Informational smallness in rational expectations equilibria

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Abstract

In an economy with asymmetric information, rational expectations equilibria (REE) need not become asymptotically incentive compatible, even if many independent replicas of the economy are merged together. We identify a sub-class of REE for which this is nevertheless the case. It consists of those REE which are focused—where close prices obtain in close state-wise posterior economies; and where the prices are rounded to whole "cents", so that very close exact prices cannot be distinguished. In this class of REE, the informational power of each agent does diminish as more and more similar agents join the market. In the limit economy with a continuum of replicas, the informational power of each individual is nullified completely in focused REE.

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1. Introduction

In a competitive equilibrium we assume that the agents take the prices as given, and do not try to influence the prices by the quantities they offer or demand in the market. A justification for the price-taking assumption may be found in the fact, that if many similar agents were playing a strategic market game, the resulting prices and allocations would not be too different from those that arise from a competitive behavior (see e.g. Mas-Colell, 1982 for a comprehensive presentation).

In a rational expectations equilibrium (REE) with asymmetric information, we also assume that the agents take the prices as given. Here, every agent starts with a private signal or type, and the utility functions as well as the prices depend on the state, i.e. the joint type...

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of the agents. Knowing the association of prices to states and observing their private type and the price, the agents maximize their utility function subject to what they can afford with their endowments. At a REE, the markets clear in every state.

What is the game-theoretic foundation for this competitive behavior in the uncertainty case? Forges and Minelli (1997) suggested this to be a self-fulfilling (SF) incentive compatible (IC) communication equilibrium of a market game. Here, the agents first report their types to a mediator, who announces a price as a function of the joint type. Equipped with the knowledge of this function, observing their private type and hearing the announcement, the agents compute their posterior expected utility functions and play a market game. At equilibrium, the agents report their type truthfully, and the price that results from the market game coincides with the mediator’s announcement.

In order for the outcome of such a strategic communication mechanism to coincide with a REE allocation, it is necessary that the REE be incentive compatible. No agent should envy the bundle he obtains when he is of a different type, because otherwise he would lie to the mediator. Therefore, we will concentrate on the following question: Is it true that when more and more similar agents join the economy the REE tend to be incentive compatible?

Forges and Minelli (1997) showed that the above communication mechanism will come close to implementing a REE if the economy is replicated many times in such a way that in all the states of the world, an agent and all his replicas always have the same type. The mediator simply looks on the majority of declarations among the replicas of each agent, so truth-telling about one’s type is an equilibrium, because no individual deviation may influence the mediator.

Without such a correlation of types between many agents, Blume and Easley (1990) have demonstrated that REE need not in general be incentive compatible, even in a large economy. This may happen if there is an agent whose type enters the utility function of all the other agents. This agent may maintain a big influence on the price by the type that he reports, even if his endowment is small relative to the overall size of the market.

Thus, in contrast to the results obtained by Roberts and Postlewaite (1976) for economies with complete information, the gains from manipulating the market allocations need not disappear asymptotically as the economy grows larger.

What happens, however, if the type of each agent enters the utility function of only few other agents? Specifically, think of a basic economy with finitely many agents, which is replicated many times independently: by observing the types in one replica nothing may be inferred about the types in other replicas. Furthermore, each agent cares only about the types of the agents in his own replica. In particular, the utility functions of agents that join the economy in new replicas are not influenced by the types of the people already in. This is the utmost way in which we may hope to gradually decrease the informational power of each specific agent, without assuming perfect correlation of types across the replicas.

Gul and Postlewaite (1992) proposed such a set-up, and argued that a useful notion of informational smallness could be demonstrated. Specifically, they were able to exhibit a mechanism which, with a large enough number of replicas, produces an allocation which is incentive compatible and almost ex-post efficient. Informational smallness is thus charac-

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1 Indeed, incentive compatibility is necessary for implementation by any mechanism. See Postlewaite and Schmeidler (1986), Palfrey and Srivastava (1989) and Wettstein (1990).
terized by the fact that, asymptotically, the trade-off between incentives and efficiency can be eliminated by means of a particular allocation mechanism.

This leaves open the question of how will the market perform in such an economy: as the economy grows, do individuals become small in the sense that the gains from manipulating the market allocations vanish asymptotically?

This was the question originally addressed by Roberts and Postlewaite, and it seems worthwhile to try to extend their program to economies with asymmetric information. The choice of the particular replica process proposed by Gul and Postlewaite is one natural way between the straightforward but stringent case of complete correlation between types, and the general but impossible case as studied by Blume and Easley (1990).²

The line of reasoning in Roberts and Postlewaite (1976) can be informally summarized by the following three arguments: first, in a large economy, when a single individual lies, the “declared” economy does not change too much; second, if the economy does not change too much, neither does the set of equilibrium prices; and third, if prices do not change too much, neither does the individual (indirect) utility.

We already know that, in economy with asymmetric information, the first argument need not hold: an individual may obtain a piece of information which is relevant for many others. This is our motivation to restrict the analysis to the Gul and Postlewaite replica process.

To get the second argument work, Roberts and Postlewaite introduced a regularity condition, and we will need a similar condition in our setting.

The third argument, which is trivial in the context of economies with complete information, becomes crucial when we discuss REE. In a REE, prices are not only signals of aggregate scarcity, but also play the role of aggregating and transmitting information originally held by individuals. Even prices which are very close in value may maintain a non negligible informational content, thereby leading to a failure of the third argument, and therefore of asymptotic incentive compatibility.

The particular role played by prices in the definition of REE thus interacts in a subtle way with the incentive structure, and may lead to a failure of market decentralization even in a setting in which other mechanisms (like the one proposed by Gul and Postlewaite) are effective in solving the trade-off between incentives and efficiency.

In Section 2, we introduce the model, and in Section 3, we discuss a series of examples which highlight the different problems one has to face in the attempt to obtain incentive compatibility in the context of decentralized markets.

The first undesirable REE that one would like to avoid are those where the prices are very different in states where the posterior optimization problems of the agents are similar. Intuitively, in such a situation there can exist REE in which, even if the economy is large, the price changes drastically with the type of a single agent. This is possible when these state-wise posterior economies have multiple equilibria, and the REE chooses different equilibria in different states. We provide an example (Example 3) in which this leads to a violation of incentive compatibility. In Section 4.1 we define a REE to be focused if such a phenomenon does not happen. The exact formulation requires the posterior economy to

² McLean and Postlewaite (1999) characterize informational smallness in a very general setting, which includes replicated economies as a special case. As in Gul and Postlewaite, the result concerns the construction of a particular mechanism, not the properties of market allocations.
be regular, so that the notion of close equilibria in close posterior economies is properly defined. 3

Another example (Example 4), though, shows that not every sequence of focused REE necessarily becomes asymptotically incentive compatible. It may just be the case that in some focused REE, with one type an agent learns the true type of his replica-mate (about which he cares), and with a different type he remains ignorant regarding that type. In such a situation, the agent may strictly prefer to be of the type that teaches him more.

However, in the example that we bring to this effect, the prices in all the states become closer and closer with the number of replicas. If we assume, therefore, that the exact prices are always rounded to the smallest unit of account in the currency, called "cent", the agents will no longer be able to distinguish between very close (exact) prices and extract information from the difference between them. 4 With this notion of cents-denominated prices, introduced in Section 4.2, we are finally in a position to state and prove, in Section 4.3 a positive result. In the sequence of replications of a basic economy, every sequence of focused, cents-denominated REE is asymptotically incentive compatible.

May one conclude, therefore, that in an appropriate limit economy the market power of each individual will become absolutely negligible? For this to be the case, each agent has to be made negligible both in terms of quantities and informational power. This leads us to consider, in Section 5, economies with a continuum of independent replicas. The definition of such a model raises some technical issues, as the aggregate demand function in it turns to be ill-defined in general. To overcome this problem, we follow the approach of Uhlig (1996) and Kajii (1991), who suggested to apply the notions of Pettis integration to this set-up.

In the resulting limit economy we find that a weak notion of focusedness, that only requires identical prices to hold in states where almost all the agents address the same posterior problems, is enough to guarantee incentive compatibility of REE. The effect of quoting only whole-cents prices is not needed any more. Such weakly focused REE turn to be constant and thus completely non-revealing. The fact that the equilibrium price function is constant in the limit means that the price becomes a complete summary not only of the demand forces due to endowments and preferences, but also to the differential information of the agents.

The proof of the asymptotic theorem in Section 4.3 shows in what sense this property is not just a feature of the mathematical formulation of the limit economy, but is actually approximated with enough replicas of the basic economy. As our example makes clear, the asymptotic result does not hold if we do not limit the ability of individuals to extract information even from prices which are very close to each other. We think that this type of limitation is quite realistic, and we provide a positive result for this modified notion of REE.

Of course, one may take the opposite stance, and argue that imposing exogenous limits on the cognitive ability of individuals goes against the spirit of REE. With such a view, our contribution is to point out that, even in the very favorable setting of the Gul and Postlewaite replica process, there is no game theoretic foundation for such a notion of equilibrium, in contrast with the case of symmetric information.

3 As we noticed above, the importance of regularity, and hence of the possibility of a continuous price selection near the limit economy was first pointed out by Roberts and Postlewaite (1976).

4 In a similar spirit to Allen (1982).
2. Model and definitions

The basic economy is \( E = \{H, L, (T^h, e^h, \mu^h)_{h \in H}, \mu\} \). Individuals, \( H \), and commodities, \( L \), are finite non-empty sets. Each individual has a finite set of possible types, \( T^h \). The set of joint types is \( T = \prod_{h \in H} T^h \), and \( \mu \) is a probability distribution on \( T \), such that each type of each individual appears with a positive probability. Individuals are also characterized by a strictly positive endowment of commodities, \( e^h \in \mathbb{R}^L_+ \), and, for each \( t \in T \), by a strictly monotonic, strictly concave \( C^2 \) utility function, \( u^h(t) : \mathbb{R}^L_+ \to \mathbb{R} \).

The \( n \)-times replicated economy \( \mathcal{E}^n \) is built as follows.

For each \( 1 \leq k \leq n \), let \( \mathcal{E}_k \) be a copy of the basic economy \( \mathcal{E} \), with \( H_k, T^h_k, T_k, \mu_k, e^h_k \) and \( u^h_k \) the copies of the corresponding ingredients.

In \( \mathcal{E}^n \), the set of agents is \( H^n = \bigcup_{k=1}^n H_k \). The set of goods remains \( L \). The set of joint types in \( \mathcal{E}^n \) is

\[
S^n = \prod_{k=1}^n T_k = \prod_{k=1}^n \prod_{h \in H} T^h_k.
\]

The probability distribution on \( S^n \) is \( \mu^n = \prod_{k=1}^n \mu_k \), the independent product of the distribution \( \mu \) on each of the copies.

The endowment of each agent \( h_k \in H_k \) is \( e^h_k = e^h \).

Finally, the utility function of agent \( h_k \in H_k \) depends only on the types in his replica \( k \):

for each \( s^n = (t_1, \ldots, t_n) \in S^n \)

\[
u_k^n(s^n) : \mathbb{R}^L_+ \to \mathbb{R}
\]

is defined by

\[
u_k^n(s^n) = u^h(t_k).
\]

**Definition 1.** A rational expectations equilibrium (REE) for the replicated economy \( \mathcal{E}^n \) is a price function

\[
p : S^n \to \Delta^{L-1}
\]

where \( \Delta^{L-1} \) is the unit simplex of prices, such that for each \( s^n \in S^n \), the solutions \( x_k^n(t_k, p(s^n)) \) to the maximization problems of the agents \( h_k \) after observing their type and the price:

\[
\max_{x \in \mathbb{R}^L_+} E_{\mu^n(\cdot | T^h_k, p)} u_k^n(s^n)(x)
\]

s.t. \( p(s^n)x \leq p(s^n)e^h_k \)

are feasible:

\[
\sum_{k=1}^n \sum_{h \in H} x_k^n(t_k, p(s^n)) = \sum_{k=1}^n e^h_k
\]
Definition 2. A REE is incentive compatible if for every $h_k \in H^n$ and every $t^h_k, \overline{t}^h_k \in T^h_k$

$$\frac{1}{\mu[T^h_k = t^h_k]} \sum_{s_n, s_{-h_k} \in S^n_{-h_k}} \mu^n(t^h_k, s^n_{-h_k})$$

$$\times [u^h_k(s^n)(x^h_k(t^h_k, p(t^h_k, s^n_{-h_k}))) - u^h_k(s^n)(x^h_k(t^h_k, p(t^h_k, s^n_{-h_k}))))] \leq 0$$

where $S^n_{-h_k}$ are the joint types of all the agents but $h_k$.

3. Examples

In this section, we motivate our approach to the problem of incentive compatibility of REE by means of several examples. The first two examples show that, in contrast to what happens in the implementation framework of Gul and Postlewaite (1992), in the REE set-up exact incentive compatibility cannot be guaranteed in large but finite economies. The third and fourth examples demonstrate the need to refine the REE notion in order to guarantee even approximate incentive compatibility in the replicated economy.

Example 1. This example demonstrates that in a replication sequence of economies such as those considered by Gul and Postlewaite (1992), REE may fail to be exactly incentive compatible. However, the REE in the example is asymptotically incentive compatible, as in Roberts and Postlewaite (1976).

In the basic economy $E$ there are two goods—$x$ and $y$ and two agents. Agent 2 has two possible types:

$$T^2 = \{0, 1\}$$

Each occurring with probability half. The utility function of agent 1 depends only on the type of agent 2 in his replica in the following way:

$$u^1(0)(x, y) = 2 \log x + \frac{1}{2} \log y$$

$$u^1(1)(x, y) = \frac{1}{2} \log x + 2 \log y$$

On the other hand, the utility function of agent 2 is independent of his type:

$$u^2(0)(x, y) = u^2(1)(x, y) = \frac{1}{2} \log x + \frac{1}{2} \log y.$$ 

The endowments of the agents of $(x, y)$ are

$$e^1 = (2, 1)$$

$$e^2 = (1, 2)$$

Denote by $p$ the relative price of $y$ with respect to $x$.\(^5\)

\(^5\) Normalizing the prices in this way and not in the unit simplex is, of course, immaterial.
Consider the following REE in the replicated economy $E^n$: a price $p'$ holds whenever the agents 2 in all the replicas are of type 0, and a different price $p''$ holds otherwise. Observing $p'$, the agents 1 in all the replicas will know they like more good $x$, and it is straightforward to compute that the resulting price is

$$p' = 0.7$$

Observing $p''$, each agent 1 will believe with probability $(2n - 1 - 1)/(2n - 1)$ that agent 2 in his replica is of type 0. His posterior utility function will therefore be

$$u_1(x, y) = \left(\frac{2n - 1 - 2/3}{2n - 1}\right) \log x + \left(\frac{2n - 1 - 1/3}{2n - 1}\right) \log y.$$ 

In this case, the resulting price will be

$$p'' = \frac{9 \times 2^n - 7}{9 \times 2^n - 10}$$

so $p' \neq p''$, as required.

Here, $p''$ converges to 1 as $n$ increases. Thus, for large $n$, the agents 2 will be happier with $p''$ because then their purchasing power will be bigger. This means that the REE is not incentive compatible: The agents 2 will all prefer to be of type 1, because that will guarantee the price $p''$.

Notice that in the previous example, as $n$ gets large, the prior probability that $p'$ occurs shrinks to zero. Therefore, when an agent 2 is of type 0, he cares less and less that he is not of type 1, because his expected gain from misrepresenting his type shrinks to zero as the number of replicas increases. Hence, the sequence is asymptotically incentive compatible in the sense that the expected utility gain from misrepresentation goes to zero as the number of replicas goes to infinity.

In the above example, the REE was symmetric with respect to the replicas: permuting the realizations of types across the replicas in a state lead to a new state with the same price. In addition, as the number of replicas increased, one price appeared with a smaller and smaller probability, while the probability of the other price converged to 1. It was also “minimally revealing”, in the sense that the price only revealed whether one specific combination of types has occurred. The following example shows that a sequence of REE may tend to be incentive compatible even without these properties.

**Example 2.** The basic economy $E$ is the same as in Example 1. In the replicated economy $E^n$, a state in $S^n$ is a sequence of $n$ 0-s and 1-s, where the $k$th digit is the type of agent 2 in the $k$th replica. We can view each such state as the binary expansion of a dyadic number in $[0, 1)$. For example, $101 \in S^3$ will be read as the binary expansion $0.101 = \frac{5}{8}$ and so forth.

Consider the following REE in $E^n$: a price $p'$ (of $y$ w.r.t. $x$) will hold in all the states in $[0, 1/3)$, and a different price $p''$ will hold in the states in $[(1/3), 1)$. In this REE, the probability that $p'$ occurs converges to 1/3, and that of $p''$ converges to 2/3. Furthermore,
the REE is clearly not symmetric between the replicas. In \( \mathcal{E}_2 \), for example, \( p' \) holds in the state 01 but \( p'' \) holds in the state 10. The agents know, of course, to which replica they belong, and make their inferences accordingly.

One can check that observing \( p' \), the posterior belief of agent 1 in the \( k \)th replica that agent 2 in his replica has type 0 is greater than 1/2 in the replicas \( k < n \), and as \( n \) grows large it converges to

\[
\mu^n(T_k^2 = 0|0, 1/3) = \frac{1}{2} + \frac{1}{2^k}.
\]

Similarly, observing \( p'' \) his belief that agent 2 in replica \( k \) has type 0 is smaller than 1/2 in the replicas \( k < n \), and when \( n \) increases it converges to

\[
\mu^n(T_k^2 = 0|1/3, 1) = \frac{1}{2} - \frac{1}{2^{k+1}}.
\]

In the first case, his posterior expected utility function approaches

\[
u^1(x, y) = \left( \frac{1}{2} + \frac{1}{3 \times 2^k} \right) \log x + \left( \frac{1}{2} - \frac{1}{3 \times 2^k} \right) \log y,
\]
so his demand for \( x \) is greater than \((1/2)(2 + p')\) in the replicas \( k < n \) and approaches \(((1/2) + 1/(3 \times 2^k))(1 + p')\). In the second case, his posterior expected utility function approaches

\[
u^1(x, y) = \left( \frac{1}{2} - \frac{1}{3 \times 2^{k+1}} \right) \log x + \left( \frac{1}{2} + \frac{1}{3 \times 2^{k+1}} \right) \log y,
\]
so his demand for \( x \) is smaller than \((1/2)(2 + p'')\) in the replicas \( k < n \) and approaches \(((1/2) - 1/(3 \times 2^{k+1}))(1 + p'')\).

As for the agents 2, their demand for \( x \) will always be

\[
x_k^2 = \frac{1}{2}(1 + 2p)
\]
when \( p = p' \) as well as when \( p = p'' \).

In the first case, comparing the supply of \( x \) to the demand we have

\[
3n > n\frac{1}{2}(2 + p') + n\frac{1}{2}(1 + 2p')
\]
which yields

\[
p' < 1,
\]
but as \( n \) grows large the demand for \( x \) approaches

\[
\sum_{k=1}^{n} \left( \frac{1}{2} + \frac{1}{3 \times 2^k} \right) (2 + p') + \frac{n}{2}(1 + 2p')
\]
while the supply is \( 3n \). Dividing both by \( n \) we find that for large \( n, p' \) is close to

\[
\frac{3 - (4/3n)(1 - (1/2^n))}{3 + (2/3n)(1 - (1/2^n))}
\]
and hence that $p'_{n \to \infty} \to 1$. Similar computations show that for large $n$

$$p''_{n \to \infty} > 1, \quad p''_{n \to \infty} \to 1.$$  

We conclude that the resulting REE is not incentive compatible for large $n$: the agents are better off with $p''$ because then their purchasing power is bigger. Therefore, agent 2 in replica $k$ strictly prefers to be of type 1, because

$$\mu^n(p''|T^2_k = 1) \to n \to \infty \mu^n([1/3, 1]) \mu^n([1/3, 1]) = \frac{(1/2 + (1/2^{k+1}))(2/3)}{1/2}$$

which is greater than

$$\mu^n(p''|T^2_k = 0) \to n \to \infty \mu^n([1/3, 1]) = \frac{(1/2 - (1/2^k))(2/3)}{1/2}.$$  

Nevertheless, this strict preference gets weaker and weaker as more and more replicas are added: both $p'$ and $p''$ approach 1, so the expected gain from switching types shrinks to zero.

Even asymptotic incentive compatibility can be precluded if there are multiple equilibria in the posterior economy, after the agents have observed their type and the price. Indeed such multiplicity allows for REE in which very different prices hold in very similar posterior economies and individual declarations may lead to substantial changes in prices, as in the following example.

**Example 3.** This example is based on an example in Mas-Colell et al. (1995, Example 15.B.2). In the original example, there are two goods, $x$ and $y$, and two agents, whose utility functions and endowments are

$$u^1(x, y) = x - \frac{1}{8} y^{-8}, \quad e^1 = (2, r)$$

$$u^2(x, y) = -\frac{1}{8} x y^{-8} + y, \quad e^2 = (r, 2)$$

where $r = 2^{8/9} - 2^{1/9}$. If $p$ is the relative price of $y$ with respect to $x$, the demand for $x$ of the agents are

$$x^1 = 2 + rp - p^{8/9} \quad (3.1)$$

$$x^2 = p^{1/9} \quad (3.2)$$

The equilibrium prices are those with which the aggregate demand for $x$ equals the aggregate endowment, i.e.

$$2 + rp - p^{8/9} + p^{1/9} = 2 + r \quad (3.3)$$

This equation has three solutions: $p = 1/2, p = 1, p = 2$. 

Introducing uncertainty, suppose now that agent 2 has two possible types—$T^2 = \{0, 1\}$, each with probability half. Let the utility function of agent 1 depend on the type of agent 2:

\begin{align*}
  u^1(0)(x, y) &= \frac{2}{5}x + \frac{1}{4}(-\frac{1}{2}y^{-8}) \\
  u^1(1)(x, y) &= \frac{1}{4}x + \frac{2}{5}(-\frac{1}{2}y^{-8})
\end{align*}

His prior expected utility is therefore

\[ u^1(x, y) = \frac{1}{2}(x - \frac{1}{8}y^{-8}). \]

Let, however, the utility function of agent 2 be independent of his type:

\[ u^2(0)(x, y) = u^2(1)(x, y) = -\frac{1}{8}x - \frac{8}{9} + y. \]

To complete the description of the basic economy $\mathcal{E}$, let the endowments of the agents remain as before:

\begin{align*}
  e^1 &= (2, r) \\
  e^2 &= (r, 2)
\end{align*}

In the replicated economy $\mathcal{E}^n$, consider a REE with only two different prices: $p'$ holds in all the states where agent 2 in the first replica is of type 0, and $p''$ holds where he is of type 1. With such a price structure, the agents in the replicas 2, . . . , $n$ learn nothing by observing the price, so their demand for $x$ remains as in (3.1) and (3.2). In the first replica, the demand of agent 2 is as in (3.2)—independent of his type. Agent 1, however, will demand more of $x$:

\[ x^1 = 2 + rp' - 2^{-1/9}(p')^{8/9} \]

when he observes $p'$, that tells him the type of agent 2 in his replica is 0; and he will demand less of $x$:

\[ x^1 = 2 + rp'' - 2^{-1/9}(p'')^{8/9} \]

when he observes $p''$, that tells him that agent 2 in his replica is of type 1. Equating demand and supply in the two cases, we get the equations

\begin{align*}
  2 + rp' - 2^{-1/9}(p')^{8/9} + (n - 1)(2 + rp' - (p')^{8/9}) + n(p')^{1/9} &= n(2 + r) \tag{3.4} \\
  2 + rp'' - 2^{-1/9}(p'')^{8/9} + (n - 1)(2 + rp'' - (p'')^{8/9}) + n(p'')^{1/9} &= n(2 + r). \tag{3.5}
\end{align*}

Dividing by $n$, for large $n$ we get equations that are very close to (3.3). It is straightforward to verify that the derivative of the left-hand side of (3.3) does not vanish at $p = 1/2$ and $p = 2$. Therefore, for large $n$ we can find a solution $p'$ to (3.4) very close to 1/2, and a solution $p''$ to (3.5) very close to 2.

The REE that we get in this way is not incentive compatible: when agent 2 in the first replica is of type 0 and the price is $p'$, he would prefer being of type 1, because then the price would be $p''$ and his utility level would be higher, as one may compute.
It is perfectly clear that the undesirable REE in the previous example is made possible by the multiplicity of equilibria in the posterior economy in each state, after the agents have revised their beliefs by observing their type and the price.

In Examples 1 and 2, the multiplicity problem does not arise, and individuals become less and less keen of having one particular type over another, either because the chance to gain something by switching types becomes smaller and smaller, or because the gain itself diminishes. We may not yet conclude, however, that, when the posterior economy in every state admits only one equilibrium, with enough replicas every REE is approximately incentive compatible, as the following example shows.

Example 4. In the basic economy there are two agents and two goods, \( x \) and \( y \). Every agent has two possible types:

\[
T_1 = \{1a, 1b\} \\
T_2 = \{2a, 2b\}
\]

Each of the four possible combinations in \( T = T_1 \times T_2 \) occurs with probability 1/4.

Each agent cares only about the type of the other:

\[
u^1(1a, 2a)(x, y) = u^1(1b, 2a)(x, y) = \frac{2}{3} \log x + \frac{1}{3} \log y
\]

\[
u^1(1a, 2b)(x, y) = u^1(1b, 2b)(x, y) = \frac{1}{3} \log x + \frac{2}{3} \log y
\]

and similarly

\[
u^2(1a, 2a)(x, y) = u^2(1a, 2b)(x, y) = \frac{2}{3} \log x + \frac{1}{3} \log y
\]

\[
u^2(1b, 2a)(x, y) = u^2(1b, 2b)(x, y) = \frac{1}{3} \log x + \frac{2}{3} \log y
\]

Finally, every agent is endowed with one unit of each of the goods \( x, y \):

\[
e^1 = e^2 = (1, 1).
\]

Consider the following REE in \( E^n \). There are three possible prices: \( p', p'' \) and \( p''' \) of \( y \) w.r.t \( x \), depending only on the types \( T_1 \) in the first replica: \( p' \) holds in \( (1a, 2a) \) in this replica, \( p'' \) holds in \( (1a, 2b) \) and \( p''' \) holds in \( (1b, 2a) \) and \( (1b, 2b) \). In words, if agent 1 in the first replica is of type 1a he can tell the type of his mate, about which he cares, but he cannot if he is of type 1b.

Whatever the price \( p \) is, the posterior utility function of all the agents in the replicas 2, \ldots, \( n \) is

\[
u(x, y) = \frac{1}{3} \log x + \frac{1}{3} \log y
\]

so their demand for \( x \) is \( 1/2(1 + p) \).

When \( p = p' \) or \( p = p'' \), agent 2 in the first replica knows that he likes more good \( x \), and demands \((2/3)(1 + p)\) of it. When \( p = p''' \) he knows he likes more good \( y \), and demands only \((1/3)(1 + p)\) of \( x \).

Finally, when \( p = p' \), agent 1 in the first replica knows he likes more good \( x \), and demands \((2/3)(1 + p)\) of it; when \( p = p'' \) he knows he likes more good \( y \), and demands
only \((1/3)(1 + p)\) of \(x\); and when \(p = p''\) he learns nothing from the price, his expected utility function is

\[
u(x, y) = \frac{1}{2} \log x + \frac{1}{2} \log y
\]

so he demands \((1/2)(1 + p)\) of \(x\).

Computing \(p', p''\) and \(p'''\) by equating demand and supply for \(x\) we get

\[
p' = \frac{3n - 1}{3n + 1}
\]

\[
p'' = 1
\]

\[
p''' = \frac{6n + 1}{6n - 1}
\]

We see that the three prices are indeed different, and that they tend to 1 as \(n\) grows large. Nevertheless, agent 1 in the first replica is much happier to be of type 1a. Then, he is able to identify which good he likes more. He ends up with about \(4/3\) of the good he likes more and about \(2/3\) of the good he likes less. When he is of type 1b, he cannot tell which good is actually more useful to him, and has to demand about 1 unit of each of the goods.

This failure of incentive compatibility relies crucially on the assumption that the agents are able to distinguish and extract information from distinct prices, even if these prices are almost the same. It makes sense, though, to assume that when the prices become very close, the agents are no longer able to distinguish between them. In the example, this would mean that for large \(n\), even with type 1a, agent 1 in the first replica would not be able to tell the type of agent 2 in his replica, and thus he would no longer end up better off with this type.

The above series of examples sets the stage for the rest of the paper. We will suggest a refinement for the notion of REE that eliminates the pathologies illustrated in Examples 3 and 4. We will prove that these refined REE do become approximately incentive compatible when the economy is replicated sufficiently many times, as illustrated by Examples 1 and 2.

4. Refinements of REE

4.1. Focused REE

In the definition of REE, one does not model explicitly the process of equilibrium formation. Nevertheless, we may imagine that when such a process whatsoever stabilizes, close prices emerge when the agents face close posterior problems. In such a case, we can say that in the relevant states the process stabilized around the same focal point. To make precise this idea of focused REE, we first recall the definition of a regular equilibrium in our context.

Given a REE in \(E^n\), the ex-post utility function of agent \(h_k\) in state \(s^n \in S^n\) is determined by his posterior belief \(\nu^h_k(s^n)\) on \(S^n\). This posterior utility function is a convex combination of the utility functions \(u^h_{k}(t)\) for the different \(t \in T\) in the basic economy \(E\). The coefficients of this convex combination are exactly the marginals of \(\nu^h_k(s^n)\) on \(T_k\)—the joint types in replica \(k\), about which agent \(h_k\) cares.
Let

\[ Z : \Delta^{L-1} \times \prod_{k=1}^{n} \Delta(T_k)^H \rightarrow R^L \]

be the aggregate excess demand as a function of the prices and the possible posterior beliefs of the agents (each on the joint types in his replica, as explained above). With posteriors \( \tilde{\nu} = ((\tilde{\nu}_k^h))_{k \leq n, h \in H} \), the price \( \tilde{\rho} = (\tilde{\rho}_1, \ldots, \tilde{\rho}_L) \) is an equilibrium of the posterior economy if \( Z(\tilde{\rho}, \tilde{\nu}) = 0 \). The equilibrium is regular if \( Z(\cdot, \tilde{\nu}) \) is \( C^1 \) at \( \tilde{\rho} \) and the \( L \times L \) matrix of price effects \( D_p Z(\tilde{\rho}, \tilde{\nu}) \) (whose \((\ell, \ell')\) entry is \( \partial Z(\tilde{\rho}, \tilde{\nu}) / \partial \rho_{\ell'} \)) has the maximal possible rank \( L - 1 \). In such a case, by the implicit function theorem there is an open neighborhood \( V \subseteq \prod_{k=1}^{n} \Delta(T_k)^H \) of \( \tilde{\nu} \) and a unique continuous equilibrium function \( \Pi : V \rightarrow \Delta^{L-1} \) satisfying

\[ Z(\nu, \Pi(\nu)) = 0 \quad \forall \nu \in V \]

with \( \Pi(\tilde{\nu}) = \tilde{\rho} \). Furthermore, \( \Pi(\nu) \) is a regular equilibrium \( \forall \nu \in V \). The posterior economy is called regular if all its equilibria are regular, and in such a case the number of equilibria is finite.

It is immediate to see that the neighborhood \( V \) may be taken to be symmetric with respect to permutations between the replicas: it only matters in how many replicas \( k \) the agents \( h_k \in H_k \) have a given \( H \)-tuple of posteriors, and it is immaterial which are these specific replicas. More precisely, what matters is the distribution of the posterior beliefs of the agents in the different copies - a distribution over \( \Delta(T)^H \). Of course, in \( \mathcal{E}^n \) this distribution may assume at most \( n \) different values (this happens if in the different replicas the agents have different \( H \)-tuples of posteriors).

In the above definition of a regular equilibrium, the domain of the excess demand function \( Z \) becomes \( \Delta^{L-1} \times \Delta(\Delta(T)^H) \), and \( \tilde{\nu} \in \prod_{k=1}^{n} \Delta(T_k)^H \) is represented by

\[ \frac{1}{n} \sum_{k=1}^{n} \delta((\tilde{\nu}_k^h))_{h \in H} \in \Delta(\Delta(T)^H), \]

where \( \delta((\tilde{\nu}_k^h))_{h \in H} \) is the unit mass at \( (\tilde{\nu}_k^h)_{h \in H} \). The resulting neighborhood \( V \) is contained in \( \Delta(\Delta(T)^H) \), which is equipped with the topology of weak convergence: \( \rho_n \rightarrow \rho \) in \( \Delta(\Delta(T)^H) \) if for every continuous \( f : \Delta(T)^H \rightarrow R \) we have

\[ \int f \, d\rho_n \rightarrow \int f \, d\rho. \]

By the law of large numbers, the fraction of replicas in which the joint type of the agents is \( t \) converges in probability to \( \mu(t) \) as the number of replicas increases. Thus, if the agents were to learn nothing from the price, with higher and higher probability the distribution of their posteriors would approach

\[ \eta = \sum_{t \in T} \mu(t) \delta(\mu(t)|^h))_{h \in H} \]
in $\Delta(\Delta(T)^H)$. We call $\eta$ the average posterior economy. In the space of characteristics of basic economies, for a generic basic economy $\xi'$ the average posterior economy $\eta$ is regular. This means that the following definition of a focused REE is generically applicable.

**Definition 3.** Suppose the average posterior economy $\eta$ is regular, where $V \subseteq \Delta(\Delta(T)^H)$ is the neighborhood of regular economies of $\eta$. Let $P : S^n \rightarrow \Delta^{L-1}$ be a REE in $\xi'$. For each $s^n \in S^n$, let $\nu(s^n) \in \Delta(\Delta(T)^H)$ be the distribution of the posterior beliefs of the agents in $s^n$, after each of them observed his type and the price. Then the REE $P$ is called focused (around $\eta$) if for some continuous equilibrium price function $\Pi : V \rightarrow \Delta^{L-1}$, whenever $\nu(s^n) \in V$ we have

$$P(s^n) = \Pi(\nu(s^n)).$$

Clearly, the REE in Example 3 is not focused for large $n$. The posterior economies are regular, and in any two states that differ only by the type of agent 2 in the first replica, the distributions of the agents’ posteriors are very close in $\Delta(\Delta(T)^H)$. Still, the equilibrium prices are very distinct. As we saw, this makes the REE not incentive compatible.

### 4.2. Cents-denominated REE

Example 4 shows that focused REE need not become even approximately incentive compatible when the economy is replicated. The example also makes clear that this is due to the extreme precision with which individuals are supposed to be able to extract information from prices. Assume, therefore, that the currency the agents are using has a smallest unit of account called “cent”, and that the prices are denominated in whole cents: the exact, real-valued price is rounded to the closest cent. As a result, the markets for the goods might not clear completely. Notice, however, that $\Delta(\Delta(T)^H)$, the domain of possible distributions of posteriors in the states, is compact in the topology of weak convergence, so the REE prices for economies parameterized by posteriors in $\Delta(\Delta(T)^H)$ will always belong to a compact subset in the interior of the price simplex $\Delta^{L-1}$ (see Mas-Colell, 1985, Proposition 5.8.2). Therefore, if the unit “cent” is small enough, the rounded prices will also be bounded away from the simplex boundary, and the excess demand over supply (or vice versa) will be small relative to the size of the market.\(^6\) Granting this, two exact prices that get closer and closer to one another will eventually be rounded to the same cents-denominated price.\(^7\)

In Example 4, the sequence of cents-denominated REE does tend to become incentive compatible. More precisely, for every $\varepsilon > 0$ it becomes $\varepsilon$-incentive compatible for $n$ large enough, where $\varepsilon$-incentive compatibility is defined as follows.

---

\(^6\) This is close in spirit to Allen (1982).

\(^7\) This will not hold if the values approach half a cent from opposite directions. But in such a case we can re-normalize the prices a little, so that the rounded prices will indeed coincide eventually.
Definition 4. A REE $P : S^n \to \Delta^{L-1}$ in $E^n$ is $\varepsilon$-incentive compatible if for every $h^k \in H^n$ and every $t^h_k, \tilde{t}^h_k \in T^h_k$

$$\frac{1}{\mu[T^h_k = t^h_k]} \sum_{s^h_{-h_k} \in S^n_{-h_k}} \mu^n(t^h_k, s^n_{-h_k})$$

$\times[u^h_k(x^h_k(th^k, P(t^h_k, s^n_{-h_k}))), - u^h_k(x^h_k(\tilde{t}^h_k, P(t^h_k, s^n_{-h_k})))] < \varepsilon$

where $x^h_k(\cdot, \cdot)$ is the posterior demand function of agent $h^k$ as a function of his type and the price.

4.3. The asymptotic theorem

Now we are finally in a position to state and prove our theorem: focused, cents-denominated REE are asymptotically incentive compatible.

Theorem 1. Suppose that for the basic economy $E$, the average posterior economy

$$\eta = \sum_{t \in T} \mu(t) \delta(\mu(\cdot \mid h))_{h \in H}$$

is regular. Then there is a small enough unit “cent”, such that for every $\varepsilon > 0$ there is a natural number $N$, for which whenever $n \geq N$, every focused, cents-denominated REE $P^n : S^n \to \Delta^{L-1}$ of $E^n$ is $\varepsilon$-incentive compatible.

The theorem will be proved in the appendix. The main idea of the proof is that in a focused, cents-denominated REE there is some price that “takes over” most of the space when there are many replicas. This price may be any rounded Walrasian equilibrium price of the average posterior economy $\eta$. Thus, the probability that an agent can alter the terms of trade by his declaration becomes smaller and smaller, and the possible gains stay bounded.

5. Incentive compatibility with a continuum of replicas

A non-atomic economy with a continuum of agents (Aumann, 1964) is a model in which the market power of each specific agent is completely negligible. In such an economy, competitive equilibria are also Nash equilibria of the corresponding market game. Is it possible to phrase a parallel limit result also for REE? Intuitively, for such a result to hold, each agent will have to be negligible not only in terms of quantities but also in terms of informational power. Thus, it is not enough to replicate our basic economy countably many times, because then each replica will still be given some positive weight. We will have to consider a continuum of replicas. We now head to define such an economy.

For a basic economy $E$, let $E_r$ be a copy of $E$ for every $r \in [0, 1]$, with $H_r, T^h_r, T_r, \mu_r, \epsilon^h_r$ and $u^h_r$ the copies of the corresponding ingredients. The indexing set $[0, 1]$ is equipped
with the Lebesgue probability measure $\lambda$. In the continuum economy $E$ the set of agents is $I = \bigcup_{r \in [0,1]} H_r$. The set of goods remains $\{1, \ldots, L\}$. The set of joint types $S = \prod_{r \in [0,1]} T_r$ is equipped with the independent product measure $\mu_c = \prod_{r \in [0,1]} \mu_r$. As before, the utility function of agent $h_r \in H_r$ depends only on the types of his replica-mates: for every $s = (t_r)_{r \in [0,1]} \in S$

$$u_h^r(s) : \mathbb{R}_+^L \to \mathbb{R}$$

is defined by

$$u_h^r(s) = u^h(t_r).$$

Prices are a measurable function $P : S \to \Delta^{L-1}$. Note that every measurable event in $S$ depends on at most countably many coordinates $r \in [0,1]$ (because by definition, the product $\sigma$-field on $S$ is the one generated by cylinders that are determined by finitely many coordinates). Since $\Delta^{L-1}$ is separable, the price function $P$ also depends on at most countably many coordinates.

Defining allocations poses a more complicated problem. We would like, of course, to allow the demand $x_h(r, \cdot)$ of the agents $h_r \in I$ to depend on their types. However, even if only one particular agent $h$ in the basic economy has two possible types $\{0, 1\}$, say, and the demand $x_h(r, \cdot)$ of his replicas $h_r$ depend non-trivially—and identically across the replicas—only on their type being 0 or 1, the aggregate demand $\int x_h(r, s) \, d\lambda(r)$ is meaningless in states $s \in S = \{0, 1\}^{[0,1]}$ which are non-measurable when viewed as functions from $[0,1]$ to $\{0, 1\}$. This is because in such states $s$, the function $r \to x_h(r, s)$ is not measurable.

To solve this difficulty, we follow the approach suggested by Uhlig (1996) and Kajii (1991), whose ideas were further used and extended by Al-Najjar (1995, 1996). Notice that in the above example, the function

$$r \to \int_S x_h(r, s) \, d\mu_c(s)$$

is constant. Furthermore, for every price function $P : S \to \Delta^{L-1}$, the average value of demand

$$r \to \int_S P(s)x_h(r, s) \, d\mu_c(s)$$

is a measurable function: except for those countably many indexes $\{r_1, r_2, \ldots\}$ on which $P$ depends,

$$\int_S P(s)x_h(r, s) \, d\mu_c(s) = \int_S P(s) \, d\mu_c(s) \int_S x_h(r, s) \, d\mu_c(s)$$

which is again constant. This leads to the following definitions.
Definition 5. A function $\psi : [0, 1] \times S \to \mathbb{R}^L$ is weakly measurable if, given any square-integrable function $P : S \to \mathbb{R}^L$, the function

$$r \to \int_S P(s) \psi(r, s) \, d\mu^c(s)$$

is measurable.

Definition 6. A function $\psi : [0, 1] \times S \to \mathbb{R}^L$ is Pettis-integrable if it is weakly measurable, and there exists a square-integrable function $F : S \to \mathbb{R}^L$ such that, given any square-integrable function $P : S \to \mathbb{R}^L$, we have

$$\int_0^1 \left[ \int_S P(s) \psi(r, s) \, d\mu^c(s) \right] \, d\lambda(r) = \int_S P(s) F(s) \, d\mu^c(s)$$

The function $F$ is the Pettis integral of $\psi$, and we write $F(s) = \int_0^1 \psi(r, s) \, d^*\lambda(r)$.

An allocation is an $H$-tuple of Pettis-integrable functions $x^h : [0, 1] \times S \to \mathbb{R}^L$, $h \in H$. An allocation $(x^h)_{h \in H}$ is feasible if, for $\mu^c$-a.e. $s \in S$,

$$\sum_{h \in H} \int_0^1 x^h(r, s) \, d^*\lambda = \sum_{h \in H} e^h$$

Example 3 may be easily generalized to this set-up. We can choose any particular replica—the replica indexed 0.17, say—and let the REE $P$ depend only on the type of agent 2 in this replica: $P = 1/2$ if he is of type 0, and $P = 2$ if he is of type 1. In all the replicas but 0.17, the agents will learn nothing from the price, and their demand will be as in Eqs. (3.1) and (3.2). The demand functions $x^1(r, s)$ and $x^2(r, s)$ are therefore constant $\lambda$-almost everywhere. In particular, they are Pettis-integrable, and together with $P$ they constitute a REE.

As before, this REE is not incentive compatible, agent 2 in replica 0.17 will always prefer to declare that he is of type 2.

To rule out such REE in the finitely-replicated economy, we needed a notion of focusedness that prohibited very distinct prices to hold in close state-wise posterior economies. The above discussion suggests, that in the continuum economy a weaker notion of focusedness may suffice, one that only prohibits different prices to hold in states where almost all the agents have exactly the same posteriors.

Definition 7. A REE $P : S \to \Delta^{L-1}$ is weakly focused if whenever for $\mu^c$-a.e. $s$, $\mu^c$-a.e. $s'$, we have

$$\mathbb{E}_{\mu^c(e^r \mid \mathcal{T}^h, P)} u^h_r(s) = \mathbb{E}_{\mu^c(e^r \mid \mathcal{T}^h, P)} u^h_r(s')$$

for $\lambda$-a.e. $r \in [0, 1]$ and all $h \in H$, we have $P(s) = P(s')$. 

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It turns out that in the limit economy $E$, this weaker refinement is sufficient to guarantee IC of REE. Even the blurring effect of quoting only whole-cents prices is not necessary any more:

**Theorem 2.** Every weakly focused REE $P : S \rightarrow \Delta^{L-1}$ is constant $\mu^c$-a.e., and therefore IC.

This is the climax of the phenomenon observed in focused REE with finitely many replicas, where one price always obtains in larger and larger portions of the space as the number of replicas increases. In the limit, the aggregation of the individual bits of information is completed: there is no more need to extract information from prices, even though the agents are perfectly capable of doing so.

6. Concluding remarks

Given the restrictions on the price function in focused, cents-denominated REE, our results may not appear very surprising from the sheer point of view of implementation theory. However, the basic motivation to search for conditions that enable the implementation of allocations, is that such allocations may eventually be decentralized under some competitive equilibrium concept. In this work, our approach is more direct: we start with the concrete solution concept under consideration—rational expectations equilibrium—and ask whether it is asymptotically implementable.

Our positive result in this respect is somewhat qualified. Indeed, it applies when the private information of individuals enters directly the utility function of only few other individuals. No doubt, such an assumption may be appropriate only in a restricted class of economic situations. Furthermore, as we have seen, incentive compatibility obtains asymptotically only for a proper sub-family of REE. Thus, our results delineate the non-trivial assumptions that are needed to justify the price-taking assumption in REE.

The fact that the equilibrium price function is constant in the limit means that the price becomes a complete summary not only of the demand forces due to endowments and preferences, but also to the differential information of the agents. The proof of the asymptotic Theorem 1 shows in what sense this property is not just a feature of the mathematical formulation of the limit economy, but is actually approximated with enough replicas of the basic economy.

We did not touch in this work, the question whether focused, cents-denominated REE always exist. Another interesting question is whether every economy with a non-incentive compatible REE has nevertheless other REE which are incentive compatible. Finally, it remains unclear at this point whether the finiteness of the type set in the basic economy is essential to our results. All these remain to be studied in future work.

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Appendix A. Proof of the theorems

To prove Theorem 1, we first need the following auxiliary theorem. It says that in a sequence of i.i.d. experiments, by observing an event with a fixed, positive probability, the number of experiments on which we can learn a lot is bounded, irrespective of the total number of experiments.

**Theorem A.1.** In the replicated economy $E^n$, for every event $A \subseteq S^n$ with

$$\mu^n(A) \geq \delta > 0$$

and for every $d > 0$, the number of copies $k \in \{1, \ldots, n\}$ for which

$$||\mu_k(\cdot|A) - \mu_k(\cdot)|| \geq d$$

does not exceed

$$1 - \sum_{t \in \mathcal{T}} \mu(t)^2/(\delta^2).$$

**Proof.** For every $t \in \mathcal{T}$ and $k \leq n$ consider the indicator function $1_{\{T_k = t\}}$ in $S^n$. The expectation of $1_{\{T_k = t\}}$ is $\mu(t)$, so

$$X_t^k = 1_{\{T_k = t\}} - \mu(t)$$

belongs to $L^2_0(S^n, \mu^n)$, the Hilbert space of square-integrable functions with mean 0. For all $k \neq \ell$, $X_t^k$ and $X_t^\ell$ are orthogonal vectors in $L^2_0(S^n, \mu^n)$, all with variance $\mu(t) - \mu(t)^2$.

Denoting by $\langle \cdot, \cdot \rangle$ the inner product in $L^2_0(S^n, \mu^n)$, for a vector $X$ in this space, we have

$$\text{Var}(X) = \int X^2 \, d\mu^n = \langle X, X \rangle.$$

Consider now the sub-$\sigma$-field

$$\mathcal{A} = \{\emptyset, A, A^c, S^n\}$$

in $S^n$. Choose $Y_t^i \in L^2_0(S^n, \mathcal{A}, \mu^n)$ with $\text{Var}(Y_t^i) = \text{Var}(X_t^i) = \mu(t) - \mu(t)^2$. The conditional expectation $E(X_t^i|A)$ is the projection of $X_t^i$ on the one-dimensional subspace $L^2_0(S^n, \mathcal{A}, \mu^n)$ of $L^2_0(S^n, \mu^n)$ (see e.g. Dudley, 1989; Theorem 10.2.9). Therefore,

$$\text{Var}(E(X_t^i|A))^{1/2} = \frac{\langle X_t^i, Y_t^i \rangle}{\text{Var}(Y_t^i)^{1/2}} = \frac{\langle Y_t^i, X_t^i \rangle}{\text{Var}(X_t^i)^{1/2}} = \text{Var}(E(Y_t^i|X_t^i))^{1/2},$$

because $E(Y_t^i|X_t^i)$ is the projection of $Y_t^i$ on the one-dimensional subspace of $L^2_0(S^n, \mu^n)$ generated by $X_t^i$.

Since the $X_t^i$-s are orthogonal, by Bessel inequality

$$\sum_{k=1}^n \text{Var}(E(X_t^i|A)) = \sum_{k=1}^n \text{Var}(E(Y_t^i|X_t^i)) \leq \text{Var}(Y_t^i) = \mu(t) - \mu(t)^2.$$

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8 Where $|| \cdot ||$ is, of course, the Euclidean norm in $R^T$. 
Summing over $t \in T$ we get
\[ \sum_{k=1}^{n} \sum_{t \in T} \text{Var}(E(X_i^k|A)) \leq \sum_{t \in T} [\mu(t) - \mu(t)^2] = 1 - \sum_{t \in T} \mu(t)^2. \]

In addition, for every $t$ and $k$
\[ \text{Var}(E(X_i^k|A)) \geq \mu_n(A) [\mu_k(T_k = t | A) - \mu(t)^2], \]
so
\[ \sum_{k=1}^{n} \sum_{t \in T} [\mu_k(T_k = t | A) - \mu(t)^2] \leq \frac{1}{\mu_n(A)} \leq \frac{1}{\delta} \sum_{t \in T} \mu(t)^2. \]

We conclude, therefore, that the number of $k \in \{1, \ldots, n\}$ for which
\[ \sum_{t \in T} [\mu_k(T_k = t | A) - \mu(t)^2] \geq d^2, \]
i.e. $||\mu_k(\cdot | A) - \mu_k(\cdot)|| \geq d$, may not exceed $1 - \sum_{t \in T} \mu(t)^2 / (\delta d^2)$, as required. \qed

A.1. Proof of Theorem 1

Choose the unit “cent” small enough, such that no price in a REE of $\mathcal{E}$ will ever be rounded to the boundary of the price simplex $\Delta^{L-1}$. This is possible due to the compactness of $\Delta(\Delta(T)^H)$, as explained in Section 4.2.

Consider the average posterior economy
\[ \eta = \sum_{t \in T} \mu(t) \delta_{\mu_k(T_k = t | A)} h \in H. \]

The collection of neighborhoods $V_r, r > 0$ constitute a base for the neighborhoods of $\eta$, where
\[ V_r = \{v \in \Delta(\Delta(T)^H) : \forall t \in T \ |v(O(t, r)) - \mu(t)| < r\} \]
and
\[ O(t, r) = \{(\tilde{\mu}^h)_{h \in H} \in \Delta(T)^H : \forall h \in H \ ||\tilde{\mu}^h(\cdot) - \mu(\cdot)|| < r\}. \]

Let $V' \subset \Delta(\Delta(T)^H)$ be the neighborhood of regular economies of $\eta$. There are finitely many continuous equilibrium price functions $\Pi' : V' \to \Delta^{L-1}$. Then for a small enough $r > 0$ (that we fix from now on), on the sub-neighborhood $V_r \subset V'$ of $\eta$ every continuous cents-denominated equilibrium price function $\Pi : V_r \to \Delta^{L-1}$ is constant.

Fix any sequence of focused, cents-denominated REE $P^n : S^n \to \Delta^{L-1}$ in $\mathcal{E}^n, n \geq 1$, with the corresponding cents-denominated (constant) equilibrium price functions $\Pi^n : V_r \to \Delta^{L-1}$. To proceed, we first prove the following claim.
Claim A.1. For every \( \delta > 0 \) we can find a large enough \( N \), such that for every \( n \geq N \) and for every whole-cents \( p^n \in \Delta^{L-1} \), \( p^n \neq \Pi^n(\eta) \), we have
\[
\mu^n(A^n_{p^n}) < \delta
\]
where
\[
A^n_{p^n} = \{ s^n \in S^n : P^n(s^n) = p^n \}.
\]

Proof. Suppose, by contradiction, that (A.1) fails for infinitely many \( n \) and \( A^n_{p^n} \). Restrict our attention to the sub-sequence for which
\[
\mu^n(A^n_{p^n}) \geq \delta. \tag{A.2}
\]
Observe that for a small enough \( d > 0 \),
\[
||\mu_k(\cdot|A^n_{p^n}) - \mu_k(\cdot)|| < d \text{ for some } k \leq n \text{ entails that}
\]
\[
||\mu_k(\cdot|A^n_{p^n}, T_k^h = t^h) - \mu_k(\cdot|T_k^h = t^h)|| < r \forall h \in H, t \in T. \tag{A.3}
\]
Using Theorem A.1, we conclude that the fraction of those \( k \in \{1, \ldots, n\} \) for which (A.3) fails tends to zero as \( n \) grows large (because the bound in Theorem A.1 does not depend on \( n \)). Hence, by (A.2) and the law of large numbers, for large enough \( n \) we can find states \( s^n = (t_1, \ldots, t_n) \in A^n_{p^n} \) such that for every \( t \in T \), the fraction of \( k \in \{1, \ldots, n\} \) for which \( t_k = t \) and (A.3) holds is very close to \( \mu(t) \) —not further than \( r \) from it. But this means that the distribution of the agents’ posteriors in \( s^n \) is in \( V_r \) (because whenever (A.3) holds, the posteriors of the agents in replica \( k \) in \( A^n_{p^n} \) belong to \( O(t, r) \)). Since \( P^n \) is focused, this implies that
\[
P^n(s^n) = \Pi^n(\eta) \neq p^n,
\]
contradicting the assumption that \( s^n \in A^n_{p^n} \), and thus proving the claim. \( \square \)

A.2. End of Proof of Theorem 1

Denote by \( C \) the finite set of whole-cents price vectors in the interior of \( \Delta^{L-1} \). For every agent \( h \in H \) there is a price vector \( p \in C \) that maximizes the value of his endowment and another \( p' \in C \) that minimizes it. Furthermore, the posterior utility function of an agent \( h \in H \) is always some average of his utility functions \( u^h(t) \) for the finitely many \( t \in T \). Together this implies that there is a finite value \( M^h \) such that the utility level that agent \( h \) may attain by exchanging his endowment belongs to the interval \([−M^h, M^h] \). Denote \( M = \max_{h \in H} M^h \) and \( m = \min\{\mu(t^h) : h \in H, t^h \in T^h\} \). We may assume that \( m > 0 \), because if \( \mu(t^h) = 0 \) for some \( t^h \in T^h \) we can a priori discard \( t^h \) from \( T^h \). From Claim A.1 and the finiteness of \( C \) it follows that for large enough \( n \)
\[
\mu(B) < \frac{m \varepsilon}{2M}
\]
where
\[
B = \{ s^n \in S^n : P^n(s^n) \neq \Pi^n(\eta) \}.
\]
For such a large $n$, for every $k \leq n$, $h \in H$ and $t^h \in T^h$

$$0 \leq \mu^n(B|T_k^h = t^h) = \frac{\mu^n(B \cap T_k^h = t^h)}{\mu(t^h)} \leq \frac{\mu^n(B)}{\mu(t^h)} \leq \frac{m \varepsilon}{2 M m} = \frac{\varepsilon}{2 M}.$$ 

Therefore, the expected gain of an agent in $E^n$ by switching types, and thus potentially causing the price to differ from $\Pi^n(\eta)$, does not exceed $\varepsilon$. This means that $P^n$ is $\varepsilon$-incentive compatible, as required.

A.3. Proof of Theorem 2

The price function $P$ is measurable with respect to the product $\sigma$-field of $S$, so it depends on at most countably many coordinates $R_P = \{r_1, r_2, \ldots\} \subset [0, 1]$. For all $r \in [0, 1] \setminus R_P$ and in every state of the world, the agents in the $r$th replica learn nothing from the price $-\mu^c(\cdot|T_r^h, P) = \mu^c(\cdot|T_r^h)$. The definition of a weakly focused REE then implies that $P$ is constant $\mu^c$-a.e. $s \in S$.

References