Algebraic Techniques in Combinatorial Geometry

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Abstract

In November 2010 the earth has shaken, when Larry Guth and Nets Hawk Katz posted a nearly complete solution to the distinct distances problem of Erdős, open since 1946. The excitement was twofold:

(a) The problem was one of the most famous problems, as well as one of the hardest nuts in the area, resisting solution in spite of many attempts (which only produced partial improvements).

(b) The proof techniques were algebraic in nature, drastically different from anything tried before.

The distinct distances problem is to show that any set of *n* points in the plane determines $\Omega\left(n/\sqrt{\log n}\right)$ distinct distances. (Erdős showed that the grid attains this bound.) Guth and Katz obtained the lower bound $\Omega\left(n/\log n\right)$.

Algebraic techniques of this nature were introduced into combinatorial geometry in 2008, by the same pair Guth and Katz. At that time they gave a complete solution to another (less major) problem, the so-called joints problem, posed by myself and others back in 1992. Since then these techniques have led to several other developments, including an attempt, by Elekes and myself, to reduce the distinct distances problem to an incidence problem between points and lines in 3-space. Guth and Katz used this reduction and gave a complete solution to the reduced problem.

One of the old-new tools that Guth and Katz bring to bear is the Polynomial Ham Sandwich Cut, due to Stone and Tukey (1942). I will discuss this tool, including a ``1-line'' proof thereof, and its potential applications in geometry. One such application, just noted by Matoušek, is an algebraic proof of the classical Szemerédi-Trotter incidence bound for points and lines in the plane.

In the talk I will review all these developments, as time will permit. Only very elementary background in algebra and geometry will be assumed.