Competition in the Presence of Social Networks: How Many Service Providers Maximize Welfare?

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Abstract. Competition for clients among service providers is a classical situation discussed in the economics literature. While better service attracts more clients, in some cases clients may prefer to keep using a low quality service if their friends are also using the same service—a phenomenon largely encouraged by the Internet and online social networks. This is evident, for example, in competition between cloud storage service providers such as DropBox, Microsoft SkyDrive and Google Drive. In such settings, the utility of a client depends on both the proposed service level and the number of friends or colleagues using the same service.

We study how the welfare of the clients is affected by competition in the presence of social connections. Quite expectantly, competition among two firms can significantly increase the clients' social welfare in comparison with the monopoly case. However, we show that a further increase in competition triggered by the entry of additional firms may be hazardous for the society (*i.e.*, to the clients), which stands in contrast to the typical situation in competition. Indeed, we show via equilibrium analysis that the social benefit of additional firms beyond the duopoly is limited, whereas the potential loss from such an addition is unbounded.

1 Introduction

Competition between firms has received much attention in the economics literature [4,16,5], and recently also in the computer science literature [13,1,12,17]. In standard models of competition, firms compete over clients (or workers), by offering a certain level of service or payoff. The utility of the firm is derived from the set of clients that select it, and can be based either purely on their number, or on a more sophisticated combinatorial function. The utility of the clients on those models is assumed to be affected only by the service they receive and by their preferences over firms. In particular, it is assumed that clients are indifferent to the decisions of other clients.

This kind of competition leads to models where the introduction of additional firms always improves the total welfare of the clients, and the introduction of

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additional clients always improves the total welfare of the firms [10,16,5]. However, this kind of competition is an oversimplification of reality. In many real world scenarios, the decisions of clients have significant positive or negative effect on the utility of other clients, an effect known as *network externalities* (see the related work section). For example, people may prefer a restaurant that has fewer clients (*e.g.*, when they wish to maintain privacy), or alternatively choose one that is highly attended (*e.g.*, if they believe it is a signal to quality, or enjoy the crowd). In quite many cases, such preference may be based on social connections, as people prefer to spend time with their friends and benefit from their presence.

The above is particularly relevant for long-term selection of online services such as cloud storage services, social networks, cellular providers and, to some extent, e-mail providers, where the benefit one generates from a service highly depends on its adoption by colleagues and friends. For example, calls within a single cellular network are sometimes cheaper or have fewer interruptions. Similarly, sharing files is easier between users who use the same cloud storage service provider.

A motivating example As a concrete example, consider two competing cloud services, Grand-docs (G) offering 3GB of storage and Medium-drive (M) offering 5GB. Suppose a client of G called Alice considers moving to M. Alice wants to be able to share a workspace with her colleague Bob, which is another user of G. Alice values the convenience of sharing a platform with each single colleague as equivalent to 1GB of storage. Thus, Bob alone would not prevent her from switching to M. However if Alice has, say, five colleagues using G, and only one who uses M, then she will (perhaps reluctantly) keep using G. Eve, on the other hand, is a freelance who uses cloud services for storage only. Thus, she prefers M regardless of the actions of other clients.

In this work we consider a model where clients' utilities depend both on the price or quality of the offered service and on the number and identity of their friends who have chosen the same firm. Our model is a two-phase game. In the first phase firms commit to a particular level of service which can be measured, for example, by bandwidth, storage capacity or accessible content of interest.[4] In the second phase, each client independently selects a firm. Each firm gains a fixed value for every client it recruits, from which it subtracts the cost of the service. The utility of a client is composed of the service level offered by her chosen firm, and an additional utility the client gains for every friend selecting the same firm.

A monopoly of a single firm guarantees that all social links are used, but gives the firm no incentive to provide a decent service level. Given a set of service providers, with fixed offers, and assuming positive externalities, the society can

[4] We consider services that are given for free, such as e-mail and some cloud services, but where service level varies. A different interpretation of the model is where service level is uniform, whereas price varies across firms. A similar analysis can be applied in the latter case of price competition, with some adjustments.

still utilize all social connections, if all clients subscribe to a single provider (say, the one offering the best service). However, there may be idiosyncratic preferences over firms (see Remark 1), in which case such a partition would not be optimal. Further, even if clients do not discriminate among providers, there may be many other partitions of clients that are also stable. Since some social connections are not exploited, these outcomes are less efficient, and we focus on the *worst case* partitions that are still stable.

Social connections may give rise to interesting dynamics in the market. For example, a new firm will have to provide a significantly higher level of service than an incumbent competitor, and even then it will only be able to attract clients who have low value for social connections. However, after this initial small wave, other clients may also move as now they might already have friends using the new service. A third and forth wave may occur, until every client achieves balance between her desired service level and social connections. While clients act myopically (as they may be unaware of the global structure of the network), firms try to predict the eventual partition of clients that will follow a given change in the service level. We, therefore, introduce an equilibrium concept that takes this behavior into consideration.

Our goal is to study the effect of competition on the *social welfare* of the clients, which is defined as the sum of their utilities, derived from both service and social connections. To that aim, we define the *clients' value of competition* of a given game as the ratio between the clients' social welfare under the *worst* equilibrium, and their social welfare under the monopoly outcome. We ask whether the value of competition increases or decreases with respect to the number of firms in the market. In particular, we are interested in how many firms should a market have to (approximately) maximize the social welfare.

Our contribution On the conceptual side, our two-phase model of competition and the corresponding solution concept allows for a focused study of the interaction between the network structure, number of firms and clients' welfare, while isolating them from other factors which are kept simple (such as production costs).

Our first result is that while a monopoly may be infinitely worse for the clients than any equilibrium under two firms or more, further increasing the number of firms in the market is not always beneficial for the clients. Surprisingly, the entry of additional firms beyond two cannot increase the value of competition by more than a constant factor. In contrast, the value of competition can *decrease* linearly with the number of firms, *i.e.*, by an unbounded factor. This demonstrates that in markets where social connections play an important role too much competition may produce an adverse result.

We further study bounds on the value of competition under the special case of a complete social graph. In particular, we show that the value of competition may still decrease with the number of firms, but only when the number of clients is sufficiently large. A complete social graph is interesting because, in some sense, it represents the opposite case of models ignoring social effects. Finally, while the value of competition refers to the clients' welfare, one can also define a complementary concept with regard to the firms' revenue. We present a preliminary result in this direction.

Due to space limitations, most proofs were deferred to Appendix B.

2 The Model

We consider a two-phase game with two types of players: firms and clients. The clients are the nodes of a graph that represents the social network. In the first phase, each firm declares a payoff or service level (*e.g.*, how much storage is allocated for clients joining it). In the second phase, the clients join firms. The objective of the firms is to get as many clients as possible, while supplying the least amount of service. The objective of the clients is to share a firm with as many of their neighbors as possible, while getting a high service level.

More formally, consider an undirected graph $G = \langle N, \Gamma \rangle$ whose nodes are the n clients. The edges of G have positive weights $w_{j,j'}$, reflecting the benefit to clients $c_j, c_{j'} \in N$ from sharing a firm with each other.[5] We denote by $\Gamma(c_j) \subseteq N$ the set of neighbors of c_j, *i.e.*, all clients $c_{j'}$ for which $w_{j,j'} > 0$. In addition, each client $c_j \in N$ is associated with a constant a_j, which is the value c_j gets from each unit of good (*e.g.*, 1 Mb of disk space) it receives from the host firm. We denote by $\mathbf{a} \in \mathbb{R}^n_+$ the vector of clients' parameters, and assume that the information on $\mathbf{a}$ is implicitly contained in N (and thus in G). In other words, whenever we have a set N of clients, or a graph G, we also have a vector $\mathbf{a}$ associated with them.

In addition, the set F consists of m firms $f_1, f_2, \ldots, f_m$. Each firm $f_i \in F$ is associated with an integer constant r_i, representing the revenue of f_i from each client joining it. We similarly denote all firms' parameters by $\mathbf{r} \in \mathbb{R}^m_+$ (where information on $\mathbf{r}$ is implicitly contained in F). An instance of our game is, therefore, represented by a pair $I = \langle G, F \rangle$, *i.e.*, a graph of clients and a collection of firms. We write $I_m = \langle G, m, r \rangle$ when F consists of m identical firms with $r_i = r$.

The strategies available to each firm are committing to a certain service level (payoff) x_i. An outcome of the game (also called *configuration*) can be written as $E = \langle \mathbf{x}, P \rangle$, where $\mathbf{x}$ is a payoff vector, and $P = (C_1, \ldots, C_m)$ is a partition of the clients to firms. We denote by $f(c, P) \in F$ the firm selected by client c, *i.e.*, $f(c, P) = f_i$ for all $c \in C_i$. Given an outcome $E = \langle \mathbf{x}, P \rangle$ in game $\langle G, F \rangle$, the utilities of the agents are as follows.

- The utility of each firm $f_i \in F$ is given by $v_i(E) = (r_i - x_i) \cdot |C_i|$.
- The utility of each client $c_j \in C_i$ is given by $u_j(E) = a_j \cdot x_i + \sum_{j' \in C_i \setminus \{c_j\}} w_{j,j'}$.

For any particular outcome E in game $\langle G, F \rangle$, we denote by $SW(G, F, E)$ (or $SW(I, E)$) the social welfare of the clients, *i.e.*, $SW(G, F, E) = \sum_{c_j \in N} u_j(E)$. In the rest of this paper, we sometimes omit the parameters I, G, F and E when

[5] In the general case, clients c_j and $c_{j'}$ may have a different value for sharing a firm with one another. See a more detailed discussion in Remark 2.

they are clear from the context. The rest of this section is devoted for defining and explaining the solution concept we use.

Remark 1 [Preferences over firms]**.** In some models of competition each client is assumed to have preferences over the different firms, reflecting differences between the products not captured by the other parameters of the model [16,17,18]. Suppose we denote by $q_{j,i} \in \mathbb{R}_+$ the preference of client c_j to firm f_i, and redefine her utility when choosing firm f_i as $u^*_j = u_j + q_{j,i}$. It is not hard to see that such preferences can be completely emulated by adding one "fixed client" c^*_i per firm, with $w(c_j, c^*_i) = q_{j,i}$. These new clients are not part of N, they cannot choose a different firm, and their utility is not counted as part of the social welfare. To simplify the exposition, we do not explicitly consider preferences or fixed clients in the remainder of the paper. However, most of our results (except in Section 5.1, where the complete graph is studied) hold even if we allow the inclusion of such fixed clients—and thus also hold if we add preferences.

2.1 Client dynamics

Suppose firms commit to some given service levels, $(x_1, \ldots, x_m)$. Clients can now choose which firms to join. Since the utility of each client is affected by the decision of her friends, there might not be dominant strategies. However, given any current partition of clients $P = (C_1, \ldots, C_m)$, every client c_j has a straight-forward best response, which is to join the firm f_i maximizing $a_j \cdot x_i + \sum_{j' \in C_i \setminus \{c_j\}} w_{j,j'}$. It is easy to see that a pure Nash equilibrium (PNE) for the clients exists (*e.g.*, when all clients join the firm offering the best service), however, there may be more than one such equilibrium. We argue that every sequence of best responses by clients must converge to some pure Nash equilibrium.

Proposition 1 *If the strategies of the firms are fixed (i.e., firms are not players), and the clients switch strategies according to a best response dynamics, then the client strategies converge into a PNE.*

Remark 2 [Symmetry]**.** The proof of Proposition 1 is due to the fact that the clients' game admits a potential function. This proof is the only place in this paper where the symmetry assumption is applied (*i.e.*, that every two clients c_j and $c_{j'}$ gain the same value from sharing a firm with each other). We note that there are other cases where Proposition 1 holds even without symmetry. For example, if c_j attributes the same positive value for every neighbor sharing a firm (say, some constant w_j). Interestingly, both symmetry and equal weight to neighbors turn out to be sufficient conditions for the existence of pure equlibrium in other models based on social connections [7,22]. It is important to emphasize that all of our results in the rest of this paper hold even without the symmetry assumption, as long as the clients are guaranteed to converge to an equilibrium.

2.2 The two-phase game and equilibria concepts

Knowing that for any profile of payoffs the clients must converge, we can define a game in extensive form. Our game proceeds in two phases as follows.

– In the first phase, each firm f_i declares a non-negative integer x_i, which is the payoff (or service level) f_i gives to each client joining it. Note that firms are not allowed to discriminate between clients. We denote the vector of firms' strategies by $\mathbf{x} \in \mathbb{N}^m$.[6]
– In the second phase, each client c_j chooses a firm. Then, the clients follow a best response dynamics until they converge to a Nash equilibrium, *i.e.*, to a partition P.

Consider an outcome $E = (\mathbf{x}, P)$ obtained from this two-phase game. Clearly, the clients have no incentive to deviate in E, however, the firms might deviate. Once a firm deviates, the clients can reach a new equilibrium, in which case we get a new outcome E'. Given a firm f_i, a strategy x'_i and two outcomes E, E' which are PNEs for the clients, we say that a outcome E' is the *projected outcome* obtained from E via the deviation x'_i of firm f_i if:

– $x'_i > x_i$, *i.e.*, firm f_i offers a higher service level.
– There is a sequence of best-responses by clients starting from the state $((x'_i, x_{-i}), P)$ that converges to E'.

The projected outcome is clearly unique if there are two firms, as clients will only join the deviating firm, until no more clients want to join. When there are three firms or more, the clients' best response dynamics might be able to reach multiple Nash equilibria. In such cases, the projected outcome is determined by one of these equilibria arbitrarily. Thus in what follows, we treat the projected outcome E' as unique given E, i and x'_i. Using the above definition of the projected outcomes, we are now ready to define our solution concept.

Definition 1. *An outcome $E = (\mathbf{x}, P)$ is a* commitment equilibrium *(CME) if: (a) E is a PNE for the clients; and (b) For any firm f_i and any $x'_i > x_i$, $v_i(E') \leq v_i(E)$, where E' is the projected outcome obtained from E via the deviation x'_i.*

In other words, the last condition states that no firm is better off increasing its payment, assuming that following this deviation the game will reach the projected outcome associated with this deviation.[7] The following section gives some theoretical and practical justifications for the definition of CME.

3 Properties of commitment equilibria

Suppose that firms announce some payoff vector $\mathbf{x}$, which results in a configuration $E = (\mathbf{x}, P)$. Moreover, suppose firm f_i deviates by announcing payoff

[6] The firms' strategies are constrained to integers. In many cases this is a realistic requirement as resources, such as storage and bandwidth, are measures in integer quantities. See Appendix A regarding more theoretical reasons for this requirement.

[7] Clearly, the selection of the projected outcome E' associated with every possible deviation may determine whether E is a CME or not. For example, if E' is assumed to be the *best outcome* for the deviator, we get the strongest (and narrowest) definition of CME. In what follows, we do not assume any particular tie-breaking scheme among projected outcomes.

$x_i' \neq x_i$ upon seeing the outcome E. In the unfolding of events, some clients may join f_i or desert it. This in turn may cause other clients to switch firms and so on. By Proposition 1, the clients will eventually reach a stable configuration $E' = ((x_i', x_{-i}), P')$. The deviation is profitable for f_i if $v_i(E') > v_i(E)$.

Note that in theory, a firm may either gain by increasing its service level, thereby triggering a cascade of new clients joining it; or by decreasing payoff and thus reducing expenses (hopefully without driving away too many clients). However, deviations of the latter type are largely impractical in most situations. Often, the service level offered by the firm is considered by the clients as a *commitment*. If Google or Dropbox will announce tomorrow that they cut the available space they offer by half, this move is likely to have a serious impact on their reputation. This is why CME considers only deviations to a higher level of service. Decreasing the level of service is not considered an option.

Other, more technical, justifications for the restrictions imposed by CMEs (on the firm strategies) are given in Appendix A. The next two properties show that CMEs exist, and that they are closely related to other solution concepts.

Observation 2 *For every game instance, best response dynamics (of the firms) must converge into a CME.*

Proof. Note that the only strategies of a firm f_i that can result from best response dynamics are $0, 1, \ldots, r_i - 1$, *i.e.*, a finite set. The observation now follows directly from Proposition 1, the fact that firms can only alternate between a finite number of strategies, and that they can never repeat a strategy by lowering payoff. □

Proposition 3 *Given a commitment equilibrium E of the extensive form game, there exists a pure sub-game perfect equilibrium in which the utilities of all clients and firms are equal to their utilities in E.*

The last proposition allows us to think of CMEs as an equilibrium selection criterion, which favors sub-game perfect equilibria that are attained via a natural iterative process: if a firm deviates, the resulting configuration of the clients can be achieved from the original one by a sequence of best responses.

4 Benefits of Competition

In this section we discuss the possible benefit to the clients from competition between firms. To do that, we first define a way to measure the effect of competition on the social welfare of the clients. Given a network G, For every instance I we denote the social welfare in the *worst* CME by $SW^*(I)$. Under a monopoly, all clients select the single firm, and in the worst case get no service. We define the *monopoly welfare* $MW(G) = SW^*(G, 1, 0)$. Note that for $I = \langle G, F \rangle$, MW only depends on G.

The *clients' value of competition* of the instance $I = \langle G, F \rangle$ is now defined as the ratio $CVC(I) = SW^*(I)/MW(G)$. Thus, values of $CVC(I)$ greater than 1 indicate that the society (of clients) gains from the competition between firms, whereas values lower than 1 mean that the competition had an adverse effect on the clients.

For reference, we also define $OPT(I)$ as the *maximal* social welfare of any outcome of I. It clearly holds that

$$OPT(I) \leq \sum_{j,j' \in N} w_{j,j'} + \max_{f_i \in F} r_i \cdot \sum_j a_j \ ,$$

and when there are no preferences over firms, then this is an equality: in the best outcome all clients share the same firm, and the firm is paying the maximal payoff.

There are several reasons for focusing on the worst equilibrium. First, firms may use non-binding agreements to settle on outcomes that are good for them and bad for the clients. Second, this is a worst case assumption allowing us to put a lower bound on the welfare in any other case. And finally, the *best* CME coincides with the optimal outcome described above, and is therefore quite trivial.

In this section we focus on the network structure, and hence assume (unless explicitly mentioned otherwise), that $a_j = a$ for all $c_j \in N$ and $r_i = r$ for all $f_i \in F$. These simplifying assumptions will be relaxed in Section 5.

The clients' value of competition measures how much the clients gain from competition. It is natural to predict that competition will improve the outcome for the clients. Indeed, a duopoly (two firms) typically yields significantly higher welfare than a monopoly (although not always, see Prop. 8). The following proposition shows that the clients value of competition can be infinite. Informally, it implies that a second firm can significantly improve the total utility of the clients.

Proposition 4 *There is a game instance I where $CVC(I) = \infty$.*

Proof. Consider a game instance with two firms f_1 and f_2 having $r > 0$, and one client with $a = 1$. In every CME of this game, there must be a firm giving payoff of $x_i \geq r - 1$. Hence, $SW^*(I) \geq r - 1$. On the other hand, if there was a single firm, the utility the client would have been 0, as the firm would have paid 0 and the client has no neighbors. Thus $CVC(I) \geq (r-1)/0 = \infty$. □

In contrast to the potentially significant improvement in the social welfare produced by a second firm, the next theorem shows that additional firms can only have a limited positive effect on the clients. We prove this strong negative result by showing that a duopoly already extracts a constant fraction of the optimal social welfare (which in itself is a positive statement on duopolies). [8]

Theorem 5. *Let $I_m = \langle G, m, r\rangle$ be an arbitrary game instance (with $a_j = a$ for all $c_j \in N$ and $r_i = r$ for all $f_i \in F$). Let B be* any *CME outcome in I_m for $m \geq 2$. Then, $SW(I_m, B) \leq \beta \cdot SW^*(I_2)$ for some constant $\beta < 7$.*

[8] We emphasize that the theorem still holds under preferences/fixed clients (see Remark 1), with some modifications of the proof.

Proof sketch. Denote by A the worst CME of I_2, and let x_1 and x_2 denote the payoff levels of firms f_1 and f_2 in A, respectively. Also, denote by $C_1, C_2 \subseteq N$ the sets of clients of outcome A corresponding to firms f_1 and f_2, respectively, and let $n_1 = |C_1|, n_2 = |C_2|$. We assume, w.l.o.g., $n_1 \geq n_2$. For every client $c_j \in C_i$, we denote $\gamma_j = \sum_{c_{j'} \in C_i} w_{j,j'}$, and $\delta_j = \sum_{c_{j'} \in N \setminus C_i} w_{j,j'}$. We also use the average values $\gamma_i^* = \frac{1}{n_i} \sum_{c_j \in C_i} \gamma_j$ and $\delta_i^* = \frac{1}{n_i} \sum_{c_j \in C_i} \delta_j$.

Observe that for *every CME* E: $SW(I_m, E) \leq OPT(I_m) = OPT(I_2)$, and in particular this inequality holds for $E = B$. Therefore to prove a constant bound, it is sufficient to bound the *price of anarchy*[9] with 2 firms, *i.e.*, to show that $OPT(I_2) \leq O(SW(I_2, A))$. Let $u_j(A), u_j(OPT)$ be the utility of client j under the configurations A and OPT (in instance I_2). For $j \in C_i$, it holds that $u_j(A) = ax_i + \gamma_j$, whereas $u_j(OPT) = r + \gamma_j + \delta_j$.

The minimal increase in x_{-i} that can convince client $c_j \in C_i$ to switch firms is $\varepsilon_j = (\gamma_j/a + x_i) - (\delta_j/a + x_{-i} + 1) \geq 0$. Assume that clients in C_i are ordered by non-decreasing ε_j. By comparing firm's utility with and without the increase, we can show that f_i cannot gain by attracting clients $c_1, \ldots, c_j$:

$$(r - x_{-i}) \cdot n_{-i} \geq (r - (x_{-i} + \varepsilon_j))(n_{-i} + j) \ . \tag{1}$$

By rearranging, we now get: $r \leq \varepsilon_j \left(1 + \frac{n_{-i}}{j}\right) + x_{-i}$.

For any non-decreasing vector $\mathbf{z} = (z_1, \ldots, z_m)$ of non-negative numbers, denote its average by $z^* = \frac{1}{m} \sum_{j \leq m} z_j$. Let $\tau \in (0,1)$ be some fraction, and let

$$\alpha_\tau = \max_{\mathbf{z} \geq \mathbf{0}} \{z_{\lceil \tau m \rceil}/z^*\}, \quad \Theta_\tau = \alpha_\tau(1 + 1/\tau) \ .$$

For example, if $\tau = 1/2$ (*i.e.*, $z_{\lceil \tau n_i \rceil}$ is the median of $\mathbf{z}$), then $\Theta_\tau = 6$.

Let $\varepsilon_i^* = \frac{1}{n_i} \sum_{c_j \in C_i} \varepsilon_j$. In what follows, we will take an arbitrary fraction τ, and prove our bound as a function of Θ_τ. We assume n_1, n_2 are sufficiently large so as to ignore rounding (*i.e.*, that $\lceil \tau n_i \rceil \cong \tau n_i$).

Applying the inequality above for $j = \tau n_i$ gives us

$$r \leq \left(1 + \frac{n_{-i}}{\tau n_i}\right) \varepsilon_{\tau n_i} + x_{-i} \leq \left(1 + \frac{n_{-i}}{\tau n_i}\right) \alpha_\tau(\gamma_i^*/a - \delta_i^*/a + x_i + 1). \tag{2}$$

In particular, for the larger firm f_1:

$$r \leq \left(1 + \frac{1}{\tau}\right) \alpha_\tau(\gamma_1^*/a - \delta_1^*/a + x_1 + 1) = \Theta_\tau(\gamma_1^*/a - \delta_1^*/a + x_1 + 1) \ , \qquad \text{and}$$

$$\sum_{j \in C_1} u_j(OPT) \leq \sum_{c_j \in C_1} (ar + \delta_j + \gamma_j) = n_1(ar + \delta_1^* + \gamma_1^*)$$

$$\leq n_1(\Theta_\tau(\gamma_1^* + ax_1 + a) + \gamma_1^*) \leq (\Theta_\tau + 1) \sum_{j \in C_1} (\gamma_j + ax_1 + a) \cong (\Theta_\tau + 1) \sum_{j \in C_1} u_j(A) \ .$$

[9] The price of anarchy of a game is the ratio between the optimal social welfare achievable by any configuration, and the worst social welfare of any Nash equilibrium.

This means that at least the clients of the larger firm f_1 cannot gain on average more than a factor of $\Theta_\tau + 1$, plus some additive term $O(na)$ that does not depend on the welfare. Moreover, this term goes to zero when we use smaller minimal units of storage a. As for the smaller firm f_2 we consider two cases.

The first case is $\gamma_2^*/a - \delta_2^*/a + x_2 \geq \gamma_1^*/a - \delta_1^*/a + x_1$. In this case, for $i = 2$

$$r \leq \Theta_\tau(\gamma_1^*/a - \delta_1^*/a + x_1 + 1) \leq \Theta_\tau(\gamma_2^*/a - \delta_2^*/a + x_2 + 1) \ ,$$

and therefore, the same arguments used above can also be used to bound $\sum_{j \in C_2} u_j(OPT)$. This concludes the first case, as

$$OPT(I_2) \leq (\Theta_\tau + 1) \sum_{j \in C_1} u_j(A) + (\Theta_\tau + 1) \sum_{j \in C_2} u_j(A) + O(na) \cong (\Theta_\tau + 1) \cdot SW(A).$$

We now consider the second case, where $\gamma_2^*/a - \delta_2^*/a + x_2 < \gamma_1^*/a - \delta_1^*/a + x_1$. Denote $\gamma_i^{**} = \gamma_i^* + ax_i$. Using this notation, the above inequality becomes: $\gamma_2^{**} - \delta_2^* < \gamma_1^{**} - \delta_1^*$. Observe that:

$$SW(A) = \sum_{j \in C_1} u_j(A) + \sum_{j \in C_2} u_j(A) = n_1\gamma_1^{**} + n_2\gamma_2^{**} \ , \tag{3}$$

and therefore:

$$\begin{aligned}
OPT(I_2) &= \sum_{j \in C_1} u_j(OPT) + \sum_{j \in C_2} u_j(OPT) = n_1(ar + \gamma_1^* + \delta_1^*) + n_2(ar + \gamma_2^* + \delta_2^*) \\
&\leq n_1(\alpha_\tau(\gamma_1^{**} - \delta_1^* + a)(1 + \frac{n_2}{\tau n_1}) + \gamma_1^* + \delta_1^*) + n_2(\alpha_\tau(\gamma_2^{**} - \delta_2^* + a)(1 + \frac{n_1}{\tau n_2}) + \gamma_2^* + \delta_2^*) \\
&\leq (\alpha_\tau + 1)n_1(\gamma_1^{**} + a) + \frac{\alpha_\tau}{\tau} n_2(\gamma_1^{**} + a - \delta_1^*) + (\alpha_\tau + 1)n_2(\gamma_2^{**} + a) + \frac{\alpha_\tau}{\tau} n_1(\gamma_2^{**} + a - \delta_2^*) \\
&\cong (\alpha_\tau + 1)SW(A) + {}^{\alpha_\tau}/_{\tau}(n_2(\gamma_1^{**} - \delta_1^*) + n_1(\gamma_2^{**} - \delta_2^*)) \ , \qquad \text{(By Eq. (3))}
\end{aligned}$$

where the first inequality holds by applying Equation (2) once with $i = 1$, and once with $i = 2$. Finally, since $w \geq x, y \geq z$ implies $wz + xy \leq wy + xz$,

$$\begin{aligned}
OPT(I_2) &\lessapprox (\alpha_\tau + 1)SW(A) + \frac{\alpha_\tau}{\tau}(n_1\gamma_1^{**} + n_2\gamma_2^{**}) = (\alpha_\tau + 1)SW(A) + \frac{\alpha_\tau}{\tau}SW(A) \\
&= (1 + \alpha_\tau(1 + {}^1/_\tau))SW(A) = (\Theta_\tau + 1)SW(A) \ ,
\end{aligned}$$

where $w = (\gamma_1^{**} - \delta_1^*);\ x = (\gamma_2^{**} - \delta_2^*);\ y = n_1; z = n_2$.

This completes the proof of the inequality $SW(I_m, B) \leq (\Theta_\tau + 1)SW(I_2, A) + O(na)$. Since by selecting the median $\tau = {}^1/_2$ we get $\Theta_\tau = 6$, the ratio is at most 7 (plus some additive term that diminishes with the resolution). □

By optimizing the value of τ in the proof of Theorem 5, it can be shown that $\beta = \Theta_\tau \leq 6.828$ (for sufficiently large n). A possible extension of the theorem, which we leave open for future research, is how the ratio changes as a function of the vector $\mathbf{r}$ in the presence of heterogeneous firms (*i.e.*, when not all the entries in r are identical). We conjecture that if the r_i values are close to one another, then the benefit of having more competing firms will still be limited.

The next theorem complements the upper bound by showing that there exist games in which introducing a third firm (or more) can increase the social welfare by a factor of 2. It remains as an open question to close the gap between these two constants.

Proposition 6 *For any $m > 2$, there exists an instance $I_m = \langle G, m, r \rangle$ s.t. $SW^*(I_m) \geq (2 - o(1))SW^*(I_2)$.*

Notice that the proof of Theorem 5 in particular shows that the price of anarchy (for clients) is upper bounded by 6.828. Any better bound on the clients' value of competition that uses a similar proof technique must also translate into an upper bound on the price of anarchy. The following proposition shows that the price of anarchy is at least 4.26 in the worst case. Thus, matching the lower bound introduced by Proposition 6 will probably require different techniques.

Proposition 7 *There exists an instance $I_2 = \langle G, 2, r \rangle$, s.t. $OPT(I_2) \geq (1 + \frac{1}{1-\ln 2})SW^*(I_2) \cong 4.26 SW^*(I_2)$.*

5 The Cost of Competition

In this section we are interested in the question: "how low can the client value of competition be?". We start with a negative example, showing that the welfare of clients can linearly degrade with the number of firms, *i.e.*, that without further restrictions the damage to clients from excessive competition is essentially unbounded.

While our construction uses a particular structure, we later show in Prop. 11 that a milder linear degradation may also occur under the complete graph.

Proposition 8 *For every $m \geq 2, \varepsilon > 0$, there exists a game instance I_m with $CVC(I_m) \leq {}^1\!/_m + \varepsilon$.*

Our next results show that Proposition 8 demonstrates the worst possible case. If the number of firms m is bounded then so is the value of competition. Interestingly, the proof of Theorem 9 provides a lower bound on the welfare not just in a CME, but in fact in any outcome where clients are stable (regardless of firms' strategies). The same is true for Theorem 10.

Theorem 9. *For every game instance $I = \langle G, F \rangle$ with m firms, $SW^*(I) \geq MW(G)/m$. That is, $CVC(I) \geq {}^1\!/_m$.*

Proof. If there is only a single firm, all clients join it and get zero payoff. Hence, the monopoly welfare is $MW(G) = \sum_{j=1}^{n} \sum_{j' \neq j} w_{j,j'}$. Let us now focus on an arbitrary CME E of I. Consider a client c_j which joins firm f_i under E. Clearly, for every other firm $f_{i'}$ it must hold that:

$$a_j \cdot x_i + \sum_{j' \in C_i \setminus \{c_j\}} w_{j,j'} \geq a_j \cdot x_{i'} + \sum_{j' \in C_{i'} \setminus \{c_j\}} w_{j,j'} \quad . \tag{4}$$

Observe that this inequality trivially holds also when $i = i'$. Hence, we can sum the inequalities for every $1 \leq i' \leq m$, and get:

$$m \cdot \left[a_j \cdot x_i + \sum_{j' \in C_i \setminus \{c_j\}} w_{j,j'} \right] \geq \sum_{i'=1}^{m} \left[a_j \cdot x_{i'} + \sum_{j' \in C_{i'} \setminus \{c_j\}} w_{j,j'} \right] \tag{5}$$

$$= \sum_{j' \neq j} w_{j,j'} + \sum_{i'=1}^{m} a_j \cdot x_{i'} \geq \sum_{j' \neq j} w_{j,j'} \ .$$

Rearranging, we get that the utility of c_j is at least $\sum_{j' \neq j} w_{j,j'}/m$. Hence, the total utility of all clients is at least: $\frac{1}{m} \cdot \sum_{j=1}^{n} \sum_{j' \neq j} w_{j,j'} = \frac{1}{m} \cdot MW(G)$. □

In the last theorem, the number of firms can also be replaced with the *maximum degree*. See Proposition 4 in the appendix.

5.1 The complete graph

The above results use examples of dense graphs to show that the clients' value of competition tend to be low. It thus makes sense to consider the complete graph, with equal edge weights (if different edge weights were allowed, non-complete graphs could be simulated by giving some edges very low weights, making them insignificant). For ease of notation, let us assume that all edge weights are 1. We note that the case of a complete graph models the situation where clients only care about the number of other clients sharing their firm, as in [15].

The main result of this section states that with complete graphs over a small set of clients, the loss due to competition (even with many firms) cannot be too high.

Theorem 10. *For any game instance $I = \langle G, F \rangle$ where G is a complete graph, it holds that $CVC(I) = \Omega(n^{-1/3})$.*

Moreover, the above bound is tight up to low order terms:

Proposition 11 *There is a family of instances $(I_m)_{m \geq 1}$, each with a complete social graph over $n(m)$ clients (where $n(m)$ is a bounded function of m), for which $CVC(I_m) = O(n^{-1/3})$.*

The instances constructed in the proof of the last proposition have the additional property that $CVC(I^n) = O(1/m)$. Hence, the proof also shows that the bound given by Theorem 9 is tight (up to lower order terms) even if we restrict ourselves to complete graphs (but allow r_i to vary, and allow $n = \Omega(m^3)$).

6 Firms' Revenue

While the main bulk of this paper is devoted to study the effect of increased competition on the welfare of clients, it is also important to understand how social connections change the revenue of the competing firms. In this section we provide a preliminary result in this direction. Given a game instance I, we define the *firms' revenue* as the sum of firms' utilities in the best CME of this game instance (best for the firms), *i.e.*, $FR(I) = \max\{\sum_{f_i \in F} v_i(E) \mid E \text{ is CME of } I\}$. The choice of the best CME in the definition of $FR(I)$ is justified by the observation that there always exists a CME with 0 utility for the firms (the one where $x_i = r_i$ for every firm f_i).

The example constructed in Proposition 6 can be analyzed to show that the addition of a third firm decreases the total revenue by half. The next theorem shows that similar examples exist for any value of m.

Theorem 12. *For any $m > 1$, there exists an instance $I_m = \langle G, m, r\rangle$ for which the total firms' revenue strictly decreases with the addition on an extra firm. Formally, $I_{m+1} = \langle G, m+1, r\rangle$ obeys $FR(I_{m+1}) \leq \frac{m-1}{m-2+\ln m} \cdot FR(I_m)$.*

The fact that increased competition can lead to lower revenue may sound trivial. However it should be noted that the marginal value of each client to a firm is *fixed*. Hence, without the network structure the number of firms does not affect the total revenue at all. It is the presence of social connections that makes competition potentially harmful for the firms. We leave for future research finding the maximal loss in revenue that may result by adding a firm. Another interesting question that we leave open is whether the firms' total revenue can also *increase* when a new firm is introduced into the game.

7 Discussion

We introduced a natural network-based model of competition with positive externalities between clients, and analyzed the effect of the number of firms on the welfare of clients.

7.1 Related work

Katz and Shapiro [15] coined the term "network externalities" to denote situations where the decision of clients effect their neighbors in the network. They described a market where consumers' utility is partly derived from the *size* of the network they select, *i.e.*, the number of other clients selecting the same firm. Subsequently, Banerji and Dutta [3] studied an extension of the model that does take into account the structure of the network, by describing the interaction among groups of clients in the limited case of two firms.

Both of these papers, as well as our work, can be classified under the "macro approach" of Economides, which seeks to understand the effect of positive externalities on consumption patterns, rather than to explain their source [11].

The utility structure of the clients in our model is similar to the one in [15], but the clients are sensitive to the identity of their peers, rather than to their number only (as in [3]). Moreover, we assume a simple myopic behavior by clients whereas in the Katz and Shapiro model clients predict the *expected size* of the firms and act accordingly. Thus our equilibrium concepts are substantially different.

Beyond the technical differences between the models, our paper brings a novel perspective. In particular, the main focus of Banerji and Dutta is on the structure of the outcome partition in the case of two firms. Whereas we study the effect of the number of firms on the the clients' social welfare under network externalities. Similarly to Economides but due to different considerations, we arrive at the conclusion that excessive competition may compromise clients' welfare.

In particular, *compatibility* and *standardization* play a major role in some models of network externalities as in [15,11]. In our model the level of compatibility among competing services is assumed to be fixed (at least in short time scales). The strategic decisions of firms are therefore simplified to setting the price/service level, as in traditional Bertrand competition [4].

Other aspects of network externalities that have been studied focused on factors such as *price discrimination* [14], or particular diffusion dynamics [13].

A two-phase framework has been suggested as a model for competition in other domains, where firms first commit to strategies, and clients follow by playing a game induced by firms' actions. Examples of such games are available in the domain of *group buying*, where each vendor commits to a discount schedule based on quantity [8,9,18]. While the utility structure of both vendors and buyers in the group buying domain is substantially different, our work demonstrates how such a two-phase framework can be applied for modeling the effect of network externalities.

7.2 Conclusions

We showed that excessive competition can fragment the network and eliminate most of the clients' utility. On the other hand, two competing firms already guarantee at least a constant fraction of the clients' maximal welfare, and thus, the positive effect of adding more competitors is bounded, whereas the potential damage is much more significant. These results complement the findings by Economides and others on the potential damage in excessive competition, and provide some formal justification to statements that unregulated competition can be inefficient or even hazardous for society. For example, the necessity of regulation *against* competition has been discussed in domains such as labor markets, banking, and others [20,6,19]. A recent formal treatment of auctions with partial information reveals a similar effect to the one we found, showing that the entry of additional auctioneers incurs a loss on the bidders' welfare [21].

A loss of welfare is quite expected when there are *negative* externalities for firms' actions, such as pollution or waste of resources [2]. We emphasize, however, that the potential negative effect of competition in our model is not due to the typical race-to-the-bottom scenario, but rather due to (positive) externalities between the clients or workers themselves.

Future work In order to focus on the effect of network externalities, we simplified some factors that are necessary for a better understanding of real markets. Possible future work will consider non-linear utilities for the firms (reflecting, *e.g.*, decreasing marginal production costs that are typical to economies of scale), partial information of the network, and far-sighted strategies used by the clients.

This work outlined bounds on the value of competition assuming either arbitrary or complete networks. Networks in the real world tend to have certain characteristics and structure, that can possibly be exploited to get better bounds on the effect of competition under real world externalities.

Finally, we introduced only a preliminary result on the firms' revenue. Much more work is needed in order to gain an understanding of this quantity. For example, we showed that there is a class of instances where the introduction of a new firm decreases the firms' total revenue by at least a given factor. We conjecture that the converse does not hold, *i.e.*, that the introduction of additional firms can never increase the total revenue. This conjecture, if true, implies that both firms and clients are prone to the effects of excessive competition, and further emphasizes the importance of regulation.

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A Relaxing equilibrium requirements

CMEs restrict the firms' strategies to be increasing and discrete. In this appendix we show the necessity of these restrictions. First, we show that relaxing each one of these restrictions can result in a game where firms' actions may not converge to a CME. Thus without these restrictions, Observation 2 does not hold.

Let us begin with the "discrete strategies" restriction. Assume firms may only increase payoff, but continuous payoffs are allowed.[10] Then, in the presence of a single client, firms may compete by offering an increasingly smaller improvement in service. This observation is summarized by the following proposition.

Proposition 1. *Assuming firm f_i is allowed to increase x_i to any larger value. Then, the game may not converge to a CME.*

Let us consider, now, what happens if we relax the "increasing strategies" restriction. The following proposition shows that in this case a CME may not even exist.

Proposition 2. *Assuming firm f_i is allowed to change x_i to any integer value. Then, there are games (even with two firms) where there is no CME.*

Proof. We have two firms with $r = 6$. There are 4 clients: $\{a, b, c\}$ form a triangle, where $w(a, b) = w(b, c) = 2$, $w(a, c) = 1$. A fourth client d is linked to b, with $w(b, d) = 1 + 1/2$ (thus, clients b, d are never indifferent between firms). The triangle clients are always together except in extreme cases (treated separately in the end of the proof), so we can treat them like a single agent A, which requires a difference of > 3 in payoff to switch firms.

Assume toward a contradiction that a CME exists. W.l.o.g., the triangle A is with firm 1, which gains either $3(6 - x_1)$, or $4(6 - x_1)$ if it also has d. In the first case, $x_1 \leq \max\{0, x_2 - 3\}$, as otherwise it can reduce service level to $x_1' = \max\{0, x_2 - 3\}$ without losing clients, and similarly in the second case $x_1 \leq \max\{0, x_2 - 1\}$.

Assume the first case. Then, since firm 2 has d, $x_2 \geq x_1 + 1.5 \geq 1.5 \Rightarrow x_2 \geq 2$, and thus, $v_2 = 1(6 - x_2) \leq 4$. If $x_1 = 0$, then by setting x_2 to 4, firm 2 will get all clients, and earn $4(6 - 4) = 8 > v_2$ - a contradiction. If $x_1 > 0$, then $x_2 \geq x_1 + 3$. Thus, firm 2 can reduce x_2 by 1 without losing its client - a contradiction. This rules out the case where firm 2 has d.

Suppose case 2, *i.e.*, firm 1 has all clients. By setting $x_2' = 5$, firm 2 cannot attract any client, and in particular client d. Thus, $x_1 \geq 5 - 1.5 = 3.5$, *i.e.*, $x_1 \in \{4, 5\}$. Then, $v_1 \leq 4(6 - 4) = 8$. by setting $x_1' = 3 = 6 - 3 \geq x_2 - 3$, firm 1 will lose at most client d (and maybe not even her). The revenue will be $v_1' \geq 3(6 - 3) = 9 > v_1$. Again a contradiction.

It remains to handle the case where the triangle is split, by showing that clients cannot be in equilibrium when this happens. If the cut separates $\{a, c\}$ from $\{b\}$, then in both sides we have clients that have more neighbors across the cut. This entails both $x_1 > x_2$ and $x_2 > x_1$, *i.e.*, a contradiction. Similarly with the cut $(\{a, b\}; \{c, d\})$. If the cut separates $\{a\}$ from $\{c\}$ (w.l.o.g., $\{a, b, d\}$ are with firm 1, $\{c\}$ is with firm 2) then $x_2 \geq x_1 + 3$. But this means that a wants to switch to firm 2. A contradiction. □

Moreover, the following simple example shows that even when a CME does exist, a sequence of best-responses by firms may result in a cycle.

Consider a single game of two clients connected by an edge of weight 1 and two firms with $r = 3$. Assume that in the initial configuration c_1 and c_2 both belong to f_1 and $x_1 = x_2 = 0$. Now, every firm

[10] A CME still exists of course, where all firms are offering the maximal service level $x_i = r_i$.

f_i in turn will increase x_i to 2 in order to acquire both clients. Right after that, the firm will decrease its service level back to 0.

Finally, let us consider what happens when we remove *both* restrictions. The next result shows that, for two firms, removing both restrictions results in a game with no interesting equilibria.

Proposition 3. *Assuming firm f_i is allowed to change x_i to any value. Then, in* every *CME (if exists) of two firms, either $x_i = 0$ for every firm that has clients, or some $x_i = r_i$. That is, either there is a firm making no profit, or all clients get no service.*

Proof. Suppose there is some client c getting a non-zero payoff. W.l.o.g., we may assume $c \in C_1$, which implies $x_1 > 0$. Since f_1 has clients, it could benefit from reducing x_1 by $\varepsilon > 0$. However, since it does not do that (E is an equilibrium), we learn that any reduction in x_1 will result in a reduction in the number of clients of f_1. That is, there is at least one client $c' \in C_1$ that will desert to the other firm f_2.

From the view point of the clients, the exact value of x_i is not important when selecting a firm. The only important thing is the difference between the payoffs given by the two firms. Assume toward a contradiction that $x_2 < r_2$. Hence, f_2 can get at least one additional client by raising x_2 by any $\varepsilon > 0$. For small enough ε, this increases the utility of f_2, which leads to a contradiction. □

B Omitted proofs

PROPOSITION 1. *If the strategies of the firms are fixed (i.e. they are not players), and the clients switch strategies according to a best response dynamics, then the client strategies converge into a pure Nash equilibrium.*

Proof. Assume the service level of the firms is fixed $\mathbf{x} = (x_1, x_2, \ldots, x_m)$. Our objective in this section is to prove that best response dynamics of the clients converge into a pure NE. Given a client c, let $x(c)$ and $C(c)$ denote the set of clients, and service level of the firm c is using, respectively. Consider the following potential function, whose value depends on the partition $P = (C_1, \ldots, C_m)$:

$$\Phi(P) = \sum_{j=1}^{n} \left[a_j \cdot x(c_j) + \frac{1}{2} \sum_{j' \in C(c_j) - \{c_j\}} w_{j,j'} \right] .$$

We claim that this potential function strictly increases with every best response move, and therefore, best response dynamics must converge to a Nash equilibrium. Consider a move of c_j. Assume, w.l.o.g., that c_j switch from firm f_1 to f_2. Then the difference in j's utility is

$$a_j \cdot x_2 + \sum_{j' \in C_2} w_{j,j'} - a_j \cdot x_1 + \sum_{j' \in C_1 - \{c_j\}} w_{j,j'} . \tag{6}$$

The change in the potential function has two parts. First, the term of j changes by:

$$a_j \cdot x_2 + \frac{1}{2} \sum_{j' \in C_2} w_{j,j'} - a_j \cdot x_1 - \frac{1}{2} \sum_{j' \in C_1 - \{c_j\}} w_{j,j'} .$$

Since the gain of c_j and $c_{j'}$ from sharing a firm with each other is equal, the change in the other terms is equal to the change due to social connections in the first part, *i.e.*:

$$\begin{aligned} \Phi(C_1 \setminus \{j\}, C_2 \cup \{j\}, C_3, \ldots) - \Phi(P) &= a_j \cdot x_2 + \frac{1}{2} \sum_{j' \in C_2} w_{j,j'} - a_j \cdot x_1 - \frac{1}{2} \sum_{j' \in C_1 - \{c_j\}} w_{j,j'} \\ &\quad - \frac{1}{2} \sum_{k \in C_1 \setminus \{j\}} w_{k,j} + \frac{1}{2} \sum_{k \in C_2} w_{k,j} \\ &= a_j \cdot x_2 + \sum_{j' \in C_2} w_{j,j'} - a_j \cdot x_1 - \sum_{j' \in C_1 - \{c_j\}} w_{j,j'}, \end{aligned}$$

i.e., exactly equals the change in utility from Eq. (6). We emphasize that the proposition holds even in the presence of fixed clients. It thus still holds, by Remark 1, for games where clients have preferecens over firms. Preferences can also be handled directly, by replacing $a_j x_i$ with $a_j x_i + q_{j,i}$ everywhere. ☐

PROPOSITION 3. *Given a commitment equilibrium E of the extensive form game, there exists a pure sub-game perfect equilibrium in which the utilities of all clients and firms are equal to their utilities in E.*

Proof. A sub-game perfect equilibrium in our context means the following.

- The strategy of each firm f_i is still a number $x_i \in \mathbb{N}$.
- The strategy of each client is a function that specifies the firm it will join given every possible combination of firm strategies. More formally, the strategy of c_j is a function $s_j : \mathbb{N}^m \to \{1, 2, \ldots, m\}$ specifying the firm c_j will join given any strategy vector $\mathbf{x} \in \mathbb{N}^m$ of the firms.
- A configuration is a sub-game perfect equilibrium if:
  - Given a fixed strategy vector $\mathbf{x} \in \mathbb{N}^m$, no client c_j can gain by changing s_j.
  - Given that the strategies of the clients are fixed, no firm f_i can gain by increasing x_i.

Consider now a CME E. We would like to construct from it a sub-game perfect equilibrium P. We define the strategies of the firms in P to be identical to the strategies of the firms in E. Let $\mathbf{x} \in \mathbb{N}^m$ be the vector of the strategies of the firms in E (and P). For every client c_j we define $s_j(\mathbf{x})$ to be equal to the strategy played by c_j in E. Consider a strategies vector $\mathbf{x}' \in \mathbb{N}^m$ that can be reached from $\mathbf{x}$ by increasing one coordinate i, let $E(\mathbf{x}')$ be the projected outcome obtained from E via deviation $\mathbf{x}'$, let f_i deviate by increasing x_i to be equal to x'_i, and then let the clients reach a new Nash equilibrium. We define $s_j(\mathbf{x}')$ to be equal to the strategy of c_j in $E(x')$. Observe that every vector $\mathbf{x}' \in \mathbb{N}^m$ can be reached in that way from $\mathbf{x}$ by at most one way, hence, this construction does not assign multiple values to $s_j(\mathbf{x}')$. Finally, all values of s_j that are not defined above are set arbitrarily.

Clearly the utility of every firm and client in E and P are equal. Let us prove that the configuration P is indeed a sub-game perfect equilibrium. First consider a deviation by a firm f_i. If f_i deviates by increasing x_i, than the a new strategies vector $\mathbf{x}' \in \mathbb{N}^m$ is only different from $\mathbf{x}$ in the i^{th} coordinate. Hence, by the above construction, for every client c_j, $s_j(\mathbf{x}')$ is equal to the strategy of c_j in $E(\mathbf{x}')$. Thus, the utility of all firms and clients after the deviation is equal to their utility in $E(\mathbf{x}')$. On the other hand, the definition of CME implies that the utility of f_i in $E(\mathbf{x}')$ is no larger than its utility in E. Hence, f_i has no incentive to deviate.

Let us now consider a deviation by a client c_j. When considering deviations of the clients, we assume the strategies of the firms are fixed. Hence, the only important thing about the new strategy of c_j is the firm f_i to which c_j goes given the fixed strategies of the firms. If, given P, c_j has an incentive to deviate to a strategy that assigns it to f_i, than by construction, it also has such an incentive to deviate to f_i given E. However, this is a contradiction since the clients are in a Nash equilibrium in E. ☐

THEOREM 5. *Let $I_m = \langle G, m, r\rangle$ be an arbitrary game instance (with $a_j = a$ for all $c_j \in N$ and $r_i = r$ for all $f_i \in F$). Let B be* any *CME outcome in I_m for $m \geq 2$. Then, $SW(I_m, B) \leq \beta \cdot SW^*(I_2)$ for some constant β.*

Proof. Denote by A the worst CME of I_2, and let x_1 and x_2 denote the payoff levels of firms f_1 and f_2 in A, respectively. Also, denote by $C_1, C_2 \subseteq N$ the sets of clients of outcome A corresponding to firms f_1 and f_2, respectively, and let $n_1 = |C_1|, n_2 = |C_2|$. We assume, w.l.o.g., $n_1 \geq n_2$. For every client $c_j \in C_i$, we denote $\gamma_j = \sum_{c_{j'} \in C_i} w_{j,j'}$, and $\delta_j = \sum_{c_{j'} \in N \setminus C_i} w_{j,j'}$. We also use the average values $\gamma^*_i = \frac{1}{n_i}\sum_{c_j \in C_i} \gamma_j$ and $\delta^*_i = \frac{1}{n_i}\sum_{c_j \in C_i} \delta_j$.

Observe that for *every CME E*: $SW(I_m, E) \leq OPT(I_m) = OPT(I_2)$, and in particular this inequality holds for $E = B$. Therefore to prove a constant bound, it is sufficient to bound the *price of anarchy* with 2 firms, *i.e.*, to show that $OPT(I_2) \leq O(SW(I_2, A))$. Let $u_j(A), u_j(OPT)$ be the utility of client j under the configurations A and OPT (in instance I_2). For $j \in C_i$, it holds that $u_j(A) = ax_i + \gamma_j$, whereas $u_j(OPT) = ar + \gamma_j + \delta_j$.

The minimal increase in x_{-i} that can convince client $c_j \in C_i$ to switch firms is $\varepsilon_j = (\gamma_j/a + x_i) - (\delta_j/a + x_{-i} + 1) \geq 0$. Assume that clients in C_i are ordered by non-decreasing ε_j. We would like to prove that f_i cannot gain by attracting clients $c_1, \ldots, c_j$:

$$(r - x_{-i}) \cdot n_{-i} \geq (r - (x_{-i} + \varepsilon_j))(n_{-i} + j) \ . \tag{7}$$

If $x_{-i} + \varepsilon_j > r$, then the right hand side is negative whereas the left hand side is non-negative (the left hand side represents the revenue of f_{-i} under B and a firm never has a negative revenue in a CME). Otherwise, observe that by offering $x'_{-i} = x_{-i} + \varepsilon_j$, firm f_{-i} can attract *at least* j more clients,[11] to a utility of at least

$$(r - x_{-i}) \cdot n_{-i} \geq (r - x'_{-i})(n_{-i} + j) = (r - x_{-i} - \varepsilon_j)(n_{-i} + j) \ .$$

Since firm $-i$ has no deviation, it must hold that the current profit is at least the new potential profit, which completes the proof of (7). By rearranging, we now get:

$$\begin{aligned}(r - x_{-i}) \cdot n_{-i} \geq (r - (x_{-i} + \varepsilon_j))(n_{-i} + j) &\iff r \cdot n_{-i} - r(j + n_{-i}) \geq -\varepsilon_j(j + n_{-i}) - jx_{-i} \\ &\iff r \leq \varepsilon_j \left(1 + \frac{n_{-i}}{j}\right) + x_{-i} \ . \end{aligned} \tag{8}$$

For any non-decreasing vector $\mathbf{z} = (z_1, \ldots, z_m)$ of non-negative numbers, denote its average by $z^* = \frac{1}{m} \sum_{j \leq m} z_j$. Let $\tau \in (0, 1)$ be some fraction, and let

$$\alpha_\tau = \max_{\mathbf{z} \geq \mathbf{0}} \{z_{\lceil \tau m \rceil} / z^*\}, \quad \Theta_\tau = \alpha_\tau (1 + 1/\tau) \ .$$

For example, if $\tau = 1/2$ (*i.e.*, $z_{\lceil \tau n_i \rceil}$ is the median of $\mathbf{z}$), then $\Theta_\tau = 6$. This is since the median equals at most twice the average (*i.e.*, $\alpha_{1/2} \leq 2$). Thus

$$\Theta_{1/2} = (1 + 1/\tau)\alpha_\tau \leq (1 + \frac{1}{1/2})2 = 3 \cdot 2 = 6 \ .$$

The inequality becomes an equality e.g. for a sequence with $\lfloor m/2 \rfloor$ zeros and $\lceil m/2 \rceil$ ones, where $z_{\lceil \tau m \rceil} = 1$ and $z^* \cong 1/2$.

Let $\varepsilon_i^* = \frac{1}{n_i} \sum_{c_j \in C_i} \varepsilon_j$. In what follows, we will take an arbitrary fraction τ, and prove our bound as a function of Θ_τ. We assume n_1, n_2 are sufficiently large so as to ignore rounding (*i.e.*, that $\lceil \tau n_i \rceil \cong \tau n_i$).

Applying inequality (8) for $j = \tau n_i$ gives us

$$\begin{aligned} r &\leq \left(1 + \frac{n_{-i}}{\tau n_i}\right) \varepsilon_{\tau n_i} + x_{-i} \leq \left(1 + \frac{n_{-i}}{\tau n_i}\right) \alpha_\tau \varepsilon_i^* + x_{-i} \\ &= \left(1 + \frac{n_{-i}}{\tau n_i}\right) \alpha_\tau (\gamma_i^*/a - \delta_i^*/a + x_i - x_{-i} + 1) + x_{-i} \\ &\leq \left(1 + \frac{n_{-i}}{\tau n_i}\right) \alpha_\tau (\gamma_i^*/a - \delta_i^*/a + x_i + 1) \ . \end{aligned} \tag{9}$$

In particular, for the larger firm f_1:

$$\begin{aligned} r &\leq \left(1 + \frac{n_2}{\tau n_1}\right) \alpha_\tau (\gamma_1^*/a - \delta_1^*/a + x_1 + 1) \leq \left(1 + \frac{1}{\tau}\right) \alpha_\tau (\gamma_1^*/a - \delta_1^*/a + x_1 + 1) \\ &= \Theta_\tau (\gamma_1^*/a - \delta_1^*/a + x_1 - x_2 + 1) + x_2 = \Theta_\tau (\gamma_1^*/a - \delta_1^*/a + x_1 + 1) \ . \end{aligned}$$

[11] After the j clients of the first wave, which immediately benefit from switching, other clients may follow.

$$
\begin{aligned}
\sum_{j\in C_1} u_j(OPT) &\le \sum_{c_j\in C_1}(ar+\delta_j+\gamma_j) = n_1(ar+\delta_1^*+\gamma_1^*) \le n_1(\Theta_\tau(\gamma_1^*-\delta_1^*+ax_1+a)+\delta_1^*+\gamma_1^*)\\
&\le n_1(\Theta_\tau(\gamma_1^*+ax_1+a)+\gamma_1^*) = n_1((\Theta_\tau+1)\gamma_1^*+ax_1+a)\\
&= \sum_{j\in C_1}((\Theta_\tau+1)\gamma_j+\Theta_\tau(ax_1+a)) \le (\Theta_\tau+1)\sum_{j\in C_1}(\gamma_j+ax_1+a)\\
&= (\Theta_\tau+1)\sum_{j\in C_1}u_j(A)+O(n_1a)\ .
\end{aligned}
$$

This means that at least the clients of the larger firm f_1 cannot gain on average more than a factor of $\Theta_\tau+1$, plus some additive term that does not depend on the welfare. Moreover, this term goes to zero when we use smaller minimal units of storage. As for the smaller firm f_2 we consider in two cases.

The first case is $\gamma_2^*/a-\delta_2^*/a+x_2 \ge \gamma_1^*/a-\delta_1^*/a+x_1$. In this case, for $i=2$

$$
r \le \Theta_\tau(\gamma_1^*/a-\delta_1^*/a+x_1+1) \le \Theta_\tau(\gamma_2^*/a-\delta_2^*/a+x_2+1)\ ,
$$

and therefore, the same arguments used above show that $\sum_{j\in C_2} u_j(OPT) \le (\Theta_\tau+1)\sum_{j\in C_2} u_j(A)+n_2a$. This concludes the first case, as

$$
\begin{aligned}
OPT(I_2) &= \sum_{j\in C_1}u_j(OPT)+\sum_{j\in C_2}u_j(OPT) \le (\Theta_\tau+1)\sum_{j\in C_1}u_j(A)+(\Theta_\tau+1)\sum_{j\in C_2}u_j(A)+O(na)\\
&= (\Theta_\tau+1)\cdot SW(A)+O(na) = O(SW(A))\ .
\end{aligned}
$$

We now consider the second case, where $\gamma_2^*/a-\delta_2^*/a+x_2 < \gamma_1^*/a-\delta_1^*/a+x_1$. Denote $\gamma_i^{**}=\gamma_i^*+ax_i$. Using this notation, the above inequality becomes: $\gamma_2^{**}-\delta_2^* < \gamma_1^{**}-\delta_1^*$. Observe that:

$$
SW(A) = \sum_{j\in C_1}u_j(A)+\sum_{j\in C_2}u_j(A) = n_1\gamma_1^{**}+n_2\gamma_2^{**}, \tag{10}
$$

and therefore:

$$
\begin{aligned}
OPT(I_2) &= \sum_{j\in C_1}u_j(OPT)+\sum_{j\in C_2}u_j(OPT) = n_1(ar+\gamma_1^*+\delta_1^*)+n_2(ar+\gamma_2^*+\delta_2^*)\\
&\le n_1(\alpha_\tau(\gamma_1^{**}-\delta_1^*+a)(1+\frac{n_2}{\tau n_1})+\gamma_1^*+\delta_1^*)+n_2(\alpha_\tau(\gamma_2^{**}-\delta_2^*+a)(1+\frac{n_1}{\tau n_2})+\gamma_2^*+\delta_2^*)\\
&= \alpha_\tau n_1(\gamma_1^{**}+a)-\alpha_\tau n_1\delta_1^*+\frac{\alpha_\tau}{\tau}n_2(\gamma_1^{**}+a)-\frac{\alpha_\tau}{\tau}n_2\delta_1^*+n_1(\gamma_1^*+\delta_1^*)\\
&\quad+\alpha_\tau n_2(\gamma_2^{**}+a)-\alpha_\tau n_2\delta_2^*+\frac{\alpha_\tau}{\tau}n_1(\gamma_2^{**}+a)-\frac{\alpha_\tau}{\tau}n_1\delta_2^*+n_2(\gamma_2^*+\delta_2^*)\\
&\le (\alpha_\tau+1)n_1(\gamma_1^{**}+a)+\frac{\alpha_\tau}{\tau}n_2(\gamma_1^{**}+a-\delta_1^*)+(\alpha_\tau+1)n_2(\gamma_2^{**}+a)+\frac{\alpha_\tau}{\tau}n_1(\gamma_2^{**}+a-\delta_2^*)\\
&= (\alpha_\tau+1)SW(A)+\frac{\alpha_\tau}{\tau}(n_2(\gamma_1^{**}-\delta_1^*)+n_1(\gamma_2^{**}-\delta_2^*))+O(na)\ , \qquad \text{(By Eq. (10))}
\end{aligned}
$$

where the first inequality holds by applying Equation (9) once with $i=1$, and once with $i=2$. Finally, since $w\ge x, y\ge z$ implies $wz+xy\le wy+xz$, we get in case 2:

$$
\begin{aligned}
OPT(I_2) &\le (\alpha_\tau+1)SW(A)+\frac{\alpha_\tau}{\tau}(n_1(\gamma_1^{**}-\delta_1^*)+n_2(\gamma_2^{**}-\delta_2^*))+O(na)\\
&\le (\alpha_\tau+1)SW(A)+\frac{\alpha_\tau}{\tau}(n_1\gamma_1^{**}+n_2\gamma_2^{**})+O(na) = (\alpha_\tau+1)SW(A)+\frac{\alpha_\tau}{\tau}SW(A)+O(na)\\
&= (1+\alpha_\tau(1+\frac{1}{\tau}))SW(A)+O(na) = (\Theta_\tau+1)SW(A)+O(na) = O(SW(A))\ ,
\end{aligned}
$$

where $w=(\gamma_1^{**}-\delta_1^*);\ x=(\gamma_2^{**}-\delta_2^*);\ y=n_1; z=n_2$.

This completes the proof of the inequality $SW(I_m, B) \leq (\Theta_\tau + 1)SW(I_2, A) + O(na)$. Since by selecting the median $\tau = 1/2$ we get $\Theta_\tau = 6$, the ratio is at most 7 (plus some additive term that diminishes with the resolution).

To find the optimal τ, we note that $\alpha_\tau \leq 1/1-\tau$ ($z_{\tau m}/z^*$ is maximized when there are $(1-\tau)m$ entries with high value $z_{\tau m}$ and the rest of the entries are 0). Thus $\Theta_\tau \leq (1 + 1/\tau)1/1-\tau$. We get the lowest upper bound for $\tau = \sqrt{2} - 1$, for which $\Theta_1 = \frac{\sqrt{2}}{3\sqrt{2}-4} \cong 5.828$, and the price of anarchy is at most $\Theta_\tau + 1 = 6.828$. □

PROPOSITION 6. *For any $m > 2$, there exists an instance $I_m = \langle G, m, r\rangle$ s.t. $SW^*(I_m) \geq (2 - o(1))SW^*(I_2)$.*

Proof. In G there are n clients arranged in two cliques of size $n/2$ with edge weights of 1. For every client c_j, we set $a = 1$. In addition, we set $r = n-3$. Consider $I_2 = \langle G, 2, r\rangle$. In configuration A, each one of the firms gets a single clique of clients and $x_1 = x_2 = 0$. The utility of each client is, thus, $u_j(A) = |\Gamma(j)| = n/2 - 1$. Note that if a firm wants to deviate, it must increase its payoff by at least $x' \geq n/2-1$ in order to change the configuration of the clients. However, such an increase will result in utility of at most $n(r-x') = n(n/2-2)$, which is lower than the current utility of the firms $n(n-3)/2 = n(n/2 - 1.5)$. Hence, A is a CME in I_2.

Observe that for any number of firms $m \geq 2$, the following configuration B is also a CME in I_m: the clients of each clique belong to a single firm (maybe the same one), and all firms pay at least $x^* = n/2 - 3$. In fact, it can be checked that for any $m > 2$, every CME of I_m *must* have this structure due to the following argument: When a clique is partitioned between two or more firms, there is always a client that wants to switch firms, and thus, it cannot be a CME. Hence, in every CME each of the two cliques belongs to a single firm, and there is at least one "empty" firm. Now, suppose that one of the non-empty firms offers $x_i < n/2 - 3$. Then an empty firm $f_{i'}$ can attract the entire clique by offering $x'_{i'} = r - 1$, since every client c_j will get

$$r - 1 = n - 4 = (n/2 - 3) + (n/2 - 1) = x^* + |\Gamma(j)| > x_i + |\Gamma(j)|.$$

The utility of each client is thus $u_j(B) \geq x^* + |\Gamma(j)| = n - 4$, which approaches $2u_j(A)$ as n grows. Thus, $SW^*(I_m) = SW(I_m, B) \geq (2 - o(1))SW(I_2, A) \geq SW^*(I_2)$. □

PROPOSITION 7. *There exists an instance $I_2 = \langle G, 2, r\rangle$, s.t. $OPT(I_2) \geq (1 + \frac{1}{1-\ln 2})SW^*(I_2) \cong 4.26SW^*(I_2)$.*

Proof. Our network will be composed of only pairs of clients. Each pair has *weight*, which is the weight of the edge linking the two clients. We construct an outcome $A = (\mathbf{x}, (C_1, C_2))$ as follows. $\mathbf{x} = (0, 0)$. We create each of C_1, C_2 from $T = n/4$ distinct pairs, with weights $w_t = \frac{2t}{n/2+2t}r = \frac{t}{T+t}r$ (assume $r = r(n)$ is large enough so weights are integers). We first show that A is a CME.

Indeed, for firm 1 the current utility is $v_1 = r \cdot n/2$. In order to attract t pairs of clients from firm 2, firm 1 will have to increase the payoff to $x'_1 \geq w_t$, for a utility of at most

$$v'_1 = (r - x'_1)(n/2 + 2t) \leq r(1 - \frac{2t}{n/2 + 2t})(n/2 + 2t) = r(n/2 + 2t - 2t) = r \cdot n/2 = v_1.$$

An identical analysis works for firm 2 as well. Next, we compute the utility of each client in OPT and in A.

$$\frac{OPT(I_2)}{SW^*(I_2)} = \frac{SW(OPT)}{SW(I_2, A)} = \frac{4\sum_{t=1}^{T}(w_t + r)}{4\sum_{t=1}^{T} w_t} = 1 + \frac{Tr}{\sum_t w_t} = 1 + \frac{T}{\sum_{t=1}^{T} \frac{t}{T+t}} \ ,$$

where,

$$\sum_{t=1}^{T} \frac{t}{T+t} = T - T\sum_{t=1}^{T} \frac{1}{T+t} = T - T\sum_{t=T+1}^{2T} \frac{1}{t} \cong T - T(\ln 2T - \ln T) = T(1 - \ln 2) \ .$$

Thus, for large n (and T), the above ratio approaches $1 + \frac{1}{1-\ln 2} \cong 4.26$, which gives us the required bound. □

Proposition 8. *For every $m \geq 2, \varepsilon > 0$, there exists a game instance I_m with $CVC(I_m) \leq 1/m + \varepsilon$.*

Proof. Consider a graph G with $n = m\ell$ clients, where $\ell > 1$. For every client $a_j = 1$. The social network of the clients is formed as following. Start with a complete graph, and partition it into m equal size parts. Next, index the clients in each part, and remove edges between clients of different partitions having the same index. More formally, let us label the clients as $\{(h,k)|1 \leq h \leq m \text{ and } 1 \leq k \leq \ell\}$. Two clients (h_1, k_1) and (h_2, k_2) have an edge between them if and only if $k_1 \neq k_2$. All edge weights are 1.

F has m firms with $r_i = r = 10$. We are interested in the CME outcome A, where every firm gives payoff of $r_i - 1$ and every client (h,k) goes to firm f_h (recall that h is in the range $1, 2, \dots, m$). The utility of each client (h,k) is $r + \ell - 2$ under this CME, because there are only $\ell - 1$ other clients using firm f_h. On the other hand, deviation to another firm $f_{h'}$ will not increase the utility of the client, because (h,k) is connected to only $\ell - 1$ other clients using $f_{h'}$. Thus, the configuration A is indeed a CME, and $SW^*(I_m) \leq SW(I_m, A) = m\ell(r + \ell - 2)$.

Let us now calculate the monopoly value of the above game. If there was only a single firm, and all clients would have joined it, then each client would have got a value of $m(\ell - 1) + 1$, equal to its degree. Thus, the social welfare of the monopoly is: $m\ell[m(\ell - 1) + 1]$. Hence,

$$CVC(I_m) = \frac{SW^*(I_m)}{MV(G)} = \frac{m\ell(r + \ell - 2)}{m\ell[m(\ell - 1) + 1]} = \frac{10 + \ell - 2}{m(\ell - 1) + 1} \quad .$$

For sufficiently large ℓ (independent of m), the last ratio approaches $1/m$. □

Proposition 4. *For every instance $I = \langle G, F \rangle$ with m firms, where the degree of every node in G is at most d, then $SW^*(I) \geq MW(G)/d$. That is, $CVC(I) \geq 1/d$.*

Proof. Consider the proof of Theorem 9. Intuitively, if client c_j has $d \leq m$ friends, which are arbitrarily divided to m sets, one of these sets must contain at least $1/d$ of the friends (*i.e.*, at least 1).

Formally, in the proof of Theorem 9, the bound on the utility of each client c_j is constructed as follows. Inequality (4) is proved for every firm $f_{i'}$. Then, all these inequalities are combined to form Inequality (5). The bound of $1/m$ follows since m inequalities are summed to produce Inequality (5). However, for the proof to work, one only needs to sum inequalities corresponding to firms having neighbors of c_j as clients, and the number of such firms cannot exceed the degree of c_j. □

B.1 Complete graphs

In this appendix we prove Theorem 10 amd Proposition 11. First, let us restate the theorem.

Theorem 10. *For any game instance $I = \langle G, F \rangle$ where G is a complete graph, it holds that $CVC(I) = \Omega(n^{-1/3})$.*

Let I be a game with the complete graph as a social network, and let E be an arbitrary CME of I. For simplicity of the exposition, we assume that E contains no firms of zero clients. This assumption simplifies the proofs, and does not effect our results.

First we show that there exists an order of firms with some useful properties.

Lemma 1. *No two firms have the same number of clients under E.*

Proof. Assume for the sake of contradiction that E contains two firms f_1 and f_2 having both $k > 0$ clients. Assume, without loss of generality, $x_1 \geq x_2$. Now, consider some client c_j of f_2. If c_j deviates to f_1, it will share firm with more neighbors, and at the same time will lose nothing in terms of the space it gets; contradicting the assumption that E is a CME. □

Lemma 2. *If firm f_1 has more clients than f_2 under E, than $x_1 < x_2$.*

Proof. Assume for the sake of contradiction that $x_1 \geq x_2$. Then every client of f_2 has an incentive to deviate to f_1, since this will increase the number of neighbors it shares a firm with, and at the same time will not decrease the amount of space it gets. $\square$

Corollary 1. *The firms can be ordered in such a way that under E they have: a strictly decreasing number of clients, and strictly increasing payments.*

Proof. Follows immediately from Lemmata 1 and 2. $\square$

Denote by $f_1, f_2, \ldots, f_m$ the order of the firms suggested by Corollary 1. Let n_i be the number of clients firm f_i has under E, and let $a_{i,\min}$ and $a_{i,\max}$ denote the minimal and maximal, respectively, a_j parameters of clients associated with f_i.

Observation 13 *Under E, for every $1 \leq i < m$, it must hold that $a_{i+1,\min}x_i + n_i \leq a_{i+1,\min}x_{i+1} + n_{i+1} - 1$, and also, $a_{i,\max}x_{i+1} + n_{i+1} \leq a_{i,\max}x_i + n_i - 1$.*

Proof. If the first inequality is not true, it is beneficial for some client of f_{i+1} to deviate to f_i, and if the second inequality is not true, it is beneficial for some client of f_i to deviate to f_{i+1}. $\square$

The last observation yields the following lower bound on the value of the $a_{i,\min}$'s.

Lemma 3. *Under E, for every $2 \leq i \leq m$, $a_{i,\min} \geq \frac{2(i-1)^2}{x_i}$.*

Proof. Rearranging the inequalities stated in Observation 13, we get for every $1 \leq j < m$:

$$a_{j,\max}(x_{j+1} - x_j) + 1 \leq n_j - n_{j+1} \leq a_{j+1,\min}(x_{j+1} - x_j) - 1 \Rightarrow a_{j,\max} \leq a_{j+1,\min} - \frac{2}{x_{j+1} - x_j} \ .$$

Adding the above inequalities for all $1 \leq j < i$, we get:

$$a_{i,\min} \geq \sum_{j=1}^{i-1} \frac{2}{x_{j+1} - x_j} \geq \frac{2(i-1)}{(x_i - x_1)/(i-1)} = \frac{2(i-1)^2}{x_i - x_1} \geq \frac{2(i-1)^2}{x_i} \ .$$

$\square$

We can now get a lower bound on the total utility of the clients.

Lemma 4. *Under E, the total utility of the clients is lower bounded by $\Omega(n^{5/3})$.*

Proof. Let us consider two cases. The first case is that $\sum_{i=1}^{\lfloor n^{1/3} \rfloor} n_i \geq n/2$ (if $\lfloor n^{1/3} \rfloor > m$, the last term is undefined. However, the proof still works with small modifications). In this case, the utility of the clients due to the social connections between them is at least:

$$\sum_{i=1}^{\lfloor n^{1/3} \rfloor} \frac{n_i(n_i - 1)}{2} \ .$$

Note that for every $x, y \in \mathbb{R}$, it holds that:

$$(x - y)^2 \geq 0 \Rightarrow x^2 + y^2 \geq \frac{(x + y)^2}{2} \Rightarrow x(x - 1) + y(y - 1) \geq 2 \cdot \frac{x + y}{2} \left(\frac{x + y}{2} - 1 \right) \ .$$

Hence, the above utility is minimized when the n_i's are all equal, *i.e.*, it can be lower bounded by:

$$\sum_{i=1}^{\lfloor n^{1/3} \rfloor} \frac{n_i(n_i - 1)}{2} \geq \sum_{i=1}^{\lfloor n^{1/3} \rfloor} \frac{0.5n^{2/3}(0.5n^{2/3} - 1)}{2} = \Omega(n^{5/3}) \ .$$

We now consider the case $\sum_{i=1}^{\lfloor n^{1/3} \rfloor} n_i \leq n/2$. In that case, at least $n/2$ clients get a payment of at least $2(n^{1/3} - 1)^2 = \Omega(n^{2/3})$. Hence, the total payments that the clients get is at least $\Omega(n^{5/3})$. □

Our upper bound follows directly from the last lemma, and completes the proof of the first part of Theorem 10. Indeed, if there was only a single firm, the total utility of the clients was $MW(G) = O(n^2)$. On the other hand, by Lemma 4, the clients get under any CME E a total utility of $\Omega(n^{5/3})$. Hence, the value of competition is always at least $\Omega(n^{-1/3})$.

PROPOSITION 11. *There is a family of instances $(I_m)_{m \geq 1}$, each with a complete social graph over $n(m)$ clients (where $n(m)$ is a bounded function of m), for which $CVC(I_m) = O(n^{-1/3})$.*

Proof. Given m, we construct a game instance I_m with m firms and n clients, where n is a function of m (which will be given later). Firm f_i has $r_i = i$, and is associated with a set F_i of $n_i = (m-i)(m+i-2)$ clients. Each client $c_j \in F_i$ has $a_j = a'_i = 2(i-1)$. We consider a CME E of this game constructed as following. Every firm f_i has a payoff of $x_i = i - 1$, and all clients of F_i use the services of f_i. Clearly, the firms do not have an incentive to deviate in this CME since any deviation must decrease their revenue to 0. Thus, we are left to show that the clients do not have an incentive to deviate. Consider a client of firm i. The value gained by this client is:

$$(m-i)(m+i-2) - 1 + 2(i-1)^2 \ .$$

If this client deviates to firm j, its value will be:

$$(m-j)(m+j-2) + 2(i-1)(j-1) \ .$$

The difference between these quantities is:

$$\begin{aligned}
&\left[(m-i)(m+i-2) - 1 + 2(i-1)^2\right] - \left[(m-j)(m+j-2) + 2(i-1)(j-1)\right] \\
&= [m^2 - i^2 - 2m + 2i] - [m^2 - j^2 - 2m + 2j] - 1 + 2(i-1)(i-j) \\
&= (j+i)(j-i) + 2(i-j) - 1 + 2(i-1)(i-j) = (j-i) \cdot [(j+i) - 2 - 2(i-1)] - 1 \\
&= (j-i)^2 - 1 \geq 0 \ ,
\end{aligned}$$

where the last inequality holds since $|i - j| \geq 1$. This completes the proof that E is indeed a CME. Next, let us calculate the utility of the clients under E.

$$SW(E) = \sum_{i=1}^{m} \left[\frac{n_i(n_i - 1)}{2} + 2n_i(i-1)^2\right] \leq \sum_{i=1}^{m} \left[\frac{(2m^2)^2}{2} + 4m^2(m)^2\right] = O(m^5) \ . \qquad (11)$$

To calculate the monopoly utility, we need first to calculate the total number n of clients in I_m.

$$\begin{aligned}
n &= \sum_{i=1}^{m} n_i = \sum_{i=1}^{m} (m-i)(m+i-2) = \sum_{i=1}^{m} (m^2 - i^2 + 2m - 2i) \\
&= m^3 - \frac{m(m+1)(2m+1)}{6} + 2m^2 - m(m-1) = \frac{2m^3}{3} + \frac{m^2}{2} + \frac{5m}{6} = \Theta(m^3) \ .
\end{aligned}$$

Thus, the monopoly welfare is $n(n-1)/2 = \Omega(m^6)$. Combining this with (11), we get $CVC(I_m) = O(1/m) = O((m^3)^{-1/3}) = O(n^{-1/3})$. □

B.2 Revenue ramifications

THEOREM 12. *For any $m > 1$, there exists an instance $I_m = \langle G, m, r \rangle$ for which the total firms' revenue strictly decreases with the addition on an extra firm. Formally, $I_{m+1} = \langle G, m+1, r \rangle$ obeys $FR(I_{m+1}) \leq \frac{m-1}{m-2+\ln m} \cdot FR(I_m)$.*

Proof. Let us fix two positive integer constants $r \gg B \gg 1$. Consider a game instance I_m constructed as following. Each firm f_i has $r_i = r$, and is associated with a pair P_i of clients. Every pair P_i of clients is connected by an edge of weight B, and no other edges appear in the graph. All clients have $a_j = 1$. The instance I_{m+1} is the same as I_m with an additional firm, identical to the firms of I_m.

We begin the proof by considering the possible CMEs of I_{m+1}. Notice that in every CME, both clients of each pair P_i must belong to the same firm. Hence, at least one firm must have no clients. Since, this is a CME, this firm must be unable to win any clients. For that to happen, every firm that does have clients must give payoff of at least $R - B$. Thus, the firms revenue from such a CME is at most:

$$FR(I_{m+1}) \leq 2m \cdot [R - (R - B)] = 2mB \ .$$

Next, we describe a CME E of I_m. For $1 \leq i < m$, x_i is given by $R - B \cdot \{1 + m/[(m-1)(i+1)]\}$. We also set $x_m = R - mB/(m-1)$. First, let us prove that this is indeed a CME. Consider a client of firm i. Notice that the value of x_i is non-decreasing as i increases. Hence, client c of f_i has no incentive to deviate to a firm of lower index. We would like to prove that it also has no incentive to deviate to firm f_j such that $i < j < k$. The utility of the client c under E is:

$$\begin{aligned} B + x_i &= B + R - B \cdot \left\{1 + \frac{m}{(m-1)(i+1)}\right\} \\ &= R - B \cdot \left\{\frac{m}{(m-1)(i+1)}\right\} \geq R - B \cdot \left\{\frac{m}{2(m-1)}\right\} \ . \end{aligned}$$

On the other hand, if c deviates to firm j, its utility will be:

$$x_j = R - B \cdot \left\{1 + \frac{m}{(m-1)(j+1)}\right\} \leq R - B \cdot \left\{1 + \frac{1}{m-1}\right\} = R - B \cdot \frac{m-2}{m-1} \ .$$

Hence, c has an incentive to deviate only when $m/2 > m - 2$. However, this inequality holds only for $m < 4/3$. Thus, it is never beneficial for a client to make such a deviation in our case. We also need to consider the possibility that c will deviate to firm m. Notice that firms $m-1$ and m are actually identical, and therefore, c does not have an incentive to deviate to firm m either, since it does not have an incentive to deviate to firm $m-1$. Thus, we can conclude that no client has an incentive to deviate. To complete the proof that the above configuration is a two phases equilibrium, we need also to consider deviations of the firms. Clearly, a deviation of a firm f_j makes sense only if x_j becomes $x_i + B + 1$, where f_i is some other firm (otherwise, f_j can win the same set of clients with a lower increase in x_j). If $i < m$, then f_j wins at most $2i$ new clients (the clients of firms $1, 2, \ldots, i$, excluding maybe its own clients if $i > j$). If $i = m-1$ or $i = m$, then f_j wins only $2(m-1) \leq 2i$ new clients because these are all the client it does not have already). Hence, the utility of f_j after the deviation is at most:

$$\begin{aligned} [R - (x_i + B + 1)] \cdot 2(i+1) &< \left[R - R + B \cdot \left\{1 + \frac{m}{(m-1)(i+1)}\right\} - B\right] \cdot 2(i+1) \\ &= B \cdot \frac{m}{(m-1)(i+1)} \cdot 2(i+1) = \frac{2Bm}{m-1} \ . \end{aligned}$$

On the other hand, before the deviation, the utility of f_j was:

$$[R - x_j] \cdot 2 = \left[R - R + B \cdot \left\{1 + \frac{m}{(m-1)(j+1)}\right\}\right] \cdot 2 \geq \left[B \cdot \left\{1 + \frac{1}{m-1}\right\}\right] \cdot 2 = \frac{2Bm}{m-1} \ .$$

Therefore, f_j does not gain from any deviation. Finally, we are left to consider deviations of f_m. However, since firms f_{m-1} and f_m are symmetric, f_m does not have an incentive to deviate because f_{m-1} does not have such an incentive. Thus, the above configuration is a CME. Let us calculate the total utility of the firms in E.

$$
\begin{aligned}
& 2 \cdot \sum_{i=1}^{m-1} (R - x_i) + 2(R - x_m) \\
&= 2 \cdot \sum_{i=1}^{k-1} \left(R - R + B \cdot \left\{ 1 + \frac{m}{(m-1)(i+1)} \right\} \right) + 2 \cdot \left(R - R + B \cdot \frac{m}{m-1} \right) \\
&= 2B \cdot \sum_{i=1}^{m-1} \left(1 + \frac{m}{(m-1)(i+1)} \right) + 2B \cdot \frac{m}{m-1} = 2B \cdot (m-1) + \frac{2Bm}{m-1} \cdot \sum_{i=1}^{m-1} \frac{1}{i+1} + \frac{2Bm}{m-1} \\
&= 2B \cdot (m-1) + \frac{2Bm}{m-1} \cdot \sum_{i=0}^{m-1} \frac{1}{i+1} \geq 2B \cdot (m-1) + \frac{2Bm}{m-1} \cdot \int_0^{m-1} \frac{dx}{x+1} \\
&= 2B \cdot (m-1) + \frac{2Bm}{m-1} \cdot \ln m \ .
\end{aligned}
$$

The last value lower bounds $FR(I_m)$. The ratio between this value and $2mB$ (our upper bound on $FR(I_{m+1})$) is:

$$
\frac{m-1}{m} + \frac{1}{m-1} \cdot \ln m \geq \frac{m-2}{m-1} + \frac{1}{m-1} \cdot \ln m = \frac{m-2+\ln m}{m-1} \ .
$$

□