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# Aggregation of Estimates when Decision By Majority is not Possible

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*R. Meir was designated as "others" (bHorayoth 13b).*

*R. Aha Bar Hanina said: It is revealed and known before Him Who spoke and the world came into existence, that in the generation of R. Meir there was none equal to him. Then why was not the halakha fixed according to his views? Because his colleagues could not fathom the depth of his mind (bEruvin 13b).*

## **Introduction**

The majority principle is a basic rule for reaching a decision in many legal systems. The principle dictates such procedural rules as creating panels consisting of an odd number of judges to ensure a majority opinion, and in some legal systems, a mandatory debate to guarantee that the majority has considered the minority opinion and rejected it. Nevertheless, decision rules are also needed for cases in which no opinion can muster a majority. The issue we analyze in this article debated between the years 200-600 C.E. by the sages of Jewish law, and it appears likely to challenge the 21st century scholars as well. The topic deals with determining the value of an asset as an aggregation of three different appraisals by professional assessors. According to the majority principle, it is clear that if two of the assessors agree on one value with the third one

disagreeing, the decision follows the majority opinion. The problem arises when each of the assessors quotes a different value. For cases in which three assessors who appraise an asset specify values of 80, 100, and 120, three Tannaim suggested various methods of aggregation. The results produced by the three methods are 100, 90, and  $93\frac{1}{3}$ . The method that produced 100 was decreed to serve as halakha, but the justification that the Babylonian Talmud gives for this decree and for the rejection of the other methods is surprising: the Amoraim avoided determining which is the "correct" method, but ruled that the alternative (which was offered twice in the course of the Talmudic debate as an halakhic ruling by Rav Huna) is obscure and incomprehensible. It was this argument that provided us with the incentive to study in-depth and examine the methods that were not accepted as halakha, especially the most challenging of the methods, the one that produced  $93\frac{1}{3}$ .

The objective of this article is to examine the rules of decision employed by the Tannaim, especially to decipher the obscure (third) decision rule, for which we propose three explanations. The opinions of the Tannaim refer to a specific case of appraisal by three assessors who quoted values distributed symmetrically around a mean. We examine a generalization of the opinions of the Tannaim to appraisals with values other than the ones mentioned in the *baraita*, especially when the distribution is not symmetrical, and the generalization of the opinion that was decreed as halakha in cases of more than three appraisals. Finally, we examine the controversy of the decisors and suggest an interpretation for the controversy.<sup>1</sup>

## The Case of the Assessors

In the Baba Batra tractate, there is a baraita dealing with the appraisal of assets:

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<sup>1</sup> In the course of the research, we had the privilege to be received by Nobel Prize laureate, Prof. Robert J. Aumann, who agreed to clarify with us the topic. As we note in the body of the article, some of the insights we present here are the result of our joint study.

Our rabbis taught: Three went to appraise. One says 100 and two 200; one says 200 and two 100. The one in the minority is overruled. One says 100, one says 80, and one says 120 it is ruled at 100. R. Eliezer Bar Zadok says: it is ruled at 90. Others say, we appraise between them and divide it by three.<sup>2</sup>

Following the baraita, a debate ensues in the Talmud:

He who says we rule 100 adopts the middle course. Rabbi Eliezer Bar Zadok says we rule 90. He is of the opinion that the land is worth 90. He who says 80 errs by 10 downward, and he who says 100 errs by 10 upward. On the contrary, the land is worth 110. He who says 100 errs by 10 downward. He who says 120 errs by 10 upward. Nevertheless, adopt the first two which do not exceed 100. Others say, we appraise between them and divide it by three. They are of the opinion that the land is worth 93 and a third. He who says 80 errs by 13 and a third downward. He who says 100 errs by 13 and a third upward. By the rule he should have said more, and reason he did not is that he thought it is enough I already exceeded my colleagues by so much. On the contrary, the land is worth 113 and a third. He who says 100 errs by 13 and a third downward. He who says 120 errs by 13 and a third upward. By the rule he should have said more and the reason he did not is that he thought it is enough I already exceeded my colleagues by so much. Nevertheless, adopt the first two which do not exceed 100. Rav Huna said: the halakha is according to the others. Rav Ashi said: we do not understand the reason of the others; shall we rule halakha according to them? The judges of the exile taught that we appraise between them and divide it by three. Rav Huna said: the halakha follows the judges of the exile. Rav Ashi said: we do not understand the reasons of the judges of the exile, shall we rule halakha according to them?<sup>3</sup>

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<sup>2</sup> For a clarification of this expression, see Shaul Lieberman, *Tosefta Kepshuta: Sotah* 8, p. 660. In his opinion, this form appears only with regard to assessors and judges, and therefore he concludes that when it is mentioned with reference to witnesses, it refers to disqualified witnesses. See also n. 22 below for the various meanings of the term *meshalshin*.

<sup>3</sup> bBaba Batra 107A. The baraita is mentioned, with changes, also in *Tosefta Ketubot* 11b. The main difference between the formulation here and the one in the *Tosefta* is in the opinion of the others: "They appraise between them and divide them (in print: it) by three."

The baraita and the Talmudic topic do not clarify why the appraisal was required, and according to most of the Rishonim, the appraisal was needed to repay a debt.<sup>4</sup>

### **The Legal Status of the Assessor**

Three assessors are asked to appraise because of the need for a minimum of three judges in a lay court.<sup>5</sup> The three who appraised for the purpose of the sale of the asset are not judges but people with knowledge or experience in appraising,<sup>6</sup> whom we shall call henceforth assessors.

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<sup>4</sup> See Ri Migash, *Innovations to Baba Batra*, 107a (Jerusalem, 1946), interpreting that this is a case of a debtor who has only land. Rashbam, *ibid*, s.v. *shlasha*, is interpreting that this is a case of assets belonging to orphans from which it is necessary to settle alimony for the wife and daughters. This is also Ri Karkusha's interpretation, *The Method of the Ancients* (New York, 1982), p. 17, who notes that some ask why is an appraisal necessary rather than selling the asset at an auction? He offers two answers to this question. One refers to a situation in which market prices are liable to change, and the appraisal will take these into consideration. The second is that the sale is of the type in which there is no bidding, as in the case of alimony and burial (see Ketubot 100b). Similarly, the interpretation of R. Avraham Abad of Narbonne (Rabad II), *The Method of the Ancients* (New York, 1982), p. 235: "Where they bid there is no appraisal." Hameiri asks what was the need for appraisal where they bid, and assumes that the meaning of "needs further investigation" is that an appraisal takes place even if there is bidding. Ritba, *Innovations to Baba Batra* (The R. Kook Institute, 2005), p. 819, quotes Rashbam's interpretation and disagrees with it, arguing that in the case of sale for the purpose of alimony for the woman and daughters, the market offers are being followed because they have no rights to receive land. Ritba quotes the interpretation of Ri that this is an appraisal for the purpose of a woman's ketubah or for repaying a debt. (This interpretation is supported by R. Shaul Lieberman, *Tosefta Kepshuta: Beur Aroch Letosefta*, Ketuvot p. 359. Based on the location of the baraita in the Tosefta, it appears that the assessors appraise the field for the woman herself, who wants to use it.) Another interpretation that appears to Ritba to be important is that the deal is between a seller and buyer who have agreed that the price would be set by three assessors. (See Avoda Zara 72a: "When [the item] has been appraised by three persons [we settle the price accordingly] even if two of the three agree [about the price.]") Rabbi Betzalel Ashkenazi, in *Shita Mekuvezet*, quoted Re'em's interpretation, whereby when the debtor has no money but has land, the creditor is given the land using this method. It is surprising that we found no Rishonim who explained the need for the appraisal for the estimation of damages (see R. Yosef Razin, *Responsa Tzafnath Paaneah*, Warsaw Edition, 245) or for value estimation or the title of the two mentioned in the first mishnah of Sanhedrin.

<sup>5</sup> See bBaba Metzia 32a: "Said Rav Yosef Bar Maniumi said Rav Nachman: A widow doesn't need a court of experts but a court of laymen." See Maimonides, *Laws of Marriage (Hilkhot Ishut)* 17, 13: "A widow, whether from engagement or marriage, may take the oath and sell land belonging to her husband and collect her ketubah in a court

The discussion about the aggregation of the assessors' appraisal raises a preliminary question about their legal status. Are assessors similar to witnesses, and should evidentiary rules apply to them? Or are assessors similar to judges, and should the procedural rules customary in court apply to them? Or perhaps their status is a special one, not similar to that of witnesses or judges, and their professional opinion is not a testimony but rather an "estimate." The baraita states at the beginning that when we have two identical appraisals and a divergent third one, "the one in the minority is overruled." Decision by majority opinion can be interpreted as a preference for the testimony of two assessors over the testimony of one assessor, "the word of one shall not stand against the word of two,"<sup>7</sup> or as the implementation of the procedural rule "incline toward the many," as is customary in court. But the Gemara has reservations. From the Temura tractate, it appears that appraisal is an estimate and not testimony or judicial verdict:

Said Ulla: They said only where two people made the appraisal, but where three made the appraisal, even if afterwards a hundred assessors disagreed with the three the appraisal of the three is not revised. But is it so? R. Safra said: Where is it said that two are on a par with a hundred? In the matter of testimony. But with regard to assessment we follow the opinion of the majority. Moreover, if there are three against three, do we not follow the latter set, so that *hekdesh* always takes precedence? Ulla holds: the need to appraise is from rabbinical law, and the rabbis adopted a lenient view.<sup>8</sup>

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of expert judges, or in a court whose judges are not experts, provided it consists of three trustworthy men who are knowledgeable in the evaluation of land... If a woman sells property without the participation of the court, a public announcement need not be made, but three trustworthy men are needed who are knowledgeable in appraisal." See R. Menachem Hameiri, *Beit Habchira*, Sanhedrin 17b: "And as long as there is only a trust two are enough, but to divide he needs an appraisal and if you need an appraisal there is never fewer than three." In our situation, we learn that in the case of an appraisal by three assessors, each one is asked to appraise by himself. There is also appraisal by ten assessors, see *Shumat Karkaot Veadam*: "And regarding land, nine and a priest (*cohen*), and regarding men a similar procedure" (Mishnah, Megila 4:3). See also n. 17 below.

<sup>6</sup> Rif on Ketubot 56b explained that the term "court of laymen" in a context of appraisal means "trustworthy people who are competent in their field."

<sup>7</sup> This rule is stated with reference to testimony rules. See, for example, bShavuot 48a regarding the testimony of two witnesses against the refuting testimony of one witness.

<sup>8</sup> bTemurah 27b.

The topic reveals that on one hand appraisal is unlike testimony because in appraisal majority opinion is being followed, whereas according to the rules of testimony "two are like a hundred." On the other hand, the topic concludes that if we were asked to assess by Torah law, it would be possible to appeal even the appraisal of three assessors if three other assessors testify that the field is worth more, because in the case of doubt matters of consecration take precedence. The finality of the appraisal by three follows from the fact that in Torah law it is possible to redeem a value of 100 by a penny (*pruta*), and the appraisal process is required only by rabbinical law. But the possibility for appeal proves that appraisal is unlike a judicial verdict because in Jewish law there is no appeals instance. The combination of the two special halakhot regarding appraisal, decision by majority opinion and the possibility for appeal, suggests that the testimony of assessors is not a testimony in the regular sense. Testimony that does not follow "two are like a hundred" but majority opinion we found only in cases in which it is possible to rely on the testimony of a single witness, or on disqualified witnesses, for "when one witness is useful a woman and a disqualified witness also testify."<sup>9</sup> Regarding this type of testimony, R. Nechemia said: "Whenever the Torah believed one witness – it followed majority opinion."<sup>10</sup> This means that wherever the Torah did not insist on the need for two valid witnesses, the testimony is valid not only with one witness but also if the witnesses are disqualified and the status of the testimony is that of an estimate, where majority opinion is followed. The contradiction between testimony laws (where two are like a hundred and majority opinion is not followed) and estimation is explicit in the Gemara: "We follow majority opinion only in the case of estimates, where the more there are, the more experts there are."<sup>11</sup> In other words, in an estimate based on majority evaluation, the larger the group of assessors, the higher the likelihood for a correct decision.<sup>12</sup>

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<sup>9</sup> Maimonides, *Laws of Witnesses (Hilkhot Edut)* 5:3.

<sup>10</sup> bSotah 31b; ibid 47b. See there two formulations in the clarification.

<sup>11</sup> bShevuot 42a.

<sup>12</sup> This intuition was proven in 1785 and it is known as the Condorcet Jury Theorem, which states that if the probability of each member of the deciding body to reach an independent "correct" decision is greater than  $\frac{1}{2}$ , then: (a) the probability of reaching a correct group decision by majority ruling is greater than the probability that an individual expert will reach a correct decision; (b) as the number of the members of the deciding body increases, the probability of obtaining a correct group decision approaches 1. Nevertheless, note that the Condorcet Jury Theorem

Despite the fact that following the majority is a universal rule derived from the verse "incline toward the many," and it is valid both in court judgments and other situations of doubt (as in nine shops),<sup>13</sup> at least in the case of estimation this is not a rule imposed by law without a reason but based on an endeavor to maximize the probability of obtaining a value as close as possible to the "real" value.<sup>14</sup>

The assessors are not witnesses in the regular sense because they do not testify about what they have seen but about an estimate derived from their evaluation and their professional knowledge. In modern conceptualization, they are expert witnesses. The assessors are not judges either, because they do not issue judgment but provide information to the court; it is the court that rules. Moreover, halakhically as well, the determination of the legal status of the assessors (and of expert witnesses) as estimators, explains the reason why the Tannaim in the baraita were required to determine aggregation rules in the case of three different appraisals. If the status of the assessors had been that of witnesses, we would have treated them as witnesses refuting each other and their testimony would have been cancelled.<sup>15</sup>

## The Problem

The baraita dealing with the aggregation of the three appraisals starts with the simple situation in which there is a majority of two against one. In this case there is no controversy between the Tannaim that the individual minority opinion is overruled. Apparently, given that the problem we are facing is not binary (as in the case of kosher v. non-kosher, etc.), and despite the fact that probably the qualifications of the assessors are similar and therefore each one has an identical chance of producing a correct or mistaken evaluation, it would have been proper to assess the

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refers to decision rule in binary problems. For a proof of the intuition in continuous problems as well, see Gerlach-Kirsten (2006).

<sup>13</sup> See bHullin 11a about meat on the market about which it is not known whether its source is a shop that sells kosher or non-kosher meat; it is possible to rely on a majority of kosher shops and allow the meat.

<sup>14</sup> It is possible that this is the intention of R. Yosef Razin, *Responsa Tzafnath Paaneah*, Warsaw Edition, 245: "In any case, in Baba Batra 107 (our topic) concerning estimation, we follow the majority although the meaning of majority [in this case] is irrelevant."

<sup>15</sup> Regarding judges who issue three different opinions see below n. 113-114 and the adjacent text.



asset in a way that weights the minority opinion as well by means of some formula.<sup>16</sup> But the baraita states that majority decision is preferred even when assessors are involved, although their legal status is that of estimators.<sup>17</sup> Rashbam and other Rishonim noted that decision by a majority of assessors is mandated by the verse about judges who must "incline toward the majority," and apparently this is imposed by the law. Another possibility is that the Tannaim understood that when an absolute majority of assessors agrees on an exact value, it is likely that the divergent

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<sup>16</sup> For example, it would have been possible to set the appraisal as the weighted average of the assessor's appraisals, in this case:  $\frac{2}{3} \times 200 + \frac{1}{3} \times 100 = 166\frac{2}{3}$ .

<sup>17</sup> See R. Yaakov Kamenetsky, *Emet Leyaakov*, Seder Moed, 1991, p. 400, which addressed the Tosefta contradiction in the Megillah tractate 23b and in the Arakhin tractate 19b, why appraise by ten, for there is no balanced court. His first argument is that our baraita (the three who appraised) must mention also the simple rule of the two who said 200 and the one who said 100, in which the opinion of the single assessor was overruled. Although the principle of following the majority is a simple one, and it apparently goes without saying, it is important to mention it when talking about assessors. In his opinion, contrary to judges who disagree whether a person is liable or is exempt, and the truth is with one or the other, in the case of assessors it is possible that they are all mistaken because there is no single truth. Therefore, there is an innovation in the baraita in that the opinion of the sole individual is disregarded if the majority has agreed on another opinion.

We believe that it is necessary to elucidate yet another point from the ruling of the baraita based on majority opinion and ignoring the minority opinion. The issue of ignoring the minority comes up also in light of the Talmudic principle of "not following the majority in matters of money" (Baba Kama 27b, *ibid* 46b), which contradicts the principle of ruling according to a majority of judges, ignoring the minority opinion. When there is a majority of judges against a minority, a debate takes place (in the language of the Mishnah, a negotiation), in which the majority and minority voice their opinions. The debate on which the decision is based strengthens the opinion of the majority in the sense that it is possible to say that the minority opinion is cancelled relative to the majority opinion. (See Tosefta, Baba Kama 27b, s.v. *kamal*). It is possible that the three who appraised also conducted a debate among themselves before they formed their opinions. This is how Rashbam understood our topic, s.v. *ubedin hu dilma*: "They negotiate the matter and each one knows what his colleague thinks and after they negotiated they set their heart on saying what their opinion is." This is what transpires also from the words of R. Yosef Migash, *Responsa Hari Even Migash* (Jerusalem, 1979) 141, who rejects appraisal by two assessors (even if they appraised unanimously) because "one can say maybe if there had been a third with them he would have disagreed with them and show them that they are mistaken and they would have adopted his opinion or one of them would have joined his opinion and the other one were left alone and his opinion would not have counted." This is how the author of *Netivot Hamishpat* 103a rules also.

opinion represents "too big an error," and therefore it ought not to be considered at all, and the individual in the minority should be overruled.

The main topic of the baraita is the case in which there is no majority representing any opinion, and each of the assessors quotes a different value, in this case 80, 100, and 120. Still, it would be in order to determine why in the case of three divergent opinions it is not possible to implement the rule of overruling the individual minority opinion. In other words, if we assume that 100 is included in 200, then the median value, 100, is included in the 120 appraisal and in this case the appraisal of 80 is overruled as the minority opinion. But the argument of overruling the individual minority opinion is stated in the baraita only when we have two identical appraisals, whereas the explanation given by the Talmud to the opinion of Tanna Kama in the case in which there are three divergent opinions is *milta metziata*, or taking the average.<sup>18</sup> It appears that the majority overrides minority opinion only when it represents a unanimous opinion, not when the majority itself is divided, because then it is not possible to say that the minority opinion is overruled.<sup>19</sup> The baraita quotes the opinions of three Tannaim. Two of them specify a numeric value without revealing the rule they used to reach their decision. The third one is formulated as an obscure rule for reaching a decision. All the opinions offer an alternative to the rule that "the one in the minority is overruled," which cannot be implemented directly in this case.<sup>20</sup>

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<sup>18</sup> See below the explanation of Tanna Kama.

<sup>19</sup> R. Shaul Lieberman, *Tosefta kifshuta: Beur arokh latosefta*, Sotah p. 660, emphasized the stylistic difference between "the one in the minority is overruled" and "it is not [to follow] the statement of one instead of two." See also R. Zacharia Agmati (Morocco, 12<sup>th</sup> century) *Sefer Haner*, p. 274, who quotes the commentary of Ri Migash and R. Meir Halevi Abulafia, *Yad Rama*, Jerusalem 1994, p. 472. The expression "the one in the minority is overruled" is used in the first part of the baraita. In the second part, which deals with the decision regarding the various opinions, the expression is "it is not [to follow] the statement of one instead of two," which appears to be based on a logical evaluation of the data.

<sup>20</sup> This conclusion must be qualified both with Bah's gloss to Rashban, which contains the argument of "the one in the minority is overruled" (the Rashbam printed in Venice (1721) contains the same formulation as the printed versions) and the commentary referred to as Rashi printed around Rif's pages s.v. *nadon bemane* (51A). "As both, he who said 30 (that is, 120 dinari) and he who said 100, acknowledge that the item is worth 100, he who said 20 (that is, 80 dinari) is alone and overruled as a minority." Regarding the identity of this commentary, especially with reference to our topic, for this portion of Baba Batra for which there is no Rashi commentary, see Y.M. Ta-Shma,

## The Opinions of the Tannaim

The opinion of Tanna Kama is that the controversy of the assessors must be settled by appraising the asset at 100. The opinion is not explained in the baraita, and at first sight it can be reached in three ways. One is the simple average that sums the three appraisals and divides the sum by three. The second is setting the value of the asset at midpoint between the two extremes of 80 and 120. The third is to use the median appraisal. In the numeric example of the baraita, each of these ways yields the same results: 100. The problem is that in other cases, each method of calculation produces a different result. Let us examine the three methods of calculation.

The ruling by average takes into account all three opinions, but it is liable to produce a result that does not coincide with the opinion of any of the assessors. Note that this method was not mentioned at all as an interpretive or halakhic possibility.<sup>21</sup>

Ruling according to the midpoint between the extremes ignores entirely the median opinion. One would have to understand why, after selecting three assessors, the median opinion is ignored, and why is it necessary to select three assessors to begin with if one of them is not taken into account. Moreover, in most cases the midpoint is not expected to coincide with the opinion of any of the assessors. This means that the appraisals are used as a sample by which we evaluate reality, but we are not committed to any individual appraisal.

Ruling by the median appraisal appears to ignore the two other opinions and is based on the appraisal of only one assessor.

The second opinion brought by the baraita is that of R. Eliezer Bar Zadok who set the appraisal at 90. This opinion appears to be based on setting the appraisal at the midpoint between the two lower appraisals, 80 and 100, so that this opinion also ignores one of the assessors entirely, in this case the assessor who quoted 120.

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"Rashi-Rif commentary and Rashi-Rosh commentary," *Rashi: Studies of His Work* (1993, p. 209); R. Avraham Havatzelet, "Rashi's commentary printed around Rif," *Moriah* 19:1-2 (1993).

<sup>21</sup> The elimination of the simple arithmetic average from the topic requires explanation and a separate study of the status of the average in halakhic thinking. See also below, notes 27 and 81, and the adjacent text.

The third opinion, of others, does not indicate a number but formulates a decision rule: we appraise between them and divide into three (*meshalshin*). The literal meaning is that an appraisal should take place between the assessors, in other words, to reach some value between them, then somehow divide it.<sup>22</sup>

## Opinions of the Tannaim in the Talmudic Topic

### Tanna Kama

Tanna Kama's opinion that the value of the asset is set at 100 is explained in the Babylonian Talmud as a midpoint (*milta metziata*). This explanation appears to suit all the methods of calculation presented above: median, average, and midpoint between the extreme points. Rashbam and Maimonides disagreed in their interpretation of the term midpoint. Maimonides accepted as halakha Tanna Kama's ruling, using the numeric example of the baraita that sets the value of the asset at 100, and added his own numeric example that illustrates how Tanna Kama's 100 should be understood:

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<sup>22</sup> The term *meshalshin* appears in the works of the sages with several meanings:

- (a) *Meshalshin* with a meaning of dividing equally. See Mishnah, Baba Batra 3:4; Mishnah, Makot 1:3; bBaba Batra 56a; bBaba Batra 146:a.
- (b) Divide into three. *Kohelet Rabbah* 9 "They divided the day in three, a third for Torah, a third for prayer, a third for labor." This is also how Rashbam interpreted, in our topic, "*tidon beshlish*."
- (c) As a proportional division that is not even. bArakhin 27b. Said Rav Hisdah: "This was taught only if he who bid forty stands by his bid, but if he who bid forty does not stand by his bid, then we divide it between them." And Rashi, *ibid*.
- (d) As giving a third in hand. Rashi, bGittin 40b, but it is possible that it is pronounced *mashlishim*.
- (e) In the sense of repeating three times: Tosefta Negaim 8b.

See also Shaul Lieberman, n. 4 above, pp. 359-360.

One says 100 and one says 90 and one says 130. We rule 110 and in this way we always place it between them.<sup>23</sup>

In other words, according to Maimonides we must ignore the median (100) and must rule according to the midrange between the two extreme points:  $110 = \frac{130+90}{2}$ . From the point of view of interpretation, Maimonides's method will always be *milta metziata*, a midpoint between the two extremes.<sup>24</sup> This method raises the following difficulties: (a) In what way are the extreme opinions that determine the ruling preferable to the other opinions? (b) The method is open to manipulation (it is not strategy-proof), and anyone adopting an extreme opinion knows that he can tip the appraisal.<sup>25</sup>

Rashbam's interpretation of Tanns

a Kama's opinion raises a difficulty. Rashbam starts with an explanation similar to that of Maimonides: "The value that reflects the decision between the two extreme values is reached by adding 5 [20, by another scale] to the lower appraisal and subtracting 5 [20] from the upper appraisal."<sup>26</sup> The Rashbam appears to be looking to rule between the two extreme assessors, and the ruling by the median assessor (100) is coincidental.<sup>27</sup> But immediately following, he says: "The ruling is according to the one who said one hundred as both, the one who said 120 and 100, agree that it is worth 100. The one who said 80 is alone and his minority opinion is overruled. And both, of 80 and 100 admit that it is not worth more than 100, and the one who said 120 is

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<sup>23</sup> *Laws of Lender and Borrower (Hilkhot Malve Velove)* 22:14. In printed versions, the word "always" is absent, but in most manuscripts it appears as quoted, or alternatively, at the end of the sentence. See n. 33 below.

<sup>24</sup> This is how R. Menahem Hameiri ruled also, *Bet HaBhira Al Baba Batra* (Sofer edition), p. 470.

<sup>25</sup> See R. Shaul Lieberman, *supra* n. 4: "We are dependent on one assessor who can exaggerate and we will take his opinion into account because of *milta metziata* against the majority."

<sup>26</sup> Rashbam, *Baba Batra* 107A, starting at *milta metziata*.

<sup>27</sup> Gilad Lipschits, a 7<sup>th</sup> grade student, suggested an additional possibility, namely that Rashbam describes a simple mathematical average, such as the ones commonly used as examples in elementary school. In other words, building from units three columns the height of which represents the values of the appraisals, and to reach the average, it is necessary to reduce the high one and add to the low one until the heights are equal, as in the case of "flattening of the land," an operation that compares the height of the land to the average. If this is the correct understanding, Rashbam was familiar with the concept of the average but did not believe that it was the proper principle to apply in our case.

found alone and his minority opinion is overruled. And in my opinion, the same is true if he said 112, which is not so much in the middle. [We find that he who said twenty (80) is alone and his minority opinion is overruled. And both, of twenty (80) and of one hundred also admit that it is not worth more than one hundred and we find that he who said thirty (120)]<sup>28</sup> is alone and his minority opinion is overruled."<sup>29</sup>

In this last quoted statement, Rashbam regards the ruling based on the value indicated by the median assessor (100) as the main law, and he bases it on the principle of the majority, a type of "median voter theorem."<sup>30</sup> According to this argument, two assessors, 100 and 120, agree that the asset is worth at least 100, and their opinion prevails over that of the third assessor (80). Similarly, following the same logic, it is possible to argue that two assessors, 80 and 100, agree that the value of the asset does not exceed 100, and their opinion prevails over the assessor who appraised it at 120. The median method is strategy-proof.

But it is necessary to elucidate how the two parts of Rashbam's statement relate to each other. It is possible to say that at the beginning of his statement, Rashbam interprets the term *milta metziata* literally, and contrary to Maimonides, the term does not express a principle for reaching a decision but the result of the numeric example at hand. Rashbam then proceeds to the argument in principle, which is not based on *milta metziata*. He adds: "And in my opinion, the same is true if he said 112, which is not so much in the middle." In the case of 80, 112, 120 the decision will be 112 despite the fact that this is not the midrange, but the principle of the majority applies here as well. Rashbam prefers this interpretation, which appears correct to him, and he is prepared to reject in its favor the Talmudic dictum of *milta metziata* as a computational description that is correct only for the numeric example of the baraita but not for other numbers.<sup>31</sup> Although it may

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<sup>28</sup> The addition between the square brackets is according to Bah's gloss, *ibid.* Prof. Aumann drew our attention to the fact that in his commentary on our topic, R. Meir Halevi Abulafia, in *Yad Rama*, supports the halakha ruling according to Tanna Kama (100) based on the explicit position that appears in Bah's version.

<sup>29</sup> *Ibid.*

<sup>30</sup> For a discussion of the median voter theorem see M. E. Schwarz, "Public Economics," (Unit 6), The Open University (forthcoming), as well as Sh. Nitzan, "Social Preference and Choice," The Open University, 2008 (Appendix to Chapter 9).

<sup>31</sup> This understanding was derived from our joint study with Prof. Robert J. Aumann.

be possible to state that according to Rashbam's opinion, *mitla metziata* means the median opinion, this does not appear to be his interpretation.<sup>32</sup>

The benefit of Rashbam's method is that the principle of decision by the median is strategy-proof. Moreover, the median represents "a type of majority" when it is not possible to use the principle of overruling the individual minority opinion. We will find that the method of ruling by the median makes the two parts of the baraita consistent. When two opinions come together, the individual minority opinion is overruled, and when we face three diverging opinions the principle of majority is expressed by the median.

### **Rabbi Eliezer Bar Zadok**

According to R. Eliezer Bar Zadok, given the numeric examples of the baraita, the ruling should be 90. This opinion is also based on the majority ruling of the two lower opinions, 80 and 100, as opposed to the minority opinion of 120. The difficulty is that the majority opinions do not agree with each other, as the assessor who quoted 80 does not agree to a value greater than 80, and the one who quoted 100 does not agree to a value smaller than 100. The innovation of R. Eliezer Bar Zadok is that as long as the gap between two appraisals is no greater than a critical value, they can be viewed as agreeing on a single value,<sup>33</sup> and the reason each assessor named a different

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<sup>32</sup> An additional possibility for understanding the change in Rashbam's statement is the following: It was important for Rashbam to mention that using the numbers in the baraita we reach 100 even without the principle of the weighted majority, as 100 is both the midrange and the opinion of the middle assessor. If we had only the two appraisals in the extremes, 80 and 120, we would reach the middle (100). In the specific case of the baraita, the midpoint is agreeable also to the middle assessor, who quoted 100. Therefore, he accepts Maimonides's principle in this case because it does not ignore the middle opinion. Rashbam proceeds to explain why, given the numbers used in the baraita, it is necessary to reach 100 also following the principle of the weighted majority. This principle explains why we ignore the opinions of the other assessors, which are perceived to be in the minority. By contrast, Maimonides does not explain why we ignore the middle opinion when other numbers are being considered.

<sup>33</sup> When the discrepancy between the opinions is greater than the critical value, it is not possible to regard the two opinions as agreeing on a single value. This is what transpires also from the statements of Rashbam regarding 90: "each one is mistaken by a little." But it is necessary to define what "a little" means. The method of R. Eliezer is therefore limited only to cases in which the discrepancy between the various opinions is not "too great." The advantage of the opinion of Tanna Kama is that it is not subject to this limitation. It is possible that Maimonides was

value is that every assessor is liable to error.<sup>34</sup> When the gap between the assessors is greater than the critical value, we must assume that the assessors are indeed disagreeing. R. Eliezer assumes that the distribution of errors for each assessor is symmetrical, and apparently, for the values of the baraita, the critical value is not smaller than 10. Therefore, it is possible to represent the 80 and 100 appraisals as a type of agreement on a value of 90, assuming that each assessor was mistaken by 10, one of them upward, the other downward. If we had only one assessor, for example the one who quoted 80, we would set the "real" value between 70 and 90, assuming that the critical value is 10. When we have two assessors, and there is some overlap in their "confidence intervals," the range of which is determined by the critical value, the representative value sets their joint opinion as the midpoint of the intersection of the two confidence intervals. Assuming a symmetrical distribution, this point is always the midpoint between the two opinions. In the numeric example of the baraita, the intersection contains only one point: 90.

Rashbam explains that R. Eliezer prefers this method of ruling over Tanna Kama's because according to Tanna Kama (in Rashbam's interpretation), the appraisal is based on a single opinion, that of the median assessor. By contrast, according to the opinion of R. Eliezer, the ruling is based on the principle of the majority that appears at the beginning of the baraita, and is perceived by him as a guiding principle for reaching a decision.

The opinion of R. Eliezer Bar Zadok contains two innovations:

- (a) R. Eliezer assumes that appraisal is not an exact science, and therefore it is in order to attribute a "reasonable" margin of error to each assessor. Contrary to Tanna Kama's opinion, who believes that the various appraisals should be accepted as is,<sup>35</sup> R. Eliezer does not view the appraisals as "opinions" among which we much choose but as a "sample" from which one must arrive as close as possible to the "correct" value.

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aware of the advantage of Tanna Kama's opinion (whereby he ruled) and hinted at it when he said "and in this way we **always** place it between them." See n. 23 above.

<sup>34</sup> Rashbam, s.v. *osin shuma* etc., explains: "And Tanna Kama believes that we must say a man is not mistaken at all, and therefore take the middle and stay lest you make a mistake. And R. Eliezer Bar Zadok believes that a man is mistaken by 10... and others believe that a man is mistaken by 13 and a third for an item worth more than 80."

<sup>35</sup> See Rashbam, s.v. *osin shuma*, etc. "And Tanna Kama believes that we must say a man is not mistaken at all."



(b) On this sample, R. Eliezer performs a "cluster analysis" and rules that the assessors who appraised the asset at 80 and 100 belong to the same cluster, the center of which is 90. This approach makes it possible to implement the principle of the majority and set the value of the asset at 90, because according to R. Eliezer's opinion, most of the assessors support this appraisal despite the fact that none of them quoted this value.

The Talmud raises a difficult and self-evident issue in this regard: "On the contrary (*adraba*), this asset is worth 110, and he who said 100 was mistaken by 10 downward, and he who said 120 by 10 upward." In other words, it would have been possible, based on the same cluster analysis, to assign the 100 and 120 appraisals to the same cluster, and thus rule by majority that the value of the asset is 110, against the minority opinion of 80. The example in the baraita is symmetrical, and therefore adding the appraisal of 100 to the same cluster as the 120 appraisal is as logical as adding it to the cluster with the 80 appraisal. In the first case, majority opinion produces 90, in the second case it produces 110. The answer of the Talmud is "Adopt the first two which do not exceed 100." In other words, there is a downward asymmetry, and therefore the appraisal of 90 is preferable to 110 because the opinions that form the 90 cluster do not cross the 100 boundary. The question is: What is the source and logic of the asymmetry?

Rashbam, s.v. *nakot miat trei kamai*, interpreted the asymmetry as follows: "In other words, according to what they say (80 and 100) we must say that close to 100 it is equal but not exactly 100, and we do not follow the upper two opinions (100 and 120), as they say that more than 100 (110), because if you take hold of too much you hold nothing (*tafasta merube lo tafasta*)."<sup>36</sup> *Tafasta merube lo tafasta* is a universal principle whereby one is always to take the lower option.<sup>37</sup> Implementing this principle in our case will result in a preference for a value of 90 over

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<sup>36</sup> See Tosfot s.v. *demetorat mana lo mapik lei*. Their opinion is also based on a preference of the lower appraisal over the higher one and not the boundary of 100, as the two high assessors, 100 and 120, also agree that the value is not lower than the 100 boundary.

<sup>37</sup> The source of this halakha is with Rabbi Akiva's statement in *Safra* (*Metzora, Parashat Zavim*, Portion 5): "Two days can be interpreted as many days. Said Rabbi Akiva, in everything that can be interpreted broadly or narrowly, you took hold of too much you hold nothing, you took hold of little you hold." The Tosafot (bYoma 70a s.v. *naavid arbaa*) debated the scope of the rule and have led to two main understandings of it. (1) Only when the dilemma is between a small and a large value that has no limit do we say the *tafasta merube lo tafasta* rule. But when the large

110, irrespective of the 100 boundary mentioned in the Talmud ("which do not exceed 100"). If so, one must ask why the 100 boundary was mentioned in the Talmud. It appears that according to Rashbam, the decision depends on the cluster to which we assign the assessor who ruled 100, the top or the bottom. Either way, his opinion is interpreted as being "close to 100," and the difference is whether it was "close from above" or "close from below." To resolve this dilemma, Rashbam needed the principle of *tafasta merube lo tafasta*, and this is how he interprets the words "it does not exceed 100" in the Talmud.<sup>38</sup>

An additional question whether the *tafasta merube lo tafasta* rule applies here, as in Rashbam's interpretation the appraisal is conducted for the purpose of selling land regarding which there are contradictory interests. The creditors are interested in a low appraisal in order to obtain a larger plot, and the owner of the land is interested in a high appraisal. In this case, we have the halakha "the burden of proof is on he who extracts from the other," and therefore it appears that one should prefer the higher appraisal in order to protect the defendant from the creditors. By contrast, in an appraisal made for estimating damages, the party extracting money is the injured one, and in this case, it is the lower cluster that fits this principle.<sup>39</sup>

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value is limited, then the large value must be taken hold of because 100 is included in the sum of 200. In some cases, *tafasta merube lo tafasta* is applied even when the large value is limited, when other rules of decision apply, as for example, *safek lehumra* (when in doubt, opt for stringency), which in this case is the small value. And when it is a question of money to which the rule "the burden of proof is on he who extracts from the other" applies, we cannot extract money at the high value but at the low one. (2) This appears to be the commonly accepted understanding: even if the large value is limited, we use the *tafasta merube lo tafasta* rule. Some of the Ahronim suggested that the use of the term "holding" (*tfisa*) in this rule is borrowed from monetary holding. In other words, hold on to the minimalist interpretation, as someone who debates whether to take hold of little or much should take hold of that which is certain, and that is little. It is possible that this is what Rashi intended in Sukkah 5A, s.v. *tafasta merube: "... vemashal hu."* See, for example, Itzhak Flexer, *Shearei Itzhak al Masekhet Sukkah*, 1999, p. 31.

<sup>38</sup> In this interpretation, Rashbam takes the word "it" (*lei*) to refer to the median assessor and not to the value of the asset. In other words, assigning the median assessor to the lower cluster prevents his removal from the range of 100.

<sup>39</sup> See n. 4 above. It appears that, according to Ritba's interpretation as well, mentioned in the note, such as "when it has been appraised by three persons," based on a contractual agreement, we do not apply "the burden of proof is on he who extracts from the other," as is apparent in the case of the worker's wages (bBaba Metzia 80a). See below text adjacent to n. 89. For the connection between the topics see n. 98 below.

Contrary to Rashbam, who did not address at all the consideration of "he who extracts from the other," Ramban (Nachmanides) says that according to the principle of "he who extracts from the other," we must always rule that the owner of the land prevails and therefore "we raise it by all we can raise." In his opinion, "this is why we ask 'on the contrary' (*adraba*), in order to raise" that is, the question of the Talmud, "on the contrary" (*adraba*) is based on the principle of "he who extracts from the other." The Talmudic justification "adopt the first two which do not exceed 100" derives from a psychological assumption that the assessors' (indeed, every person's) conception of value is not continuous but discrete by orders of magnitude, and a psychological block prevents assessors from crossing the boundaries of an order of magnitude (such as tens, hundreds, or thousands) from below upward. In the words of Ramban, "People are competent between 100 and more than 100 more so than they are in other appraisals (when they do not cross an order of magnitude) because people are more precise between 100 and more than 100."<sup>40</sup> <sup>41</sup> Therefore, in his opinion, the joining of the 100 appraisal to the 80 takes precedence because this cluster and the lower one are united in that they do not cross the 100 boundary, whereas joining the 100 appraisal to the upper one raises the median assessor above this boundary.<sup>42</sup> But we must still ask what the basis for this psychological presumption is, and why is it sufficiently strong to extract money from the defendant.<sup>43</sup> Moreover, Ramban's suggestion is applicable only in cases such as that of the baraita, in which the assignment of an assessor to one of the clusters involves crossing the boundary of an order of magnitude. It is not clear how Ramban would interpret the opinions of R. Eliezer and others in cases in which the assignment

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<sup>40</sup> *Innovations of Ramban to Baba Batra* (Maarva edition, 1993), p. 127.

<sup>41</sup> There are many halakhot based on psychological assumptions, such as "the presumption that nobody would dare [act in such a way toward] his creditor" (bBaba Metzia 3a), "the presumption that no woman would dare [act in such a way toward] her husband" (*Shulchan Aruch*, Even Haezer 17:2), "the presumption that no man commits a sin unless he stands to profit by it" (bBaba Metzia 5b).

<sup>42</sup> In Ramban's interpretation, the words "which do not exceed 100" refer to the assessors whose appraisal does not cross the 100 boundary (80 and 100). Compare with Rashbam's interpretation, n. 38 above.

<sup>43</sup> It is possible that in Ramban's understanding, the psychological assumption in question is like a halakhic presumption based on which it is possible to extract money. See, for example, *Shulchan Aruch*, Hoshen Mishpat 78:1; R. Shlomo Ganzfried, *Pnei Shlomo* (Jerusalem Institute, 1999, p. 225): "And Ramban was forced to justify this." See *ibid* a brilliant analysis of our case.

of the median assessor to the upper cluster does not require crossing the boundary of an order of magnitude.

Ritba offers a different interpretation. In his opinion, the preference of the appraisal of 90 over 110 follows from the fact that this decision is supported by the majority opinion, whereas the 110 decision is supported only by the assessor who appraised at 120. The difficulty with this explanation is that R. Eliezer's opinion is based on the assumption that the assessor's ruling incorporates a deviation from the correct value and does not reflect a precise valuation to begin with. Ritba's interpretation, however, in the second stage, assumes that the original appraisals were exact and based on them the appraisal of 90 must be preferred over 110. It appears that Ritba himself tries to cope with this difficulty: "Maybe there is no mistake and we find that [the assessor] does not adopt this appraisal (110) but only the one (120)."<sup>44</sup> But according to this assumption, "if there is no mistake here" the assessor who appraised at 100 does not necessarily adopt the 90 appraisal.

In sum, the case described in the baraita is symmetrical. The final appraisal according to R. Eliezer (and as we shall see below, according to the others as well), breaks the symmetry and rules for the appraisals below 100. As we have seen, Rishonim have tried to explain the asymmetry. Ramban noted that the downward asymmetry is contrary to the reverse asymmetry following from the principle of "the burden of proof is on he who extracts from the other" (for the benefit of the defendant). An argument for an upward asymmetry is offered by R. Shlomo Ganzfried in his contribution to the topic. According to him, the upper cluster should have preference over the lower one because in clusters of equal radius, the higher the center of the cluster the lower the rate of error of each assessor when expressed as a percentage of the value of the asset.<sup>45</sup>

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<sup>44</sup> R. Yom-Tov Bar Avraham Al-Sevili, (Ritba), *Innovations of Ritba to Baba Batra* (Jerusalem, The R. Kook Institute), p. 819.

<sup>45</sup> This is how R. Ganzfried explains, *ibid*, the use of the term "on the contrary" (*adraba*) that appears twice in the topic in relation with the upper cluster. According to R. Ganzfried, the term *adraba* indicates a preference for the option being suggested, whereas in the case of a balanced doubt, the Talmud's way to raise doubt is using the term *dilma* (lest, maybe).

## The "Others"

The third opinion is that of the "others," and it is the only one that does not specify a numeric figure but offers a rule for reaching a decision: "we appraise between them and divide it by three." This is the most challenging of the methods.

Two questions arise from the proposal of the others: to whom do the others refer by the term "between them" and what does "divide by three" mean? "Between them" can be interpreted as referring to the three assessors, to the two extreme assessors, or to the assessors of the lower cluster who have already been mentioned in the opinion of R. Eliezer Bar Zadok. Below we present three propositions for understanding the method of the others, based on different interpretations of the term "between them." The meaning of the term "divide by three" will be derived from each of the possibilities we propose.

The symmetry of the Talmudic discussion indicates that the Talmud perceived the method of the others as an elaboration of the method of R. Eliezer Bar Zadok. The Talmud questions both methods with the expressions "on the contrary" (*adraba*), and justifies the preference of the lower cluster with "adopt the first two." Both R. Eliezer and the others join two divergent opinions to form one cluster. In the center of this cluster is the "correct" value from which each of the assessors in the cluster has deviated. As a result, these two appraisals are considered to form a majority opinion that overrules the third appraisal (the minority opinion). According to both R. Eliezer and the others, the value that determines the final appraisal as the center of the majority cluster created around the lower value.<sup>46</sup>

The differences between the method of the others and that of R. Eliezer Bar Zadok, according to the Talmudic explanation, are the following:

- (a) The others suggest a formula for finding the center of the cluster and for the calculation (of its radius). In the numeric example of the baraita, the Talmud sets the radius at  $13\frac{1}{3}$ .

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<sup>46</sup> This is how Rabad explained (*quoted in Shita Mekuvetzet*) the method of the two Tannaim, R. Eliezer and the others, as being based on the same principle: "Therefore we take the average of the first two assessors who are actually of the same opinion, and this is the method of the others."

- (b) The others encounter a difficulty in interpretation that causes them to take into account a factor that has not been mentioned before. To accept the agreement between quotes of 80 and 100 for an asset whose value is  $93\frac{1}{3}$ , we must assume that just as 80 deviates downward by  $13\frac{1}{3}$  so is 100 liable to deviate upward by  $13\frac{1}{3}$ , and therefore we would expect an appraisal of  $106\frac{2}{3}$ , except that such an appraisal has not been made. How can we then assume an agreement between the appraisals of 80 and 100 at  $93\frac{1}{3}$ ?
- (c) To settle this difficulty, the Talmud suggests that in the method of the others there is yet another innovation, in addition to that of R. Eliezer Bar Zadok. The others assume that the appraisals of the assessors are affected also by strategic considerations. In other words, assessors update their appraisals as new information becomes available, as for example the appraisals of their colleagues. Thus, it is reasonable to assume that the original appraisal of the assessor (before he learns about the appraisals of his colleagues) was different from the one he quoted at the end. In the example before us, it is possible that the assessor who appraised the asset at 100 intended to say  $106\frac{2}{3}$ , but out of fear of exaggerating, updated his appraisal to 100 under the influence of his colleague who appraised the asset at 80.<sup>47</sup>

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<sup>47</sup> See Ramban, *Innovations to Baba Batra* (Maarava editon, 1993) p. 127, who asked, regarding the opinion of the others, why not calculate the point from which every opinion deviates by  $13\frac{1}{3}$  from the starting point of 100? Accordingly, we ought to say that the value of the asset is  $86\frac{2}{3}$  and the assessor who appraised at 80 updated upward under the influence of the assessor who appraised at 100, although initially he intended to indicate a value lower than 80. In response to this, Ramban, Ritba (in his *Innovations*), and Ran (in *Shita Mekuvethzet*) offer a psychological assumption whereby the principle of taking into account another opinion ("I already exceeded my colleagues by so much") is voiced only in the upward direction, not in the downward one. But an assessor who appraises at a lower value than his colleagues has no difficulty (shame) to voice his opinion even if it is far from the opinion above it, and therefore will voice his exact opinion and not revise it upward. R. Shlomo Ganzfried justified this question by the principle of minimizing the error, see n. 45 above. According to him, once the preference of the lower cluster has been established, its center must be raised as much as possible in order to minimize the rate of error of each assessor in percentages relative to the value of the asset. Note that the above issue raised by the Rishonim does not arise at all according to the interpretation in the Tosfot. See the text adjacent to n. 48 below.

(d) The Talmudic explanation of the method of others is based on a hidden assumption of communication between the assessors. So far we considered three independent assessors, each one using private information. But according to the method of the others, it is clear that the assessor who appraised at 100 knows about the appraisal of 80. The Tosfot<sup>48</sup> indicate that the order of the appraisal is "that the two lesser ones speak first." The author of *Torat Haim*<sup>49</sup> proposes an order similar to that of an English auction, "as to the order, at the beginning they ask whether it is worth something that is simply more, such as 60 and 70 and 80 (in other words, an obvious value such as 60, 70, 80). And when they ask whether it is worth more than what is called, one of them quits, and when they ask about more than 100 the other quits, and when they ask about more than 120 the third one quits." Thus, gradually, the appraisals are delivered from the bottom up, each one being influenced by the one below it.<sup>50</sup> By contrast, Rashbam does not specify the order in which the assessors speak but assumes that the appraisals of the assessors are common knowledge: "And although we do now know which one spoke first, they negotiated the matter, and each one knows what the opinion of his colleague is. And after they negotiated the matter, they say what is right in their opinion. And because he knew that his colleague wanted to say 80, he was afraid to say more than 100."<sup>51</sup>

The Talmud asks about the method of others (as about the method of R. Eliezer Bar Zadok) why, by the same token, not say that the value of the asset is  $113\frac{1}{3}$ , and let us assume that the assessor appraising at 100 deviated downward by the rate we assumed ( $13\frac{1}{3}$ ), whereas the assessor who appraised 120 intended to say  $126\frac{2}{3}$  but revised his appraisal to 120. The answer of the Talmud,

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<sup>48</sup> Tosfot, *ibid*, s.v. *mestiya deka tafina*.

<sup>49</sup> R. Avraham Haim Shor, *Torat Haim* (Galitzia, 16<sup>th</sup>-17<sup>th</sup> century), Baba Batra 107.

<sup>50</sup> It would have been possible to suggest the order of a Dutch auction, in which they ask who agrees to 1000? Who agrees to 900? and so on. But it is clear that this type of bidding does not fit our topic, because in our case the assessor who appraised at 100 clearly revised his appraisal under the influence of the appraisal of 80, which means that he knew about it. About the theory of classical bidding see Krishna Vijay, "Auction Theory," *Academic Press*, 2000.

<sup>51</sup> Rashbam, 107b, s.v. *ubadin hu deleima*.

as in the discussion about R. Eliezer above, is that the method reflecting the agreement of two opinions that sets the value below 100 takes precedence over a value that exceeds 100.<sup>52</sup>

Note that the Talmud does not explain the calculation of the deviation: how it was set at  $13\frac{1}{3}$  and how this calculation is derived from the words of the others, "we appraise between them and divide it by three." The Rishonim explained that the calculation is derived from the difference between the extremes ( $120 - 80 = 40$ ) divided equally between the three assessors:  $\frac{40}{3} = 13\frac{1}{3}$ . There seems to be some difficulty in applying this calculation to the expression "we appraise between them." Therefore, we must first and foremost understand the mathematical logic upon which it is built.<sup>53</sup>

### **Proposals for Interpreting the Others' Method**

Given that the suggestion of the others, according to the Talmudic topic (whereby 100 is in fact  $106\frac{2}{3}$ ), is pure speculation, how is it possible to base a halakhic ruling on a speculation that carries no necessity? First, how do we know that the assessor who appraised at 100 intended to say exactly  $106\frac{2}{3}$ ? Moreover, by the same logic it is possible to claim that even the assessor who appraised at 120 intended to say more, and if so the basis of the calculation  $\frac{40}{3} = 13\frac{1}{3}$  is not clear. It is possible to say that adhering to a decision of  $93\frac{1}{3}$  stands on its own and does not follow from attributing strategic thinking to the assessors. This numeric result is derived directly from what the others said, "we appraise between them and divide it by three," whereas the strategic argument is interpreted only as a "defense" of the main thesis of  $93\frac{1}{3}$ , allowing it to be based on

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<sup>52</sup> See the discussion adjacent to n. 36 above.

<sup>53</sup> Lieberman, n. 4 above, suggested that the division by three has to do with the fact that we have three opinions. Following this logic, if there is a large number of assessors, the difference is divided by the number of assessors.



a valid rule for reaching a decision. According to this proposal, unlike in the case of R. Eliezer's method, with the others' method we must distinguish between two stages:

- (a) "Others say, we appraise between them and divide it by three. They are of the opinion that the land is worth 93 and a third." The number  $93\frac{1}{3}$  is derived from the formula "appraise between them and divide it by three." We must explain how they arrived at this value and what is the logic behind it, which is not explicit in the topic. And even if we accept this value as a logical numeric conclusion, we still lack a valid rule for reaching a decision that can support the logic of the others because the statements of the others in the baraita are not formulated as a rule for reaching a legal decision but as an obscure mathematical formula.

To this end, the Talmud offered the next stage.

- (b) The value of  $93\frac{1}{3}$  can be supported by the majority principle, as understood by R. Eliezer Bar Zadok, in other words, through a cluster of the opinions of most of the assessors, the center of which is this value. But we must regard majority opinion flexibly and pay heed to the claim that the appraisal of 100 can also reflect  $106\frac{2}{3}$ , as noted above.

### **Proposal No. 1**

In an attempt to understand the first stage in the method of the others, Prof. Aumann, in the course of our joint study, offered an understanding based on the statement of the baraita "we appraise between them and divide it by three" which also arrives at  $93\frac{1}{3}$ . According to this proposal, and contrary to what has been said above, the method of the others is not an elaboration of R. Eliezer's method but it is based actually on Tanna Kama's method, as understood by Maimonides.

As we have seen, in the interpretation of Maimonides the method of decision is the midrange. The difficulty in Tanna Kama's method is that it ignores entirely the median value. The method of the others overcomes this difficulty. The appraisal is set between the extreme opinions, but we must "divide by three," that is, divide the weight of each extreme proportionally with all the

opinions. If we obtain the point of origin of the asymmetry that is implied in the topic, we must adjust the appraisal of 100 downward (despite the fact that it is positioned "in the middle") and join it to the appraisal of 80. The result is that the appraisal that is to be set between the extremes grants the appraisal of 80 twice the weight of the appraisal of 120, in other words,  $\frac{2}{3} \times 80 + \frac{1}{3} \times 120 = 93\frac{1}{3}$ . According to this proposal, "between them" is interpreted as referring to the two extreme assessors.

Computationally, this method is identical with the calculation proposed by Rashbam in his interpretation of the range of 40 that must be divided by 3 and added to the low value that is not being called into question:  $80 + 13\frac{1}{3} = 93\frac{1}{3}$ .<sup>54</sup> Contrary to Rashbam, it displays legal logic as well. But we must also understand whether the legal logic on which the proposal is based is consistent with the understanding of the Talmud. According to the Talmudic topic, a ruling of  $13\frac{1}{3}$  appears to be based on ignoring the appraisal of 120, because taking this appraisal into account would lead to the assumption that he too lowered his original appraisal under the influence of the two assessors who appraised lower than himself.<sup>55</sup> But the present proposal relies clearly on the declared value stated by the assessor who appraised at 120.<sup>56</sup>

Following this line of thought, it is possible to give a similar explanation to R. Eliezer Bar Zadok's method as well, in a manner that is different from that suggested by the Talmud, and to say that the result of 90 is not the midpoint between 80 and 100. This result can be represented in a ruling between the extremes as follows:  $\frac{1}{4} \times 120 + \frac{3}{4} \times 80 = 90$ . According to this proposal, in the first stage R. Eliezer calculates differently from Tanna Kama (according to Maimonides's

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<sup>54</sup> It is possible to see that Rashbam's calculation and that of this proposal are similar. Let us designate the three appraisals  $x$ ,  $y$ , and  $z$  ( $x < y < z$ ). According to Rashbam, the calculation is  $\frac{z-x}{3} + x$ , and according to this proposal it is  $\frac{2}{3}x + \frac{1}{3}z$ . It is easy to show that  $\frac{z-x}{3} + x = \frac{2}{3}x + \frac{1}{3}z$ .

<sup>55</sup> See *Torat Haim*, n. 49 above, s.v. *adraba*.

<sup>56</sup> An additional question that is outside the scope of the present discussion is: assuming that we had  $n$  assessors producing appraisals  $a_1, a_2, \dots, a_n$  so that  $a_1 < a_2 < \dots < a_n$ , how would we assign weights to the opinions of the assessors according to this proposal?

understanding) the relative weights of the extremes. In his opinion it is not possible to assign the same weight to the two extremes, and 80 must be assigned three times greater weight than 120. During the second stage, the Talmudic explanation validates the ruling of 90 based on the rule of decision by majority opinion because it is possible to achieve a majority around the appraisal of 90. The others disagree with the ratio proposed by R. Eliezer and suggest a ratio of 1 to 2 between the extremes. In the Talmudic explanation, their method still allows treating the result of  $93\frac{1}{3}$  as an agreed-upon value. The difficulty is that it is not clear whence did R. Eliezer derive these relative weights.

This understanding of the topic results in a coherent view of the three Tannaitic opinions, each presenting a ruling based on a midpoint between the extremes. According to Tanna Kama, this point is the midrange (in the Maimonides's understanding of Tanna Kama); according to R. Eliezer, this point falls between the extremes at a ratio of 1:3; and according to the others, the ratio is 1:2. This understanding provides insight into Maimonides's method, who interpreted the opinion of Tanna Kama as a division of the extremes, and not in the way Rashbam interpreted it. It could be said that the understanding of the other Tannaim, and especially the opinion of the others, as proposed here, is what made Maimonides interpret Tanna Kama's opinion as well in light of the Talmudic text as a "middle course," which is a ruling in the middle, between the extremes. Nevertheless, according to this explanation, we must still investigate how the Tannaim obtained the above relative ratios and what lies at the bottom of the controversy.

This understanding results in a coherent view of the Tannaitic opinions also with regard to the cluster analysis, even in Maimonides's understanding of Tanna Kama. According to this view, Tanna Kama sees all the appraisals as one large cluster, and therefore sets the "center" of the cluster at the midpoint of the range. The intention of Maimonides in providing a numeric example of his own<sup>57</sup> is to emphasize that in any case Tanna Kama disagrees with R. Eliezer and the others, and not only in the case of a symmetrical distribution of the appraisals around a certain value. In the numeric example of Maimonides (90, 100, 130), Tanna Kama would rule (according to Maimonides) 110, R. Eliezer would rule 95 in the center of the cluster (90, 100),

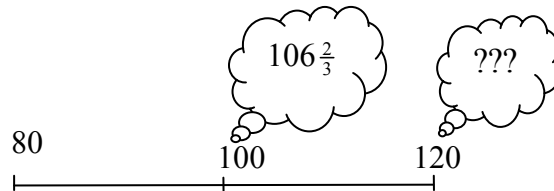
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<sup>57</sup> See n. 23 above.

and the others, according to the present understanding, would rule  $90 \times \frac{2}{3} + 130 \times \frac{1}{3} = 103\frac{1}{3}$ . This value is the center of the  $(90, 116\frac{2}{3})$  cluster and expresses the Talmudic estimate that the assessor who appraised at 100 intended to say  $116\frac{2}{3}$ .

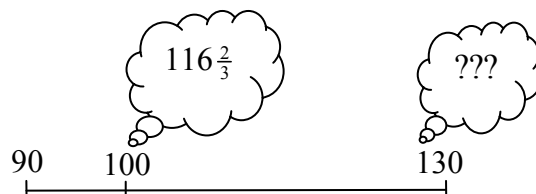
In Figure 1, the appraisal of 100 is the result of the revision of the appraisal of the assessor who intended to quote  $106\frac{2}{3}$  but "updated" his appraisal after hearing the first assessor's appraisal of 80. The original discrepancy between the appraisals of the first two assessors was  $26\frac{2}{3}$ , and the discrepancy after the update is 20.

*Figure 1*



In Figure 2, based on Maimonides's numeric example, the original discrepancy between the appraisals of the assessors was also  $26\frac{2}{3}$ , but after the revision it was reduced to 10. In this presentation of the method of others, the revision of the appraisals does not appear consistent. In the example illustrated in Figure 1, the discrepancy between the appraisals of the assessor after the revision is 75% of the original discrepancy, whereas in the example shown in Figure 2, the discrepancy after the revision is 37.5% of the original discrepancy.

*Figure 2*



We must ask a general question at this point. According to the interpretation of the Rishonim, the Talmudic "guess" that the assessor who appraised at 100 intended to say  $106\frac{2}{3}$  is based on the division of the range between the three assessors. The difficulty here is that just as we assume that the assessor who appraised at 100 intended to say more, the same must be assumed about the assessor who appraised at 120, and if so, it is not clear why it is that we divide the 80-120 range, given that the value of 120 reflects a "revision" and not necessarily the "true" opinion of the third assessor. The same is true for the example illustrated in Figure 2.

In our opinion, the greatest difficulty with this method is that if we tilt the weights upward instead of downward, and assign a weight of  $\frac{2}{3}$  to the upper extreme of the range and  $\frac{1}{3}$  to the lower extreme, we obtain  $\frac{1}{3} \times 80 + \frac{2}{3} \times 120 = 106\frac{2}{3}$ . But this calculation is not consistent with the course of the topic, because subsequently the Talmud asks: "On the contrary, the land is worth 113 and a third." In other words, the counter-value that the Talmud presents with its question is significantly different from the counter-value obtained according to this method. Therefore this proposal is likely to stand as an independent explanation of the opinion of the others, which is not consistent with the Talmudic interpretation of the others' method.

## **Proposal No. 2**

We now propose a different way of understanding the first stage using the method of the others (the stage when the average deviation is established), which is also different from Rashbam's interpretation. This proposal is different both in the manner in which the deviation is calculated and in the underlying logic. Our proposal is to interpret the opinion of the others, "we appraise between them and divide it by three," as applying to what R. Eliezer said. The relation between R. Eliezer's method and that of the others is clearly implied already in the Talmudic topic.<sup>58</sup> Based on this relation, we propose interpreting "between them" in the expression "we appraise between them and divide it by three" as applying to the two assessors of the lower cluster (80, 100), on whom R. Eliezer based his ruling. The others as well, because of the asymmetry arguments of R. Eliezer, perceive the (80, 100) cluster as deciding the appraisal, but in their

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<sup>58</sup> See above, at the beginning of the discussion, in the section "The Others."

opinion we must **divide by three** between them and not settle for the center of the cluster, as R. Eliezer ruled. In other words, the appraisal of 100 is assigned twice the weight of the 80 appraisal:  $80 \times \frac{1}{3} + 100 \times \frac{2}{3} = 93\frac{1}{3}$ .

According to our proposal, despite the fact that the others agree that the appraisal is based on the lower cluster, they assign a weight of  $\frac{2}{3}$  to the 100 appraisal, taking into account the third assessor who set an even higher value (120). On one hand, the others do not take into account a value higher than 100 because such a value is outside the selected cluster, and it is therefore considered an "error." On the other hand, they do not entirely ignore the opinion of the third assessor, in the sense that its presence grants to the extremity that is near it in the selected cluster a double weight.<sup>59</sup> This rule for reaching a decision is coherent also in the simple case in which two assessors say 100 and one of them says 200.<sup>60</sup>

It follows that, according to the others, the same result is obtained when the appraisals are (80, 100, 120) as when they are (80, 100, 130). In other words, an "exaggeration" in the appraisal of the upper assessor does not affect the final result. A similar characteristic is present also in Tanna Kama's method according to Rashbam, and a manipulation of the extreme appraisals does not affect the final appraisal. But whereas Tanna Kama's method is entirely robust, the method of the others is immune only to upward exaggeration.

According to this understanding, in the Tannaitic controversy about ruling on an appraisal when there are no two opinions in agreement, Tanna Kama (according to Rashbam) rules by the median assessor, R. Eliezer according to the two lower assessors, and the method of the others improves upon R. Eliezer's method and weights into the appraisals of the lower assessors the appraisal of the upper assessor.<sup>61</sup>

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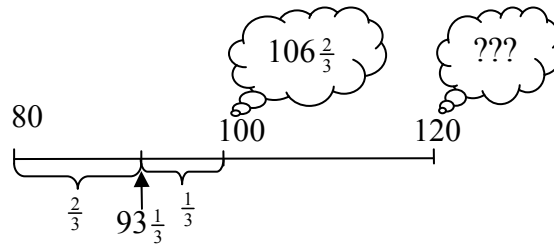
<sup>59</sup> It is possible that this idea is hinted at by Rabad in *Shita Mekuvtzet*.

<sup>60</sup> Applying the rule in this case would give  $100 \times \frac{1}{3} + 100 \times \frac{2}{3} = 100$ .

<sup>61</sup> Again we note that the even weighting of all the opinions of the assessors as a simple arithmetic average is not being considered. See n. 21 above.

This explanation of the opinion of the others yields the result  $93\frac{1}{3}$  for the numeric example in the baraita (Figure 3). But for the numeric example brought by Maimonides (90, 100, 130), we obtain a different result from the one produced by Rashbam's calculation and by our Proposal No. 1. The 90-100 range that is disputed by the two lower assessors is divided proportionally, and the final result is  $96\frac{2}{3}$ . This value can be supported by the majority principle as understood by R. Eliezer and the others, in other words, as the center of a cluster of opinions that includes most of the assessors' opinions, assuming that the 100 appraisal reflects an initial intention of saying  $103\frac{1}{3}$ , similarly to what is said in the Talmudic topic. In this calculation, the discrepancy between the appraisals of the assessors after the revision is 75% of the original discrepancy, as in the example of the baraita. The result is a general one.<sup>62</sup>

**Figure 3**




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<sup>62</sup> Let us designate the original appraisals of the assessors as  $x < y_0 < z_0$  and the appraisals they stated as  $x < y_1 < z_1$ . The cluster of our interest is the lower of the two  $(x, y_1)$ , and the final appraisal is determined by the formula  $a = \frac{1}{3}x + \frac{2}{3}y_1$ . The discrepancy in the original appraisals of the assessors in the lower cluster is  $\Delta_0 \equiv y_0 - x = 2(a - x) = 2\left(\frac{1}{3}x + \frac{2}{3}y_1 - x\right)$ , and the discrepancy after the revision is  $\Delta_1 \equiv y_1 - x$ . The relation between the discrepancies is therefore  $\frac{\Delta_1}{\Delta_0} = \frac{y_1 - x}{2\left(\frac{1}{3}x + \frac{2}{3}y_1 - x\right)} = \frac{3}{4}$ . By contrast, it is possible to show that according to Proposal No. 1 the ratio of the discrepancies is  $\frac{\Delta_1}{\Delta_0} = \frac{3(y-x)}{2(z-x)}$ , which means that the ratio depends on  $z$  as well.

Our proposal explains why the extreme points of the range, 80 and 120, can serve to anchor the calculation, as it would have been possible to ask whether the assessor who appraised at 120 also intended to say "more." According to our proposal, if he had tried to tilt the final appraisal upward, he would have said 100, and any upward deviation would only lower the final appraisal. It follows that if he quoted a value of 120, this appears to be a value he holds true. Moreover, the assessor who appraised at 120 has no incentive to revise his appraisal because, according to our proposal, revisions by the upper assessor will not change the final appraisal.

It is possible to demonstrate that the method of others is not monotonic: according to our proposal, if the assessor who appraised at 120 had said 100 we would have said that the one in the minority (80) is overruled and rule that the final appraisal is 100. The fact that this assessor quoted a value higher than 100 lowered the final appraisal according to the method of the others to  $93\frac{1}{3}$ . The Tosfot point to a similar paradox:<sup>63</sup> if we had only two appraisals, one of 80 and one of 120, we would have placed the appraisal at 110.<sup>64</sup> If so, how is it possible that adding a third opinion that is lower (80) will raise the value of the appraisal to  $113\frac{1}{3}$ , as the Talmud assumed in its question about the opinion of the others?<sup>65</sup> Ramban, who mentions this question as "some ask" (*iesh makshim*) argues that there is an error in the basic assumption of the questioners because when there are only two appraisals, the final appraisal is determined by the principle of "the burden of proof is on he who extracts from the other" (bBaba Kama 46a). By contrast, when there are three appraisals we follow majority opinion.<sup>66</sup>

The author of *Torat Haim* points out that the question in the Tosafot is based on an error. Based on the assumption that the assessors are influenced by the appraisals of their colleagues, were it not for the assessor who appraised 80, the two other assessors would have quoted higher values. He draws attention to an additional indication of the non-monotonicity of the others' method: if

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<sup>63</sup> Ibid, s.v. *adraba hai ara'a*... The Tosfot did not justify this question.

<sup>64</sup> The Tosafot assume that everybody agrees with this. Both R. Eliezer Bar Zadok and the others would be able to say that there is agreement here on 110, and Tanna Kama would also see 110 as the midpoint.

<sup>65</sup> See text adjacent to and before n. 52 above.

<sup>66</sup> *Innovations to Baba Batra* (Maarva edition, 1993), p. 127.



we had only the appraisals of 80 and 120, the final appraisal would have been at least 100. With the addition of the appraisal of 100, the final appraisal is down to  $93\frac{1}{3}$ . His argument is that if the qualifications of the assessors are identical, there is no reason to assume that there is an assessor who makes no mistakes. Therefore, when we have two assessors, we can assign identical errors to both. But when we have three or more assessors, we cannot assign identical errors to all because there is no value on a one-dimensional number line that is equidistant from three or more points. Therefore, the others attribute unequal errors to the three assessors.<sup>67</sup> The above discussion was intended to explain how this division is determined.

This proposal is not consistent with the Talmudic question "On the contrary, the land is worth 113 and a third," because deflecting the appraisal upward, according to the proposal, would yield  $\frac{1}{3} \times 120 + \frac{2}{3} \times 100 = 106\frac{2}{3}$ . Therefore, this proposal is also likely to stand as an independent explanation of the opinion of the others, which is not compatible with the Talmudic understanding.

### Proposal No. 3

As noted above, the method of the others is based on R. Eliezer's method of cluster analysis.<sup>68</sup> The Talmudic clarification of the others' method is based on two assumptions: (1) the error attributed to each assessor in the lower cluster is  $13\frac{1}{3}$ ; and (2) even professional assessors are influenced by the opinions of their colleagues and revise their professional appraisals accordingly. From the first assumption, the Talmud infers that the center of the lower cluster is  $80 + 13\frac{1}{3} = 93\frac{1}{3}$ . The value  $93\frac{1}{3}$  is not the center of cluster (80, 100) but the center of cluster

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<sup>67</sup> The argument of *Torat Haim* explains a similar surprise expressed by R. Yosef Hutner, *Hevel Yosef*, Olam Hamishpat, about *Shulchan Aruch* Hol Hamoed, 103, p. 58: in Maimonides's understanding of Tanna Kama, if we had three assessors, two of whom quoted 100 and one 130, we would rule according to the majority and set the appraisal at 100. But if one of the assessors quoted a lower price (90), the final appraisal goes up according to Maimonides to 110!? According to the explanation in *Torat Haim*, when two assessors quote identical values, we assume that they aimed at the truth. But when there is no agreement about any value, we must assume that they all erred.

<sup>68</sup> See n. 46 above and the adjacent text.

$(80, 106\frac{2}{3})$ . Based on the second assumption, it is clear that we can attribute to the assessor who quoted 100 a higher initial appraisal of  $106\frac{2}{3}$ . In this section, we propose a mathematical formula that clarifies the values that the Talmud attributes to the method of the others, as well as the meaning of the expression "we appraise between them and divide it by three."

R. Eliezer begins with cluster analysis. On the selected cluster he estimates the error of each assessor in the cluster (in this case, the [80, 100] cluster and an error of 10), and ignores the assessor who is not included in the cluster. The others disagree and believe that the error for each assessor should be estimated first, and the clusters based on the estimated errors. The logic is that all three assessors have identical skills and therefore there is no reason to assume *a priori* that the error of one assessor is greater than that of his colleagues.<sup>69</sup> The preliminary stage of error estimation is expressed in the words of the others: "we appraise between them and divide it by three," with emphasis on "between them," and with reference to **all** the assessors.<sup>70</sup> According to our proposal, the error estimate per assessor used by the others is the measure of the average deviation.

The average deviation of each assessor is expressed as the average distance from the median,<sup>71</sup> that is, the average deviation of the appraisal from the median (in absolute values). Let us

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<sup>69</sup> As the author of *Torat Haim* says: "It is assumed that we must attribute the error to all three, as one is not superior over his colleague." Based on this opinion, *Torat Haim* justifies an apparent paradox. If we had two appraisals, (80 and 120), we would set the final appraisal at 100. With the addition of an appraisal of 100, the result according to the other is  $93\frac{1}{3}$ . Because we must attribute an error to each assessor, when there are three assessors, the error is attributed to each and there is no reason to say that the assessor who quoted 100 did not err, as it follows from Tanna Kama's method (especially in Rashbam's interpretation). See also the interpretation of *Minhat Bikurim* (R. Shmuel Avigdor) about the Tosefta, Ketubot 11:3.

<sup>70</sup> This proposal suits better the version in the Tosefta, Ketubot 11:2: "We appraise between them and estimate by a third." We shall see that this proposal is based on two computational stages, as it transpires from the formulation in the Tosefta.

<sup>71</sup> Because according to Rashbam, the preferred central tendency measure is the median, the appropriate measure of dispersion used by the others is the average deviation from the median (as opposed to the deviation from the

designate the number of assessors as  $n$ , the median appraisal as  $med(a^i)$ , the appraisals of assessors  $i = 1, 2, 3$  as  $a^i$ , and the mean deviation as  $md$ . The mean deviation is given by the formula:

$$(1) \quad md = \frac{1}{n} \sum_{i=1}^n |a^i - med(a^i)|$$

One of the two basic assumptions of the method of the others is that the assessors take into account the opinions of their colleagues and revise their appraisals accordingly. Let us designate the original appraisal of assessor  $i$  as  $a_0^i$ , and let us assume that the revised appraisal is described by the following formula:

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average), as one of the known differences between the mean and the median is that  $\bar{x} = \min_x \sum_{i=1}^n (x_i - x)^2$ ,

whereas  $med(\mathbf{x}) = \min_x \sum_{i=1}^n |x_i - x|$ . In other words, the mean minimizes the sum of squared deviations (and therefore minimizes the standard deviation), whereas the median minimizes the sum of absolute deviations, and therefore it minimizes the average deviation (from the median). It follows that an explanation of the opinion of others based on the measure of average deviation from the median is coherent with the use of the median as the measure of the central tendency according to Rashbam's understanding of Tanna Kama, although the more accepted measure of dispersion nowadays is the standard deviation based on the average of squared deviations proposed by Gauss. The measure of average deviation proposed by Laplace measures the average absolute deviation. Both measures refer to deviations from the average. Although the Gauss measure has several advantages over Laplace, we assume that the Laplace measure is closer to the Talmudic thinking both because it is easier to compute and because it is more intuitive.

$$(2) \quad a_1^i = \begin{cases} a_0^i & a_1^j \geq a_0^i \\ a_0^i - \beta^i md & a_1^j < a_0^i \end{cases}$$

where  $\beta^i$  is the "correction coefficient" of assessor  $i$ , which measures the degree to which the assessor took into consideration the professional opinion of his colleagues. In our case, given that the skills of the assessors are identical, it is reasonable to assume that every assessor attributes to himself and his colleagues half the potential error ("two are holding on to the error" of sorts), and therefore  $\beta^i = \frac{1}{2}$ ,  $\forall i$ . According to formula (2), assessor  $i$  revises his appraisal only if he finds that it is higher than that of another assessor ( $j$ ). But it is possible that assessor  $i$  does not know the mean deviation,  $md$ , so we assume that in the case of a revision, he reduces his original appraisal at most by  $\beta md$ , and therefore  $a_0^i \leq a_1^i + \beta md$ . In other words,  $a_0^i$  is the upper limit of the estimated value of the original appraisal quoted by assessor  $i$ , but in practice it is possible that his original appraisal was lower. The "reconstructed" cluster is  $(a_1^i, a_1^j)$ , and the final appraisal is found in the center of this cluster. Thus, it is clear why the Talmud says "by the rule he should have said more," without indicating a precise number.

If we insert the numbers that appear in the baraita into formula (1) we obtain  $md = \frac{|-20| + |0| + |20|}{3} = 13\frac{1}{3}$ . It follows that according to formula (2),  $a_0^2 \leq 100 + \frac{1}{2} \times 13\frac{1}{3} = 106\frac{2}{3}$ , the "reconstructed" cluster is  $(80, 106\frac{2}{3})$ , and the center of the reconstructed cluster,  $93\frac{1}{3}$ , is the final appraisal.

The Talmud asks the following about this calculation: "On the contrary (*adraba*), the land is worth 113 and a third." In other words, if we applied this procedure to the two assessors who quoted the high appraisals, we would obtain a reconstructed cluster of  $(100, 126\frac{2}{3})$ , and the final appraisal would be set at its center, that is, at  $113\frac{1}{3}$ . The answer of the Talmud, "adopt the first

two which do not exceed 100," is identical with the answer proposed in the discussion of R. Eliezer's method.

This method is different from those proposed above, and with different numbers from those that appear in the baraita it produces different results. For example, with the numbers suggested by Maimonides (90, 100, 130), the result according to Rashbam will be 100, according to Proposal No. 1  $103\frac{1}{3}$ , and according to Proposal No. 2  $96\frac{2}{3}$ . According to proposal No. 3, we calculate the mean deviation from the median (which in the numeric example of Maimonides is also 100) as follows:  $md = \frac{|-10|+|0|+|30|}{3} = 13\frac{1}{3}$ . According to the mean deviation, we "reconstruct" the original appraisal of the median assessor and obtain  $a_0^2 \leq 100 + \frac{1}{2} \times 13\frac{1}{3} = 106\frac{2}{3}$ . The reconstructed cluster is therefore  $(90, 106\frac{2}{3})$ , and the final appraisal (the center of the reconstructed cluster) is  $98\frac{1}{3}$ .<sup>72</sup>

## The Halakha Ruling

In two instances, R. Huna's opinion that halakha follows the others is presented in our topic, and twice R. Ashi rejects the ruling with the argument: "we do not understand the reason of the others; shall we rule halakha according to them?" The second time, R. Huna relies on the judges of the exile, monikers for the Amoraim Shmuel and Karna, but R. Ashi rejects their ruling despite the prevailing rule that the halakha follows Shmuel in monetary law.<sup>73</sup> Apparently it was

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<sup>72</sup> Similarly to the calculation presented in n. 62, it is possible to show that  $\frac{\Delta_1}{\Delta_0} = \frac{y_1 - x}{2[(y_1 + \beta md) - x]}$ . It is easy to demonstrate that as  $md$  increases, the ratio decreases.

<sup>73</sup> See Ephraim A. Orbach, *Halakha: Its Sources and Development* (1984), p. 219. According to him, this ruling by R. Ashi is characteristic of his opinion in principle about the central position of rational inference (*svara*) in halakhic ruling. In other words, even if the great Amoraim rule with the others, R. Ashi will not rule as they did if he does not understand their reasons. See a similar explanation by Rif on the topic of one who was married to three wives, Ketubot 51 in Rif's pages.

clear to the decisors that the choice was between Tanna Kama and the others, and once the method of the others was rejected, it was clear that the ruling is not according to R. Eliezer Bar Zadok but according to Tanna Kama. Following R. Ashi's ruling, all the decisors agreed that the halakha follows Tanna Kama, but they disagreed about how to interpret his method, as explained above. The questions that arise from this ruling are:

- (a) What does R. Ashi mean by "we do not understand the reason of the others?"
- (b) Having ruled that the halakha follows Tanna Kama, how is his opinion to be interpreted?
- (c) How should we rule in cases of multiple appraisals and when the distribution of the appraisals is not symmetric around the average?

We discuss these questions in order.

### **Clarification of R. Ashi's Argument**

We found four explanations in the Rishonim to the expression "we do not understand the reason of the others."

- (1) Rabbenu Gershom raises two possible doubts about the method of the others:
  - a. "Because one may ask the same question that the Talmud asked: On the contrary (*adraba*)." In other words, the argument "adopt the first two which do not exceed 100," which is at the basis of the preference of the lower cluster, is not convincing even in a symmetric case, especially when it works against the principle whereby the burden of proof is on he who extracts from the other.<sup>74</sup> It is possible that Rabbenu Gershom aimed at an even stronger objection, namely, how is it possible to say "that do not exceed 100" when we assume that the assessor who quoted 100 intended originally to say more, from which it follows that that there is a clear majority to the opinion that the asset is worth more than 100.<sup>75</sup>

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<sup>74</sup> See text adjacent to n. 39 above.

<sup>75</sup> See text adjacent to n. 38 above. It appears that Rashbam's understanding of the "*adraba*" question and of the "adopt the first two" justification in the Talmud settles this question.

- b. "Actually, he could say more, but the exact sum is not specified." This sentence can be interpreted in several ways. His intention may be to say that the very assumption that assessors revise their professional appraisals under the influence of their colleagues is a speculation about the nature of cognitive process taking place in the head of the assessors, which has not been substantiated. Alternatively, he may have intended to say that even if the assumption was acceptable, the others did not explain from where they derived the rate of revision and what the assessors intended to say initially.<sup>76</sup>
- (2) Rashbam interpreted as follows: "their reasoning is unreasonable," and Ritba wrote in the same vein: "their reasoning is insufficient." From these statements it is apparent that the meaning of the others is known and even understood by R. Ashi, but that R. Ashi found in the method of the others flaws that were not clarified later in the Talmud nor in the interpretations of Rashbam and of Ritba.
- (3) In the interpretation of Rabad II (author of *Haeshkol*), the method of the others is based on attributing an error to all the assessors, and in his opinion, it is not reasonable to say that they all made mistakes.<sup>77</sup> Rabad adds that this is also the reason for rejecting R. Eliezer's method.<sup>78</sup>

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<sup>76</sup> Note, however, that according to Proposal No. 3, the value that the assessor intended to quote initially is of no consequence; what matters is the upper limit of the appraisal that can be attributed to him.

<sup>77</sup> Rashbam, s.v. *osim shuma*, explains that it was on this very point that Tanna Kama disagreed with R. Eliezer and the others. According to Tanna Kama, there is no necessity to say that all the assessors erred equally. On the contrary, in his opinion one is to assume that one of the assessors is closer to the "truth" than his colleagues. The method of the median adopted by Rashbam is used not only for reaching a decision (based on the majority principle) but also for identifying the assessor who is closest to the "truth." According to R. Eliezer and the others, as noted above, assuming that the skills of the assessors are identical, we must attribute an identical deviation to all of them. But from what Rashbam says, it is not clear whether this is indeed the reason for R. Ashi's rejection of the opinion of others (and of R. Eliezer) as the basis for halakha.

<sup>78</sup> R. Avraham Abad, *Shitat Hakadmonim: Baba Batra Tractate*, New York, 1981, p. 235.

- (4) R. Ishmael Ben Hachmon explains that the method of the others was rejected because it attributes strategic considerations and revision of appraisals under the influence of other assessors, "which is not a God-fearing way."<sup>79</sup>

The following additional doubts attributed to R. Ashi could be added to those raised by the Rishonim:

- (1) It is possible to attribute R. Ashi's claim that "we do not understand the reason of the others" to the basis for the calculation of  $93\frac{1}{3}$ . The Talmud assumes that the assessors of the lower cluster err by  $13\frac{1}{3}$ , and the Rishonim explain that this value is calculated as follows:  $\frac{120-80}{3} = 13\frac{1}{3}$ . The Rishonim do not explain their position, but it is reasonable to say that the range between 80 and 120 contains the correct appraisal, and it is the range of the controversy. It is not clear why the division of the range by the number of assessors yields the error that is to be attributed to each of them. It is reasonable to argue that this is what made R. Ashi wonder, although we showed above that it is possible to propose three different methods that would yield a final appraisal of  $93\frac{1}{3}$ . Even if we assume that R. Ashi was aware of all these possibilities, as long as it is not possible to reach a decision which of them is truly at the foundation of the method of the others, the conclusion must be that "we do not understand the reason of the others." Indeed, with different numbers each method produces different results.
- (2) The method of the others is based on a hidden assumption, noted by Ramban, who explains why there is no fear that the assessor who quoted 80 intended initially to say less and revised his appraisal upward under the influence of the other appraisals. According to Ramban, "An assessor may underestimate, but he is ashamed to overestimate." In other words, the hidden assumption is that assessors are concerned about exaggerating upward, but not downward. It is possible that R. Ashi wonders what is the basis of this assumption. Following Ramban's statement, we note that assuming bi-directional revision of appraisals (both upward and downward), the only assessor about whom it makes sense to assume that he quoted a value representing his initial opinion is the assessor who

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<sup>79</sup> R. Ishmael Ben Hachmon, *Perush Kadmon: Baba Batra* (Jerusalem 1971), p. 176.



quoted 100, because if he wanted to take into consideration the opinions of his colleagues, then on his right was the assessor who quoted 120 and on his left the one who quoted 80, so it is reasonable to assume that their influences offset each other.

### Halakha Ruling Following Tanna Kama: The Maimonides-Rashbam Controversy

The Halakha ruling on our topic, which is agreed upon by all the decisors, is to adopt a "middle course," according to the opinion of Tanna Kama in the Talmudic interpretation. The expression "middle course" (*milta metziata*) hints to the fact that the appraisal of the court should be based on some measure of a central tendency of the assessors' appraisals. Measures of central tendency that were probably known in the ancient world are the mode, the midrange, and the median. The mean is not mentioned at all by the decisors with reference to our topic.<sup>80</sup> It is not clear whether the sages were familiar with the mean,<sup>81</sup> although it is likely that Pythagoras and his students, in the 6<sup>th</sup> century BCE, were familiar with the arithmetic, geometric, and harmonic mean, which are

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<sup>80</sup> Let us assume that we have a sample  $\mathbf{x} = \{x_i\}_{i=1}^n$ . The arithmetic mean is  $\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i$ . To calculate the median, we rearrange the sample so that  $x_i \leq x_{i+1}$  and define  $k = \text{int}\left(\frac{n+1}{2}\right)$  where  $\text{int}(n+1)$  is the closest integer to  $\frac{n+1}{2}$ . Then the median is  $\text{med}(\mathbf{x}) = \frac{1}{2}(x_k + x_{n+1-k})$ . The midrange is  $\text{mid}(\mathbf{x}) = \frac{1}{2}(x_{\max} - x_{\min})$ .

<sup>81</sup> Following Hershkovitz, Rosenbaum claims that from the Talmudic topic in Pesahim 89b it follows that the sages knew the weighted average and used it. See R. Rosenbaum: *The Wisdom of Fraction (tishboret) Theory: Jewish Reflections of Mathematics*, The R. Kook Institute, Jerusalem 2005, and D. Hershkovitz, *Mathematics and Halakha*, Technion, Haifa (200). In our opinion, Hershkovitz's interpretation is neat but it is not necessary in the case of that topic. It follows from Chebyshev's inequality that the distance between the median and the mean is no greater than one standard deviation, that is  $|\bar{x} - \text{med}(\mathbf{x})| \leq \sigma$ .

known as the "Pythagorean means."<sup>82</sup> Nicomachus (2<sup>nd</sup> century BCE) names ten types of average.<sup>83</sup>

According to Maimonides, the "middle course" is the midrange. According to Rashbam, it refers to the median appraisal. The Ahronim disagreed whether to rule according to Rashbam or according to Maimonides. The *Shulchan Aruch* ruled according to Maimonides, whereas *Sema* (*Sefer Meirat Eynaim*) and subsequent Ahronim ruled according to Rashbam.<sup>84</sup>

We pointed out above the difficulties in Maimonides's method. First, the result is affected greatly by exceptional appraisals and it is open to manipulation. Second, the final appraisal is liable to be a value that was not quoted by any of the assessors, which is a problematic result whether we treat the appraisal as testimony or as a type of judicial process.<sup>85</sup> Another difficulty with Maimonides's method is that based on the rule derived from the topic, it transpires that the final appraisal is the result of a valid rule for reaching a decision. At the beginning of the baraita, when two assessors quoted an identical value, it is said that "the one in the minority is overruled." Thus, in the Talmudic discussion, the methods of both R. Eliezer and of the others are presented as being based on the decision of a majority of assessors. Tanna Kama's method, according to Rashbam, is also based on the majority principle. Contrary to all these,

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<sup>82</sup> Pythagoras left no written theory, and most of his teaching is known from relatively late traditions. One of the important sources of Pythagoras's theories is the book by Nicomachus of Gerasa (today Jerash, in Jordan), n. 83 below.

<sup>83</sup> Nicomachus of Gerasa, *Introduction to Arithmetic*. Martin Luther D'Ooge, trans, Annapolis MD. St. John's College Press, 28II. See also, Heath, T. L., *A History of Greek Mathematics*. Oxford Clarendon Press, 1921, I, 87.

<sup>84</sup> *Sema* (Hoshen Mishpat 103, 100:5) rules according to Rashbam and points for clarification to his *drisha* about *Tur* (ibid). He claims to prove there that Rif, Rosh, and *Tur* ruled according to Rashbam. *Beer Hagola* (ibid) disagreed with *Sema* and noted that Rabad did not disagree with Maimonides, mentioning also that he did not get to see that passage in the *drisha*. (Nor do we have the *drisha* on chapter 103 of the *Tur*). It is possible, however, to infer from the language of the *Tur* "we follow **him** who says 100," that the reference is to the median assessor, according to Rashbam. Mahari Ben Lev wrote in a responsum (Part 2, chapter 29) that all the decisors disagreed with Maimonides. The other Ahronim who ruled according to *Sema* are *Urim Vetumim*, *Netivot Hamishpat*, and *Aruch Hashulchan*.

<sup>85</sup> Although in our opinion the appraisal should not be considered as one of these; see the section on the legal status of the assessors above.

Maimonides's method is not based on a majority,<sup>86</sup> and furthermore, R. Jonathan Eybeschutz points out that in certain cases Maimonides's method is contrary to the majority.<sup>87</sup> It is possible to settle these issues assuming that in Maimonides's opinion, the appraisal is not a testimony and not a judicial process but some type of statistical sample of expert opinions according to which the value of the asset is determined. It appears that this is how Maimonides understood the appraisal, and he applied the preferred measure of central tendency to the sample, which in his opinion is the midrange.

Rashbam's opinion, by contrast, is that Tanna Kama's method involves selecting the median opinion. We noted above that the median method is strategy-proof.<sup>88</sup>

We find a controversy similar to that between Maimonides and Rashbam concerning the proper method of aggregation in yet another topic. It is said in a baraita: "He engages a laborer and says to him [I will pay you] as one or two townspeople [are paid] and pays him the lowest wage. This

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<sup>86</sup> R. Yosef Hutner, *Hevel Yosef*, Olam Hamishpat about *Shulchan Aruch* Chof Hamoed 103, p. 58, suggests viewing Maimonides's method as a rule for reaching a decision based on the majority. The midpoint of the cluster that contains the two extreme assessors represents two assessors against the middle one. In his opinion, Rashbam's method is problematic because it is based, in his understanding, on the principle of "halakha according to the arbitrator (*hamachria*)" (see, for example, bShabbat 40A). But the principle of "halakha according to the arbitrator" is valid when the decisive opinion uses a distinction that is relevant according to the dissenting opinions as well, but not when this opinion is a third, new opinion. See bBaba Kama 116A "There is no decisive third ruling," Rashi and Tosfot, *ibid*. In our opinion, Maimonides's method is not based at all on the majority principle. By contrast, Rashbam's method is based on the majority principle and not on "halakha according to the arbitrator," as it appears from his writings. The principle of the "arbitrator," as it appears in the Talmud, is relevant in cases of qualitative controversies, and it is possible that it is also based on majority ruling in some respects.

<sup>87</sup> R. Jonathan Eybeschutz, *Urim Vetumim* (Jerusalem, 1975) Hoshen Mishpat 103a. In the example of Maimonides himself (90, 100, 130), according to Maimonides, the final appraisal is 110 despite the fact that most of the assessors agree that the asset is not worth more than 100. According to him, Maimonides's method is refuted in the Gemara in bKetubot 100A, which explains the validity of the court's appraisal by the argument that the court represents many, whereas Maimonides is alone in skewing the appraisal to a value higher than 100.

<sup>88</sup> When the skills of the assessors are not identical, the method of the median is preferable over the mean as well, as has been proven recently: Petra Gerlach-Kristen "Monetary Policy Committees and Interest rate setting" EER 50 (2006) p. 487.

is R. Yehoshua's view. But the sages say: "We appraise between them."<sup>89</sup> The wage defined in the labor agreement reflects the wage paid to one or two residents of the town. Apparently, R. Yehoshua's method is based on the rule that the burden of proof is on he who extracts from the other. But according to the sages, this rule should not be applied, and as it is said in the Jerusalem Talmud: "Said R. Hoshai: this means that custom overrules halakha. Said R. Imi: usually, he who extracts from the other must prove his claim, but this case is exceptional."<sup>90</sup> This means that the wages of the laborer must be set according to the custom of the state, but in the state there is more than one custom and several rates for wages. The sages ruled that in this case they appraise among themselves, that is, they aggregate the various rates. Maimonides ruled with the sages and interpreted: "He engages a laborer and says to him [I will pay you] as one or two townspeople [are paid] we take the lowest wage and the highest wage [in town] and appraise between them."<sup>91</sup> It is easy to see that Maimonides, true to his method,<sup>92</sup> adopted an estimate based on the midrange.<sup>93</sup>

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<sup>89</sup> bBaba Metzia 87a.

<sup>90</sup> jBaba Metzia 7A; 11b.

<sup>91</sup> Maimonides, *Laws of Hire (Hilkhot Skhirut)* 9, Halakha 2. In print, the term is "*meshamnin*" (they appraise), but Maimonides, *Perush Hamishnayot on Behorot 2:6* distinguishes between "*meshmin*" and "*meshamnin*." See also Mishnah, Baba Batra 7:4 and the comment by R. Yosef Kapach. It is the same in the edition of R. Nahum Eliezer Rabinovich, *Yad Pshuta*, Mishpatim 1, p. 194, as well as in the version that appears in the Makbili edition.

<sup>92</sup> Maimonides also followed Aristotle's *Ethics* and regarded the middle course a preferable golden path in many areas. See *Halakhot Deot* 1-2. Maimonides's introduction to his commentary on the Avot tractate, *Shmona Prakim*, chapter 4.

<sup>93</sup> Maimonides followed here the commentary of Rabbenu Hananel, Domb edition (1989), p. 156. This is also the interpretation of R. Jonathan of Lunel, presented in *Shita Mekuvetzet*, with emphasis on the fact that the matter does not concern the midrange between the most expensive and the cheapest, but the midrange between the "high average" and the "low average." *Tur* notes in chapter 331 that Rama ruled the same way. *Beit Yosef*, *ibid*, understood from *Magid Mishneh* on the ruling of Maimonides that Ramban and Rashba also adopted the interpretation "*meshmin*" as being Maimonides's method. But from the text of *Magid Mishneh* that is before us, it appears that the reference to the writings of Ramban and Rashba was intended to point out their innovation that we ought to apply the rule of "*meshmin*" to additional Talmudic topics, where this rule has not been mentioned explicitly. This is what transpires also from Ramban's explicit words, who did not interpret the meaning of "*meshmin*" at all.

Rashi interpreted "*meshamnin beynehem*" (they appraise between themselves) as follows: "Not by less and not by more but as is the custom of the state, average."<sup>94</sup> *Tur* quotes Rashi differently: "That they give them as they give to the average ones."<sup>95</sup> The emphasis in Rashi's writing is on the rate paid "to the average ones," whereas according to Maimonides the midrange can be a value that does not reflect the rate paid to any worker. It appears, therefore, that Rashi hints at setting the laborer's wage according to the median wage and not according to the midrange.<sup>96</sup> Ritba is more explicit: "In other words, pays average wage, and if there are two averages or three, the wage is at their center."<sup>97</sup> Clearly, Ritba did not follow Maimonides because if the meaning of average is the midrange, it means that it is a single value, and two or three averages are not possible.

In this case, as in the case of the assessors, the Rishonim disagreed about how to reflect by means of a single value many distributed values.<sup>98</sup> One faction of Rishonim prefers the median, whereas another faction prefers the midrange. This controversy has halakhic significance primarily when the values are not distributed symmetrically around the mean. The characteristics of these measures are likely to serve as indications of the decisors' priorities and to suggest the relevant criteria for generalizing the ruling.

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<sup>94</sup> Rashi, *ibid*, s.v. *meshamnin beynehem*.

<sup>95</sup> *Hoshen Mishpat* 331. A similar version appears in Rashi printed with Rif's commentary. "Not like the less and not like the more but as the custom of the average ones who are between them."

<sup>96</sup> This is the understanding of *Lehem Mishne*, in its commentary on Maimonides's *Halakhot Schirut*, *ibid*, and of the *Prisha* commentary on *Tur*, *Hoshen Mishpat*, Section 331.

<sup>97</sup> *Innovations of Ritba*, Baba Metzia 87a, s.v. *vehachamim omrim meshamnin beynehem*.

<sup>98</sup> The understanding that the controversy of the Rishonim regarding the case of the assessors is related to the controversy of the Rishonim in the case of the laborers' wages transpires clearly from the Vilna Gaon's (Gra) gloss to Baba Metzia 87a, as well as from the commentary of the *Tur*, *Hoshen Mishpat* 331. It appears that already Ramban (Baba Metzia 87) and Rashba (who is mentioned in *Magid Mishneh*, n. 93 above), hinted that the principle of "*meshamnin beynehem*" applies to other cases, including "as estimated by three persons" (see *Avodah Zarah* 72a, that is, the seller obligates himself to sell at a price appraised by three assessors, but the three assessors quoted different values). See also n. 4 above, about Ritba interpreting the case of the assessors in this manner.

Without relation to the opinions of the decisors mentioned above, the choice between the median and the midrange, as a universal rule for reaching a verdict, must take into account the chance vis-à-vis the risk inherent in the application of the selected rule. The question is not about the chance of finding the "truth" or the type and magnitude of the error in a specific case, but about the chance and risk inherent in the very choice of a certain rule and its application in all cases.

It is customary to evaluate the degree of fitness of measures of central tendency of the original observations, and the risk inherent in their use by the loss function. The loss function is intended to evaluate the best suited measure of central tendency of data in a way that minimizes the loss of data. The four useful loss functions are:<sup>99</sup>

(1) **Number of errors:** Measures the number of observations that are not identical with the selected measures of central tendency (henceforth, the measure). The distance of the observations from the measure is of no consequence. The greater the number of observations that are identical with the measures, the smaller the loss of data as a result of using the measure. The measure that minimizes this function is the mode.

(2) **Maximal deviation:** Checks the distance between the measure and the most distant observation. The measure that minimizes maximal deviation minimizes in fact the greatest contingent "error" in estimating the central tendency of the data. The measure that minimizes this function is the midrange, that is,  $mid(\mathbf{x}) = \min_x \max |x_i - x|$ .

(3) **The sum of absolute deviations:** Sums the absolute deviations from the measure of all observations. The measure that minimizes the sum of the absolute deviations minimizes also the mean deviation. The measure that minimizes this function is the median, that is,

$$med(\mathbf{x}) = \min_x \sum_{i=1}^n |x_i - x|.$$

(4) **The sum of the squared deviations:** Sums the squares of the differences between observations and the measure. The measure that minimizes this function minimizes also

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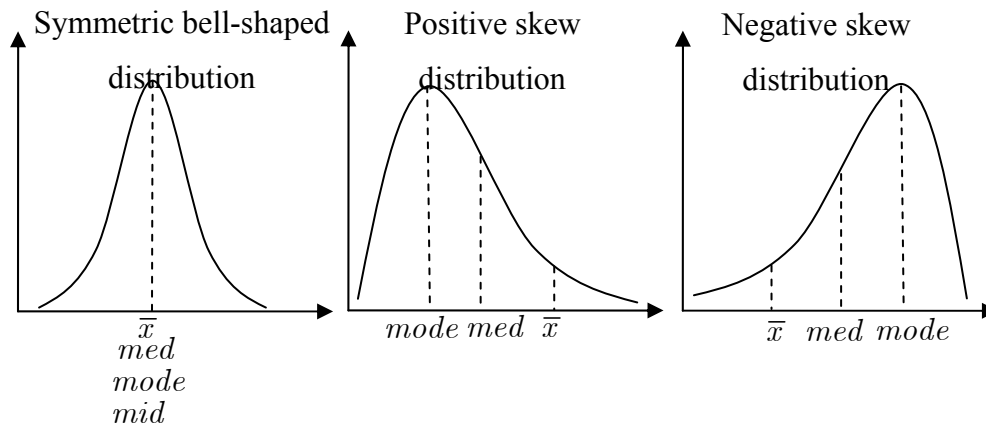
<sup>99</sup> See, for example, S. Zamir and R. Beit-Marom, *Introduction to Statistics for the Social Sciences*, The Open University of Israel 1993, Unit 2, p. 88ff.

the standard deviation. The measure that minimizes this function is the mean, that is,

$$\bar{x} = \min_x \sum_{i=1}^n (x_i - x)^2.$$

An additional consideration in the choice of measure of central tendency is the degree of its sensitivity to the distribution characteristics of the data. For example, in a symmetric, bell-shaped distribution, the median, mode, midrange, and mean are identical and converge on the center of the distribution. In a positive skew distribution, the mode is the smallest measure from among the ones we discuss here, the mean is the largest, and the median is located between the mode and the mean. In a negative skew distribution the largest measure is the mode, followed by the median and the mean (Figure 4). The only measure that is not affected by the shape of the distribution is the midrange. The median is only minimally sensitive to the shape of the distribution, whereas the two other measures, the mean and the mode, are highly sensitive to the shape of the distribution.<sup>100</sup>

**Figure 4**



<sup>100</sup> The degree of sensitivity of the measure of central tendency to the distribution of the data between the clusters has no halakhic significance because in Tanna Kama's opinion, contrary to those of R. Eliezer Bar Zadok and the others, the determination of the appraisal is not based on cluster analysis.

Figure 4 shows that when the distribution is skewed, as in the case of the price of assets, the median is a better estimate of the measure of central tendency of the population than are the mean and the mode. By contrast, the midrange is a highly effective measure for estimating the central tendency with small samples if the distribution is sufficiently flat (platykurtic). For example, the midrange is highly effective in small samples of uniform distribution, especially when the extremes of the distribution are not known. In this case, the midrange is an unbiased minimal mean variance estimator of the central tendency. But the midrange is not an effective estimator of mesokurtic distributions, such as the normal distribution.

We found a clear example of a halakhic argument about the effectiveness of the use of the midrange of a sample as an estimator of central tendency in the Mishnah, Keilim 17:5-6: "The pomegranate of which they have spoken refers to one that is neither small nor big but of average size... The size of an egg which they have spoken is neither big nor small but of average size. R. Yehuda ruled: The largest and the smallest must be brought and put in water and the displaced water is then divided. Said R. Yossi: But who can tell me which is the largest and which is the smallest? All depends on the observer's estimate." According to R. Yehuda, it is necessary to determine the size of the "average" as the midrange of the sample. R. Yossi's argument is that the "largest" and the "smallest" mentioned by R. Yehuda are from the sample and do not necessarily represent the extreme values in reality. In nature, the distribution of the eggs' volume is normal, that is, mesokurtic, and therefore the midrange of the sample is indeed not an effective estimator. The halakha follows R. Yossi,<sup>101</sup> and Rashbam's opinion in our case is understandable, especially if we assume that the distribution of the prices of the assets is positive-skewed. By contrast, Maimonides, although he ruled with R. Yossi, in the topics we examined, prefers the principle of the midrange. It is possible to explain Maimonides's opinion with regard to these topics by the fact that the samples are small and the distribution is uniform. In the case of the assessors, the distribution of interest to us is not that of the prices of the assets but of the appraisals around a certain value that we want to estimate. In the absence of concrete data about each assessor, the working assumption is that their skills are quite similar if not identical. (Even if in reality the skills of the assessors show normal distribution, 98% of assessors are within one standard

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<sup>101</sup> See Maimonides, *Mishneh Torah*, Halakhot Keilim, Chapter 6, Halakha 2.



deviation at most from the mean.) In the case of the laborer's wages, the nature of the distribution is again of no significance because the objective of the estimate is to interpret the labor agreement and not to estimate the average wage in town.

Moreover, in choosing the suitable measure, it is appropriate to take into account its degree of sensitivity to extreme values (which also reflects the degree to which it is strategy-proof). This is an important consideration if we assume that a relatively "large" deviation reflects an error, or even deliberate manipulation. Measures that are less sensitive to extreme values and to strategic manipulations are the mode and the median; the more sensitive ones are the midrange and the mean.

As noted, the mean is not mentioned at all by the decisors in our topic.<sup>102</sup> It is possible that the explanation has to do with the fact that the mean suffers from all the disadvantages of the alternative measures (sensitivity to extreme values, to manipulation, and to the skew of the distribution), whereas the loss function that it minimizes, the sum of the squared deviations, has no halakhic meaning. The two measures that are left (in the absence of the mode) are the median and midrange.<sup>103</sup> Choosing the median is likely to bring the court closer to the "truth" than choosing the midrange, and it minimizes the mean error of the court in the overall number of cases it hears. Choosing the midrange is more effective than the median only if the assumptions listed above are true (small samples, relatively "flat" distribution with unknown extremes); it does not minimize the mean error, however, but rather the court's maximum error. It follows that choosing the median reflects "daring" in that it takes a risk for the sake of a better chance, whereas choosing the midrange reflects a solid approach and aversion to greater risk.

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<sup>102</sup> Except for the initial assumption (*hava amina*) in Rashbam, see Gilad Lipschits's suggestion in n. 27 above, as well as n. 81 and the adjacent text.

<sup>103</sup> About the possibility of reaching a decision based on the mode see the text adjacent to n. 105 below.

## Generalization of the Ruling in the Case of the Assessors

Our topic deals with reaching a decision about three assessors whose opinions are distributed symmetrically around the median opinion. In this section, we discuss the generalization of the decision for cases in which the distribution of opinions is not symmetric.

### Generalization of Tanna Kama's Method

Tanna Kama's method can be generalized simply, both according to Rashbam's interpretation (median) and Maimonides's interpretation (midrange). Rashbam and Maimonides themselves illustrated their methods for cases in which the distribution of the evaluations is not symmetric.<sup>104</sup> According to the interpretations of Maimonides and Rashbam, Tanna Kama's method can also be generalized to more than three assessors because the median or midrange calculation is not limited to any number of observations.<sup>105</sup>

### Generalization of R. Eliezer Bar Zadok's Method

The generalization of R. Eliezer's method to a general number  $n$  of assessors requires cluster analysis, which is the topic of a separate article. To generalize R. Eliezer's method for an asymmetric case, let us examine the extreme example of the following three appraisals (80, 100, 101). The appraisal of 100 can be joined either to that of 80 or to that of 101. The center of the (100, 101) cluster is  $100\frac{1}{2}$ , which means that the deviation attributed to each assessor in this cluster does not exceed  $\frac{1}{2}$ . By contrast, the center of the (80, 100) cluster is 90, and the deviation attributed to each assessor is 20 times greater than in the (100, 101) cluster. Which is the preferred combination in this case?

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<sup>104</sup> Rashbam, n. 29 above and Maimonides, n. 23 above.

<sup>105</sup> When the number of assessors is even, the median quote is the average of the two mean appraisals. A question not raised in this article is how would Tanna Kama rule in the case of more than three assessors, a minority of which quotes an identical value. The question is whether it is possible to apply individually, to each of the assessors who disagree with the minority group, the principle of "the one in the minority is overruled," or is an absolute majority required, so that it is necessary to implement the median and midrange decision rule in this case as well.

The two main possibilities are:

- (1) R. Eliezer always prefers to build clusters of opinions in such a way that the majority is formed by the lower value (following the principle of "adopt the first two").
- (2) The Talmud needs an argument to justify R. Eliezer's preference to join the deciding quote to the lower cluster only in a symmetric system. In an asymmetric system the preferred combination joins the median appraisal to the more concentrated cluster even if this happens to be the higher one<sup>106</sup> because this combination minimizes the radius of the cluster, in other words, the average deviation of each assessor in the cluster.

Halakhic literature did not address the generalization of R. Eliezer's method because it was rejected as halakha, and therefore there are no sources discussing this issue. We will investigate separately various algorithms to perform cluster analysis on this problem according to R. Eliezer's method. In the present article, we limit ourselves to examining the issue in light of the explanation of the Rishonim: Rashbam and Ramban.<sup>107</sup>

According to Rashbam, the question is whether the principle of "take hold of too much you hold nothing" (*tafasta merube lo tafasta*) can prevail over the weight we must assign to the fact that the upper cluster is more compact. In our opinion, Rashbam used two expressions that indicate that the use of the rule of "take hold of too much" is a "default" of sorts for cases in which we lack a reason for tilting the scale in either direction.<sup>108</sup> Thus, the expression "as both sides are equally reasonable" indicates that we are dealing with a symmetric and even case. Further he

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<sup>106</sup> An additional theoretical possibility, which is clearly inconsistent with R. Eliezer's method, is to begin assembling the clusters based on the principle of minimizing deviations. In our example, we obtain the cluster (100, 101) vis-à-vis 80. Next, the decision is a weighted average of the centers  $100\frac{1}{2}$  and 80. The question is what relative weight to assign to each value and how this relative weight is determined.

<sup>107</sup> Ritba's method requires a separate inquiry. See text adjacent to n. 44 above.

<sup>108</sup> Regarding the rule of "take hold of too much" see n. 37 above. See Rashi, *Hagiga* 17a s.v. *tafasta merube lo tafasta*: "Wherever you find two alternatives, one takes hold of a lot and one takes hold of little, you should take hold of a little because even if you had to take hold of the large amount but you took hold of the small amount, your hold is a hold, as the small amount is in the large amount, but if you take hold of the large amount and you had to take hold of the small amount, you will find that you took hold not according to law."

says: "They all admit that it is close to an even hundred," that is, the two possible clusters lead to the conclusion that the "true" value is close to 100, from either the right or the left, equally. In asymmetric cases the rule of the default turns out to be inferior to other possible rules for reaching a decision, and therefore the assessor who appraised 100 should be added to the more compact cluster, even if it is the upper cluster.

According to Ramban, the question is whether the psychological assumption about the boundaries of the "order of magnitude" can prevail over the weight we must assign to the compactness of the upper cluster. In asymmetric cases, especially in the extreme case of (80, 100, 101), it is possible to say on one hand that, based on the psychological assumption of Ramban, the lower cluster must be preferred because the two appraisals it contains do not cross the 100 boundary. On the other hand, our objective is to determine the true value of the asset with the greatest possible precision. Thus, it is possible that the assessor who appraised at 100 also wanted to quote a little more than 100 (as did the assessor who quoted 101), but did not dare. Preference of the upper, compact cluster ( $100\frac{1}{2}$ ) expresses a value that is closer to the truth. The alternative possibility (90) attributes a greater error to the assessor who appraised at 100.<sup>109</sup>

### **Generalization of the Method of the Others**

Because the method of the others is based on the method of R. Eliezer, its generalization for  $n$  assessors requires a discussion of the algorithms of cluster analysis, which is outside the scope of the present article. We discuss the generalization of the method of others for three assessors based on the three proposals presented above.

**Proposal No. 1.** Given that this proposal is based on weighting the extreme opinions, it is possible to implement it with any collection of three different appraisals. The guiding logic, as in Maimonides's method, is that the median appraisal does not affect the final one. But to which end of the range the weights are assigned ( $\frac{1}{3}, \frac{2}{3}$ ), depends on additional considerations.

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<sup>109</sup> Minimizing errors as a guiding principle is emphasized in the innovations of R. Shlomo Ganzfried, *Pnei Shlomo*, text adjacent to n. 45 above.

**Proposal No. 2.** According to this proposal, the final appraisal is based on a selected cluster, one end of which receives a double weight under the influence of the third appraisal that pulls the center of the cluster toward itself. The question that arises here is which is the selected cluster in an asymmetric system, a point over which there seems to be no disagreement between R. Eliezer and the others.<sup>110</sup> If the selected cluster is the upper one (unlike the example discussed in our topic), the bottom end receives the double weight.

**Proposal No. 3.** The generalization of this proposal is also dependent on the preliminary choice of the cluster on which the final appraisal is based. After choosing the cluster, the calculation is analogous to the one described above.

## Summary and Conclusions

The baraita presented in the Baba Batra tractate (107a) deals with the aggregation of three appraisals. The baraita presents the opinions of three Tannaim who addressed the specific case described in the baraita, involving symmetrically distributed appraisals. Although in the course of the Talmudic debate R. Huna twice recommends ruling according to the third opinion (that of the "others"), R. Ashi rejects the recommendation with the argument: "We do not understand the reason of the others; shall we rule halakha according to them?" After the opinions of the others are rejected, the Rishonim agree that the halakha follows Tanna Kama, but they disagree about how to interpret his method.

The method of the others, as well as that of R. Eliezer, is based on halakhic considerations that violate the mathematical symmetry and tip the aggregation toward the lower appraisal. In the first part of the article, we sought to discover the meaning of the others' opinion, which appears to be an interesting one and poses a challenge to understand it. We recommended three ways of understanding this method, only one of which appears to be consistent with the course of the Talmudic discussion. The three proposals are based on three different principles: the midrange,

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<sup>110</sup> See the generalization of R. Eliezer Bar Zadok's method above.

cluster analysis, and the measure of average deviation. We showed that these ways of understanding disagree about the meaning of the others' reference to "we divide between them." Does "between them" refer to the two assessors Tanna Kama referred to in the opinion of Maimonides, to the two assessors mentioned in R. Eliezer's method who is mentioned together with the others, or to the three assessors mentioned in the "Three went to appraise" baraita? When R. Ashi says that "We do not understand the reason of the others," we can assume that he has an interpretation of the reason of the others but he nevertheless rejects it as a basis for halakha. The ways of understanding this method that we proposed indicate that the opinion of the others is subject to various interpretations, all of which produce the same result using the numbers of the baraita, but different results in other cases. It may be that this is the reason why R. Ashi concludes that "We do not understand the reason of the others; shall we rule halakha according to them?"

In the second part of the article we showed that the controversy of the decisors about the "middle course" (*milta metziata*), whether it refers to the midrange or the median, is also present in another context surrounding the wages of laborers, which also involves the aggregation of various values. Our conclusion is that Maimonides, consistent in his method, prefers the midrange, whereas the schools of Rashi and Rashbam prefer the median. We showed that each method has its advantages and disadvantages, based on which loss function we wish to minimize.

In the last part of the article we raised the question of the generalization of our topic to values that are different from those brought in the baraita, although we noted above that in this article we do not discuss the generalization of the ruling to a general number  $n$  of assessors because generalizing the methods of R. Eliezer and of the others requires a discussion of algorithms of cluster analysis, a topic intended for a separate article. Nevertheless, we note that the need to generalize the rule used to reach a decision to a general number  $n$  of assessors is rooted in Jewish law itself, which in at least one case, appraisal in matters of consecration (*hekdesh*) of land and humans, demands appraisal by a large number of assessors: "And regarding land, nine and a priest (*cohen*), and the case of a human is similar to them."<sup>111</sup> The need to make decisions

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<sup>111</sup> See n. 5 above. See *Aruch Hashulchan*, Hoshen Mishpat 103, Section 1, which also addresses situations in which there are many assessors.

based on a broad pool of data arises in many other areas. To discuss the generalization of each method, it is necessary to answer two questions of principle for the halakha ruling, which touch upon the boundaries of the Tannaitic controversy surrounding the baraita:

- (1) Is the rule "The one in the minority is overruled" valid only in a case of majority vs. minority or also for the mode, which represents a relative rather than an absolute majority? For example, if we have five appraisals, two of which are identical and the rest different from each other, and an absolute majority is required, the Tannaitic controversy is still in effect because none of the opinions enjoys an absolute majority. But if the mode, as a relative majority prevails over each individual opinion, by all accounts the decision follows the two identical appraisals based on "the one in the minority is overruled."
- (2) Is it possible to claim that the halakha ruling in the Tannaitic controversy refers specifically to the case described in the baraita, that is, a case in which we are dealing with three assessors? In the case of many assessors, it is possible that the Tannaim do not disagree and even Tanna Kama adopts one of the other opinions. It is possible, if for example hundreds of appraisals must be aggregated, that Tanna Kama prefers a method of cluster analysis because the method of the median is not necessarily the most effective one. Thus, the halakha ruling that follows Tanna Kama is valid only in the case of three appraisals, and in the case of a larger number of appraisals there is room to allow halakha decisors to debate the issue.

We did not discuss these issues in the present article because such a discussion must be preceded by the generalization of the opinions of the Tannaim to a large number of appraisals, which requires a separate discussion, as noted above.

An additional question that touches upon the boundaries of the Tannaitic controversy is the question of the analogy. In the baraita, the Tannaim addressed the issue of aggregating the assessor's appraisals. The question is, to which additional areas is it possible to apply the controversy by analogy? For example, in no democratic country is the rule of the "middle course" (*milta metziata*) applied in political controversies. Arrow (1963) raised the question of

aggregation of voter preferences and showed that it has no satisfactory answer.<sup>112</sup> The question of analogy with the Tannaitic controversy surrounding our topic was raised with respect to controversy between rabbinical judges who do not serve as assessors but rule based on the arguments of the litigating parties. How are we to rule in cases in which no single opinion enjoys a majority? It is reasonable to argue that even Maimonides will admit that one ought not decide here by means of the midrange, if only because, owing to the disadvantage of this method, it is possible to arrive to a decision that none of the judges named. It stands to reason that a halakhic ruling by a rabbinical court should reflect the opinion of the majority of the judges, and it is not possible to issue a ruling that none of the judges supported. *Mahari Ben Lev* believes that the Maimonides and Rashbam controversy has to do only with the matter of the baraita dealing with appraisal. When the issue involves three judges who disagree about a verdict, by all accounts it is necessary to rule according to the majority principle proposed by Rashbam following the opinion of Tanna Kama.<sup>113</sup> We found that other Ahronim also made this distinction.<sup>114</sup>

The implications of this topic for other cases that involve the aggregation of various values should be handled with caution. We have shown at the outset of the article that the understanding of the aggregation in our case as an "estimate" is clear in light of the legal halakhic status of the assessors as expert witnesses, as opposed to judges and witnesses. In other cases it may not be possible to rely on estimates, and the present Tannaitic controversy may not apply.

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<sup>112</sup> Arrow, K. J., *Social Choice and Individual Values* (2<sup>nd</sup> Ed.), New Haven: Yale University Press, 1963.

<sup>113</sup> See R. Yosef Even Lev, *Responsa Mahari Ben Lev*, 2:29.

<sup>114</sup> See *Halakhot Dayanim im Halakha Psuka* (Harry Fischel Institute, 1986), p. 414, n. 30. Similar ideas are raised by R. Zacharia Agmati (12<sup>th</sup> century, Morocco), *Sefer Haner*, p. 274, who clarifies the issue in the name of R. Yosef Halevi (Ri Migash), and implies that the appraisals of the assessors are viewed as testimony before a court. ("And it follows that he withdrew both appraisals of  $93\frac{1}{3}$ , and he who said 30, which is actually 120, is one appraisal. And one appraisal is never preferred over two.") See also *Yad Rama*, n. 19 above, which implies that the status of the assessors in court is that of witnesses, and according to our proposition in the section about the status of the assessors, they should be treated as expert witnesses.