cdot \overline{Q} \cdot \left[\ln\left(\overline{Q}\right) - \ln(Q)\right] + (\lambda - 1 + \beta) \cdot \overline{Q} - Q \cdot \beta .$$

From (B.1) it follows that the sign of $C_{\overline{Q}}\left(Q, \overline{Q}\right)$ is the sign of $f\left(\overline{Q}\right)$. Therefore, it is sufficient to prove that $f\left(\overline{Q}\right)$ monotonically rises from negative to positive values within the range $Q \le \overline{Q} \le \overline{\beta} \cdot Q$ in order to prove the same for $C_{\overline{Q}}\left(Q, \overline{Q}\right)$ and thus to prove that $C(Q, \overline{Q})$ is a u-shape function of $\overline{Q}$.

At the left end of this range of values for $\overline{Q}$, i.e., when $\overline{Q}$ equals *Q*:

(B.3) $$f(Q) = (\lambda - 1) \cdot Q < 0 ,$$

where the inequality follows from $\lambda < 1$.

At the right end of this range of values for $\overline{Q}$, i.e., when $\overline{Q}$ equals $\overline{\beta} \cdot Q$:

(B.4) $$f\left(\overline{\beta} \cdot Q\right) = g(\beta) \cdot \lambda \cdot \overline{\beta} \cdot Q > 0,$$

where $g(\beta) \equiv 1 - (\beta - 1) \cdot \ln\left(\overline{\beta}\right)$ and the inequality follows from $g(\beta) > 0$ which was already introduced in the proof of proposition 2.

It remains to show that there is a single value of $\overline{Q}$ for which $f\left(\overline{Q}\right) = 0$. To do so we show now that its first derivative is positive throughout the relevant range. By (B.2):

(B.5) $$f'\left(\overline{Q}\right) = -(\beta - 1) \cdot \lambda \cdot \left[\ln\left(\overline{Q}\right) - \ln(Q)\right] + (2 - \beta) \cdot \lambda + \beta - 1,$$

and from (B.5)

(B.6) $$f''\left(\overline{Q}\right) = -(\beta - 1) \cdot \lambda \cdot \frac{1}{\overline{Q}} < 0.$$

At the right end of this range of values for $\overline{Q}$, i.e., when $\overline{Q}$ equals $\overline{\beta} \cdot Q$:

(B.7) $$f'\left(\overline{\beta} \cdot Q\right) = -\lambda \cdot (\beta - 1) \cdot \ln\left(\overline{\beta}\right) + (2 - \beta) \cdot \lambda + \beta - 1$$

$$> -\lambda + (2 - \beta) \cdot \lambda + \beta - 1 = (\beta - 1) \cdot (1 - \lambda) > 0,$$

where the first inequality follows from $g(\beta) > 0$, and the second equality follows from $\beta > 1$ and $\lambda < 1$.

Thus, $f'(\overline{Q})$ is positive at the right end of the relevant range and decreasing throughout this range. This implies that $f'(\overline{Q})$ is positive throughout its definition range. Together with (B.3) and (B.4), this proves that there is a single level of $\overline{Q}$ for which $f(\overline{Q})$, and therefore $C_{\overline{Q}}(Q, \overline{Q})$, equals 0.

Appendix C

In this appendix we prove *Proposition 4* which states that welfare under licensing is larger than welfare under free entry for each level of $\overline{Q}$ satisfying $\overline{Q} > Q$.

From the general form of the welfare function, given by (26), which is relevant to both cases, it is clear that comparing welfare can be focused at comparing the function $C(Q, \overline{Q})$ between the two cases. For the purpose of this appendix, in the case of free entry we denote this function by $C^{FE}(Q, \overline{Q})$ and in the case of licensing we denote it by $C^{R}(Q, \overline{Q})$. Under these notations, to prove the proposition it is sufficient to show that $C^{R}(Q, \overline{Q}) > C^{FE}(Q, \overline{Q})$, for each level of $\overline{Q}$. To do so we define the function:

(C.1) $$D(\overline{Q}) \equiv C^{R}(Q, \overline{Q}) - C^{FE}(Q, \overline{Q}),$$

and prove that $D(\overline{Q}) > 0$ for each level of the cap, $\overline{Q}$. First, we will show it for the range $\overline{Q} > \overline{\beta} \cdot Q$. Then we will show that it also holds within the range

In the range $\overline{Q} > \overline{\beta} \cdot Q$, applying (35) for $C^{R}(Q, \overline{Q})$ and (30) for $C^{FE}(Q, \overline{Q})$ in (C.1), and simplifying, yields:

(C.2) $$D\left(\overline{Q}\right) = \frac{K}{\overline{Q}^{\beta-1}} \cdot \left[\lambda \cdot u(\beta) + 1\right] > 0,$$

where:

(C.3) $$u(\beta) \equiv g(\beta) \cdot \overline{\beta}^{\beta} - \overline{\beta}$$

$g(\beta)$ was introduced in the proof of *Proposition 2* and analyzed in detail in appendix D. The inequality follows from $0 < \lambda < 1$ taken together with $u(\beta) > -1$ which is established in Appendix E.[3]

To show that $D\left(\overline{Q}\right) > 0$ also in the range $Q < \overline{Q} < \overline{\beta} \cdot Q$, we return to (C.1) and now we apply (28) in it for $C^{FE}\left(Q, \overline{Q}\right)$, together with, once again, (35) for $C^{R}\left(Q, \overline{Q}\right)$.

From (28), (35) and (C.1) it immediately follows that when the cap, $\overline{Q}$, is at its lowest possible level, i.e., at the current level of Q:

(C.4) $$D(Q) = 0.$$

In addition, by continuity, and since it was already established that $D\left(\overline{Q}\right) > 0$ in the range $\overline{Q} > \overline{\beta} \cdot Q$:

[3] $u(\beta)$ is a function of β alone, with no free parameters. Therefore, it is possible to see that $u(\beta) > 0$ throughout the range $\beta > 1$ numerically, by plotting this function. Readers interested in an analytical proof will find it in Appendix E.

(C.5) $$D\left(\overline{\beta}\cdot Q\right) > 0 .$$

Thus, $D\left(\overline{Q}\right)$ equals zero at the left end of the range $Q < \overline{Q} < \overline{\beta}\cdot Q$ and strictly positive at the right end of this range. Therefore, the only manner by which $D\left(\overline{Q}\right)$ can be negative at some sub-range of this range is if within that sub-range it has a local minimum point, i.e., a point in which $D'\left(\overline{Q}\right) = 0$ and $D''\left(\overline{Q}\right) > 0$. This, however, is impossible, because, as we shall now show, within the range $Q < \overline{Q} < \overline{\beta}\cdot Q$, if $D'\left(\overline{Q}\right) = 0$ at a certain point then at that point $D''\left(\overline{Q}\right) > 0$. To show this, we return to (C.1), apply (28) and (35) in it, differentiate and simplify. This yields that within the range $Q < \overline{Q} < \overline{\beta}\cdot Q$:

(C.6) $$D'\left(\overline{Q}\right) = (\beta - 1)\cdot\left[\begin{array}{l}\dfrac{\overline{\beta}\cdot\lambda - 1 - \overline{\beta}^{\beta}\cdot(\beta - 1 + \lambda)}{\overline{Q}^{\beta}} + \\ \qquad\qquad \overline{\beta}^{\beta}\cdot(\beta - 1)\cdot\lambda\cdot\dfrac{\ln\left(\overline{Q}\right) - \ln(Q)}{\overline{Q}^{\beta}} + \dfrac{\overline{\beta}^{\beta}\cdot\beta\cdot Q}{\overline{Q}^{\beta+1}}\end{array}\right]$$

Applying (C.6) in $D'\left(\overline{Q}\right) = 0$ and simplifying , yields that at that point:

(C.7) $$\overline{\beta}^{\beta}\cdot(\beta - 1)\cdot\lambda\cdot\left[\ln\left(\overline{Q}\right) - \ln(Q)\right] = \overline{\beta}^{\beta}\cdot(\beta - 1 + \lambda) - \left(\overline{\beta}\cdot\lambda - 1\right) - \frac{\overline{\beta}^{\beta}\cdot\beta\cdot Q}{\overline{Q}} .$$

Returning to (C.6), differentiating again, and simplifying, yields:

(C.8) $$D''\left(\overline{Q}\right)=\frac{\beta\cdot(\beta-1)}{\overline{Q}^{\beta+1}}\cdot\left\{\begin{matrix}-\left(\overline{\beta}\cdot\lambda-1\right)+\overline{\beta}^{\beta}\cdot(\beta-1+\lambda)\\ \\ +\overline{\beta}^{\beta-1}\cdot\lambda-\dfrac{\overline{\beta}^{\beta}\cdot(\beta+1)\cdot Q}{\overline{Q}}\\ \\ -\overline{\beta}^{\beta}\cdot(\beta-1)\cdot\lambda\cdot\left[\ln\left(\overline{Q}\right)-\ln(Q)\right]\end{matrix}\right\}.$$

Applying (C.7) in (C.8) and simplifying, yields that within the range $Q<\overline{Q}<\overline{\beta}\cdot Q$, when $D'\left(\overline{Q}\right)=0$:

(C.9) $$D''\left(\overline{Q}\right)=\frac{(\beta-1)\cdot\beta}{\overline{Q}^{\beta+1}}\cdot\overline{\beta}^{\beta-1}\cdot\left(\lambda-\frac{\overline{\beta}\cdot Q}{\overline{Q}}\right)<\frac{(\beta-1)^{2}}{\overline{Q}^{\beta+1}}\cdot\overline{\beta}^{\beta}\cdot(\lambda-1)<0,$$

where the first inequality follows from $\overline{Q}<\overline{\beta}\cdot Q$ and the second from $0<\lambda<1$. □

Appendix D

In this appendix we show that the following function:

(D.1) $$g(\beta)\equiv 1-(\beta-1)\cdot\ln\left(\overline{\beta}\right),$$

is positive throughout the definition range of β, namely the range $\beta>1$.

To do so we start by recalling that

(D.2) $$\overline{\beta}\equiv\frac{\beta}{\beta-1},$$

and therefore:

(D.3) $$\frac{d\overline{\beta}}{d\beta}=\frac{1\cdot(\beta-1)-\beta\cdot 1}{(\beta-1)^2}=\frac{-1}{(\beta-1)^2}\,.$$

(D.1), (D.2) and (D.3) help us calculate the following limit:

(D.4) $$\lim_{\beta\to\infty} g(\beta)=1-\lim_{\beta\to\infty}\left[\frac{\ln(\overline{\beta})}{\frac{1}{\beta-1}}\right]=1-\lim_{\beta\to\infty}\left[\frac{\frac{1}{\overline{\beta}}\cdot\frac{-1}{(\beta-1)^2}}{\frac{-1}{(\beta-1)^2}}\right]=1-\lim_{\beta\to\infty}\tfrac{1}{\overline{\beta}}=0\,,$$

where the second equality follows from L'Hospital's rule.

Differentiating (D.1) yields that for each $\beta > 1$:

(D.5) $$g'(\beta)=-\ln(\overline{\beta})+\frac{1}{\beta}\,.$$

(D.5) together with (D.2), leads to:

(D.6) $$\lim_{\beta\to\infty} g'(\beta)=0\,,$$

and (D.5), together with (D.2) and (D.3), imply that for each $\beta > 1$:

(D.7) $$g''(\beta)=\frac{1}{\beta^2\cdot(\beta-1)} > 0.$$

From (D.6) and (D.7) it follows that $g'(\beta)$ < 0 for each β > 1. This, taken together with (D.4) establishes that $g(\beta) > 0$ for each β > 1.

Appendix E

Lemma 1: $u(\beta) > -1$ for all β > 1.

Proof: Applying (C.2) and rearranging terms reveals that $u(\beta) > -1$ is equivalent to:

(E.1) $$\frac{\beta^{\beta}}{(\beta-1)^{\beta-1}} \cdot g(\beta) > 1.$$

To show that (E.1) holds we define its LHS by $h(\beta)$. The following two characteristics of $h(\beta)$ lead directly to $h(\beta)$ > 1.

(a) $$\underset{\beta \to 1}{Lim}\, h(\beta) = 1$$

(b) $$h'(\beta) > 0 \qquad \forall \beta > 1.$$

To prove (a): we calculate the following limits:

(E.2) $$\lim_{\beta \to 1} (\beta - 1)^{\beta - 1} = \lim_{\beta \to 1} e^{\ln(\beta-1)^{\beta-1}} = \lim_{\beta \to 1} e^{(\beta-1)\cdot\ln(\beta-1)} = \lim_{\beta \to 1} e^{\frac{\ln(\beta-1)}{\frac{1}{\beta-1}}}$$

$$= \lim_{\beta \to 1} e^{\frac{\frac{1}{\beta-1}}{\frac{-1}{(\beta-1)^2}}} = \lim_{\beta \to 1} e^{-(\beta-1)} = 1$$

(E.3) $$\lim_{\beta\to 1} g(\beta) = 1 - \lim_{\beta\to 1}\left[\frac{\ln(\bar{\beta})}{\frac{1}{\beta-1}}\right] = 1 - \lim_{\beta\to 1}\left[\frac{\frac{1}{\bar{\beta}}\cdot\frac{-1}{(\beta-1)^2}}{\frac{-1}{(\beta-1)^2}}\right] = 1 - \lim_{\beta\to 1}\frac{\beta-1}{\beta} = 1,$$

which, together with (E.1), lead directly to (a).

The following derivative is useful in proving (b):

(E.4) $$\frac{d\left[\frac{\beta^{\beta}}{(\beta-1)^{\beta-1}}\right]}{d\beta} = \frac{de^{\ln\frac{\beta^{\beta}}{(\beta-1)^{\beta-1}}}}{d\beta} = \frac{de^{\beta\cdot\ln\beta\ -(\beta-1)\cdot\ln(\beta-1)}}{d\beta}$$

$$= e^{\beta\cdot\ln\beta-(\beta-1)\cdot\ln(\beta-1)}\cdot\left[\ln\beta+1-\ln(\beta-1)-1\right]$$

$$= \frac{\beta^{\beta}}{(\beta-1)^{\beta-1}}\cdot\ln(\bar{\beta})$$

Using (E.4) in differentiating $h(\beta)$, as given by (E.1), yields:

(E.5) $$h'(\beta) = \frac{\beta^{\beta}}{(\beta-1)^{\beta-1}}\cdot\ln(\bar{\beta})\cdot g(\beta) + \frac{\beta^{\beta}}{(\beta-1)^{\beta-1}}\cdot g'(\beta)$$

$$= \frac{\beta^{\beta}}{(\beta-1)^{\beta-1}}\cdot\left[\ln(\bar{\beta})\cdot g(\beta) + g'(\beta)\right].$$

$$= \frac{\beta^{\beta}}{(\beta-1)^{\beta-1}} \cdot \left\{ \frac{1}{\beta} - (\beta-1) \cdot \left[\ln(\bar{\beta})\right]^2 \right\},$$

where the third equality is based on (D.1) and (D.5). To prove (b) it now remains to show that the expression in the square brackets is positive. For that purpose we define it as the following function:

(E.6) $$\phi(\beta) \equiv \frac{1}{\beta} - (\beta-1) \cdot \left[\ln(\bar{\beta})\right]^2,$$

calculate the following limit:

(E.7) $$\lim_{\beta\to\infty} \phi(\beta) = \lim_{\beta\to\infty} \left\{ 0 - \frac{\left[\ln(\bar{\beta})\right]^2}{\frac{1}{\beta-1}} \right\} = -\lim_{\beta\to\infty} \frac{2 \cdot \left[\ln(\bar{\beta})\right] \cdot \frac{1}{\bar{\beta}} \cdot \frac{-1}{(\beta-1)^2}}{\frac{-1}{(\beta-1)^2}}$$

$$= -\lim_{\beta\to\infty} \frac{2 \cdot \ln(\bar{\beta})}{\bar{\beta}} = 0,$$

and the derivative:

(E.8) $$\phi'(\beta) = -\frac{1}{\beta^2} - 1 \cdot \left[\ln(\bar{\beta})\right]^2 - (\beta-1) \cdot 2 \cdot \ln(\bar{\beta}) \cdot \frac{1}{\bar{\beta}} \cdot \frac{-1}{(\beta-1)^2}$$

$$= -\frac{1}{\beta^2} - \left[\ln(\bar{\beta})\right]^2 + 2 \cdot \ln(\bar{\beta}) \cdot \frac{1}{\beta} = -\left[\frac{1}{\beta} - \ln(\bar{\beta})\right]^2 < 0.$$

(E.8) and (E.7) imply that $\phi(\beta) > 0$ and therefore, by (E.6) and (E.5), imply that $h'(\beta) > 0$, for all $\beta > 1$. This establishes (b). □